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ASYMPTOTIC MATCHING FOR INTERPLANETARY FLYBYS

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ABSTRACT. The matching of planet-centered hyperbolas with heliocentric elliptic arcs is explained, following the treatment by Breakwell and Perko (1965). Application is described to the computation of accurate multi-flyby trajectories, following Granstrom (1971), and to the guidance law for in-flight trajectory corrections.

I. **Introduction.** The first application of matched asymptotic expansions in celestial mechanics was the pioneering paper of Lagerstrom and Kevorkian [4] in 1963, which considered Earth-to-Moon trajectories having small geocentric angular momentum.

A later paper, in 1965, by Breakwell and Perko [1] removed the restrictions to small angular momentum, and hence was applicable also to the matching of heliocentric ellipses with local planetocentric hyperbolas. In the Earth-to-Moon context, particular attention was paid to the "aiming problem," i.e., to the effect of "small" changes in geocentric initial conditions on the subsequent Moon-centered hyperbola, more precisely the osculating hyperbola at closest approach. Here "small" meant relative changes in initial position and velocity of the order of the mass-ratio ν of Moon to Earth. It was found that, to first order in ν, the effect of the Moon's attraction on the approach asymptote to the Moon-centered hyperbola, and on the time of closest approach could be represented by certain "gross biases," independent of initial conditions and obtainable by quadrature, together with a "local" time correction proportional to the logarithm of the hyperbolic eccentricity. In other words, the location of the approach asymptote, as well as the "velocity at infinity" along it, vary essentially linearly with initial conditions over the assumed small range, whereas the time of closest approach includes a highly non-linear, but easily calculable, term.

Corresponding results apply to the matching of planet-centered hyperbolas with "initial conditions" on heliocentric ellipses before and after planetary encounter, and to the matching of the departure hyperbola from one planet to the arrival hyperbola at the next planet.

It is the purpose of this paper to recapitulate this analysis and to present certain features of its application in 1971 by Granstrom [3] to the calculation of accurate interplanetary multi-flyby trajectories, as well as to the guidance laws necessary for in-flight trajectory corrections.

II. **The Outer Expansion.** Suppose that $\vec{r}^{(0)}(t)$, $t_0 \leqq t \leqq t_1$, represents a geocentric Keplerian orbit, either elliptic, parabolic or hyperbolic, which would intersect a *massless* Moon at time t_1, the massless Moon being presumed to follow the true path of the moon. If the actual orbit, including small initial condition changes and perturbation by the Moon and by other more distant bodies (as well as by Earth oblateness) is:

$$\vec{r}(t) = \vec{r}^{(0)}(t) + \vec{\rho}(t), \tag{1}$$

and if the orbit difference $\vec{\rho}(t)$ is separated by orders of $\nu = \mu_{☾}/\mu_{\oplus}$: ($\vec{\rho}^{(2)}$, for example, is of order ν^2, at least for t not close to t_1; it will be found later to be of a different order in a matching region close to t_1)

$$\vec{\rho}(t) = \vec{\rho}^{(1)}(t) + \vec{\rho}^{(2)}(t) + \cdots, \tag{2}$$

then:

$$\begin{aligned} \ddot{\vec{\rho}}^{(1)}(t) &= \Gamma_{\oplus}(t)\vec{\rho}^{(1)}(t) + \vec{f}(\vec{r}^{(0)}(t), t) \\ \ddot{\vec{\rho}}^{(2)}(t) &= \Gamma_{\oplus}(t)\vec{\rho}^{(2)}(t) + (\partial\vec{f}/\partial\vec{r}^{(0)}(t))\vec{\rho}^{(1)}(t) \\ &\quad + \text{TERMS WITH } (\vec{\rho}^{(1)}(t))^2, \text{ etc.,} \end{aligned} \tag{3}$$

in which $\vec{f}(\vec{r}(t), t)$ denotes the perturbing acceleration, and $\Gamma_{\oplus}(t)$ denotes the geocentric gravity-gradient tensor $(\mu_{\oplus}/r^3)(3\hat{r}\,\hat{r}^T - 1)$, $\hat{r}$ denoting the unit-vector $\vec{r}/r$ and $\mu_{\oplus}$ the Earth's gravitational parameter.

The effect of initial condition changes can be included entirely in $\vec{\rho}^{(1)}(t)$ so that:

$$\begin{aligned} \vec{\rho}^{(1)}(t) &= [\Phi^{\oplus}_{rr_0}(t, t_0), \Phi^{\oplus}_{rv_0}(t, t_0)] \begin{pmatrix} \delta\vec{r}_0 \\ \delta\vec{v}_0 \end{pmatrix} \\ &\quad + \int_{t_0}^{t} \Phi^{\oplus}_{rv'}(t, t')\vec{f}(\vec{r}^{(0)}(t'), t')\, dt' \\ \vec{\rho}^{(2)}(t) &= \int_{t_0}^{t} \Phi^{\oplus}_{rv'}(t, t')[(\partial\vec{f}/\partial\vec{r}^{(0)}(t'))\vec{\rho}^{(1)}(t') \\ &\quad + \text{TERMS WITH } (\vec{\rho}^{(1)}(t'))^2]\, dt', \end{aligned} \tag{4}$$

where $\Phi^{\oplus}_{rr_0}(t, t_0)$, $\Phi^{\oplus}_{rv_0}(t, t_0)$ denote the 3×3 portions $\partial\vec{r}^{(0)}(t)/\partial\vec{r}(t_0)$ and $\partial\vec{r}^{(0)}(t)/\partial\vec{v}(t_0)$ of the unperturbed geocentric 6×6 transition matrix $\Phi^{\oplus}(t, t_0)$, for which various analytical formulas are available.

Now, as $t' \to t_1$, the perturbing acceleration $\vec{f}(\vec{r}^{(0)}(t'), t')$ "blows up" in the manner:

$$\vec{f}(\vec{r}^{(0)}(t')t') \sim \mu_{☾}\vec{v}_1/v_1{}^3(t_1 - t')^2, \tag{5}$$

where $\vec{v}_1$ is the relative velocity with which the unperturbed trajectory $\vec{r}^{(0)}(t)$ meets the massless Moon, and v_1 denotes its magnitude. Also

$$\Phi^{\oplus}_{rv}(t, t') \sim (t - t') \times (\text{identity matrix}). \tag{6}$$

The quadrature in $(4)^{(1)}$ is thus *singular* as $t \to t_1$, but the singularity can be separated out:

$$\begin{aligned} &\int_{t_0}^{t} \Phi^{\oplus}_{rv'}(t, t')\vec{f}(\vec{r}^{(0)}(t'), t')\, dt' \\ &\quad = \int_{t_0}^{t} (t - t') \frac{\mu_{\text{☾}}\vec{v}_1}{v_1^3(t_1 - t')^2} dt' \\ &\quad + \int_{t_0}^{t} \left[\Phi^{\oplus}_{rv'}(t, t')\vec{f}(\vec{r}^{(0)}(t'), t') - (t - t' \frac{\mu_{\text{☾}}\vec{v}_1}{v_1^3(t_1 - t')^2}\right] dt'. \end{aligned} \tag{7}$$

The first term on the right of (7) contains the singular part, while the second term, which with its time derivative remains finite as $t \to t_1$, may be denoted by $\vec{\rho}_F^{(1)}(t)$. Evaluating the first term and substituting into $(4)^{(1)}$, we find:

$$\vec{\rho}^{(1)}(t) = \frac{\mu_{\text{☾}}\vec{v}_1}{v_1^3} \ln(\tau_0/\tau) + \vec{\rho}_B^{(1)}(t), \tag{8}$$

where τ denotes $t_1 - t$, and

$$\begin{aligned} \vec{\rho}_B^{(1)}(t) &= \frac{\mu_{\text{☾}}\vec{v}_1}{v_1^3}\left(\frac{\tau}{\tau_0} - 1\right) + \vec{\rho}_F^{(1)}(t) \\ &+ [\Phi^{\oplus}_{rr_0}(t, t_0), \Phi^{\oplus}_{rv_0}(t, t_0)]\begin{pmatrix} \delta\vec{r}_0 \\ \delta\vec{v}_0 \end{pmatrix}. \end{aligned}$$

The second-order position perturbation $\vec{\rho}^{(2)}(t)$ contains further singularities:

$$\begin{aligned} \vec{\rho}^{(2)}(t) &= \frac{\mu_{\text{☾}}^2\vec{v}_1}{v_1^6\tau}\left(\ln\frac{\tau_0}{\tau} - 3/2\right) \\ &+ \frac{\mu_{\text{☾}}}{2v_1^3\tau}\left(3\frac{\vec{v}_1\vec{v}_1^T}{v_1^2} - 1\right)\vec{\rho}_B^{(1)}(t_1) + \text{LESSER TERMS}, \end{aligned} \tag{9}$$

where the omitted terms include terms like $\nu^2 \ln \tau$ as well as bounded terms of order ν^2.

It remains to obtain the resulting expression for the position $\vec{r}(t) - \vec{r}_{\text{☾}}(t)$ relative to the Moon for small τ. Firstly, the unperturbed relative position is:

$$\vec{r}^{(0)}(t) - \vec{r}_{☾}(t) = -\vec{v}_1\tau - (1/6)\Gamma_{\oplus}(t_1)\vec{v}_1\tau^3 + O(\nu\tau^2, \tau^4), \tag{10}$$

where the error term $O(\nu\tau^2)$ arises from the inclusion of the Moon's mass in the Moon's acceleration relative to the Earth. Combining (2), (8), (9), (10), and writing

$$\vec{\rho}_B^{(1)}(t) = \vec{\rho}_B^{(1)}(t_1) - \tau\dot{\vec{\rho}}_B^{(1)}(t_1) + O(\nu\tau^2),$$

we obtain the *outer expansion*:

$$\begin{aligned}\vec{r}(t) - \vec{r}_{☾}(t) = [-\vec{v}_1\tau] &+ \left[\frac{\mu_{☾}\vec{v}_1}{v_1^3}\ln\frac{\tau_0}{\tau} + \vec{\rho}_B^{(1)}(t_1)\right] \\ &+ \left[-\tau\dot{\vec{\rho}}_B^{(1)}(t_1) - (1/6)\Gamma_{\oplus}(t_1)\vec{v}_1\tau^3 + \frac{\mu_M{}^2\vec{v}_1}{v_1{}^6\tau}\left(\ln\frac{\tau_0}{\tau} - 3/2\right.\right. \\ &+ \left.\frac{\mu_{☾}}{2v_1{}^3\tau}\left(3\frac{\vec{v}_1\vec{v}_1{}^T}{v_1^2} - 1\right)\vec{\rho}_B^{(1)}(t_1)\right] + O(\nu^2),\end{aligned} \tag{11}$$

where successive groups [ ] are of orders $\nu^{1/2}$, ν, $\nu^{3/2}$, *if, as we now assume*, $\tau/\tau_0 = (\nu^{1/2})$. Here and later no distinction is made between $O(\nu^k)$ and $O(\nu^k \ln \nu)$. The $O(\nu^2)$ in (11) includes the remainder following truncation of the expansion of the position perturbation $\vec{\rho}$ at $\vec{\rho}^{(2)}$; this remainder, unlike the $\vec{\rho}^{(i)}$*'s*, satisfies a *non-linear* differential equation, but it is straightforward (see Perko, 1967, [5]) to verify that it is $O(\nu^2)$ when τ/τ_0 is $O(\nu^{1/2})$.

III. **The Inner Expansion.** Our basic results will be obtained by comparing, in the "matching region" represented by $\tau/\tau_0 = O(\nu^{1/2})$, the outer expansion (11) with an "inner expansion" representing a Moon-centered hyperbola perturbed by the Earth.

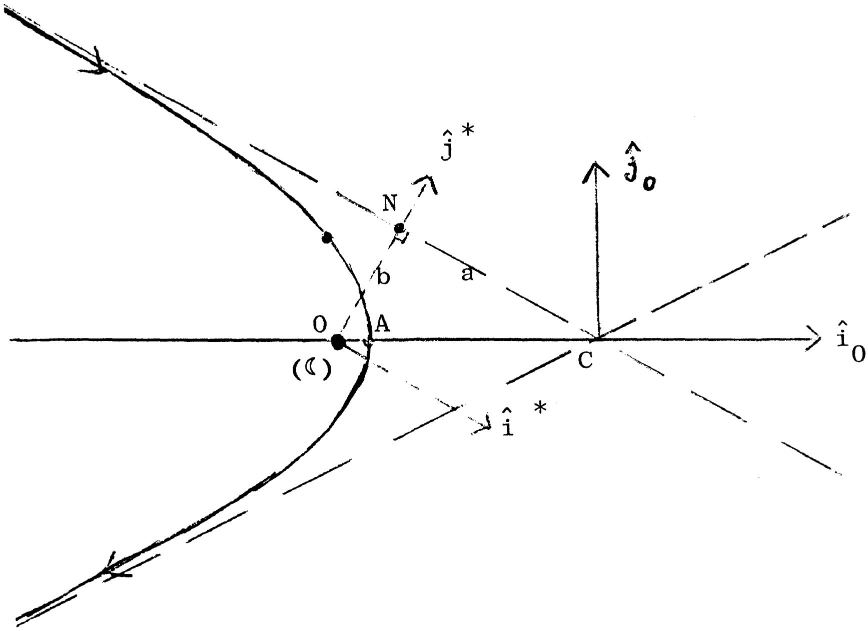
Relative to the center C of the hyperbola, the unperturbed position is expressible as:

$$\vec{r} = a\cosh F\,\hat{\imath}_0 - b\sinh F\,\hat{\jmath}_0,$$

F being the hyperbolic anomaly, b the length of the perpendicular ON from the focus 0 (the Moon) onto an asymptote, a the semi-transverse axis AC, equal to CN, and $\hat{\imath}_0, \hat{\jmath}_0$ being unit vectors along the symmetry axes of the hyperbola.

The eccentricity of the hyperbola is $e_H = \sqrt{1 + (b^2/a^2)}$, and the length a is related to the velocity v_∞ at infinity by: $v_\infty^2 = \mu_{☾}/a$, so that:

$$e_H = \sqrt{1 + b^2 v_\infty^4/\mu_{☾}^2}. \tag{12}$$



If we introduce unit-vectors $\hat{\imath}^*, \hat{\jmath}^*$ along and perpendicular to the incoming asymptote (see figure), the unperturbed position relative to the focus 0 (the Moon) is easily expressed as:

$$\vec{r}^{*(0)} = \frac{\mu_{☾}}{v_\infty^2}\hat{\imath}^*\left(-e_H\sinh\xi + 1 - \frac{1}{e_H}e^{-\xi}\right) + b\hat{\jmath}^*\left(1 - \frac{1}{e_H}e^{-\xi}\right), \tag{13}$$

where ξ denotes $-F$. If, furthermore, τ^* denotes the time $t_p - t$ prior to perilune passage (closest approach) at time t_p, the "time equation" is:

$$\tau^* = \frac{\mu_M}{v_\infty^3}(e_H\sinh\xi - \xi). \tag{14}$$

The perturbing acceleration due to the Earth is $\Gamma_\oplus\vec{r}^{*(0)}$, and if τ^* (like τ in the outer expansion) is $O(\nu^{1/2})$, so that e^ξ is $O(\nu^{-1/2})$, this perturbing acceleration is predominantly $-\Gamma_\oplus(t_1)v_\infty\tau^*\hat{\imath}^*$, and the perturbed position, to first order, is:.

$$\vec{r}^* = \vec{r}^{*(0)} - (1/6)\Gamma_\oplus(t_1)v_\infty\tau^{*3}\hat{\imath}^* + \text{ LESSER TERMS.}$$

This yields the *inner expansion:*

$$\vec{r}^* = \left[-\frac{\mu_{☾} e_H}{2v_\infty^2} \hat{i}^* e^{\xi} \right] + \left[\frac{\mu_{☾} \hat{i}^*}{v_\infty^2} + b\hat{j}^* \right]$$

$$(15) \qquad + \left[\frac{\mu_{☾}}{v_\infty^2} \left(\frac{e_H}{2} - \frac{1}{e_H} \right) \hat{i}^* e^{-\xi} - \frac{b}{e_H} \hat{j}^* e^{-\xi} \right.$$

$$\left. - (1/6)\Gamma_\oplus(t_1) v_\infty \tau^{*3} \hat{i}^* \right] + O(\nu^2)$$

successive term groups [ ] being again of orders $\nu^{1/2}$, ν, $\nu^{3/2}$, and the $O(\nu^2)$ including the remainder following truncation of the perturbation at first order.

IV. **The Matching.** To compare the inner and outer expansions (15) and (11), we guess (and later verify) that $t_p - t_1$ and b are both $O(\nu)$. Denoting $t_p - t_1$ by δ, we may write:

$$\tau = \tau^* - \delta = \left[\frac{\mu_{☾} e_H}{2v_\infty^3} e^{\xi} \right] + \left[-\delta - \frac{\mu_{☾} \xi}{v_\infty^3} \right] + \left[-\frac{\mu_{☾} e_H}{2v_\infty^3} e^{-\xi} \right], \tag{16}$$

and

$$\ln(\tau/\tau_0) = \left[\xi + \ln \frac{\mu_{☾} e_H}{2v_\infty^3 \tau_0} \right] + \left[\frac{2}{e_H} e^{-\xi} \left(\xi + \frac{v_\infty^3 \delta}{\mu_M} \right) \right] + O(\nu). \tag{17}$$

On substituting (16) and (17) into (11), the outer expansion may be written:

$$\vec{r}(t) - \vec{r}_M(t) = \left[-\frac{\mu_{☾} e_H}{2v_\infty^3} [\vec{v}_1 + \dot{\vec{\rho}}_B^{(1)}(t_1)] e^{\xi} \right]$$

$$(18) \qquad + \left[\vec{v}_1 \delta + \frac{\mu_{☾} \vec{v}_1}{v_1^3} \ln \frac{2v_\infty^3 \tau_0}{\mu_{☾} e_H} + \vec{\rho}_B^{(1)}(t_1) \right]$$

$$+ \left[\text{TERMS } \tau^3, \frac{\nu^2}{\tau}, \frac{\nu^2 \xi}{\tau} \right] + O(\nu^2).$$

Comparison with the inner expansion (15) gives agreement in the constant and e^{ξ} terms, provided that

$$v_\infty \hat{i}^* = \vec{v}_1 + \dot{\vec{\rho}}_B^{(1)}(t_1) + O(\nu^{3/2}) \tag{19}$$

and

$$-\vec{v}_1 \delta + b\hat{j}^* = \frac{\mu_{☾} \vec{v}_1}{v_1^3} \ln \frac{2v_\infty^3 \tau_0}{\mu_{☾} e_H} + \vec{\rho}_B^{(1)}(t_1) - \frac{\mu_{☾} \hat{i}^*}{v_\infty^2} + O(\nu^2),$$

which may be rewritten, by virtue of (19), as:

$$(20)\qquad -\vec{v}_1\delta + b\hat{j}^* = \frac{\mu_M\vec{v}_1}{v_1{}^3}\left(\ln\frac{2v_\infty{}^3\tau_0}{\mu_{☾}e_H} - 1\right) + \vec{\rho}_B{}^{(1)}(t_1) + O(\nu^2).$$

The agreement of the remaining terms in (15) and (18) is quite easily verified with the aid of (19), (20). This agreement is, of course, an inevitable consequence of the validity of both expansions in the matching region $\tau/\tau_0 = O(\nu^{1/2})$. Note that δ and b, according to (20), are now verified to be $O(\nu)$. The error $O(\nu^{3/2})$ in (19) is, moreover, replaceable by $O(\nu^2)$, since the evaluation of the $O(\nu^2)$ terms in the inner and outer expansions would introduce no new terms linear in τ or τ^*.

It is perhaps worth pointing out that the choice, $\tau/\tau_0 = O(\nu^{1/2})$, for the matching region was quite arbitrary, although convenient. A higher exponent than 1/2 would require further terms in the outer expansion in order to obtain an error not exceeding $O(\nu^2)$, while a lower exponent would require further terms in the inner expansion.

In a recent paper [2], Breakwell and Perko carried both expansions essentially through $O(\nu^{5/2})$ instead of $O(\nu^{3/2})$, for the particular case of the circular restricted 3-body problem, and arrived at corrections of order ν^2 to the matching formulas (19) and (20). The second-order velocity correction was particularly tedious.

Recalling that $\vec{\rho}_B{}^{(1)}(t_1)$ includes initial conditions in a term

$$[\Phi^{\oplus}_{rr_0}(t, t_0), \Phi^{\oplus}_{rv_0}(t, t_0)]\begin{pmatrix}\delta\vec{r}_0\\ \delta\vec{v}_0\end{pmatrix},$$

so that $\dot{\vec{\rho}}_B{}^{(1)}(t_1)$ includes them in a term

$$[\Phi^{\oplus}_{vr_0}(t, t_0), \Phi^{\oplus}_{vv_0}(t, t_0)]\begin{pmatrix}\delta\vec{r}_0\\ \delta\vec{v}_0\end{pmatrix},$$

the matching formulas (21) and (22) may be combined into:

$$(21)\qquad \begin{pmatrix}\left[t_1 - t_p + \dfrac{\mu_{☾}}{v_1{}^3}\ln e_H\right]\vec{v}_1 + \vec{b}^-\\ \vec{v}_\infty{}^- - \vec{v}_1\end{pmatrix} = \begin{pmatrix}\vec{B}_1\\ \vec{B}_1{}'\end{pmatrix} + \Phi^{\oplus}(t, t_0)\begin{pmatrix}\delta\vec{r}_0\\ \delta\vec{v}_0\end{pmatrix} + O(\nu^2)$$

where $\vec{v}_{\infty}^-, = \vec{v}_{\infty}^-{}^*$, is the arrival velocity at infinity, $\vec{b}^- = b\hat{j}^*$, is the perpendicular onto the arrival asymptote, and the "gross biases"

$\vec{B}_1, \vec{B}_1{}'$, are independent of initial condition changes $\delta\vec{r}_0, \delta\vec{v}_0$.

A similar formula relates the osculating hyperbola around a planet with "initial conditions" on a heliocentric ellipse either prior to or following planetary encounter, and these may be combined [2] to relate successive planetary encounters as follows:

$$\begin{pmatrix} \left[t_i - t_{p_i} + \frac{\mu_i}{(v_i^-)^3}\ln e_{H_i}\right]\vec{v}_i^- + \vec{b}_i^- \\ \vec{v}_{\infty_i}^- - \vec{v}_i^- \end{pmatrix} = \begin{pmatrix} \vec{B}_i \\ \vec{B}_i{}' \end{pmatrix}$$

$$\text{(22)} \quad + \Phi^{\odot}(t_i, t_{i-1}) \begin{pmatrix} \left[t_{i-1} - t_{p_{i-1}} - \frac{\mu_{i-1}}{(v^+_{i-1})^3}\ln e_{H_{i-1}}\right]\vec{v}^+_{i-1} + \vec{b}^+_{i-1} \\ \vec{v}^+_{\infty_{i-1}} - \vec{v}^+_{i-1} \end{pmatrix}$$

+ 2nd ORDER TERMS.

Here t_{i-1}, t_i denote times of departure and arrival on an unperturbed heliocentric elliptic arc from a massless planet "$i - 1$," with gravitational parameter μ_{i-1}, to the massless planet "i," with parameter μ_i; $\vec{v}^+_{i-1}$, $\vec{v}_i^-$ denote the relative velocities of departure and arrival, $t_{p_{i-1}}$ and t_{p_i} denote actual times of closest approach, and $\vec{b}^+_{i-1}$, $\vec{v}^+_{\infty_{i-1}}$, $e_{H_{i-1}}$ refer to the osculating hyperbola around planet $i - 1$ at time $t_{p_{i-1}}$ while $\vec{b}_i^-$, $\vec{v}^-_{\infty_i}$, e_{H_i} refer to that around planet i at time t_{p_i}. $\Phi^{\odot}(t_i, t_{i-1})$ is the transition matrix along the unpertrubed heliocentric ellipse, and $\vec{B}_i$, $\vec{B}_i{}'$ represent certain gross biases due to all the planets including planets $i - 1$ and i.

The quantities $\vec{b}_i^+$, $\vec{v}^+_{\infty_i}$ related to the departure asymptote from planet i are related to $\vec{b}_i^-$, $\vec{v}^-_{\infty_i}$ simply by a rotation through an angle $\delta_i = 2\,\mathrm{Sin}^{-1}(1/e_{H_i})$. Thus:

$$\text{(23)} \quad \begin{pmatrix} \vec{b}_i^+ \\ \vec{v}^+_{\infty_i} \end{pmatrix} = \frac{1}{1 + v^4_{\infty_i} b_i^2/\mu_i^2} \begin{pmatrix} \frac{v^4_{\infty_i} b_i^2}{\mu_i^2} - 1 & \frac{2v_{\infty_i} b_i^2}{\mu_i} \\ -\frac{2v^3_{\infty_i}}{\mu_i} & \frac{v^4_{\infty_i} b_i^2}{\mu_i^2} - 1 \end{pmatrix} \begin{pmatrix} \vec{b}_i^- \\ \vec{v}^-_{\infty_i} \end{pmatrix}$$

V. **The Computation of Accurate Multi-Flyby Trajectories.** The first step, and a laborious one, in the calculation of a multi-flyby trajectory, is to search for a succession of unperturbed heliocentric elliptic arcs, passing from one massless planet to another, such that the magnitudes v_i^-, v_i^+ of the relative velocities $\vec{v}_i^-$, $\vec{v}_i^+$ of departure and arrival at planet i are approximately equal, and that the change δ_i in the direction of the relative velocity is compatible with a flyby above the planet's surface and appreciable atmosphere. This search is usually aided by Lambert's Theorem for determining a (heliocentric) ellipse passing between two points in a given time interval.

Now formula (12) for hyperbolic eccentricity, reapplied in the planetary context, shows that a "zero order" estimate of δ_i is sufficient to determine b_i to first order in $\nu_i(=\mu_i/\mu_\odot)$. Denoting $(1/2)(v_i^- + v_i^+)$ by v_i^*, $\vec{v}_{\infty i}^- - v_i^*(\vec{v}_i^-/v_i^-)$ by $\vec{\Delta} v_{\infty i}^-$, $\vec{v}_{\infty i}^+ - v_i^*(\vec{v}_i^+/v_i^+)$ by $\vec{\Delta} v_{\infty i}^+$, and $t_{pi} - t_i$ by Δt_i, formula (22) may be written

$$(24)\qquad \begin{pmatrix} -v_i^* \dfrac{\vec{v}_i^-}{v_i} \Delta t_i \\ \\ \vec{\Delta} v_{\infty i}^- \end{pmatrix} = \Phi^{\oplus}(t_i, t_{i-1}) \begin{pmatrix} -v_{i-1}^* \dfrac{\vec{v}_{i-1}^+}{v_{i-1}^+} \Delta t_{i-1} \\ \\ \vec{\Delta} v_{\infty i-1}^+ \end{pmatrix} + \begin{pmatrix} \vec{B}_i^* \\ \\ \vec{B}_i^{*\prime} \end{pmatrix}$$

where the bias B_i^* includes the terms with $\vec{b}_i^-$, $\vec{b}_i^+$, $\ln e_{Hi}$, $\ln e_{Hi-1}$, all replaced by their zero-order estimates, while the bias $\vec{B}_i^{*\prime}$ includes corrections for the small differences between v_i^* and v_i^-, v_i^+.

If the component of $\vec{\Delta} v_{\infty i}^-$ along $\vec{v}_i^-/v_i$ is denoted by $\Delta v_{\infty i}$ (it is just the magnitude correction $v_{\infty i} - v_i^*$), and if the two-vector part of $\vec{\Delta} v_{\infty i}^-$ orthogonal to $\vec{v}_i^-/v_i^-$ is denoted by η_i^-, and the part of $\vec{\Delta} v_{\infty i}^+$ orthogonal to $\vec{v}_i^+/v_i^-$ by η_i^+, then formula (24) can be partitioned as follows:

$$(25)\qquad \begin{pmatrix} 0 \\ \xi_i \\ \eta_i^- \end{pmatrix} = \begin{pmatrix} M_{12}^{(i)} & M_{13}^{(i)} \\ M_{22}^{(i)} & M_{23}^{(i)} \\ M_{32}^{(i)} & M_{33}^{(i)} \end{pmatrix} \begin{pmatrix} 0 \\ \xi_{i-1} \\ \eta_{i-1}^+ \end{pmatrix} + \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix},$$

where ξ_i denotes the two-vector:

$$(26)\qquad \xi_i = \begin{pmatrix} -v_i^* \Delta t_i \\ \\ \Delta v_{\infty i} \end{pmatrix}.$$

It follows from (25) that

$$(27)\qquad \eta_{i-1}^+ = -(M_{13}^{(i)})^{-1}[M_{12}^{(i)} \xi_{i-1} + \alpha_i],$$

so that

$$\text{(28)} \qquad \xi_i = [M_{22}^{(i)} - M_{23}^{(i)} (M_{13}^{(i)})^{-1} M_{12}^{(i)}] \xi_{i-1} + [\beta_i - M_{23}^{(i)} (M_{13}^{(i)})^{-1} \alpha_i],$$

while η_i^- is similarly expressible in terms of ξ_{i-1}, α_i, γ_i. Repeated application of (28) yields:

$$\text{(29)} \qquad \xi_N = \underset{2\times 2}{A}\ \xi_0 + \underset{2\times 1}{\lambda},$$

N representing the final destination planet, so that Δv_{∞_0} and Δv_{∞_N} are easily expressible linearly in terms of Δt_0, Δt_N and the biases λ_1, λ_2, as are also all intermediate Δv_{∞_i}, Δt_i and all the asymptote direction corrections $\eta_i^{\pm}/v_i^*$. This provides the planetary encounter information for a two-parameter family of multi-flyby trajectories, the two parameters being time of departure from Earth and time of arrival at the final planet, these times being close, it must be assumed, to their zero-order estimates t_0, t_N.

The numerical values of the original gross biases $\vec{B}_i$, $\vec{B}_i{}'$, which are related (see [1]) to quadratures such as $\vec{\rho}_F{}^{(1)}(t_1)$ in (7), may be obtained by comparison of the preliminary unperturbed elliptic arc between t_i and t_{i-1} with some numerically integrated "exact" trajectory from a close approach to planet $i - 1$ at a time close to t_{i-1} to a close approach to planet i at a time close to t_i. After the selection of particular Δt_0 and Δt_N and the calculation from (25)–(29), of the corresponding first order planetary encounter information, "exact" legs of the multi-flyby trajectory should be numerically integrated from one planet to the next, and the relatively small mismatches easily reduced (see [3]) by a re-application of the matching formulas, until a single essentially continuous "nominal" multi-flyby trajectory is obtained.

VI. **Mid-Course Guidance.** Suppose, now, that small trajectory deviations from the nominal trajectory have been estimated during flight, and that a velocity correction is planned at some time t_c between t_{pi-1} and t_{pi}, but not too close to either. If $\delta\hat{\vec{r}}(t_c)$, $\delta\hat{\vec{v}}\ (t_c)$ are the estimated small position and velocity deviations, the next b_i can be restored by a small velocity impulse $\delta\vec{v}_D$ satisfying:

$$\text{(30)} \qquad \begin{pmatrix} -\vec{v}_{\infty_i}^{\,-}\delta t_{p_i} \\ \delta\vec{v}_{\infty_i}^{\,-} \end{pmatrix} = \Phi^{\odot}(t_{p_i}, t_c) \begin{pmatrix} \delta\hat{\vec{r}}(t_c) \\ \delta\hat{\vec{v}}\ (t_c) + \delta\vec{v}_D \end{pmatrix},$$

from which $\delta\vec{v}_D$ is easily expressible linearly in terms of $\delta\hat{\vec{r}}(t_c)$,

$\delta\hat{\vec{v}}\,(t_c)$, *and* δt_{p_i}, as are also $\delta v_{\infty i}$, $\delta\eta_i^-$ and all subsequent $\delta\xi_j$, $\delta\eta_j^\pm$.

This provides considerable flexibility in choosing the velocity correction $\delta\vec{v}_D$. Possible criteria are:

(i) $\delta t_{p_i} = 0$,

(ii) a later $\delta t_{p_j} = 0$,

(iii) $\mathrm{Min}_{t_{p_i}} |\delta\vec{v}_D|$.

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