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ASYMPTOTIC NORMALITY OF LINEAR COMBINATIONS OF FUNCTIONS OF ORDER STATISTICS[1]

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1. Introduction. Let $X_1, \cdots, X_N$ be i.i.d. uniform (0, 1) r.v.'s defined on a probability space $(\Omega, \mathfrak{A}, P)$. Let F_N denote the empirical df, and let $X_{N1} \leqq \cdots \leqq X_{NN}$ denote the ordered $X_1, \cdots, X_N$. We wish to consider statistics of the form

$$T_N = N^{-1} \sum_{i=1}^N c_{Ni} g(X_{Ni}) \tag{1.1}$$

where g is a specified function and $\{c_{Ni}: 1 \leqq i \leqq N, N \geqq 1\}$ is a set of specified constants.

REMARK. We may suppose the X_i's to have an arbitrary continuous df F provided we replace g by $g(F^{-1})$.

We define inverses of df's to be left continuous; thus

$$F_N^{-1}(t) = \inf \{x: F_N(x) \geqq t\};$$

and we write $g \circ F_N^{-1}$ for the composed function $g(F_N^{-1})$. Note that $T_N = \int_0^1 g \circ F_N^{-1} \, d\nu_N$ when the signed measure ν_N puts mass c_{Ni}/N at i/N for $i = 1, \cdots, N$ and puts 0 mass elsewhere. Let ν denote a signed measure on (0, 1). (The signed measures ν_N will not be used, but ν is in some sense their limit.) For technical reasons to be displayed below, we bound ourselves away from 0 and 1 by an amount β_N where $\beta_N \to 0$ as $N \to \infty$ at a rate to be specified later. Let

$$\mu_N = \int_{\beta_N}^{1-\beta_N} g \, d\nu \tag{1.2}$$

where $\int_\alpha^\beta \cdot \, d\nu = \int_{(\alpha,\beta]} \cdot \, d\nu$. Let J_N be defined on (0, 1] by

$$J_N(t) = c_{Ni} \quad \text{for} \quad (i-1)/N < t \leqq i/N, \qquad 1 \leqq i \leqq N.$$

Let $I(t) = t$ be the identity function on [0, 1] and let $\int \cdot \, dI$ denote integrals with respect to Lebesgue measure. Let

$$T_N^* = N^{\frac{1}{2}}(T_N - \mu_N). \tag{1.3}$$

Define stochastic processes $\{L_N(t): 0 < t < 1\}$ for $N \geqq 1$ by

$$L_N(t) = N^{\frac{1}{2}}[g \circ F_N^{-1}(t) - g(t)]. \tag{1.4}$$

Then

$$T_N^* = S_N^* + \gamma_N \tag{1.5}$$

where

$$S_N^* = \int_{\beta_N}^{1-\beta_N} L_N \, d\nu \tag{1.6}$$

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and

(1.7) $$\gamma_N = N^{-\frac{1}{2}}[\sum_{i=1}^N c_{Ni} g(X_{Ni}) - N \int_{\beta_N}^{1-\beta_N} g \circ F_N^{-1}\, d\nu].$$

We wish to obtain asymptotic normality of statistics of the form $N^{\frac{1}{2}}(T_N - \mu)$ where

(1.8) $$\mu = \int_0^1 g\, d\nu.$$

However, our technique is tailor-made for S_N^*. Thus, after preliminaries in Sections 2 and 3, we obtain asymptotic normality of S_N^* in Section 4 by first showing that the L_N-processes converge to a natural limiting L_0-process. Lemma 4.3 may be of interest in its own right. In Section 5 asymptotic normality of T_N^* is obtained by giving conditions under which $\gamma_N \to_p 0$; from this we go quickly to $N^{\frac{1}{2}}(T_N - \mu)$. The theorem of Section 5 does not apply to the Windsorized mean; in Section 6 results are extended to include statistics of this type. Section 7 extends the results to get joint asymptotic normality of a vector of statistics of the type considered. Section 8 extends all results to the case that N_v is a random sample size satisfying $N_v/v \to_p 1$ as $v \to \infty$.

Related work is contained in [1], [3], [4], [5], [6], [7] and [10]. See Section 9 for a short discussion of these.

2. The basic identity. Let

(2.1) $$\rho(f_1, f_2) = \sup_{0<t<1} |f_1(t) - f_2(t)|$$

for arbitrary functions f_1 and f_2.

LEMMA 2.1. *We have*

(2.2) $$\rho(L_N, -A_N U_N(F_N^{-1}) + \delta_N) = 0$$

where

(2.3) $$U_N = N^{\frac{1}{2}}(F_N - I),$$

(2.4) $$A_N = (g \circ F_N^{-1} - g)/(F_N^{-1} - I)$$

and

(2.5) $$\delta_N = A_N N^{\frac{1}{2}}(F_N \circ F_N^{-1} - I).$$

(In (2.4) A_N is defined by left continuity at any otherwise undefined points.)

PROOF. Now at any fixed t

$$L_N = [(g \circ F_N^{-1} - g)/(F_N^{-1} - I)] N^{\frac{1}{2}}[-(F_N \circ F_N^{-1} - F_N^{-1}) + (F_N \circ F_N^{-1} - I)]$$
$$= A_N[-U_N(F_N^{-1}) + N^{\frac{1}{2}}(F_N \circ F_N^{-1} - I)];$$

where in dividing by $F_N^{-1} - I$ for fixed t we are dividing by a function which is a.s. never 0. Since the quantity in (2.2) is determined by its values for rational t, the result follows. □

3. Some definitions. Let $\{U_0(t):0 \leqq t \leqq 1\}$ denote a separable tied-down Wiener process; that is a Gaussian process with $E[U_0(t)] = 0$ and $E[U_0(s)U_0(t)] = s(1-t)$ for all $0 \leqq s \leqq t \leqq 1$. Let $g'(t) = (d/dt)g(t)$ whenever this derivative exists. When defined let

$$L_0(t) = -g'(t)U_0(t), \qquad 0 < t < 1. \tag{3.1}$$

It is well-known that the U_N–processes (2.3) converge weakly to the U_0–process. However, in Pyke and Shorack (1968a) the U_0– and U_N–processes are replaced by $\tilde{U}_0$– and $\tilde{U}_N$–processes on a new probability space $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{P})$; these new processes have the same finite dimensional distributions as do the old ones, but in addition satisfy the strong condition

$$\rho(\tilde{U}_N, \tilde{U}_0) \rightarrow_{\text{a.s.}} 0 \quad \text{as} \quad N \rightarrow \infty. \tag{3.2}$$

Also $\tilde{F}_N = N^{\frac{1}{2}}\tilde{U}_N + I$ is a.s. a df having exactly N jump discontinuities each of magnitude N^{-1}. It is these new processes on this new probability space with which we work below; however, we drop the symbol $\sim$ from the notation.

Remark. Results obtained below which involve convergence stronger than convergence in law may apply only to the specially constructed process. Thus the conclusion $S_N^* \rightarrow_p N(0, \sigma^2)$ in Corollary 4.1 yields only that S_N^* converges in law to a $N(0, \sigma^2)$ rv when viewed on the original probability space.

Let L_N^*, $U_N^*(F_N^{-1})$, A_N^*, δ_N^* equal L_N, $U_N(F_N^{-1})$, A_N, δ_N respectively for $\beta_N \leqq t \leqq 1-\beta_N$ and let them equal 0 otherwise. Let

$$\rho_\phi(f_1, f_2) = \sup_{0<t<1} |f_1(t) - f_2(t)|/\phi(t)$$

for arbitrary functions f_1, f_2 and any fixed positive function ϕ.

Definition 3.1. Let $\mathbf{Q}$ denote the class of all non-negative continuous functions q defined on [0, 1] which are bounded below by functions $\bar{q}$ non-decreasing (non-increasing) on $[0, \frac{1}{2}]$ (on $[\frac{1}{2}, 1]$) with $\int_0^1 [\bar{q}(t)]^{-2}\, dt < \infty$.

Two examples are worth noting: $q(t) = [t(1-t)]^{\frac{1}{2}-\delta}$ for any $\delta > 0$ and $q(t) = -t^{\frac{1}{2}} \log t$.

Let q_0 denote a fixed function in the class $\mathbf{Q}$ for which $q_0(t)/t$ increases as $t \searrow 0$. We now suppose that β_N satisfies $\beta_N \geqq 1/N$, $N\beta_N$ is an integer and

$$q_0(\beta_N)q_0(1/N)/\beta_N \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty. \tag{3.3}$$

Remark. One useful choice is $q_0(t) = -t^{\frac{1}{2}} \log t$ and $\beta_N = [N^\delta]/N$, where [ ] denotes the greatest integer function and where $\delta > 0$.

4. Convergence of the L_N–processes in integral metrics.

Lemma 4.1. *Let $q \in \mathbf{Q}$. Then $\rho_q(U_N^*(F_N^{-1}), U_0) \rightarrow_p 0$ as $N \rightarrow \infty$.*

Proof. This is essentially Theorem 2.2 of [8]; though slight modifications in the proofs of Lemmas 2.3, 2.4, 2.5 and Theorem 2.2 of [8] must be made to make the present proof completely rigorous. A proof based on 13.17 and 13.8 of Breiman (1968) is also possible. □

Let $|\nu|$ denote the total variation measure for ν.

LEMMA 4.2. *If* $|\nu|$ $(\{t:g'(t)$ *fails to exist*$\}) = 0$, *then* a.s. P *we have*

$$A_N \to g' \text{ a.s. } |\nu| \quad as \quad N \to \infty.$$

PROOF. Now $A_N = (g \circ F_N^{-1} - g)/(F_N^{-1} - I)$ where $\rho(F_N^{-1}, I) \leqq \rho(F_N^{-1}, F_N \circ F_N^{-1}) + \rho(F_N \circ F_N^{-1}, I) \leqq \rho(F_N, I) + 1/N \to 0$ a.s. P. Thus $P[\omega : A_N(t) \to g'(t)$ for all t such that $g'(t)$ exists$] = 1$. □

For $q \varepsilon \mathbf{Q}$ let

$$\rho_q^*(f_1, f_2) = \sup_{\beta_N \leqq t \leqq 1-\beta_N} |f_1(t) - f_2(t)|/q(t). \tag{4.1}$$

We also write $Z_N = o_p(b_N)$ if $Z_N/b_N \to_p 0$ as $N \to \infty$ and $Z_N = O_p(b_N)$ if for all $\epsilon > 0$ there exists M_ϵ and N_ϵ such that for each $n > N_\epsilon$ $|Z_n/b_n|$ exceeds M_ϵ with probability less than ϵ. We use the fact that $o_p(1)O_p(1) = o_p(1)$.

LEMMA 4.3. *Given* $\delta, \epsilon > 0$ *and* $0 < \lambda < 1$ *there exists an integer* N_ϵ *and sets* $S_{N,\epsilon}$ *having* $P(S_{N,\epsilon}) > 1 - \epsilon$ *such that*

$$t - |F_N^{-1}(t) - t| \geqq \lambda t \quad \text{for all} \quad \beta_N \leqq t \leqq \tfrac{1}{2}$$

provided $N > N_\epsilon$ *and* $\omega \varepsilon S_{N,\epsilon}$.

PROOF. Now $(N^{\frac{1}{2}}q_0(1/N))^{-1} \to 0$ as $N \to \infty$. Thus

$$\rho_{q_0}^*(F_N^{-1}, I) \leqq N^{-\frac{1}{2}}[\rho_{q_0}^*(U_N(F_N^{-1}), U_0) + \rho_{q_0}^*(U_0, 0)] + [Nq_0(1/N)]^{-1}$$
$$= o_p(q_0(1/N)).$$

Thus for N exceeding some N_ϵ and $\beta_N \leqq t \leqq \frac{1}{2}$

$$|F_N^{-1}(t) - t| \leqq q_0(t)q_0(1/N)$$

on some set $S_{N,\epsilon}$ whose probability exceeds $1 - \epsilon$.
Thus for $N > N_\epsilon$, $\omega \varepsilon S_{N,\epsilon}$ and $\beta_N \leqq t \leqq \frac{1}{2}$ we may assume by (3.3) that

$$\begin{aligned} t - |F_N^{-1}(t) - t| &\geqq t[1 - q_0(t)q_0(1/N)/t] \\ &\geqq t[1 - q_0(\beta_N)q_0(1/N)/\beta_N] \\ &\geqq \lambda t. \ \square \end{aligned}$$

For a given function h and $0 < \lambda < 1$ define

$$\begin{aligned} h_\lambda(t) &= h(\lambda t) &&\text{for } 0 < t \leqq \tfrac{1}{2} \\ &= h(1 - \lambda(1 - t)) &&\text{for } \tfrac{1}{2} < t \leqq 1. \end{aligned} \tag{4.2}$$

For functions f on $(0, 1)$ define

$$\|f\|_\nu = \int_0^1 |f| \, d|\nu|. \tag{4.3}$$

THEOREM 4.1. *If*

(C1) g *is absolutely continuous on* $(\epsilon, 1 - \epsilon)$ *for all* $\epsilon > 0$ *and* $|g'| \leqq h$ a.e. $|\nu|$ *where the function* h *is increasing (decreasing) on* $[\frac{1}{2}, 1)$ *(on* $(0, \frac{1}{2}]$*)*

and

(C2) $\int_0^1 h_\lambda q \, d|\nu| < \infty$ *for some* $q \varepsilon \mathbf{Q}$ *and some* $0 < \lambda < 1$, *then*

$$\|L_N^* - L_0\|_\nu \to_p 0 \quad as \quad N \to \infty.$$

PROOF. Now

$$\begin{aligned}\|L_N^* - L_0\|_\nu &\leqq \rho_q(U_N^*(F_N^{-1}), U_0) \int_0^1 |A_N^*| q \, d|\nu| + \rho_q(U_0, 0) \int_0^1 |A_N^* - g'| q \, d|\nu| \\ &\quad + N^{\frac{1}{2}} \rho_q^*(F_N \circ F_N^{-1}, I) \int_0^1 |A_N^*| q \, d|\nu| \\ &= \int_0^1 |g'| q \, d|\nu| o_p(1) + \int_0^1 |A_N^* - g'| q \, d|\nu| O_p(1) \\ &= o_p(1) + \int_0^1 |A_N^* - g'| q \, d|\nu| O_p(1),\end{aligned}$$

using Lemma 4.1 and Lemma 2.2 of [8] in the first equality and $\int_0^1 |g'| q \, d|\nu| < \infty$ by hypotheses in the second. (Everything so far in the present proof is also true if the symbol * restricted the range of a quantity to $1/N \leqq t \leqq 1 - 1/N$. What follows is not.) Since an absolutely continuous function is the integral of its derivative, (C1) gives $|A_N(t)| \leqq \max [h(t), h(F_N^{-1}(t))]$. Thus by Lemma 4.3 for $N > N_\epsilon$, $\omega \varepsilon S_{N,\epsilon}$ and $\beta_N \leqq t \leqq \frac{1}{2}$ we have $|A_N(t)| \leqq h(\lambda t) = h_\lambda(t)$; with a symmetric result for $\frac{1}{2} \leqq t \leqq 1 - \beta_N$. Thus we may suppose that for $N > N_\epsilon$, $\omega \varepsilon S_{N,\epsilon}$ and $\beta_N \leqq t \leqq 1 - \beta_N$ $|A_N(t)| \leqq h_\lambda(t)$. Now a.s. P we have $A_N \to g'$ a.e. $|\nu|$ by Lemma 4.2. Thus

$$I_{S_{N,\epsilon}} \int_0^1 |A_N^* - g'| q \, d|\nu| \to_{\text{a.s.}} 0$$

by applying the dominated convergence theorem, with dominating function $2h_\lambda q$ by (C2), separately to each ω point. Thus

$$\int_0^1 |A_N^* - g'| q \, d|\nu| \to_p 0. \qquad \square$$

REMARK. If for some M and $\theta > 0$ the function $M[t(1-t)]^\theta$ can be used for h in Theorem 4.1, then $h_\lambda \leqq \bar{M} h$ for some $\bar{M} > 0$. This is very useful because often q and a bound on the Radon-Nikodym derivative of $|\nu|$ with respect to Lebesgue measure can also be taken to be of this same functional form. Thus, often, the integral in (C2) can be bounded by a constant times $\int_0^1 [t(1-t)]^{1-\delta} \, dt$ for some $\delta > 0$.

PROPOSITION 4.1. *If* $\|L_N^* - L_0\|_\nu \to_p 0$ *as* $N \to \infty$ *and if* $\int_0^1 |g'| q \, d|\nu| < \infty$ *for some* $q \varepsilon \mathbf{Q}$, *then*

$$S_N^* \to_p \int_0^1 L_0 \, d\nu \qquad as \quad N \to \infty.$$

The limiting rv is $N(0, \sigma^2)$ where

$$\sigma^2 = 2 \int_0^1 \int_0^v g'(u) g'(v) u(1-v) \, d\nu(u) \, d\nu(v). \tag{4.4}$$

If, moreover, $\gamma_N \to_p 0$ as $N \to \infty$, then $T_N^* \to_p \int_0^1 L_0 \, d\nu$.

PROOF. Now $|S_N^* - \int_0^1 L_0 \, d\nu| \leqq \| L_N^* - L_0 \|_\nu \to_p 0$, by (1.6). The limiting rv is normal since $\sigma^2 = E[(\int_0^1 L_0 \, d\nu)^2]$ is finite by the hypotheses. The final claim follows from (1.5). $\square$

The remainder of this section constitutes a digression.

THEOREM 4.1′. *If*

(C1′) *g is absolutely continuous on* $(\epsilon, 1-\epsilon)$ *for all* $\epsilon > 0$ *and* g' *exists* a.e. $|\nu|$, *and if*

(C2′) (a) $\int_0^1 |g'| q \, d|\nu| < \infty$ *for some* $q \, \varepsilon \, \mathbf{Q}$

(b) *for some* $0 < \lambda < 1$ *there exists* $0 < M < \infty$ *such that*

$$|g'(u)/g'(t)| < M$$

whenever $\lambda t < u < t/\lambda \leqq 1/2\lambda$ *or whenever* $\lambda(1-t) < 1-u < (1-t)/\lambda \leqq 1/2\lambda$, *then*

$$\|L_N^* - L_0\|_\nu \to_p 0 \quad as \quad N \to \infty.$$

PROOF. From the proof of Theorem 4.1 it suffices to show that $I_{S_{N,\epsilon}}|A_N^* - g'|$ times q is a.s. bounded by a $|\nu|$-integrable function. But by (C1′)

$$A_N(t) - g'(t) = \int_t^{F_N^{-1}(t)} g'(u)\, du/[F_N^{-1}(t) - t] - g'(t)$$

so that on the set $S_{N,\epsilon}$ of the proof of Theorem 4.1 we have

$$|A_N^*(t) - g'(t)| \leqq |g'(t)|(|\int_t^{F_N^{-1}(t)} |g'(u)/g'(t)|\, du/[F_N^{-1}(t) - t]| + 1)$$
$$\leqq |g'(t)|(M + 1);$$

where the last inequality uses (C2′) (b). Finally (C2′) (a) shows that $|g'|q$ is the desired $|\nu|$-integrable function. □

THEOREM 4.1″. *If*

(C1″) *g satisfies a first order Lipschitz condition on* $(\epsilon, 1-\epsilon)$ *for all* $\epsilon > 0$ *and* g' *exists* a.e. $|\nu|$ *and*

(C2″) (a) $\int_0^1 |g'| q \, d|\nu| < \infty$ *for some* $q \, \varepsilon \, \mathbf{Q}$,

(b) *there exists* $\delta > 0$ *such that for some* $0 < \lambda < 1$ *there exists* $0 < M < \infty$ *such that* $|g'(u)/g'(t)| < M$ *whenever* $\lambda t < u < t/\lambda \leqq \delta/\lambda$ *or whenever* $\lambda(1-t) < 1-u < (1-t)/\lambda \leqq \delta/\lambda$. *Or there exists* $\delta_0 > 0$ *such that* $c_{Ni} = 0$ *for* $i \leqq N\delta_0$ *and* $i \geqq N(1-\delta_0)$ *and N exceeding some* N_0,

(c) $|\nu|([\delta_1, 1-\delta_1]) < \infty$ *for some* $0 < \delta_1 < \delta$,

then

$$\|L_N^* - L_0\|_\nu \to_p 0 \quad as \quad N \to \infty.$$

PROOF. As in the previous proofs, we seek to bound $|A_N^* - g'|$ on $S_{N,\epsilon}$ by a $|\nu|$-integrable function. (C1″) and (C2″) (c) yield the bound on $[\delta, 1-\delta]$ with probability exceeding $1 - 2\epsilon$ for large N and (C1″) and (C2″) (b) yield the bound on $(0, \delta)$ and $(1-\delta, 1)$ by the argument of Theorem 4.1′. □

REMARK. This form of the theorem corresponds closely to Theorem 3 of Chernoff, et al (1967) which gives asymptotic normality under conditions labeled A*, B** and E. Our (C1″) is essentially slightly weaker than their A*, our (C2″) (a) (c) is essentially slightly stronger than their B** and our (C2″) (b) is essentially their E. The statistics to which the conclusions apply are different.

REMARK. The proof of Theorem 4.1 shows that $\|L_N^* - L_0\|_\nu \to_p 0$ as $N \to \infty$ provided we have g' exists a.e. $|\nu|$, (C2′) (a) and $\int_0^1 |A_N^* - g'|q\, d|\nu| \to_p 0$ as $N \to \infty$.

5. Limiting distribution of T_N^*. Throughout this section we assume $\nu = \nu_J$ for some function J; that is ν_J is given by $\nu_J((a, b]) = \int_a^b J\, dI$ for all $(a, b] \subset (0, 1)$. Note that $|\nu_J|((a, b]) = \int_a^b |J|\, dI$. Now by (1.7) $\gamma_N = \gamma_{N1} + \gamma_{N2} + \gamma_{N3}$ where

$$\gamma_{N1} = N^{-\frac{1}{2}} \sum_{i=1}^{N\beta_N} c_{Ni} g(X_{Ni}), \qquad \gamma_{N3} = N^{-\frac{1}{2}} \sum_{i=N(1-\beta_N)+1}^{N} c_{Ni} g(X_{Ni}),$$

$$\gamma_{N2} = N^{-\frac{1}{2}} \sum_{i=N\beta_N+1}^{N(1-\beta_N)} [c_{Ni} - N \int_{(i-1)/N}^{i/N} J\, dI] g(X_{Ni}).$$

THEOREM 5.1. *Suppose that* $\|L_N^* - L_0\|_\nu \to_p 0$ *as* $N \to \infty$ *(see Section 4),* $\int_0^1 |g'|q\, d|\nu| < \infty$ *for some* $q \in \mathbf{Q}$ *and*

(C3) $N^{\frac{1}{2}} \int_{\beta_N}^{1-\beta_N} |J_N - J| k_\lambda\, dI \to 0$ *as* $N \to \infty$ *for some* $0 < \lambda < 1$, *where* $|g| \leqq k$, k *is increasing on* $[\frac{1}{2}, 1]$ *and* k *is symmetric about* $\frac{1}{2}$;

(C4) (a) *For some* K *and* $\theta > 0$ $|g(u)| \leqq K[u(1-u)]^{-\theta}$ $0 < u < 1$ *and*

$$\text{(b)} \quad N^{-\frac{1}{2}} \sum_{i=1}^{N\beta_N} |c_{Ni}| (i/N)^{-\theta} \to 0$$

and $N^{-\frac{1}{2}} \sum_{i=N(1-\beta_N)+1}^{N} |c_{Ni}| (1 - i/N)^{-\theta} \to 0$ *as* $N \to \infty$; *and*

(C5) $N^{\frac{1}{2}} \int_{(\beta_N, 1-\beta_N]^c} g\, d\nu \to 0$ *as* $N \to \infty$.

Then $N^{\frac{1}{2}}(T_N - \mu)$ *is asymptotically* $N(0, \sigma^2)$ *with* $\mu = \int_0^1 g\, d\nu$ *and* σ^2 *given by* (4.4).

REMARK. (C4) (b) may be replaced by

(C4) (b′) *there exists* K, Δ *and a sequence* β_N *(see Section 3) such that* $|c_{Ni}| \leqq K[(i/N)(1 - i/N)]^\Delta$ *for* $i \leqq N\beta_N$ *and* $i > N(1 - \beta_N)$ *and* $N^{\frac{1}{2}} \beta_N^{1+\Delta-\theta} \to 0$ *as* $N \to \infty$.

REMARK. (C4) (a) implies $\int_0^1 |g|^m\, dI \leqq K^m \int_0^1 [u(1-u)]^{-m\theta}\, du$ and hence implies the existence of absolute mth moments of $g(X)$ provided $m < \theta^{-1}$ (C4) (b) and (b′) shows that if the constants c_{Ni} for extreme values of i are of a small enough order, then certain moments of $g(X)$ need not exist.

PROOF. Now by definition of J_N in Section 1

$$\gamma_{N2} = N^{\frac{1}{2}} \int_{\beta_N}^{1-\beta_N} [J_N - J] g \circ F_N^{-1}\, dI.$$

Thus, as in the proof of Theorem 4.1, for $N > N_\epsilon$ and $\omega \in S_{N,\epsilon}$ with $P(S_{N,\epsilon}) > 1 - \epsilon$ we have

$$I_{S_{N,\epsilon}} |\gamma_{N2}| \leqq N^{\frac{1}{2}} \int_{\beta_N}^{1-\beta_N} |J_N - J| k_\lambda\, dI;$$

so that $\gamma_{N2} \to_p 0$ by (C3). $\gamma_{N1} + \gamma_{N3} \to_p 0$ under (C4) as in the proof of Theorem 4 of Stigler (1967); which is a short, self-contained theorem. Thus by Proposition 4.1 we have $T_N^* \to_p \int_0^1 L_0\, d\nu$ as $N \to \infty$. Under (C5) the constant μ may replace μ_N. □

6. Point mass in the limiting measure. In this section we apply Theorem 4.1 to statistics of the form

$$T_N' = T_N + \tau_N \tag{6.1}$$

with T_N of the form (1.1) treated in Theorem 5.1 and

$$\tau_N = \sum_{i=1}^{s} d_{Ni} g(X_{N,[Np_i]+1}) \tag{6.2}$$

where the constants d_{Ni} satisfy $(d_{Ni} - d_i) = o(N^{-\frac{1}{2}})$ with $0 < d_i < \infty$ for $1 \leqq i \leqq s$, where $0 < p_1 < p_2 < \cdots < p_s < 1$ and where [ ] denotes the greatest integer function. We suppose that $g'(p_i)$ exists for $1 \leqq i \leqq s$.

THEOREM 6.1. *Under the hypotheses of Theorem* 5.1 *and the assumptions of this section* $N^{\frac{1}{2}}(T_N' - \mu - \mu_s)$ *is asymptotically distributed as a* $N(0, \sigma^2)$ rv *where*

$$\mu_S = \sum_{i=1}^{s} d_i g(p_i)$$

and

$$\begin{aligned}\sigma^2 &= \operatorname{Var}\left[\textstyle\int_0^1 L_0 J\, dI + \sum_{i=1}^{s} d_i L_0(p_i)\right] \\ &= 2\textstyle\int_0^1\int_0^v g'(u)g'(v)u(1-v)J(u)J(v)\,du\,dv \\ &\quad + 2\textstyle\sum_{i=1}^{s} d_i \int_0^1 g'(u)g'(p_i)[\min(u, p_i) - up_i]J(u)\,du \\ &\quad + \textstyle\sum_{i=1}^{s}[d_i g'(p_i)]^2 p_i(1-p_i) + 2\sum_{i<j} d_i d_j g'(p_i)g'(p_j)p_i(1-p_j).\end{aligned}$$

PROOF. Now

$$\begin{aligned}N^{\frac{1}{2}}(\tau_N - \mu_S) = N^{\frac{1}{2}} &\textstyle\sum_{i=1}^{s}(d_{Ni} - d_i)g(X_{N,[Np_i]+1}) \\ &+ N^{\frac{1}{2}}\textstyle\sum_{i=1}^{s} d_i[g(X_{N,[Np_i]+1}) - g(p_i)].\end{aligned}$$

Now

$$\begin{aligned}N^{\frac{1}{2}}[g(X_{N,[Np_i]+1}) - g(p_i)] &= N^{\frac{1}{2}}[g\circ F_N^{-1}(([Np_i]+1)/N) - g(p_i)] \\ &= N^{\frac{1}{2}}[g\circ F_N^{-1}(p_i) - g(p_i)] \\ &\to_p L_0(p_i),\end{aligned}$$

by Lemma 6.1 below and (1.4). Thus

$$N^{\frac{1}{2}}(\tau_N - \mu_S) \to_p \sum_{i=1}^{s} d_i L_0(p_i).$$

Thus

$$N^{\frac{1}{2}}(T_N' - \mu - \mu_S) \to_p \int_0^1 L_0 J\, dI + \sum_{i=1}^{s} d_i L_0(p_i);$$

where the limiting rv is $N(0, \sigma^2)$ with σ^2 as given. □

LEMMA 6.1. *If* $g'(t_0)$ *exists for some* t_0 *in* $(0, 1)$, *then*

$$L_N^*(t_0) \to_p L_0(t_0) \quad as \quad N \to \infty.$$

PROOF. Now at t_0 we have

$$\begin{aligned}|L_N^* - L_0| &\leqq |A_N^*||U_N^*(F_N^{-1}) - U_0| + |A_N^* - g'||U_0| + N^{-\frac{1}{2}}|A_N^*| \\ &= |A_N^* - g'|O_p(1) + o_p(1) = o_p(1)\end{aligned}$$

where the first equality uses Lemma 4.1 and the second equality uses Lemma 4.2. □

7. Joint asymptotic normality. Suppose the conclusion of Theorem 6.1 applies to each to the statistics T'_{Nk}, $k = 1, \cdots, K$ of the form (6.1). Then the vector

$$(N^{\frac{1}{2}}(T'_{N1} - \mu_1 - \mu_{s_1}), \cdots, N^{\frac{1}{2}}(T'_{NK} - \mu_K - \mu_{s_K}))$$

is asymptotically multivariate normal $N(\mathbf{0}, \Sigma)$ where the j, kth element of Σ is

$\sigma_{jk} = \text{Cov}\,[\int_0^1 L_0 J_j\,dI + \sum_{i=1}^{s_j} d_i^{(j)} L_0(p_i^{(j)}), \int_0^1 L_0 J_k\,dI + \sum_{i=1}^{s_k} d_i^{(k)} L_0(p_i^{(k)})]$;

the meaning of the above notation is clear. This is true since the convergence in probability of a vector is implied by the convergence in probability of each of its coordinates.

8. Random sample size. All results of Sections 1–7 carry over if N is replaced by N_v and $N \to \infty$ is replaced by $v \to \infty$ where the stochastic process $\{N_v : v > 0\}$ satisfies $N_v/v \to_p 1$. (All of the previous sections may be recopied except for Lemma 4.1. To prove Lemma 4.1 in the case of random sample size we cannot modify Theorem 2.2 of [8]; rather we must make slight modifications in the random sample size version of it which is given in [9].)

9. Comparison with other results. The results of this paper are a straightforward application of the results and techniques of [8] in which the value of replacing weakly convergent processes by a.s. convergent equivalent processes is demonstrated. However, bounding the coefficient A_N (see the proof of Theorem 4.1) introduced a problem not encountered in [8]; and Lemma 4.3 provided the key to a solution. But in order to prove Lemma 4.3 we found it necessary to introduce β_N. This in turn required conditions in Theorem 5.1 which would make $\gamma_{N1} + \gamma_{N3}$ negligible.

In the expository paper [7] Pyke also uses the technique of [8] to obtain some results for this problem; however a different basic identity was exploited.

Bickel [1] also uses an approach in which the statistic is viewed as a functional over a stochastic process. His technique uses weak convergence of the stochastic processes, rather than the stronger versions in [8]; and he uses the metric corresponding to $q \equiv 1$. He proves asymptotic normality when $\sum_{i<tN} c_{Ni}$ converges to a function $J(t)$ of bounded variation, F has a continuous density positive on its support, and either no weight is put on observations below the pth and above the qth percentile, or the more extreme observations are not weighted more than in the sample mean.

The techniques used by other authors are quite different from those in the papers cited above. Charnoff et al [4] use a device of Rényi to express T_N as a linear combination of exponential rv's plus a remainder term. Stigler [10] uses a projection technique due to Hájek. The results of these two papers and the present one seem to be of approximately equal strength. (A remark in Section 4 compares our results with [4].) Moore's [6] result is an elegant proof of a special case. Mark Brown [3], by a neat application of a theorem of Sethuraman on the convergence of stochastic integrals, also obtains a special case.

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