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Asymptotic ordinal inefficiency of random serial dictatorship

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We establish that the fraction of preference profiles for which the random serial dictatorship allocation is ordinally efficient vanishes for allocation problems with many object types. We consider also a probabilistic setting where in expectation agents have moderately similar preferences reflecting varying popularity across objects. In this setting we show that the probability that the random serial dictatorship mechanism is ordinally efficient converges to zero as the number of object types becomes large. We provide results with similarly negative content for allocation problems with many objects of each type. One corollary is that ordinal efficiency is a strict refinement of ex-post efficiency at most preference profiles.

Keywords. Allocation problem, ex-post efficiency, ordinal efficiency, probabilistic serial, random serial dictatorship.

JEL classification. D6.

1. Introduction

We consider the problem of allocating n indivisible objects to n agents, with each agent entitled to receive one object.[1] In many applications the assignment is based on the agents' ordinal preferences over objects because it is infeasible to extract preferences over all possible lottery allocations. Fairness considerations preclude monetary transfers and motivate random assignments. The random serial dictatorship mechanism is widely used in practical allocations, most notably in the assignment of university housing, and sometimes in the context of school choice.[2] In random serial dictatorship each possible ordering of the agents is drawn with equal probability and, for each realization of the ordering, the first agent is assigned his most preferred object, the next agent is assigned his most preferred object among those remaining, and so on. Abdulkadiroglu and Sonmez (1998) prove that the random serial dictatorship and the core from random endowments are equivalent allocation mechanisms, and argue that the equivalence validates the wide use of random serial dictatorship observed in practice.

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[1] This problem, also known as the house allocation problem, has been introduced by Hylland and Zeckhauser (1979), and is closely related to the housing market of Shapley and Scarf (1974).

[2] For instance, random serial dictatorship is implemented in the third round of the student placement mechanism in New York City (Abdulkadiroglu et al. 2005).

Random serial dictatorship is ex-post Pareto optimal. However, Bogomolnaia and Moulin (2001) show that this mechanism may result in unambiguous efficiency loss. They provide an example with $n = 4$ in which the random serial dictatorship allocation is first-order stochastically dominated with respect to the preferences of each agent by another random allocation. Motivated by the example, Bogomolnaia and Moulin call a random allocation ordinally efficient if it is not first-order stochastically dominated under the preferences of all agents by any other random allocation. Clearly, any ordinally efficient random allocation is ex-post Pareto optimal. In the context of allocation mechanisms based solely on ordinal preferences, ordinal efficiency is perhaps the most compelling efficiency notion.

In the main framework (Sections 2–4) we assume that there is only one object of each type. We prove that the fraction of preference profiles for which the random serial dictatorship allocation is ordinally efficient vanishes for large allocation problems (Theorem 1). Therefore, instances of ordinally inefficient random serial dictatorship allocations like the one identified by Bogomolnaia and Moulin are prevalent for large allocation problems. Since the random serial dictatorship allocation is ordinally efficient if and only if every ex-post efficient allocation lottery is ordinally efficient (Proposition 3), it follows that ordinal efficiency is a strict refinement of ex-post efficiency at most preference profiles.

If equal probability is assigned to each preference profile, a restatement of Theorem 1 is that the probability that the random serial dictatorship allocation is ordinally efficient vanishes for large allocation problems. However, in applications objects typically differ in popularity. Some objects are more likely than others to be highly ranked by all agents. For instance, in the assignment of university housing we expect that some features of the rooms, such as rent and size, affect the preferences of all students in a similar fashion. The assumption that the preferences of all agents are uniformly distributed over the set of all preference profiles does not capture such similarities. We consider a general probabilistic setting where in expectation agents may have similar preferences reflecting different popularity across objects, as modeled by Immorlica and Mahdian (2005). We establish that in this setting the probability that the random serial dictatorship mechanism is ordinally efficient converges to zero as the number of object types becomes large (Theorem 2). This result requires that the similarity in the agents' preferences be moderate, that is, the ratio of the popularity weights of any two objects be bounded across all possible allocation problems.

The assumption that there is only one object of each type is a logical first theoretical step, and may be reasonable in the context of university housing.[3] We recognize that it does not apply in the context of school choice where there are many identical seats at each school. Nevertheless, Theorems 1 and 2 extend to settings where the quota of

[3]In the context of university housing, random serial dictatorship is often implemented by randomly assigning each student or group of students a time window when they may select their rooms or apartments. Even when there are many rooms with identical rent and size in the same building, students may have strict preferences over the rooms available at their draw time taking into account variables such as floor number, distance to the nearest bathroom, kitchen or lounge, light or noise exposure, and gender or substance-free designation.

each object is held constant across a collection of allocation problems with *many object types*. Therefore their negative conclusions apply to school choice environments in which the number of seats available at any school is considerably smaller than the number of schools.

In order to accommodate school choice settings in which the number of seats available at any school is considerably larger than the number of schools, we consider in the spirit of Kojima and Manea (2006) a framework with a fixed set of m object types, and analyze the ordinal efficiency of random serial dictatorship in allocation problems with *many objects of each type* (Section 5). For a number of interesting restrictions on the supply and popularity of the objects, we obtain the sharp asymptotic prediction that in large allocation problems the probability that random serial dictatorship is ordinally efficient approaches either 0 or 1. As a means to quantify the similarity in preferences necessary or sufficient for asymptotic ordinal efficiency of random serial dictatorship, the main result for this setting (Theorem 4) assumes that all objects are in equal supply, and that each object is $s \geq 1$ times more popular than the next most popular object. Under these conditions, the random serial dictatorship mechanism is asymptotically ordinally inefficient if $s \leq m-2$ and ordinally efficient if $s > m-1$. We view this as an inefficiency result because in most applications m is fairly large, and it is unlikely that the preferences of the agents are similar to the extent that $s > m-2$.

Bogomolnaia and Moulin define a new solution to the random assignment problem, the probabilistic serial mechanism, which always attains ordinal efficiency. Kojima and Manea (2006) show that if the quota of each object is sufficiently large then truthful reporting of ordinal preferences is a weakly dominant strategy for the agents in the probabilistic serial mechanism. This non-manipulability property and the ordinal efficiency of the probabilistic serial mechanism, which by the present results is asymptotically nonexistent for random serial dictatorship, support the use of the probabilistic serial mechanism in many circumstances rather than the random serial dictatorship mechanism. Simulations based on real preferences also suggest that the probabilistic serial mechanism generates an efficiency gain over the random serial dictatorship mechanism in large allocation problems. Using the data of student placement in public schools in New York City, Pathak (2008) compares the resulting random assignments for each student with respect to first order stochastic dominance in the two mechanisms. He finds that about 50% of the students are better off under the probabilistic serial mechanism, and about 6% are better off under the random serial dictatorship mechanism.[4]

In the framework with many objects of each type, Che and Kojima (2008) show that the random serial dictatorship and the probabilistic serial mechanisms are asymptotically equivalent, and conclude that the inefficiency of random serial dictatorship becomes small in large allocation problems. It should be emphasized that their interpretation applies only to environments in which the quota of each object is considerably larger than the number of object types (no bound for the expected inefficiency as a

[4]For the rest of the students, the random assignments generated by the two mechanisms are not comparable with respect to first order stochastic dominance.

function of the quotas is provided). A similar result is not available for settings in which the number of object types is considerably larger than the quota of each object.

There exists a growing literature on ordinal efficiency. Bogomolnaia and Moulin (2001) provide characterizations of ordinal efficiency in terms of the acyclity of a binary relation and of an algorithm in which agents "eat" probability shares of available objects at varying speeds over a time interval. Abdulkadiroglu and Sonmez (2003) provide a characterization of ordinal efficiency in terms of one-to-one domination by sets of possibly infeasible assignments that are frequency equivalent to sets of feasible allocations. McLennan (2002) and Manea (2008a) allow for indifferences, and prove using different methods that any ordinally efficient random allocation is a welfare maximizing outcome with respect to some vector of von Neumann–Morgenstern utilities that is consistent with the ordinal preferences. Katta and Sethuraman (2006) extend Bogomolnaia and Moulin's results to the case of indifferences. Manea (2008b) shows that when preferences are commonly known the agents can write an ordinally efficient contract specifying ordering exchanges that they all prefer to the random serial dictatorship allocation with respect to first-order stochastic dominance.

The rest of the paper is organized as follows. The next section describes the main framework and defines the key concepts. Sections 3 and 4 present the frequency result and its probabilistic generalization, with proofs relegated to Appendices A and B, respectively. We state and prove the two results separately because the investigation of the forces driving the ordinal inefficiency of random serial dictatorship is more transparent when unaccompanied by the technical details of moderate similarity probability distributions over preferences. Section 5 analyzes the framework with many objects of each type, with the preliminary proofs presented in Appendix C. Section 6 discusses the challenges of the theoretical approach to measuring the ordinal inefficiency of random serial dictatorship.

2. Main framework

In an *n-allocation problem* each of the *agents* in the set $N = \{1, 2, \dots, n\}$ is entitled to exactly one *object* from the set $O = \{o_1, o_2, \dots, o_n\}$. An *allocation* is a vector α indexed by N, with α_i representing the object allocated to agent $i \in N$ ($i \neq j \Rightarrow \alpha_i \neq \alpha_j$). Each allocation α can be associated with a *permutation matrix* (a matrix with entries in $\{0, 1\}$, with each row and each column containing exactly one 1), π^α, with $\pi^\alpha_{ij} = 1$ if $\alpha_i = o_j$, and $\pi^\alpha_{ij} = 0$ otherwise. Let $\mathscr{A}$ be the set of *all allocations*. An *allocation lottery* p is a probability distribution over $\mathscr{A}$. We associate to each allocation lottery p a *random allocation* Π, which is a *bistochastic* $n \times n$ *matrix* (a matrix with non-negative entries, with each row and column summing to 1) that describes the probabilities that each agent (associated with a row) receives each object (associated with a column),

$$\Pi = \sum_{\alpha \in \mathscr{A}} p(\alpha) \pi^\alpha. \tag{1}$$

By the Birkhoff–von Neumann theorem (Pulleyblank 1995, pp. 187–188), any bistochasic matrix can be written (not necessarily uniquely) as a convex combination of permutation matrices, thus the set of random allocations is identical to the set of $n \times n$ bistochastic matrices. Any representation of a random allocation Π as in (1) (and the corresponding p) is a *lottery decomposition*.

Each agent $i \in N$ has a *strict preference relation* $\succ_i$ over O.[5] We denote by $\succ = (\succ_i)_{i \in N}$ the preference profile of all agents, and by $\mathscr{P}^n$ the set of all strict preference profiles $\succ$. An allocation α *Pareto dominates* an allocation α' at $\succ$ if $\alpha \neq \alpha'$, and $\alpha_i \succ_i \alpha'_i$ or $\alpha_i = \alpha'_i$ for all $i \in N$. An allocation $\alpha \in \mathscr{A}$ is *Pareto optimal* at $\succ$ if there is no allocation that Pareto dominates it at $\succ$. An *allocation lottery* is *ex-post Pareto optimal* at $\succ$ if all allocations in its support are Pareto optimal at $\succ$. A *random allocation* is *ex-post Pareto optimal* at $\succ$ if it admits a lottery decomposition that is ex-post Pareto optimal at $\succ$.

A random allocation Π *ordinally dominates* another random allocation Π' at $\succ$ if for each agent i the lottery Π_i first-order stochastically dominates the lottery Π'_i with respect to $\succ_i$,

$$\sum_{k:o_k \succ_i o_j} \Pi_{ik} \geq \sum_{k:o_k \succ_i o_j} \Pi'_{ik}, \forall i, j,$$

with strict inequality for some i, j. The random allocation Π is *ordinally efficient* at $\succ$ if it is not ordinally dominated at $\succ$ by any other random allocation; otherwise, it is *ordinally inefficient* at $\succ$. In an expected utility world, if Π ordinally dominates Π' at $\succ$, then every agent, irrespective of his von Neumann-Moregenstern utility index consistent with $\succ$, prefers Π to Π'.

An *ordering* of the agents is a one-to-one function from $\{1, 2, \ldots, n\}$ to N. The *serial dictatorship allocation* at $\succ$ for the ordering f, denoted $\delta^f(\succ)$, is generated as follows: agent $f(1)$ receives his most preferred object (according to $\succ_{f(1)}$), agent $f(2)$ receives his most preferred object among those remaining (according to $\succ_{f(2)}$), and so on. Denote the set of *orderings* by $\mathscr{F}^n$. The *random serial dictatorship allocation* at $\succ$, denoted $\Delta^n(\succ)$, corresponds to the allocation lottery assigning probability $1/n!$ to each serial dictatorship allocation $\delta^f(\succ)$, $f \in \mathscr{F}^n$,

$$\Delta^n(\succ) = \sum_{f \in \mathscr{F}^n} \frac{1}{n!} \pi^{\delta^f(\succ)}.$$

Proposition 1 (Corollary of Theorem 2, Abdulkadiroglu and Sonmez 1998). *An allocation α is Pareto optimal at $\succ$ if and only if it is the serial dictatorship allocation at $\succ$ for some ordering $f \in \mathscr{F}^n$, $\alpha = \delta^f(\succ)$.*

Example 1. Bogomolnaia and Moulin note that the random serial dictatorship allocation is not always ordinally efficient. The following example is a simplified version of theirs. Let $n = 4$ and the preferences of the agents be given by

$$i = 1, 2: \; o_1 \succ_i o_2 \succ_i o_3 \succ_i o_4$$
$$i = 3, 4: \; o_2 \succ_i o_1 \succ_i o_3 \succ_i o_4.$$

[5]We assume that for each agent there are no externalities from the object assignment for the other agents.

It is easy to check that

$$\Delta^4(\succ)=\begin{pmatrix} \frac{5}{12} & \frac{1}{12} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{12} & \frac{1}{12} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{12} & \frac{5}{12} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{12} & \frac{5}{12} & \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

which is ordinally dominated by

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}. \qquad \Diamond$$

Bogomolnaia and Moulin prove that the *ordinal binary relation* $\rhd[\Pi,\succ]$ on O defined by

$$o_j \rhd[\Pi,\succ]\, o_k \iff \exists i \in N, o_j \succ_i o_k \;\&\; \Pi_{ik} > 0$$

may be used to test Π's ordinal efficiency.

Proposition 2 (Lemma 3, Bogomolnaia and Moulin 2001). *The random allocation* Π *is ordinally efficient at* $\succ$ *if and only if the relation* $\rhd[\Pi,\succ]$ *is acyclic.*

The intuition for the "only if" part is as follows. For a preference profile and a random allocation, an object dominates another object according to the ordinal binary relation if there exists one agent who prefers the first object to the second and receives the second object with positive probability. The agent would be willing to move probability weight away from the less preferred object to the more preferred one, and a cycle in the ordinal binary relation enables all corresponding agents to trade probability weights so that all of them are made better off in the sense of first-order stochastic dominance.

3. The frequency result

We show that the fraction of preference profiles for which the random serial dictatorship allocation is ordinally efficient vanishes for large allocation problems.

Theorem 1. *The fraction of preference profiles* $\succ \in \mathscr{P}^n$ *for which the random serial dictatorship allocation* $\Delta^n(\succ)$ *is ordinally efficient at* $\succ$ *converges to zero as* $n \to \infty$.

The detailed proof is in Appendix A. We provide some intuition here.

Definition 1 (Special objects). An object o is *special* at $\succ$ if for any allocation α and any agent i such that α is Pareto optimal at $\succ$ and $\alpha_i = o$, o is ranked first under $\succ_i$.

An object is special at a preference profile if any Pareto optimal allocation assigns it to an agent who ranks it first. One important observation, proven in Appendix A, is that if no object is special then the ordinal binary relation generated by the random serial dictatorship allocation is cyclic, hence by Proposition 2 the random serial dictatorship allocation is ordinally inefficient.

Lemma 1. *If $\Delta^n(\succ)$ is ordinally efficient at $\succ$, then there is a special object at $\succ$.*

The proof of Theorem 1 proceeds by showing that the fraction of preference profiles at which special objects exist becomes small in large allocation problems.

Fix an n-allocation problem and consider the probability space $(\mathscr{P}^n, \mathbb{P})$ with the measure $\mathbb{P}$ assigning equal probability to each event $\succ \in \mathscr{P}^n$. We denote by $\mathbb{P}(\cdot \mid \cdot)$ and $\mathbb{E}(\cdot \mid \cdot)$ the probabilities and expectations on this space conditional (when well defined) on events or random variables.[6] Let $O_S(\succ)$ denote the set of special objects at $\succ$. We can view O_S as a random variable on the space $(\mathscr{P}^n, \mathbb{P})$ with realizations in the set of subsets of O.

Fix an object $o \in O$. By Lemma 1, the fraction of preference profiles $\succ \in \mathscr{P}^n$ for which the random serial dictatorship allocation $\Delta^n(\succ)$ is ordinally efficient at $\succ$ is smaller than or equal to

$$\mathbb{P}(O_S \neq \varnothing) = \mathbb{P}(\cup_{j=1}^n \{\succ \mid o_j \in O_S(\succ)\}) \leq \sum_{j=1}^n \mathbb{P}(o_j \in O_S) = n\mathbb{P}(o \in O_S),$$

where the last equality follows by symmetry. The delicate part of the proof is to develop an upper bound for $\mathbb{P}(o \in O_S)$. The exercise involves non-trivial combinatorial arguments.

Let $(r_i(\succ))_{i \in N}$ be the vector specifying the ranks assigned by each agent to o under $\succ$. Note that the random vector $(r_i)_{i \in N}$ has i.i.d. components, each distributed with equal probability over $\{1, 2, \ldots, n\}$. Let $(i_1, i_2, \ldots, i_n)$ be the random vector ordering N increasingly with respect to r, with ties broken in favor of lower indexed players.[7]

There is some intuition that if the number of agents who rank o as their most preferred object is not very large, yet there is a large number of agents who rank o as one of their most preferred objects, but not most preferred, then one of the latter agents may receive o at some serial dictatorship allocation, yielding o non-special. Hence, we divide the preference profiles into two sets, according to the truth value of the condition $r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1$ (which means that at most x agents rank o as their most preferred object, and at least y agents rank o as one of their $z+1$ most preferred objects, but not most preferred), where (x, y, z) is a triplet of integers explicitly defined in Appendix A. The goal is to show that the preference profiles at which the condition is true most likely

[6] Many of the random variables we consider are not real-valued, but rather set- or vector-valued; measurability is not a concern because all domains and ranges are finite measure spaces, where all events are assumed to be measurable.

[7] Formal definitions are given in Appendix A.

yield o non-special, and that the probability that the condition is false is small. Therefore, we write

$$\mathbb{P}(o \in O_S) \leq \mathbb{P}(o \in O_S \mid r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1) + \mathbb{P}(r_{i_{x+1}} = 1 \text{ or } r_{i_{x+y}} > z+1). \tag{2}$$

In order to bound the first term on the right-hand side of (2), define the (random) orderings f_k for $k = 1, \ldots, y$ by

$$(f_k(1), f_k(2), \ldots, f_k(n)) = (i_{x+y+1}, \ldots, i_n, i_{x+k}, i_{x+1}, \ldots, i_{x+k-1}, i_{x+k+1}, \ldots, i_{x+y}, i_1, \ldots, i_x).$$

Conditional on $r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1$, these orderings first list the agents who do not necessarily rank o as one of their most preferred objects, then the agents who rank o as one of their most preferred objects, but not most preferred ($1 < r_{i_{x+k}} \leq z+1$ for $k = 1, \ldots, y$), with choice over the first agent within this group, and in the end agents who may rank o first. For $k = 1, \ldots, y$, the serial dictatorship allocations corresponding to f_k are identical for agents $i_{x+y+1}, \ldots, i_n$. Denote the (random) set of objects allocated to these agents by O_0. Let $\underline{O}_k$ be the (random) set of $r_{i_{x+k}} - 1$ objects that i_{x+k} prefers to o. The definitions are illustrated in the table below (and formalized in Appendix A), where columns represent the preferences of the agents, ordered according to f_k, and boxed objects correspond to the (potential) serial dictatorship allocation for f_k.

	i_{x+y+1}	$\ldots$	i_n	i_{x+k}	i_{x+1}	$\ldots$	i_{x+k-1}	i_{x+k+1}	$\ldots$	i_{x+y}	i_1	$\ldots$	i_x
1	$\ldots$	$\ldots$	$\ldots$	$\uparrow$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\boxed{o \notin^{?} O_0}$	$\vdots$	$\underline{O}_k$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_{i_{x+k}} - 1$	$\vdots$	$\vdots$	$\vdots$	$\downarrow$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r_{i_{x+k}}$	$\vdots$	$\vdots$	$\vdots$	$\boxed{o}^{?}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

If $r_{i_{x+1}} > 1$ and o is a special object then the serial dictatorship allocation for the ordering f_k, which is Pareto optimal by Proposition 1, cannot allocate o to any agent in $\{i_{x+y+1}, \ldots, i_n, i_{x+k}\}$ (o is not the favorite object of any of these agents since $r_{i_{x+1}} > 1$). Hence if $r_{i_{x+1}} > 1$ and $o \in O_S$ then $o \notin O_0$ and $\underline{O}_k \not\subset O_0, \forall k = 1, \ldots, y$. Based on this fact, in Appendix A we show that the first term on the right-hand side of (2) is smaller than or equal to

$$\mathbb{E}(\mathbb{P}(\underline{O}_k \not\subset O_0, \forall k = 1, \ldots, y \mid r, O_0) \mid o \notin O_0, r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1).$$

A key observation is that conditional on r, O_0, the sets $(\underline{O}_k)_{k=1,\ldots,y}$ are independently distributed, with $\underline{O}_k$ placing equal probability on all $r_{i_{x+k}} - 1$ element subsets of $O \setminus \{o\}$. Also, note that $|O_0| = n - x - y$. Therefore, in the event $o \notin O_0$,[8]

$$\mathbb{P}(\underline{O}_k \not\subset O_0, \forall k = 1, \ldots, y \mid r, O_0) = \prod_{k=1}^{y} \big(1 - \mathbb{P}(\underline{O}_k \subset O_0 \mid r, O_0)\big) = \prod_{k=1}^{y} \left(1 - \frac{\binom{n-x-y}{r_{i_{x+k}}-1}}{\binom{n-1}{r_{i_{x+k}}-1}}\right).$$

[8] $\binom{n}{k}$ denotes the binomial coefficient, $n!/(k!(n-k)!)$, the number of k-element subsets of an n-element set.

Indeed, in the event $o \notin O_0$, conditional on r, O_0, the set $\underline{O}_k$ can take $\binom{n-1}{r_{i_{x+k}}-1}$ values with equal probability, and exactly $\binom{n-x-y}{r_{i_{x+k}}-1}$ of them are included in O_0. If $r_{i_{x+y}} \leq z+1$ we can prove that

$$\frac{\binom{n-x-y}{r_{i_{x+k}}-1}}{\binom{n-1}{r_{i_{x+k}}-1}} > 1 - \frac{(x+y-1)z}{n-z}.$$

Therefore, in the event $o \notin O_0, r_{i_{x+y}} \leq z+1$, we obtain the bound

$$\mathbb{P}(\underline{O}_k \not\subset O_0, \forall k = 1, \ldots, y \mid r, O_0) < \left(\frac{(x+y-1)z}{n-z}\right)^y,$$

which leads to

$$\mathbb{P}(o \in O_S \mid r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1) < \left(\frac{(x+y-1)z}{n-z}\right)^y.$$

In order to find a rough bound for the second term on the right-hand side of (2), we write

$$\mathbb{P}(r_{i_{x+1}} = 1 \text{ or } r_{i_{x+y}} > z+1) < \mathbb{P}(r_{i_x} = 1) + \mathbb{P}(r_{i_{x+y+1}} \geq z+1).$$

The event $r_{i_x} = 1$ is a (non-disjoint) union of $\binom{n}{x}$ events selecting an x-element subset of N in which all agents rank o first; the probability of each such event is $1/n^x$. Therefore,

$$\mathbb{P}(r_{i_x} = 1) < \binom{n}{x}\frac{1}{n^x}.$$

In Appendix A we use Sterling's approximation formulae to prove that

$$\binom{n}{x} < \frac{n^{n+1/2}}{x^{x+1/2}(n-x)^{n-x+1/2}},$$

which eventually leads to[9]

$$\mathbb{P}(r_{i_x} = 1) < \left(\frac{e}{x}\right)^{x+1/2}.$$

Analogously, we show that

$$\mathbb{P}(r_{i_{x+y+1}} \geq z+1) < \left(\frac{ne}{x+y}\right)^{x+y+1/2} e^{-z}.$$

Putting everything together, the fraction of preference profiles $\succ \in \mathscr{P}^n$ for which the random serial dictatorship allocation $\Delta^n(\succ)$ is ordinally efficient at $\succ$ is smaller than

$$n\left(\left(\frac{(x+y-1)z}{n-z}\right)^y + \left(\frac{e}{x}\right)^{x+1/2} + \left(\frac{ne}{x+y}\right)^{x+y+1/2} e^{-z}\right).$$

In Appendix A we provide values for x, y, z as functions of n such that the bound above converges to 0 as $n \to \infty$.

[9] $e \approx 2.718$ denotes the base of the natural logarithm.

Proposition 3. *$\Delta^n(\succ)$ is ordinally efficient at $\succ$ if and only if every ex-post Pareto optimal random allocation at $\succ$ is ordinally efficient at $\succ$.*

Proof. Let Π be an ex-post Pareto optimal random allocation at $\succ$. By Proposition 1, Π can be written as a convex sum of $\delta^f(\succ)$ for $f \in \mathscr{F}^n$. Since

$$\Delta^n(\succ) = \sum_{f \in \mathscr{F}^n} \frac{1}{n!} \pi^{\delta^f(\succ)},$$

it follows that every non-zero entry in Π corresponds to a non-zero entry in $\Delta^n(\succ)$. Hence $\triangleright[\Pi, \succ]$ is acyclic for all ex-post Pareto optimal Π's at $\succ$ if and only if $\triangleright[\Delta^n(\succ), \succ]$ is acyclic, which together with Proposition 2 proves our claim. □

Remark 1. An equally simple proof based on Abdulkadiroglu and Sonmez's (2003) dominance characterization is possible, since a set is not dominated if and only if none of its subsets is dominated.

Proposition 3 establishes that ordinal efficiency is a strict refinement of ex-post Pareto optimality for any preference profile at which random serial dictatorship is ordinally inefficient. Hence, by Theorem 1, ordinal efficiency is a stronger efficiency criterion than ex-post Pareto optimality for most preference profiles in large allocation problems.

4. The probabilistic result

If equal probability is assigned to each preference profile, a restatement of Theorem 1 is that the probability that the random serial dictatorship allocation is ordinally efficient vanishes for large allocation problems. However, in applications objects typically differ in popularity; some objects are more likely than others to be highly ranked by all agents. For example, in the assignment of university housing we expect that some features of the rooms, such as rent and size, affect the preferences of all students in the same direction. Assuming that the preferences of all agents are uniformly distributed over the set of all preference profiles cannot accommodate such similarities.

In the extreme case of similar preferences all agents have identical preferences. For that case the outcome of random serial dictatorship is ordinally efficient (by Proposition 2). Hence the conclusion that random serial dictatorship is asymptotically ordinally inefficient in large allocation problems does not hold when the support of the distribution over preferences includes only profiles with extremely similar preferences. It is important to identify the extent of (non-extreme) similarity in preferences that preserves the negative probabilistic interpretation of Theorem 1.

The following stochastic generation of preferences, based on Immorlica and Mahdian (2005), embeds the idea that in expectation agents may have similar preferences. Let $w_1^n, w_2^n, \ldots, w_n^n$ be positive weights summing to 1, corresponding to the objects o_1, $o_2, \ldots, o_n$, respectively; w_j^n is interpreted as a measure of o_j's *popularity*. Consider the discrete probability space over the set of preference profiles $\mathscr{P}^n$ with measure $\mathbb{P}^n$ determined by the following procedure. The preferences of each agent are generated independently, from the top to the bottom of the ranking. For each agent, the most preferred object is drawn from the distribution specified by w^n, and conditional on the top t objects having been determined, the $t+1^{st}$ ranked object is drawn from the remaining $n-t$ objects with probabilities proportional to their w^n weights (conditional on the top t objects drawn being $o_{j_1},\ldots,o_{j_t}$, the probability that an object o_j not drawn yet is ranked $t+1^{st}$ is $w_j^n/(1-w_{j_1}^n-\cdots-w_{j_t}^n)$). Interpreting w_j^n as o_j's popularity, each agent is more likely to prefer more popular objects to less popular ones. If all objects are equally popular, i.e., $w_1^n=\cdots=w_n^n=1/n$, then all preference profiles are equally probable, as in the probabilistic interpretation of Theorem 1.

Recall that $\Delta^n(\succ)$ denotes the random serial dictatorship allocation at $\succ$. We can view Δ^n as a random variable on the space $(\mathscr{P}^n,\mathbb{P}^n)$ with realizations in the set of random allocations. We identify a restriction on the sequence of popularity weights $(w^n)_{n\geq 1}$ such that the extension of Theorem 1,

$$\lim_{n\to\infty}\mathbb{P}^n(\Delta^n(\succ)\text{ is ordinally efficient at }\succ)=0, \tag{3}$$

is true. Some restrictions are obviously necessary. For, let p be any real in $(0,1)$ and $\succ^n$ be the preference profile with $o_1\succ_i^n o_2\succ_i^n\ldots\succ_i^n o_n$ for all $i\in N$. For every n-allocation problem, we may choose w^n such that the probability that agent i has preference $\succ_i^n$ is larger than $p^{1/n}$ (let $w_j^n=a^{n-j}/(1+a+a^2+\cdots+a^{n-1}), j=1,\ldots,n$ for sufficiently large a). Then the corresponding $\mathbb{P}^n$-probability of the preference profile $\succ^n$, for which $\Delta^n(\succ^n)$ is ordinally efficient at $\succ^n$, is larger than p, hence the limiting value of the left-hand side term in (3) is at least p. In order to preclude such choices of extremely similar preferences we require that the ratio of the popularity weights of any two objects be bounded.

Definition 2 (Moderate similarity). A sequence of popularity weights $(w^n)_{n\geq 1}$ satisfies *moderate similarity with ratio s* if

$$w_j^n/w_{j'}^n\leq s,\forall j,j'\in\{1,\ldots,n\},\forall n\geq 1.$$

A sequence of popularity weights $(w^n)_{n\geq 1}$ satisfies *moderate similarity* if it satisfies moderate similarity with ratio s for some s.

Remark 2. Note that a special class of sequences of popularity weights satisfying moderate similarity stems from sequences of allocation problems with the following properties. Objects are partitioned into (a fixed number of) categories (tiers) such that all objects within the same category are equally popular, while the relative popularity of any pair of categories is constant across all allocation problems. For instance, public schools in a city may be roughly partitioned into three tiers, with students being more likely to prefer top tier schools. However, some students may be interested in a lower tier school due to its convenient location, siblings or friends already attending it, or its excellence in particular subjects.

Moderate similarity is a sufficient condition for the conclusion (3).

Theorem 2. *Suppose that the sequence of popularity weights $(w^n)_{n\geq 1}$ satisfies moderate similarity. The probability—in the space $(\mathscr{P}^n, \mathbb{P}^n)$, generated by w^n—that the random serial dictatorship allocation Δ^n is ordinally efficient converges to zero as $n \to \infty$:*

$$\lim_{n\to\infty} \mathbb{P}^n(\Delta^n(\succ) \text{ is ordinally efficient at } \succ) = 0.$$

The proof appears in Appendix B. The approach is similar to that of Theorem 1, but computing bounds for the probability of various events is technically more involved due to the details of moderate similarity probability distributions.

Remark 3. Theorems 1 and 2 extend to settings where the quota of each object is held constant across a set of allocation problems with large numbers of agents and object types.

Remark 4. The conclusion of the theorem remains unchanged if for every n-allocation problem there exists a set of objects that all agents rank below or above all other objects (we can think of these objects as having popularity weights in $\{0, \infty\}$; moderate similarity is violated), as long as the cardinality of the complementary set of objects goes to infinity as $n \to \infty$, and satisfies moderate similarity. More generally, the conclusion does not change if for every size of the allocation problem objects are divided into categories, and all agents agree on the relative ranking of the categories, but may disagree on the ranking of objects within the same category, as long as moderate similarity with a common similarity ratio is satisfied within each category, and either of the following two sequences (or their term by term maximum) goes to infinity as $n \to \infty$: (1) the cardinality of the largest category or (2) the number of categories containing at least 4 objects.[10] The categories may be determined by qualities of the objects that enter lexicographically (and identically) in agents' preferences. Examples of such qualities are location or condition of rooms in university housing, and safety or performance ranking of high schools in school choice.

Remark 5. The theorem can be extended to allow for higher degrees of preference similarity in larger allocation problems. Specifically, fix $\zeta < 1/5$ and a sequence $(s_n)_{n\geq 1}$ such that $\lim_{n\to\infty} s_n/n^{\zeta} = 0$. Suppose that $(w^n)_{n\geq 1}$ is a sequence of popularity weights satisfying the following conditions:

$$w^n_j / w^n_{j'} \leq s_n, \forall j, j' \in \{1, \ldots, n\}, \forall n \geq 1.$$

Then the probability—in the space $(\mathscr{P}^n, \mathbb{P}^n)$, generated by w^n—that the random serial dictatorship allocation Δ^n is ordinally efficient converges to zero as $n \to \infty$. The minor modification of the proof necessary for this extension is shown at the end of Appendix B.

[10] Example 1 and the proof of Theorem 2 imply the existence of a uniform upper bound $\varepsilon_s < 1$ such that for every n-allocation problem with $n \geq 4$ and every probability distribution over $\mathscr{P}^n$ that satisfies moderate similarity with ratio s, the probability that Δ^n is ordinally efficient is smaller than ε_s. Thus if there are c categories containing at least 4 objects and moderate similarity with ratio s is satisfied within each category then the probability that Δ^n is ordinally efficient is smaller than ε_s^c, which converges to 0 as $c \to \infty$.

5. Framework with many objects of each type

The assumption that there is only one object—or, as per Remark 3, a constant number of objects—of each type may be inappropriate in the context of school choice where there are many identical seats at each school.[11] In order to accommodate this application, we consider in the spirit of Kojima and Manea (2006) a framework with a fixed set of object types, and analyze the ordinal efficiency of random serial dictatorship in allocation problems with many objects of each type. The definitions and notation are recycled from the previous sections and adapted accordingly. There is a fixed set of m *object types*, $O=\{o_1,o_2,\dots,o_m\}$.[12] With each o_j we associate two positive numbers, q_j and w_j, such that $w_1/q_1 \geq w_2/q_2 \geq \cdots \geq w_m/q_m$ and $\sum_{j=1}^m q_j = \sum_{j=1}^m w_j = 1$; q_j represents the *fraction* of objects to be allocated that are of type o_j, and w_j measures the *relative popularity* of o_j. Both definitions are spelled out below.

We consider a sequence of allocation problems indexed by the number of *agents* n. In an n*-allocation problem* each of the agents $1,2,\dots,n$ needs to be allocated one object from a collection that includes nq_j copies of o_j.[13] An *allocation* is a vector α, with α_i representing the object allocated to agent i, satisfying the endowment constraints $|\{i \mid \alpha_i = o_j\}| = nq_j$.

Each agent i has a *strict preference relation* $\succ_i$ over O. We denote by $\succ = (\succ_i)_{i\in N}$ the preference profile of all agents, and by $\mathscr{P}^n$ the set of all strict preference profiles $\succ$. As in Section 4, consider the discrete probability space over the set $\mathscr{P}^n$ with measure $\mathbb{P}^n$ determined by the following procedure. The preferences of each agent are generated independently, from the top to the bottom of the ranking. For each agent, the most preferred object is drawn from the distribution specified by w, and conditional on the top t objects having been determined, the $t+1^{st}$ ranked object is drawn from the remaining $m-t$ objects with probabilities proportional to their w weights (conditional on the top t objects drawn being $o_{j_1},\dots,o_{j_t}$, the probability that an object o_j not drawn yet is ranked $t+1^{st}$ is $w_j/(1-w^n_{j_1}-\cdots-w^n_{j_t})$).

Serial dictatorships, random serial dictatorship, ordinal efficiency, ordinal binary relations, and the associated notation are defined analogously to Section 2. The conclusion of Proposition 2 extends to the present setting.

For a number of interesting restrictions on the supply fractions q and the popularity weights w we obtain the sharp asymptotic prediction that the corresponding $\mathbb{P}^n$-probability that the random serial dictatorship for the n-allocation problem is ordinally efficient converges to either 0 or 1 as $n\to\infty$.[14] In some cases we are able to identify conditions that are "almost" necessary and sufficient for asymptotic ordinal efficiency of random serial dictatorship.

[11]The same observation does not necessarily apply to university housing. See footnote 3.

[12]When there is no risk of confusion, we simply write "object" for "object type," and "o_j" for "object of type o_j."

[13]It is assumed that $q_1,q_2,\dots,q_m$ are rational numbers and n belongs to the set of multiples of the least common denominator of $q_1,q_2,\dots,q_m$, so that nq_j is an integer for $j=1,2,\dots,m$.

[14]In the present framework, the operator $\lim_{n\to\infty}$ applies for n restricted to the set described in footnote 13.

The following notation is necessary for our analysis. For every n-allocation problem and every preference profile $\succ \in \mathscr{P}^n$ denote by $v_j^n(\succ)$ and $v_{jk}^n(\succ)$ the numbers of agents who rank o_j and respectively o_j, o_k (with o_j above o_k) at the top under $\succ$, by $v_{\_j\_k}^n(\succ)$ the number of agents who prefer o_j to o_k under $\succ$, and by $\mathscr{E}^n(\succ)$ the set of random allocations that are ordinally efficient at $\succ$; $v_j^n, v_{jk}^n, v_{\_j\_k}^n$, and $\mathscr{E}^n$ denote the corresponding random variables on the space $(\mathscr{P}^n, \mathbb{P}^n)$.

Note that v_j^n and v_{jk}^n have binomial distributions corresponding to n Bernoulli trials with means w_j and $w_j w_k/(1-w_j)$, respectively. In Appendix C we show that $\mathbb{P}^n(o_j \succ_i o_k) = w_j/(w_j + w_k)$. Thus $v_{\_j\_k}^n$ has a binomial distribution corresponding to n Bernoulli trials with mean $w_j/(w_j + w_k)$. By the weak law of large numbers,

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}^n\left(\left|\frac{v_j^n}{n} - w_j\right| > \varepsilon\right) = \lim_{n\to\infty} \mathbb{P}^n\left(\left|\frac{v_{jk}^n}{n} - \frac{w_j w_k}{1-w_j}\right| > \varepsilon\right)$$
$$= \lim_{n\to\infty} \mathbb{P}^n\left(\left|\frac{v_{\_j\_k}^n}{n} - \frac{w_j}{w_j + w_k}\right| > \varepsilon\right) = 0. \qquad (4)$$

The lemmata below are used in the proofs of the main results. The proofs of all results except Theorem 4 appear in Appendix C.

Lemma 2. *Fix n and $\succ \in \mathscr{P}^n$. If $o_j \rhd [\Delta^n(\succ), \succ]\, o_k$ then*

(i) there exists $l \neq k$ such that $v_l^n(\succ) \geq nq_l$

(ii) $v_{\_j\_k}^n(\succ) \geq nq_j$.

Lemma 3. *If $\lim_{n\to\infty} \mathbb{P}^n(o_j \rhd [\Delta^n(\succ), \succ]\, o_k) = 0, \forall j > k$ then $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 1$.*

One immediate consequence of Lemma 2 is that if $m = 2$ then for every n and $\succ \in \mathscr{P}^n$ the relation $\rhd[\Delta^n(\succ), \succ]$ is acyclic, hence $\Delta^n(\succ)$ is ordinally efficient at $\succ$. Hereafter we assume that $m \geq 3$, which is an unstated hypothesis for Proposition 4 and Theorems 3 and 4.

Part (i) of Proposition 4 below states that "$w_2 > q_2$ or $w_3/q_3 > 1 - w_1$" is a sufficient condition for asymptotic ordinal inefficiency of random serial dictatorship. The condition involves only the supply fractions and popularity weights of o_2 and o_3 (modulo the convention that $w_1/q_1 \geq w_2/q_2 \geq \cdots \geq w_m/q_m$), and requires that either o_2 or o_3 has a popularity to supply ratio above a corresponding threshold. Part (ii) claims that for $m = 3$ the condition is "almost" necessary for asymptotic ordinal inefficiency of random serial dictatorship in that "$w_2 < q_2$ and $w_3/q_3 < 1 - w_1$" is a sufficient condition for asymptotic ordinal efficiency of random serial dictatorship. Whether analogous conditions exist for settings with $m \geq 4$ is an open question.

Proposition 4. *(i) If $w_2 > q_2$ or $w_3/q_3 > 1 - w_1$ then*

$$\lim_{n\to\infty} \mathbb{P}^n(\Delta^n(\succ) \textit{ is ordinally efficient at } \succ) = 0.$$

(ii) *Suppose that* $m=3$. *If* $w_2<q_2$ *and* $w_3/q_3<1-w_1$ *then*

$$\lim_{n\to\infty}\mathbb{P}^n(\Delta^n(\succ) \text{ is ordinally efficient at } \succ)=1.$$

The intuition for the proof is as follows. For the first half of part (i), the condition $w_2 > q_2$ implies that $\lim_{n\to\infty}\mathbb{P}^n(v_j^n > nq_j) = 1$ for $j=1,2$ (by (4)). We show that for $j=1,2$, conditional on the event $\{\succ \mid v_j^n(\succ) > nq_j\}$, there is a serial dictatorship that allocates o_{3-j} to an agent whose favorite object is o_j with probability approaching 1 as $n\to\infty$. Then $\lim_{n\to\infty}\mathbb{P}^n(o_1 \rhd[\Delta^n(\succ),\succ]\, o_2 \;\&\; o_2 \rhd[\Delta^n(\succ),\succ]\, o_1) = 1$. For the second half of part (i), we prove that if[15] $w_2 < q_2$ and $w_3/q_3 > 1-w_1$ then $\lim_{n\to\infty}\mathbb{P}^n(v_1^n > nq_1, v_j^n + v_{1j}^n > nq_j, v_1^n + v_j^n > n(q_1+q_j)) = 1$ for $j=2,3$ (by (4)). We show that for $j=2,3$, conditional on the event $\{\succ\mid v_1^n(\succ) > nq_1, v_j^n(\succ)+v_{1j}^n(\succ) > nq_j, v_1^n(\succ)+v_j^n(\succ) > n(q_1+q_j)\}$, there is a serial dictatorship that allocates o_{5-j} to an agent whose favorite object is o_j with probability approaching 1 as $n\to\infty$. Then $\lim_{n\to\infty}\mathbb{P}^n(o_2 \rhd[\Delta^n(\succ),\succ]\, o_3 \;\&\; o_3 \rhd[\Delta^n(\succ),\succ]\, o_2)=1$. For either half, Proposition 2 leads to $\lim_{n\to\infty}\mathbb{P}^n(\Delta^n\in\mathscr{E}^n)=0$.

For part (ii), by Lemma 2,

$$\mathbb{P}^n(o_2 \rhd[\Delta^n(\succ),\succ]\, o_1 \text{ or } o_3 \rhd[\Delta^n(\succ),\succ]\, o_1) \le \mathbb{P}^n(v_2^n \ge nq_2 \text{ or } v_3^n \ge nq_3)$$
$$\mathbb{P}^n(o_3 \rhd[\Delta^n(\succ),\succ]\, o_2) \le \mathbb{P}^n(v_{\_3\_2}^n \ge nq_3).$$

The inequalities $w_2 < q_2$ and $w_3/q_3 < 1-w_1$ imply that the right-hand sides of the inequalities above converge to 0 as $n\to\infty$ (by (4)), hence $\lim_{n\to\infty}\mathbb{P}^n(o_2 \rhd[\Delta^n(\succ),\succ]\, o_1$ or $o_3 \rhd[\Delta^n(\succ),\succ]\, o_1$ or $o_3 \rhd[\Delta^n(\succ),\succ]\, o_2)=0$. By Lemma 3, $\lim_{n\to\infty}\mathbb{P}^n(\Delta^n\in\mathscr{E}^n)=1$.

Even for a setting with $m=3$, Proposition 4 leaves out the cutoff cases $w_2=q_2$ and $w_3/q_3=1-w_1$. The analysis of all such cases is lengthy and unrevealing (two subcases are necessary in order to address each of the two cases above, w_1 equal to/greater than q_1 and respectively w_2 equal to/smaller than q_2). One salient cutoff case leading to a surprising conclusion is one where the popularity of each object is proportional to its supply, that is, $w_j=q_j$ for $j=1,2,\dots,m$. In this case, as the allocation problem becomes large, the random serial dictatorship mechanism is asymptotically ordinally inefficient despite the fact that the expected fraction of agents receiving their favorite object in this mechanism goes to 1.

The framework with $w_j=q_j$ for $j=1,2,\dots,m$ accommodates allocation problems where, in contrast to university housing and school choice, the social planner may set the supply of the objects to match their expected popularity (but adjusting the supply to each realization of the preferences is costly or infeasible). Under these conditions, agents frequently receive their favorite objects in the random serial dictatorship. Nonetheless, the random serial dictatorship outcome is asymptotically ordinally inefficient in large allocation problems.

Theorem 3. *Suppose that* $w_j=q_j$ *for* $j=1,2,\dots,m$. *Then*

$$\lim_{n\to\infty}\mathbb{P}^n(\Delta^n(\succ) \text{ is ordinally efficient at } \succ)=0.$$

[15] The case $w_2>q_2$ has been addressed in the first half, and the case $w_2=q_2$ needs separate treatment.

For a sketch of the proof, note that the probability that there are exactly nq_j agents whose favorite object is o_j, i.e., $v_j^n = nq_j$, converges to 0 as $n \to \infty$. It follows that, with probability approaching 1 as $n \to \infty$, either there exist g_1, g_2 such that $v_{g_1}^n > nq_{g_1}$ and $v_{g_2}^n > nq_{g_2}$, or there exists g such that $v_g^n > nq_g$ and $v_h^n < nq_h, \forall h \neq g$. For all $j \neq k \in \{1, 2, \dots, m\}$, the set of conditions (4) and $w_j = q_j$ imply that $\lim_{n\to\infty} \mathbb{P}^n(v_{jk}^n > v_j^n - nq_j) = 1$. By ideas similar to the proof of the first and second half of part (i) of Proposition 4, we argue that

$$\lim_{n\to\infty} \mathbb{P}^n(o_{g_1} \rhd [\Delta^n(\succ), \succ]\, o_{g_2} \;\&\; o_{g_2} \rhd [\Delta^n(\succ), \succ]\, o_{g_1} \mid v_{g_1}^n > nq_{g_1}, v_{g_2}^n > nq_{g_2}) = 1$$

and respectively

$$\lim_{n\to\infty} \mathbb{P}^n(o_{h_1} \rhd [\Delta^n(\succ), \succ]\, o_{h_2} \;\&\; o_{h_2} \rhd [\Delta^n(\succ), \succ]\, o_{h_1} \mid v_g^n > nq_g, v_h^n < nq_h, v_{gh}^n > v_g^n - nq_g, \forall h \neq g) = 1$$

for all $h_1, h_2 \neq g$. Then we can use Proposition 2 to show that $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n \mid v_{g_1}^n > nq_{g_1}, v_{g_2}^n > nq_{g_2}) = 0$ and $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n \mid v_g^n > nq_g, v_h^n < nq_h, \forall h \neq g) = 0$, hence $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 0$.

In the setting of Section 4 we argue that if preferences are extremely similar then random serial dictatorship attains asymptotic ordinal efficiency. An analogous statement is true for the present setting. Suppose that preferences are similar in the sense that every object is significantly more popular than the next most popular object, specifically that $w_k/w_j > (1 - q_j)/q_j$ for all $j > k$. Fix $j > k$; $w_k/w_j > (1 - q_j)/q_j$ is equivalent to $q_j > w_j/(w_j + w_k)$. Since $\lim_{n\to\infty} \mathbb{P}^n(|v_{\_j\_k}^n/n - w_j/(w_j + w_k)| > \varepsilon) = 0, \forall \varepsilon > 0$, it must be that $\lim_{n\to\infty} \mathbb{P}^n(v_{\_j\_k}^n(\succ) \geq nq_j) = 0$, which coupled with part (ii) of Lemma 2 leads to $\lim_{n\to\infty} \mathbb{P}^n(o_j \rhd [\Delta^n(\succ), \succ]\, o_k) = 0$. By Lemma 3, $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 1$.

In order to quantify the similarity in preferences necessary or sufficient for asymptotic ordinal (in)efficiency of random serial dictatorship, we restrict attention to a setting in which similarity is parametrized by a single variable, $s \geq 1$. In Theorem 4 we assume that all objects are in equal supply, and each object is s times more popular than the next most popular object, that is, $q_j = 1/m, w_j = s^{m-j}/(1 + s + \cdots + s^{m-1})$ for $j = 1, 2, \dots, m$. We show that $s \leq m - 2$ and $s > m - 1$ are sufficient conditions for asymptotic ordinal inefficiency and respectively ordinal efficiency of random serial dictatorship; the only gray area is $s \in (m - 2, m - 1]$. We view Theorem 4 as an inefficiency result because in most applications m is fairly large, and it is unlikely that the preferences of the agents are similar to the extent that $s > m - 2$.

Theorem 4. *Suppose that $q_j = 1/m, w_j = s^{m-j}/(1 + s + \cdots + s^{m-1})$ for $j = 1, 2, \dots, m$.*

(i) If $s \in [1, m - 2]$, then

$$\lim_{n\to\infty} \mathbb{P}^n(\Delta^n(\succ) \text{ is ordinally efficient at } \succ) = 0.$$

(ii) If $s \in (m - 1, \infty)$, then

$$\lim_{n\to\infty} \mathbb{P}^n(\Delta^n(\succ) \text{ is ordinally efficient at } \succ) = 1.$$

Proof. We first prove part (i). For $m=3$ the conclusion follows from Theorem 3. Given the running assumption that $m \geq 3$, we need to treat only the case $m \geq 4$ henceforth. For $s=1$, the statement is a special case of Theorem 3. Suppose that $s \in (1, m-2]$. We prove that $w_2 > q_2$, and then Proposition 4 delivers the conclusion that $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 0$.

After straightforward algebraic manipulation, the inequality $w_2 > q_2$ is equivalent to

$$s^m - ms^{m-1} + ms^{m-2} - 1 < 0.$$

Consider the polynomial Q in z, given by $Q(z) = z^m - mz^{m-1} + mz^{m-2} - 1$. We need to prove that $Q(s) < 0$. As Q has 3 changes of sign, Descartes' rule of signs implies that Q has at most 3 positive roots (accounting for multiplicities). Since $Q(1) = Q'(1) = 0$, $z = 1$ is a root of multiplicity 2. As $m \geq 4$, we obtain

$$\begin{aligned} Q(m-2) &= (m-2)^m - m(m-2)^{m-1} + m(m-2)^{m-2} - 1 = (4-m)(m-2)^{m-2} - 1 < 0 \\ Q(m-1) &= (m-1)^m - m(m-1)^{m-1} + m(m-1)^{m-2} - 1 = (m-1)^{m-2} - 1 > 0. \end{aligned}$$

Thus Q has also a root $z^* \in (m-2, m-1)$. Since Q has at most 3 positive roots (accounting for multiplicities), it follows that the set of positive roots of Q are 1 (with multiplicity 2) and z^* (with multiplicity 1). In particular, Q has no roots in $(1, m-2]$, so Q has no sign change in $(1, m-2]$. Then $Q(m-2) < 0$ implies that $Q(s) < 0$ for all $s \in (1, m-2]$, and the proof is concluded as outlined above.

We next prove part (ii). Suppose that $s > m-1$. Fix $j > k$. Then $w_j/w_k \leq 1/s < 1/(m-1)$, which implies that $w_j/(w_j + w_k) < 1/m$. One consequence of $\lim_{n\to\infty} \mathbb{P}^n(|\nu^n_{\_j\_k}/n - w_j/(w_j + w_k)| > \varepsilon) = 0, \forall \varepsilon > 0$ is

$$\lim_{n\to\infty} \mathbb{P}^n(\nu^n_{\_j\_k} \geq n/m) = 0. \tag{5}$$

By part (ii) of Lemma 2,

$$\mathbb{P}^n(o_j \rhd [\Delta^n(\succ), \succ]\, o_k) \leq \mathbb{P}^n(\nu^n_{\_j\_k} \geq nq_j). \tag{6}$$

Keeping in mind that $q_j = 1/m$, (5) and (6) imply that $\lim_{n\to\infty} \mathbb{P}^n(o_j \rhd [\Delta^n(\succ), \succ]\, o_k) = 0$. We showed that $\lim_{n\to\infty} \mathbb{P}^n(o_j \rhd [\Delta^n(\succ), \succ]\, o_k) = 0, \forall j > k$. Therefore, $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 1$ by Lemma 3. □

6. Conclusion

We do not attempt to find tight bounds for the rate of convergence to zero of the probability that random serial dictatorship is ordinally efficient as the allocation problem becomes large. For a fixed preference profile, ordinal efficiency of random serial dictatorship depends on the structure of the set of Pareto optimal allocations (indeed, by Propositions 1 and 2, the ordinal binary relation is determined by the set of Pareto optimal allocations). The set of possible sets of Pareto optimal allocations at various preference profiles is intractable to many attempts of counting, particularly for the purpose of measuring the probability of instances of acyclicity in the ordinal binary relation. Due to this intractability, our method of proof considers only cycles that involve agents willing to move weight towards their most preferred object, which obviously leads to overestimating the probability of efficiency. Even in counting these special cycles, the lack of a tractable structure of the possible sets of Pareto optima at different preference profiles permits us to work only with the small number of Pareto optima that are outcomes of some particular serial dictatorships. It would be desirable to develop more precise evaluations of the asymptotically vanishing probability of ordinal efficiency and of the distribution of the efficiency loss under random serial dictatorship in large allocation problems.

Appendix A

Proof of Lemma 1. We proceed by contradiction. Assume that $\Delta^n(\succ)$ is ordinally efficient at $\succ$ and that no object is special at $\succ$. Set $j_1 = 1$. Since o_{j_1} is not a special object at $\succ$, there exists an agent i_1 who does not rank o_{j_1} first, but is assigned o_{j_1} at some Pareto optimal allocation α_1. By Proposition 1, α_1 is the serial dictatorship allocation $\delta^{f_1}(\succ)$ for some ordering f_1. For $k \geq 1$, proceed inductively to define $i_{k+1}, j_{k+1}, \alpha_{k+1}, f_{k+1}$ as follows. Let $o_{j_{k+1}}$ be agent i_k's most preferred object (hence $j_k \neq j_{k+1}$). Since $o_{j_{k+1}}$ is not a special object at $\succ$, there exists an agent i_{k+1} who does not rank $o_{j_{k+1}}$ first, but is assigned $o_{j_{k+1}}$ at some Pareto optimal allocation α_{k+1}. By Proposition 1, α_{k+1} is the serial dictatorship allocation $\delta^{f_{k+1}}(\succ)$ for some ordering f_{k+1}. Note that, by definition, agent i_k ranks object $o_{j_{k+1}}$ first, in particular, $o_{j_{k+1}} \succ_{i_k} o_{j_k}$. Since $\delta^{f_k}(\succ)$ receives weight $1/n!$ in the serial dictatorship lottery, $\Delta^n_{i_k, j_k}(\succ) > 0$. Hence,

$$o_{j_{k+1}} \rhd [\Delta^n(\succ), \succ]\, o_{j_k}, \forall k \geq 1.$$

Since O is finite, it follows that $\rhd[\Delta^n(\succ), \succ]$ is cyclic, and Proposition 2 implies that $\Delta^n(\succ)$ is ordinally inefficient at $\succ$, which is a contradiction. □

Proof of Theorem 1. Define the sequences of positive integers $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, (z_n)_{n\geq 1}$ by[16]

$$x_n = y_n = [n^{1/5}], \quad z_n = [n^{2/5}]. \tag{A.1}$$

Fix $n \geq 3$. Recall the definition of the probability space $(\mathscr{P}^n, \mathbb{P})$ from Section 3. As n is fixed until the last step of the proof, where the limit $n \to \infty$ is considered, we write without risk of confusion x, y, z for x_n, y_n, z_n, respectively.

Let $O_S(\succ)$ denote the set of special objects at $\succ$. We view O_S as a random variable on the space $(\mathscr{P}^n, \mathbb{P})$. Fix an object $o \in O$. By Lemma 1, the fraction of preference profiles $\succ \in \mathscr{P}^n$ for which the random serial dictatorship allocation $\Delta^n(\succ)$ is ordinally efficient at $\succ$ is smaller than or equal to

$$\mathbb{P}(O_S \neq \varnothing) = \mathbb{P}(\cup_{j=1}^n \{\succ \mid o_j \in O_S(\succ)\}) \leq \sum_{j=1}^n \mathbb{P}(o_j \in O_S) = n\mathbb{P}(o \in O_S),$$

where the last equality follows by symmetry.

[16]We denote by $[x]$ the integer part of x, the largest integer that is not larger than x.

Let $(r_i(\succ))_{i\in N}$ be the random vector specifying the ranks assigned by each agent to o under $\succ$, $r_i(\succ)=|\{o' \mid o' \succ_i o\}|+1$. Note that $(r_i)_{i\in N}$ has i.i.d. components, each distributed with equal probability over $\{1,2,\ldots,n\}$. Let $(i_1,i_2,\ldots,i_n)$ be the random vector ordering N increasingly with respect to r, with ties broken according to the indexing of N, i.e.,

$$\cup_{k=1}^{n}\{i_k\}=N$$
$$r_{i_1}\le r_{i_2}\le\cdots\le r_{i_n}$$
$$r_{i_k}=r_{i_l}\Rightarrow(i_k-i_l)(k-l)\ge 0.$$

By definition, $o\in O_S\Rightarrow r_{i_1}=1$.

Motivated by the intuition provided after the statement of the theorem, we write

$$\mathbb{P}(o\in O_S)\le\mathbb{P}(o\in O_S \mid r_{i_{x+1}}>1, r_{i_{x+y}}\le z+1)+\mathbb{P}(r_{i_{x+1}}=1 \text{ or } r_{i_{x+y}}>z+1), \tag{A.2}$$

and define the (random) orderings f_k for $k=1,\ldots,y$ by

$$(f_k(1),f_k(2),\ldots,f_k(n))=(i_{x+y+1},\ldots,i_n,i_{x+k},i_{x+1},\ldots,i_{x+k-1},i_{x+k+1},\ldots i_{x+y},i_1,\ldots,i_x).$$

Note that for all $k=1,\ldots,y$ the serial dictatorship allocations corresponding to f_k are identical for the agents $i_{x+y+1},i_{x+y+2},\ldots,i_n$. Denote the (random) set of objects allocated to these agents by O_0. Let $\underline{O}_k$ be the (random) set of $r_{i_{x+k}}-1$ objects that i_{x+k} prefers to o. Mathematically, for $k=1,\ldots,y$,

$$O_0(\succ)=\{\delta_t^{f_k(\succ)}(\succ) \mid t=i_{x+y+1}(\succ),i_{x+y+2}(\succ),\ldots,i_n(\succ)\}$$
$$\underline{O}_k(\succ)=\{o' \mid o' \succ_{i_{x+k}(\succ)} o\}.$$

If $r_{i_{x+1}}>1$ and o is a special object then the serial dictatorship allocation for the ordering f_k, which is Pareto optimal by Proposition 1, cannot allocate o to any agent in $\{i_{x+y+1},\ldots,i_n,i_{x+k}\}$, hence

$$r_{i_{x+1}}>1 \text{ and } o\in O_S\Rightarrow o\notin O_0 \text{ and } \underline{O}_k\not\subset O_0, \forall k=1,\ldots,y.$$

Thus the first term on the right-hand side of (A.2) satisfies

$$\begin{aligned}
&\mathbb{P}(o\in O_S \mid r_{i_{x+1}}>1, r_{i_{x+y}}\le z+1)\\
&\qquad\le\mathbb{P}(o\notin O_0,\underline{O}_k\not\subset O_0,\forall k=1,\ldots,y \mid r_{i_{x+1}}>1,r_{i_{x+y}}\le z+1)\\
&\qquad\le\mathbb{P}(\underline{O}_k\not\subset O_0,\forall k=1,\ldots,y \mid o\notin O_0, r_{i_{x+1}}>1,r_{i_{x+y}}\le z+1)\\
&\qquad\quad=\mathbb{E}(\mathbb{P}(\underline{O}_k\not\subset O_0,\forall k=1,\ldots,y \mid r,O_0) \mid o\notin O_0, r_{i_{x+1}}>1,r_{i_{x+y}}\le z+1).
\end{aligned}$$

A key observation is that conditional on r,O_0, the sets $(\underline{O}_k)_{k=1,\ldots,y}$ are independently distributed, with $\underline{O}_k$ placing equal probability on all $r_{i_{x+k}}-1$ element subsets of $O\setminus\{o\}$. Also, note that $|O_0|=n-x-y$. Therefore, in the event $o\notin O_0$,

$$\mathbb{P}(\underline{O}_k\not\subset O_0,\forall k=1,\ldots,y \mid r,O_0)=\prod_{k=1}^{y}\left(1-\mathbb{P}(\underline{O}_k\subset O_0 \mid r,O_0)\right)=\prod_{k=1}^{y}\left(1-\frac{\binom{n-x-y}{r_{i_{x+k}}-1}}{\binom{n-1}{r_{i_{x+k}}-1}}\right).$$

Indeed, in the event $o \notin O_0$, conditional on r, O_0, the set $\underline{O}_k$ can take $\binom{n-1}{r_{i_{x+k}}-1}$ values with equal probability, and exactly $\binom{n-x-y}{r_{i_{x+k}}-1}$ of them are included in O_0.

In the event $1 < r_{i_{x+y}} \leq z+1$,

$$\begin{aligned}
\frac{\binom{n-x-y}{r_{i_{x+k}}-1}}{\binom{n-1}{r_{i_{x+k}}-1}} &= \frac{(n-x-y)!(n-r_{i_{x+k}})!}{(n-x-y-r_{i_{x+k}}+1)!(n-1)!} \\
&= \frac{n-x-y}{n-1}\,\frac{n-x-y-1}{n-2}\cdots\frac{n-x-y-r_{i_{x+k}}+2}{n-r_{i_{x+k}}+1} \\
&= \left(1-\frac{x+y-1}{n-1}\right)\left(1-\frac{x+y-1}{n-2}\right)\cdots\left(1-\frac{x+y-1}{n-r_{i_{x+k}}+1}\right) \\
&\geq \left(1-\frac{x+y-1}{n-r_{i_{x+k}}+1}\right)^{r_{i_{x+k}}-1} \\
&\geq \left(1-\frac{x+y-1}{n-z}\right)^{z} \\
&> 1-\frac{(x+y-1)z}{n-z},
\end{aligned}$$

where the first inequality follows from term by term minoration, the second follows from the restriction $r_{i_{x+y}} \leq z+1$, and the last follows from Bernoulli's inequality.[17,18]

Therefore, in the event $o \notin O_0, r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1$, we obtain the bound

$$\mathbb{P}(\underline{O}_k \not\subset O_0, \forall k = 1,\ldots,y \mid r, O_0) < \left(\frac{(x+y-1)z}{n-z}\right)^y,$$

which leads to

$$\mathbb{P}(o \in O_S \mid r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1) < \left(\frac{(x+y-1)z}{n-z}\right)^y.$$

In order to bound the second term on the right-hand side of (A.2), for notational convenience, we replace the probability of the event $r_{i_{x+1}} = 1$ with the probability of the encompassing event $r_{i_x} = 1$, and the probability of the event $r_{i_{x+y}} > z+1$ with the probability of the encompassing event $r_{i_{x+y+1}} \geq z+1$, to obtain

$$\mathbb{P}(r_{i_{x+1}} = 1 \text{ or } r_{i_{x+y}} > z+1) < \mathbb{P}(r_{i_x} = 1) + \mathbb{P}(r_{i_{x+y+1}} \geq z+1).$$

[17] Bernoulli's inequality states that

$$(1+t)^a > 1+at, \forall t > -1, a > 1$$

(Mitrinović 1970, pp. 34-36).

[18] The argument is necessary only for $(x+y-1)/(n-z) < 1$, since otherwise

$$\frac{\binom{n-x-y}{r_{i_{x+k}}-1}}{\binom{n-1}{r_{i_{x+k}}-1}} > 0 \geq 1-\frac{(x+y-1)z}{n-z}.$$

The event $r_{i_x}=1$ is a (non-disjoint) union of $\binom{n}{x}$ events selecting an x-element subset of N in which all agents rank o first; the probability of each such event is $1/n^x$. Therefore,

$$\mathbb{P}(r_{i_x}=1)<\binom{n}{x}\frac{1}{n^x}.$$

Analogously,

$$\mathbb{P}(r_{i_{x+y+1}}\geq z+1)<\binom{n}{n-x-y}\left(\frac{n-z}{n}\right)^{n-x-y}.$$

We can obtain good asymptotic bounds for $\binom{n}{x}$ and $\binom{n}{n-x-y}$ using Sterling's approximation formulae,[19]

$$\binom{n}{x}<\frac{n^{n+1/2}}{x^{x+1/2}(n-x)^{n-x+1/2}}$$
$$\binom{n}{n-x-y}<\frac{n^{n+1/2}}{(x+y)^{x+y+1/2}(n-x-y)^{n-x-y+1/2}}.$$

Therefore,

$$\begin{aligned}\mathbb{P}(r_{i_x}=1)&<\frac{n^{n+1/2}}{x^{x+1/2}(n-x)^{n-x+1/2}n^x}\\&=\left(\frac{1}{x}\right)^{x+1/2}\left(\frac{n}{n-x}\right)^{n-x+1/2}\\&\leq\left(\frac{1}{x}\right)^{x+1/2}e^{\left(\frac{x}{n-x}\right)(n-x+1/2)}\\&<\left(\frac{1}{x}\right)^{x+1/2}e^{x+1/2}\\&=\left(\frac{e}{x}\right)^{x+1/2},\end{aligned}$$

where the second inequality follows from the fact that $1+t\leq e^t$ for all reals t, in particular $n/(n-x)\leq e^{x/(n-x)}$, and the third from simple algebraic manipulation ($x<n/2$).

By similar methods,

$$\begin{aligned}\mathbb{P}(r_{i_{x+y+1}}\geq z+1)&<\frac{n^{n+1/2}}{(x+y)^{x+y+1/2}(n-x-y)^{n-x-y+1/2}}\left(\frac{n-z}{n}\right)^{n-x-y}\\&<\left(\frac{n}{x+y}\right)^{x+y+1/2}\left(\frac{n-z}{n-x-y}\right)^{n-x-y}\\&\leq\left(\frac{n}{x+y}\right)^{x+y+1/2}e^{x+y-z}\\&<\left(\frac{ne}{x+y}\right)^{x+y+1/2}e^{-z}.\end{aligned}$$

[19] One set of Sterling approximation formulae is given by the inequalities

$$\sqrt{2\pi}\,t^{t+1/2}e^{-t}<t!<\sqrt{2\pi}\,t^{t+1/2}e^{-t+1/12},\forall t\geq 1,$$

where e denotes the base of the natural logarithm (Mitrinović 1970, pp. 181–185).

Putting everything together and reinstating the index n for x, y, z, the fraction of preference profiles $\succ \in \mathscr{P}^n$ for which the random serial dictatorship allocation $\Delta^n(\succ)$ is ordinally efficient at $\succ$ is smaller than

$$n\left(\left(\frac{(x_n+y_n-1)z_n}{n-z_n}\right)^{y_n}+\left(\frac{e}{x_n}\right)^{x_n+1/2}+\left(\frac{ne}{x_n+y_n}\right)^{x_n+y_n+1/2}e^{-z_n}\right),$$

which obviously converges to zero as n tends to infinity for x_n, y_n, z_n specified by (A.1). □

Appendix B

Proof of Theorem 2. Let $(w^n)_{n\geq 1}$ be a sequence of popularity weights that satisfies moderate similarity with ratio $s \geq 1$. Recall the definition of $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ in (A.1).

Fix $n \geq 3$. Consider the probability space $(\mathscr{P}^n, \mathbb{P}^n)$ generated by the weights w^n defined in Section 4. As n is fixed until the last step of the proof, where the limit $n \to \infty$ is considered, we write without risk of confusion $\mathbb{P}$ for $\mathbb{P}^n$ (and denote by $\mathbb{E}$ the corresponding expectation operator), w for w^n, and x, y, z for x_n, y_n, z_n, respectively.

Let O_S be the (random) set of special objects. By Lemma 1,

$$\mathbb{P}(\Delta^n(\succ) \text{ is ordinally efficient at } \succ) \leq \mathbb{P}(O_S \neq \varnothing) \leq \sum_{o\in O}\mathbb{P}(o \in O_S).$$

Fix an object o. Redefine $(r_i)_{i\in N}$, $(i_1, i_2, \ldots, i_n)$, $(f_k, \underline{O}_k)_{k=1,\ldots,y}$, and O_0 analogously to Appendix A, for the new probability space.

As in Appendix A, we write

$$\mathbb{P}(o\in O_S) \leq \mathbb{P}(o \in O_S \mid r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1) + \mathbb{P}(r_{i_x} = 1) + \mathbb{P}(r_{i_{x+y+1}} \geq z+1). \tag{B.1}$$

The following lemma enables us to find bounds for each of the three terms on the right-hand side of (B.1), similar to those for the case of the uniform probability distribution over preferences from Appendix A. We omit the trivial proof.

Lemma B.1. *Suppose that $s \geq 1$ and $J \subset N$. The unique solution to the linear program*

$$\max \sum_{j\in J} v_j \text{ subject to } \sum_{l\in N} v_l = 1, v_l > 0, v_l/v_{l'} \leq s, \forall l, l' \in N$$

is $v_l = s/(n+(s-1)|J|)$ for $l \in J$ and $v_l = 1/(n+(s-1)|J|)$ for $l \notin J$. The unique solution to the linear program identical to the one above, except that max *is replaced by* min*, is $v_l = 1/(s(n-|J|)+|J|)$ for $l \in J$ and $v_l = s/(s(n-|J|)+|J|)$ for $l \notin J$.*

By an argument similar to the analogous step in Appendix A, an upper bound for the first term on the right-hand side of (B.1) is

$$\mathbb{E}(\mathbb{P}(\underline{O}_k \not\subset O_0, \forall k=1,\ldots,y \mid r, O_0) \mid o \notin O_0, r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1).$$

Fix a realization $(\tilde r, \tilde O_0)$ of (r, O_0) with $o \notin \tilde O_0, \tilde r_{\tilde i_{x+1}} > 1, \tilde r_{\tilde i_{x+y}} \le z+1$ (note that the random vector i is measurable with respect to the random vector r, hence $r = \tilde r$ determines i uniquely; the corresponding vector value is denoted $\tilde i$). Conditional on $r = \tilde r, O_0 = \tilde O_0$, the sets $(\underline{O}_k)_{k=1,\dots,y}$ are independently distributed; however, unlike in Appendix A, $\underline{O}_k$ does not place equal probability on all $\tilde r_{\tilde i_{x+k}} - 1$ element subsets of $O \setminus \{o\}$. We can still write

$$\mathbb{P}(\underline{O}_k \not\subset O_0, \forall k = 1, \dots, y \mid r = \tilde r, O_0 = \tilde O_0) = \prod_{k=1}^{y} (1 - \mathbb{P}(\underline{O}_k \subset O_0 \mid r = \tilde r, O_0 = \tilde O_0)). \tag{B.2}$$

Lemma B.2. *If* $o \notin \tilde O_0$ *and* $\tilde r_{\tilde i_{x+y}} \le z+1$ *then*

$$\mathbb{P}(\underline{O}_k \subset O_0 \mid r = \tilde r, O_0 = \tilde O_0) > 1 - \frac{sz(x+y+z)}{n}.$$

Proof. Since the preferences of all agents are generated independently,

$$\mathbb{P}(\underline{O}_k \subset O_0 \mid r = \tilde r, O_0 = \tilde O_0) = \mathbb{P}(\underline{O}_k \subset \tilde O_0 \mid r_{\tilde i_{x+k}} = \tilde r_{\tilde i_{x+k}}) = \frac{\mathbb{P}(\underline{O}_k \subset \tilde O_0 \;\&\; r_{\tilde i_{x+k}} = \tilde r_{\tilde i_{x+k}})}{\mathbb{P}(r_{\tilde i_{x+k}} = \tilde r_{\tilde i_{x+k}})}.$$

It follows that $\mathbb{P}(\underline{O}_k \subset O_0 \mid r = \tilde r, O_0 = \tilde O_0)$ equals[20,21]

$$\frac{\displaystyle\sum_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}} \in \tilde O_0} w_{j_1} \frac{w_{j_2}}{1-w_{j_1}} \cdots \frac{w_{j_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j_1}-\cdots-w_{j_{\tilde r_{\tilde i_{x+k}}-2}}} \frac{w_o}{1-w_{j_1}-\cdots-w_{j_{\tilde r_{\tilde i_{x+k}}-1}}}}{\displaystyle\sum_{\neq o_{j'_1},\dots,o_{j'_{\tilde r_{\tilde i_{x+k}}-1}} \in O\setminus\{o\}} w_{j'_1} \frac{w_{j'_2}}{1-w_{j'_1}} \cdots \frac{w_{j'_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j'_1}-\cdots-w_{j'_{\tilde r_{\tilde i_{x+k}}-2}}} \frac{w_o}{1-w_{j'_1}-\cdots-w_{j'_{\tilde r_{\tilde i_{x+k}}-1}}}},$$

which is greater than or equal to

$$\min_{\substack{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}} \in \tilde O_0 \\ \neq o_{j'_1},\dots,o_{j'_{\tilde r_{\tilde i_{x+k}}-1}} \in O\setminus\{o\}}} \left(\frac{\dfrac{w_o}{1-w_{j_1}-\cdots-w_{j_{\tilde r_{\tilde i_{x+k}}-1}}}}{\dfrac{w_o}{1-w_{j'_1}-\cdots-w_{j'_{\tilde r_{\tilde i_{x+k}}-1}}}} \right) \frac{\displaystyle\sum_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}} \in \tilde O_0} w_{j_1} \frac{w_{j_2}}{1-w_{j_1}} \cdots \frac{w_{j_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j_1}-\cdots-w_{j_{\tilde r_{\tilde i_{x+k}}-2}}}}{\displaystyle\sum_{\neq o_{j'_1},\dots,o_{j'_{\tilde r_{\tilde i_{x+k}}-1}} \in O\setminus\{o\}} w_{j'_1} \frac{w_{j'_2}}{1-w_{j'_1}} \cdots \frac{w_{j'_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j'_1}-\cdots-w_{j'_{\tilde r_{\tilde i_{x+k}}-2}}}},$$

[20] We abuse notation in writing w_o for the popularity weight of object o; if j is such that $o = o_j$, then $w_o := w_j$. The symbol "$\neq$" in the indexing of various summations stands for "sequence with distinct terms."

[21] For example, the term

$$w_{j_1} \frac{w_{j_2}}{1-w_{j_1}} \cdots \frac{w_{j_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j_1}-\cdots-w_{j_{\tilde r_{\tilde i_{x+k}}-2}}} \frac{w_o}{1-w_{j_1}-\cdots-w_{j_{\tilde r_{\tilde i_{x+k}}-1}}}$$

in the numerator represents the probability that the most preferred objects under $\succ_{\tilde i_{x+k}}$ are in order $o_{j_1}, o_{j_2}, \dots, o_{j_{\tilde r_{\tilde i_{x+k}}-1}}$ and o. Hence the numerator represents $\mathbb{P}(\underline{O}_k \subset \tilde O_0 \;\&\; r_{\tilde i_{x+k}} = \tilde r_{\tilde i_{x+k}})$ and the denominator represents $\mathbb{P}(r_{\tilde i_{x+k}} = \tilde r_{\tilde i_{x+k}})$.

which after canceling the w_o term becomes

$$\frac{\min_{\neq o_{j'_1},\dots,o_{j'_{\tilde r_{\tilde i_{x+k}}-1}}\in O\setminus\{o\}} 1-(w_{j'_1}+w_{j'_2}+\cdots+w_{j'_{\tilde r_{\tilde i_{x+k}}-1}})}{\max_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}}\in\tilde O_0} 1-(w_{j_1}+w_{j_2}+\cdots+w_{j_{\tilde r_{\tilde i_{x+k}}-1}})}\times \frac{\sum_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}}\in\tilde O_0} w_{j_1}\frac{w_{j_2}}{1-w_{j_1}}\cdots\frac{w_{j_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j_1}-\cdots-w_{j_{\tilde r_{\tilde i_{x+k}}-2}}}}{\sum_{\neq o_{j'_1},\dots,o_{j'_{\tilde r_{\tilde i_{x+k}}-1}}\in O\setminus\{o\}} w_{j'_1}\frac{w_{j'_2}}{1-w_{j'_1}}\cdots\frac{w_{j'_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j'_1}-\cdots-w_{j'_{\tilde r_{\tilde i_{x+k}}-2}}}}.$$

The denominators satisfy the following inequalities:

$$\sum_{\neq o_{j'_1},\dots,o_{j'_{\tilde r_{\tilde i_{x+k}}-1}}\in O\setminus\{o\}} w_{j'_1}\frac{w_{j'_2}}{1-w_{j'_1}}\cdots\frac{w_{j'_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j'_1}-\cdots-w_{j'_{\tilde r_{\tilde i_{x+k}}-2}}}=\mathbb{P}(r_{\tilde i_{x+k}}\geq\tilde r_{\tilde i_{x+k}})\leq 1$$

$$\max_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}}\in\tilde O_0} 1-(w_{j_1}+w_{j_2}+\cdots+w_{j_{\tilde r_{\tilde i_{x+k}}-1}})\leq 1.$$

To bound the second numerator,

$$\begin{aligned}\sum_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}}\in\tilde O_0} w_{j_1}\frac{w_{j_2}}{1-w_{j_1}}\cdots\frac{w_{j_{\tilde r_{\tilde i_{x+k}}-1}}}{1-w_{j_1}-\cdots-w_{j_{\tilde r_{\tilde i_{x+k}}-2}}} &\geq \sum_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}}\in\tilde O_0} w_{j_1}\dots w_{j_{\tilde r_{\tilde i_{x+k}}-1}}\\ &= \sum_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-2}}\in\tilde O_0} w_{j_1}\dots w_{j_{\tilde r_{\tilde i_{x+k}}-2}} \sum_{o_{j_{\tilde r_{\tilde i_{x+k}}-1}}\in\tilde O_0\setminus\{o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-2}}\}} w_{j_{\tilde r_{\tilde i_{x+k}}-1}}.\end{aligned}$$

Since the set $\tilde O_0\setminus\{o_{j_1},o_{j_2},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-2}}\}$ has at least $n-x-y-z+1$ elements ($|\tilde O_0|=n-x-y$, $\tilde r_{\tilde i_{x+k}}\leq z+1$), it follows by Lemma B.1 that

$$\sum_{o_{j_{\tilde r_{\tilde i_{x+k}}-1}}\in\tilde O_0\setminus\{o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-2}}\}} w_{j_{\tilde r_{\tilde i_{x+k}}-1}}\geq 1-\frac{s(x+y+z-1)}{n+(s-1)(x+y+z-1)}\geq 1-\frac{s(x+y+z-1)}{n},$$

so

$$\sum_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-1}}\in\tilde O_0} w_{j_1}\dots w_{j_{\tilde r_{\tilde i_{x+k}}-1}}\geq\left(1-\frac{s(x+y+z-1)}{n}\right)\sum_{\neq o_{j_1},\dots,o_{j_{\tilde r_{\tilde i_{x+k}}-2}}\in\tilde O_0} w_{j_1}\dots w_{j_{\tilde r_{\tilde i_{x+k}}-2}},$$

and reiterating the argument $\tilde r_{\tilde i_{x+k}}-1$ times, the second numerator is larger than or equal to

$$\left(1-\frac{s(x+y+z-1)}{n}\right)^{\tilde r_{\tilde i_{x+k}}-1}\geq\left(1-\frac{s(x+y+z-1)}{n}\right)^{z}>1-\frac{sz(x+y+z-1)}{n},$$

where the first inequality follows from $\tilde{r}_{\tilde{i}_{x+k}} \leq z+1$ and the second from Bernoulli's inequality (footnote 17).[22]

The following bound for the first numerator is obtained from Lemma B.1:

$$\min_{\neq o_{j'_1},\dots,o_{j'_{\tilde{r}_{\tilde{i}_{x+k}}-1}} \in O\setminus\{o\}} 1-(w_{j'_1}+w_{j'_2}+\cdots+w_{j'_{\tilde{r}_{\tilde{i}_{x+k}}-1}}) \geq 1-\frac{sz}{n+(s-1)z} \geq 1-\frac{sz}{n}.$$

Therefore,

$$\mathbb{P}(\underline{O}_k \subset O_0 \mid r=\tilde{r}, O_0=\tilde{O}_0) \geq \left(1-\frac{sz}{n}\right)\left(1-\frac{sz(x+y+z-1)}{n}\right) > 1-\frac{sz(x+y+z)}{n},$$

which finishes the proof of Lemma B.2. □

By (B.2) and Lemma B.2, the first term on the right-hand side of (B.1) satisfies

$$\mathbb{P}(o \in O_S \mid r_{i_{x+1}} > 1, r_{i_{x+y}} \leq z+1) < \left(\frac{sz(x+y+z)}{n}\right)^y. \tag{B.3}$$

In order to bound the second term on the right-hand side of (B.1), note that the event $r_{i_x}=1$ is a (non-disjoint) union of $\binom{n}{x}$ events selecting an x-element subset of N in which all agents rank o first; the probability of each such event is $(w_o)^x$. By Lemma B.1, $w_o \leq s/(n+s-1) \leq s/n$. Then using the inequalities in the analogous step of Appendix A,

$$\mathbb{P}(r_{i_x}=1) < \binom{n}{x}\left(\frac{s}{n}\right)^x \leq s^x \binom{n}{x}\frac{1}{n^x} < \left(\frac{es}{x}\right)^{x+1/2}. \tag{B.4}$$

Analogously, to bound the third term on the right-hand side of (B.1),

$$\mathbb{P}(r_{i_{x+y+1}} \geq z+1) < \binom{n}{n-x-y}\mathbb{P}(r_1 \geq z+1)^{n-x-y},$$

where the probability that all agents in a (fixed) $n-x-y$-element subset of N do not rank o among the top z objects has been evaluated to be $\mathbb{P}(r_1 \geq z+1)^{n-x-y}$ by symmetry and independence of the preferences of the agents.

Lemma B.3.

$$\mathbb{P}(r_1 \geq z+1) \leq \left(1-\frac{1}{sn}\right)^z.$$

Proof. Note that

$$\mathbb{P}(r_1 \geq t+1) = \mathbb{P}(r_1 \geq t)\mathbb{P}(r_1 \neq t \mid r_1 \geq t) \leq \mathbb{P}(r_1 \geq t)(1-w_o). \tag{B.5}$$

[22] The argument is necessary only for $s(x+y+z-1)/n < 1$, since otherwise

$$\mathbb{P}(\underline{O}_k \subset O_0 \mid r=\tilde{r}, O_0=\tilde{O}_0) \geq 0 > 1-\frac{sz(x+y+z)}{n}.$$

Indeed, taking into account the stochastic generation of $\succ_1$, $\mathbb{P}(r_1 \neq t \mid r_1 \geq t) \leq 1 - w_o$ because for any possible draw $o_{j_1}, \ldots, o_{j_{t-1}}$ of the top ranked $t-1$ objects out of $O \setminus \{o\}$, the probability that o will be drawn as the t^{th} ranked object is $w_o/(1 - w_{j_1} - \cdots - w_{j_{t-1}}) \geq w_o$, hence the conditional probability that o will not be drawn as the t^{th} ranked object is not larger than $1 - w_o$.

Multiplying the inequalities (B.5) for $t = 1, \ldots, z$ we obtain $\mathbb{P}(r_1 \geq z+1) \leq (1-w_o)^z$. Since $w_o \geq 1/(s(n-1)+1) \geq 1/(sn)$ by Lemma B.1, it follows that

$$\mathbb{P}(r_1 \geq z+1) \leq \left(1 - \frac{1}{sn}\right)^z. \qquad \square$$

By Lemma B.3,

$$\mathbb{P}(r_{i_{x+y+1}} \geq z+1) < \binom{n}{n-x-y}\left(1 - \frac{1}{sn}\right)^{z(n-x-y)}.$$

By inequalities similar to those used in the analogous step of Appendix A,

$$\begin{aligned}
\mathbb{P}(r_{i_{x+y+1}} \geq z+1) &< \frac{n^{n+1/2}}{(x+y)^{x+y+1/2}(n-x-y)^{n-x-y+1/2}}\left(1-\frac{1}{sn}\right)^{z(n-x-y)} && \text{(B.6)}\\
&< \left(\frac{n}{x+y}\right)^{x+y+1/2}\left(\frac{n}{n-x-y}\right)^{n-x-y}\left(1-\frac{1}{sn}\right)^{z(n-x-y)} && \text{(B.7)}\\
&\leq \left(\frac{n}{x+y}\right)^{x+y+1/2} e^{\frac{x+y}{n-x-y}(n-x-y)} e^{-\frac{1}{sn}z(n-x-y)} && \text{(B.8)}\\
&< \left(\frac{ne}{x+y}\right)^{x+y+1/2} e^{-\frac{z(n-x-y)}{sn}}. && \text{(B.9)}
\end{aligned}$$

Substituting (B.3), (B.4), and (B.6)–(B.9) in (B.1) and reinstating the index n for x, y, z, we obtain

$$\mathbb{P}(o \in O_S) \leq \left(\frac{sz_n(x_n+y_n+z_n)}{n}\right)^{y_n} + \left(\frac{es}{x_n}\right)^{x_n+1/2} + \left(\frac{ne}{x_n+y_n}\right)^{x_n+y_n+1/2} e^{-z_n\frac{n-x_n-y_n}{sn}},$$

so the probability that the random serial dictatorship allocation $\Delta^n(\succ)$ is ordinally efficient at $\succ$ is smaller than

$$n\left(\left(\frac{sz_n(x_n+y_n+z_n)}{n}\right)^{y_n} + \left(\frac{es}{x_n}\right)^{x_n+1/2} + \left(\frac{ne}{x_n+y_n}\right)^{x_n+y_n+1/2} e^{-z_n\frac{n-x_n-y_n}{sn}}\right),$$

which converges to zero as n tends to infinity for x_n, y_n, z_n specified by (A.1). $\square$

Proof for Remark 5. The details of the proof of Theorem 2 remain unchanged up to the last step. We obtain that the probability that $\Delta^n(\succ)$ is ordinally efficient at $\succ$ is smaller than

$$n\left(\left(\frac{s_n z_n(x_n+y_n+z_n)}{n}\right)^{y_n} + \left(\frac{es_n}{x_n}\right)^{x_n+1/2} + \left(\frac{ne}{x_n+y_n}\right)^{x_n+y_n+1/2} e^{-\frac{z_n}{s_n}\frac{n-x_n-y_n}{n}}\right).$$

For x_n, y_n, z_n specified according to (A.1), if $\lim_{n\to\infty} s_n/n^{\zeta} = 0$ for some $\zeta < 1/5$, then the expression above converges to zero as n tends to infinity. $\square$

Appendix C

Proof that $\mathbb{P}^n(o_j \succ_i o_k) = w_j/(w_j + w_k)$. Note that

$$\mathbb{P}^n(o_j \succ_i o_k) = \sum_{l_1,\dots,l_r \neq j,k} w_{l_1} \frac{w_{l_2}}{1-w_{l_1}} \cdots \frac{w_{l_r}}{1-w_{l_1}-\cdots-w_{l_{r-1}}} \frac{w_j}{1-w_{l_1}-\cdots-w_{l_r}}$$
$$= w_j \sum_{l_1,\dots,l_r \neq j,k} w_{l_1} \frac{w_{l_2}}{1-w_{l_1}} \cdots \frac{w_{l_r}}{1-w_{l_1}-\cdots-w_{l_{r-1}}} \frac{1}{1-w_{l_1}-\cdots-w_{l_r}},$$

and

$$\mathbb{P}^n(o_k \succ_i o_j) = \sum_{l_1,\dots,l_r \neq j,k} w_{l_1} \frac{w_{l_2}}{1-w_{l_1}} \cdots \frac{w_{l_r}}{1-w_{l_1}-\cdots-w_{l_{r-1}}} \frac{w_k}{1-w_{l_1}-\cdots-w_{l_r}}$$
$$= w_k \sum_{l_1,\dots,l_r \neq j,k} w_{l_1} \frac{w_{l_2}}{1-w_{l_1}} \cdots \frac{w_{l_r}}{1-w_{l_1}-\cdots-w_{l_{r-1}}} \frac{1}{1-w_{l_1}-\cdots-w_{l_r}},$$

where summations are over all sequences $l_1,\dots,l_r$ of distinct numbers in $\{1,2,\dots,m\} \setminus \{j,k\}$.[23] Hence

$$\frac{\mathbb{P}^n(o_j \succ_i o_k)}{1-\mathbb{P}^n(o_j \succ_i o_k)} = \frac{\mathbb{P}^n(o_j \succ_i o_k)}{\mathbb{P}^n(o_k \succ_i o_j)} = \frac{w_j}{w_k},$$

which leads to $\mathbb{P}^n(o_j \succ_i o_k) = w_j/(w_j + w_k)$. □

Proof of Lemma 2. Suppose that $o_j \rhd[\Delta^n(\succ),\succ]\, o_k$. Then there exists an agent who receives o_k with positive probability at $\Delta^n(\succ)$, but prefers o_j to o_k. It follows that there exists an ordering $f \in \mathscr{F}^n$ such that the copies of o_j are exhausted before the copies of o_k in the serial dictatorship allocation $\delta^f(\succ)$.

To prove part (i), let o_l be the object whose copies are exhausted before the copies of any other object at $\delta^f(\succ)$. It must be that $l \neq k$, as the copies of o_j are exhausted before the copies of o_k at $\delta^f(\succ)$. No agent who does not rank o_l as his most preferred object receives o_l at $\delta^f(\succ)$,[24] so only one of the $\nu_l^n(\succ)$ agents ranking o_l as most preferred may receive o_l at $\delta^f(\succ)$. But nq_l agents receive o_l at $\delta^f(\succ)$, hence $\nu_l^n(\succ) \geq nq_l$.

To prove part (ii), note that since the copies of o_j are exhausted before the copies of o_k at $\delta^f(\succ)$, no agent who prefers o_k to o_j receives o_j at $\delta^f(\succ)$,[25] so only one of the

[23]For example, the term

$$w_{l_1} \frac{w_{l_2}}{1-w_{l_1}} \cdots \frac{w_{l_r}}{1-w_{l_1}-\cdots-w_{l_{r-1}}} \frac{w_j}{1-w_{l_1}-\cdots-w_{l_r}}$$

in the expression for $\mathbb{P}^n(o_j \succ_i o_k)$ represents the probability that the most preferred objects under $\succ_i$ are in order $o_{l_1}, o_{l_2}, \dots, o_{l_r}$ and o_j.

[24]If an agent who does not rank o_l as his most preferred object picks o_l at $\delta^f(\succ)$ then at the time of his choice not all copies of o_l are exhausted, thus by the definition of o_l a copy of his most preferred object must also be available. However, it is not optimal for the agent to pick o_l if his most preferred object is available.

[25]If an agent who prefers o_k to o_j picks o_j at $\delta^f(\succ)$ then at the time of his choice not all copies of o_j are exhausted, thus a copy of o_k must also be available. However, it is not optimal for the agent to pick o_j if o_k is available.

$v^n_{\_j\_k}(\succ)$ agents ranking o_j higher than o_k in their preferences may receive o_j at $\delta^f(\succ)$. But nq_j agents receive o_j at $\delta^f(\succ)$, hence $v^n_{\_j\_k}(\succ) \geq nq_j$. □

Proof of Lemma 3. If $\lim_{n\to\infty} \mathbb{P}^n(o_j \rhd[\Delta^n(\succ),\succ]\, o_k) = 0, \forall j > k$ then

$$\lim_{n\to\infty} \mathbb{P}^n(o_j \not\rhd[\Delta^n(\succ),\succ]\, o_k, \forall j > k) = 1.$$

However, note that the condition $o_j \not\rhd[\Delta^n(\succ),\succ]\, o_k, \forall j > k$ implies that $\rhd[\Delta^n(\succ),\succ]$ is acyclic, which by Proposition 2 is equivalent to $\Delta^n(\succ) \in \mathscr{E}^n(\succ)$. Therefore, $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 1$. □

Proof of Proposition 4. Recall that

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}^n\left(\left|\frac{v^n_j}{n} - w_j\right| > \varepsilon\right) = \lim_{n\to\infty} \mathbb{P}^n\left(\left|\frac{v^n_{jk}}{n} - \frac{w_j w_k}{1 - w_j}\right| > \varepsilon\right) = 0. \tag{C.1}$$

Proof of Part (i), $w_2 > q_2$. Suppose that $w_2 > q_2$. The inequalities $w_1/q_1 \geq w_2/q_2 > 1$ and (C.1) imply that $\lim_{n\to\infty} \mathbb{P}^n(v^n_j > nq_j, v^n_{j(3-j)} \geq 1, \forall j \in \{1,2\}) = 1$, thus[26]

$$\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = \lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n \;\&\; v^n_j > nq_j, v^n_{j(3-j)} \geq 1, \forall j \in \{1,2\}). \tag{C.2}$$

Fix n and let $\succ \in \mathscr{P}^n$ be a preference profile such that $v^n_j(\succ) > nq_j, v^n_{j(3-j)}(\succ) \geq 1$ for $j = 1,2$. Let $j \in \{1,2\}$. Then there exists an ordering of the n agents such that the top ranked objects under $\succ$ are o_j for the first nq_j agents (possible because $v^n_j(\succ) > nq_j$), and o_j, o_{3-j}, with o_j preferred to o_{3-j}, for the $nq_j + 1^{st}$ agent (possible because $v^n_{j(3-j)}(\succ) \geq 1$). In the table below columns represent the preferences of the agents, listed according to the ordering.

nq_j					
o_j	o_j	$\cdots$	o_j	o_j	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	o_{3-j}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

In the serial dictatorship for this ordering the first nq_j agents receive o_j, and the $nq_j + 1^{st}$ agent receives o_{3-j}. Since the $nq_j + 1^{st}$ agent prefers o_j to o_{3-j}, it follows that $o_j \rhd[\Delta^n(\succ),\succ]\, o_{3-j}$. Therefore, $o_1 \rhd[\Delta^n(\succ),\succ]\, o_2$ and $o_2 \rhd[\Delta^n(\succ),\succ]\, o_1$. Then Proposition 2 implies that $\Delta^n(\succ)$ is not ordinally efficient at $\succ$. Thus

$$\mathbb{P}^n(\Delta^n \in \mathscr{E}^n \;\&\; v^n_j > nq_j, v^n_{j(3-j)} \geq 1, \forall j \in \{1,2\}) = 0,$$

which together with (C.2) leads to $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 0$.

[26]Throughout the proofs of Proposition 4 and Theorems 3 and 4, when existence of the considered limits is not immediately obvious, it is a consequence of subsequent arguments.

Proof of Part (i), $w_3/q_3 > 1 - w_1$. Suppose that $w_3/q_3 > 1 - w_1$. If $w_2 > q_2$ then $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 0$, as already argued. Hence we may focus on the situation in which $w_2 \le q_2$. We solve only the case $w_2 < q_2$, as the case $w_2 = q_2$ can be approached using similar ideas (cf. Theorem 3).

Let $j \in \{2,3\}$. The condition $w_2/q_2 \ge w_3/q_3 > 1 - w_1$ implies that $w_j + w_1 w_j/(1-w_1) > q_j$, which by (C.1) leads to

$$\lim_{n\to\infty} \mathbb{P}^n(v_j^n + v_{1j}^n > nq_j) = 1.$$

The inequalities $1 > w_2/q_2 \ge \cdots \ge w_m/q_m$ (here we use the assumption that $w_2 < q_2$), $\sum_{k=1}^m w_k = \sum_{k=1}^m q_k = 1$ and $m \ge 3$ imply that $w_1 > q_1$ and $w_1 + w_j > q_1 + q_j$, which by (C.1) lead to

$$\lim_{n\to\infty} \mathbb{P}^n(v_1^n > nq_1) = 1$$
$$\lim_{n\to\infty} \mathbb{P}^n(v_1^n + v_j^n > n(q_1 + q_j)) = 1.$$

Also by (C.1),

$$\lim_{n\to\infty} \mathbb{P}^n(v_{j(5-j)}^n \ge 1) = 1.$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = \lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n \;\&\; v_1^n > nq_1, v_j^n + v_{1j}^n > nq_j, v_1^n + v_j^n > n(q_1+q_j), v_{j(5-j)}^n \ge 1, \forall j \in \{2,3\}). \tag{C.3}$$

Fix n and let $\succ \in \mathscr{P}^n$ be a preference profile such that $v_1^n(\succ) > nq_1, v_j^n(\succ) + v_{1j}^n(\succ) > nq_j, v_1^n(\succ) + v_j^n(\succ) > n(q_1 + q_j), v_{j(5-j)}^n(\succ) \ge 1$ for $j = 2,3$. Let $j \in \{2,3\}$. Then there exists an ordering of the n agents such that the top ranked objects under $\succ$ are

- o_1 for the first nq_1 agents (possible because $v_1^n(\succ) > nq_1$)
- o_1, o_j, with o_1 preferred to o_j, for the next $\min(v_{1j}^n(\succ), v_1^n(\succ) - nq_1, nq_j)$ agents (possible because if $v_1^n(\succ) - nq_1 \ge v_{1j}^n(\succ)$ then the first nq_1 agents may be selected among those who rank o_1 first, but do not rank o_j second under $\succ$, so that there are $v_{1j}^n(\succ)$ agents left who rank o_1 first and o_j second under $\succ$; and if $v_{1j}^n(\succ) > v_1^n(\succ) - nq_1$ then the first nq_1 agents may be selected so that o_j is ranked second by all other $v_1^n(\succ) - nq_1$ agents who rank o_1 first under $\succ$)
- o_j for the next $nq_j - \min(v_{1j}^n(\succ), v_1^n(\succ) - nq_1, nq_j)$ agents (possible because $v_j^n(\succ) > nq_j - v_{1j}^n(\succ)$ and $v_j^n(\succ) > nq_j - (v_1^n(\succ) - nq_1)$)
- o_j, o_{5-j}, with o_j preferred to o_{5-j}, for the $n(q_1 + q_j) + 1^{st}$ agent (possible because $v_{j(5-j)}^n(\succ) \ge 1$).

In the table below columns represent the preferences of the agents, listed according to the ordering.

nq_1			$\min(v^n_{1j}(\succ), v^n_1(\succ)-nq_1, nq_j)$			$nq_j-\min(v^n_{1j}(\succ), v^n_1(\succ)-nq_1, nq_j)$				
$\boxed{o_1}$	$\cdots$	$\boxed{o_1}$	o_1	$\cdots$	o_1	$\boxed{o_j}$	$\cdots$	$\boxed{o_j}$	o_j	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\boxed{o_j}$	$\cdots$	$\boxed{o_j}$	$\vdots$	$\vdots$	$\vdots$	$\boxed{o_{5-j}}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

In the serial dictatorship for this ordering the first nq_1 agents receive o_1, the next nq_j agents receive o_j, and the $n(q_1+q_j)+1^{st}$ agent receives o_{5-j}. Since the $n(q_1+q_j)+1^{st}$ agent prefers o_j to o_{5-j}, it follows that $o_j \rhd[\Delta^n(\succ),\succ]\, o_{5-j}$. Therefore, $o_2 \rhd[\Delta^n(\succ),\succ]\, o_3$ and $o_3 \rhd[\Delta^n(\succ),\succ]\, o_2$. Then Proposition 2 implies that $\Delta^n(\succ)$ is not ordinally efficient at $\succ$. Thus

$$\mathbb{P}^n(\Delta^n \in \mathscr{E}^n \,\&\, v^n_1 > nq_1, v^n_j + v^n_{1j} > nq_j, v^n_1 + v^n_j > n(q_1+q_j), v^n_{j(5-j)} \geq 1, \forall j \in \{2,3\}) = 0,$$

which together with (C.3) leads to $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 0$.

Proof of Part (ii). Suppose that $m=3$, $w_2 < q_2$ and $w_3/q_3 < 1-w_1$. Fix n and a preference profile $\succ \in \mathscr{P}^n$. By part (i) of Lemma 2,

$$\mathbb{P}^n(o_2 \rhd[\Delta^n(\succ),\succ]\, o_1 \text{ or } o_3 \rhd[\Delta^n(\succ),\succ]\, o_1) \leq \mathbb{P}^n(v^n_2 \geq nq_2 \text{ or } v^n_3 \geq nq_3).$$

But by (C.1), since $1 > w_2/q_2 \geq w_3/q_3$, it follows that $\lim_{n\to\infty} \mathbb{P}^n(v^n_2 \geq nq_2 \text{ or } v^n_3 \geq nq_3) = 0$, hence

$$\lim_{n\to\infty} \mathbb{P}^n(o_2 \rhd[\Delta^n(\succ),\succ]\, o_1 \text{ or } o_3 \rhd[\Delta^n(\succ),\succ]\, o_1) = 0. \tag{C.4}$$

By part (ii) of Lemma 2, as $v^n_{\_3\_2} = v^n_{13} + v^n_3$ $(m=3)$,

$$\mathbb{P}^n(o_3 \rhd[\Delta^n(\succ),\succ]\, o_2) \leq \mathbb{P}^n(v^n_{13} + v^n_3 \geq nq_3).$$

Since $w_3/q_3 < 1-w_1$ can be rewritten as $w_1 w_3/(1-w_1) + w_3 < q_3$, (C.1) implies that $\lim_{n\to\infty} \mathbb{P}^n(v^n_{13} + v^n_3 \geq nq_3) = 0$. Thus

$$\lim_{n\to\infty} \mathbb{P}^n(o_3 \rhd[\Delta^n(\succ),\succ]\, o_2) = 0. \tag{C.5}$$

As the conditions (C.4) and (C.5) hold, the hypothesis of Lemma 3 is satisfied $(m=3)$, and therefore $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 1$. □

Proof of Theorem 3. Recall that

$$\forall \varepsilon > 0, \lim_{n\to\infty} \mathbb{P}^n\left(\left|\frac{v^n_j}{n} - w_j\right| > \varepsilon\right) = \lim_{n\to\infty} \mathbb{P}^n\left(\left|\frac{v^n_{jk}}{n} - \frac{w_j w_k}{1-w_j}\right| > \varepsilon\right) = 0. \tag{C.6}$$

Since $\lim_{n\to\infty} \mathbb{P}^n(v^n_j = nq_j) = 0$ for $j = 1, 2, \ldots, m$, it follows that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) &= \lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n \,\&\, \exists g_1, g_2 \text{ such that } v^n_{g_1} > nq_{g_1} \text{ and } v^n_{g_2} > nq_{g_2}) \\ &\quad + \lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n \,\&\, \exists g \text{ such that } v^n_g > nq_g \text{ and } v^n_h < nq_h, \forall h \neq g). \end{aligned} \tag{C.7}$$

As $\lim_{n\to\infty}\mathbb{P}^n(v^n_{jk}\geq 1)=1$ for all j,k (by (C.6)), the first term on the right-hand side of (C.7) equals

$$\lim_{n\to\infty}\mathbb{P}^n(\Delta^n\in\mathscr{E}^n\ \&\ \exists g_1,g_2 \text{ such that } v^n_{g_1}>nq_{g_1} \text{ and } v^n_{g_2}>nq_{g_2} \text{ with } v^n_{g_1g_2},v^n_{g_2g_1}\geq 1).$$

Fix n and let $\succ\in\mathscr{P}^n$ be a preference profile such that $v^n_{g_1}(\succ)>nq_{g_1}, v^n_{g_2}(\succ)>nq_{g_2}$ and $v^n_{g_1g_2}(\succ), v^n_{g_2g_1}(\succ)\geq 1$ for some g_1 and g_2. Then there exists an ordering of the n agents such that the top ranked objects under $\succ$ are o_{g_1} for the first nq_{g_1} agents (possible because $v^n_{g_1}(\succ)>nq_{g_1}$), and o_{g_1}, o_{g_2}, with o_{g_1} preferred to o_{g_2}, for the $nq_{g_1}+1^{st}$ agent (possible because $v^n_{g_1g_2}(\succ)\geq 1$). In the table below columns represent the preferences of the agents, listed according to the ordering.

$$\begin{array}{cccccc}
\multicolumn{4}{c}{\overbrace{\hspace{6cm}}^{nq_{g_1}}} & & \\
\boxed{o_{g_1}} & \boxed{o_{g_1}} & \cdots & \boxed{o_{g_1}} & o_{g_1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \boxed{o_{g_2}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

In the serial dictatorship for this ordering the first nq_{g_1} agents receive o_{g_1}, and the $nq_{g_1}+1^{st}$ agent receives o_{g_2}. Since the $nq_{g_1}+1^{st}$ agent prefers o_{g_1} to o_{g_2}, it follows that $o_{g_1}\rhd[\Delta^n(\succ),\succ]\,o_{g_2}$. By symmetry, $o_{g_2}\rhd[\Delta^n(\succ),\succ]\,o_{g_1}$. Then Proposition 2 implies that $\Delta^n(\succ)$ is not ordinally efficient at $\succ$. Thus

$$\mathbb{P}^n(\Delta^n\in\mathscr{E}^n\ \&\ \exists g_1,g_2 \text{ such that } v^n_{g_1}>nq_{g_1} \text{ and } v^n_{g_2}>nq_{g_2} \text{ with } v^n_{g_1g_2},v^n_{g_2g_1}\geq 1)=0,$$

which proves that the first term on the right-hand side of (C.7) is zero.

For all j,k the set of conditions (C.6) and $w_j=q_j$ imply that $\lim_{n\to\infty}\mathbb{P}^n(v^n_{jk}>v^n_j-nq_j)=1$; also by (C.6), $\lim_{n\to\infty}\mathbb{P}^n(v^n_{jk}\geq 1)=1$. Then the second term on the right-hand side of (C.7) equals

$$\lim_{n\to\infty}\mathbb{P}^n(\Delta^n\in\mathscr{E}^n\ \&\ \exists g \text{ such that}$$
$$v^n_g>nq_g \text{ and } v^n_h<nq_h, v^n_{gh}>v^n_g-nq_g, \forall h\neq g, v^n_{h_1h_2}\geq 1, \forall h_1,h_2\neq g).$$

Fix n and let $\succ\in\mathscr{P}^n$ be a preference profile such that there exists g with $v^n_g(\succ)>nq_g$, and $v^n_h(\succ)<nq_h, v^n_{gh}(\succ)>v^n_g(\succ)-nq_g$ for all $h\neq g$ and $v^n_{h_1h_2}(\succ)\geq 1$ for all h_1, $h_2\neq g$.

Fix $h_1,h_2\neq g$. The conditions

$$v^n_h(\succ)<nq_h, \forall h\neq g, \sum_{j=1}^m v^n_j(\succ)=\sum_{j=1}^m nq_j, \text{ and } m\geq 3$$

lead to $v^n_g(\succ)+v^n_{h_1}(\succ)>nq_g+nq_{h_1}$. Then there exists an ordering of the n agents such that the top ranked objects under $\succ$ are o_g for the first nq_g agents (possible because $v^n_g(\succ)>nq_g$); o_g, o_{h_1}, with o_g preferred to o_{h_1}, for the next $\min(v^n_g(\succ)-nq_g, nq_{h_1})$

agents (possible because $v^n_{gh_1}(\succ) > v^n_g(\succ) - nq_g$); o_{h_1} for the next $nq_{h_1} - \min(v^n_g(\succ) - nq_g, nq_{h_1})$ agents (possible because $v^n_{h_1}(\succ) > nq_{h_1} - (v^n_g(\succ) - nq_g)$); and o_{h_1}, o_{h_2}, with o_{h_1} preferred to o_{h_2}, for the $n(q_g + q_{h_1}) + 1^{st}$ agent (possible because $v^n_{h_1h_2}(\succ) \geq 1$). In the table below columns represent the preferences of the agents, listed according to the ordering.

nq_g			$\min(v^n_g(\succ)-nq_g, nq_{h_1})$			$nq_{h_1} - \min(v^n_g(\succ)-nq_g, nq_{h_1})$				
$\boxed{o_g}$	$\cdots$	$\boxed{o_g}$	o_g	$\cdots$	o_g	$\boxed{o_{h_1}}$	$\cdots$	$\boxed{o_{h_1}}$	o_{h_1}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\boxed{o_{h_1}}$	$\cdots$	$\boxed{o_{h_1}}$	$\vdots$	$\vdots$	$\vdots$	$\boxed{o_{h_2}}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

In the serial dictatorship for this ordering the first nq_g agents receive o_g, the next nq_{h_1} agents receive o_{h_1}, and the $n(q_g + q_{h_1}) + 1^{st}$ agent receives o_{h_2}. Since the $n(q_g + q_{h_1}) + 1^{st}$ agent prefers o_{h_1} to o_{h_2}, it follows that $o_{h_1} \rhd [\Delta^n(\succ), \succ]\, o_{h_2}$. By symmetry, $o_{h_2} \rhd [\Delta^n(\succ), \succ]$ o_{h_1}. Then Proposition 2 implies that $\Delta^n(\succ)$ is not ordinally efficient at $\succ$. Thus

$$\mathbb{P}^n(\Delta^n \in \mathscr{E}^n \;\&\; \exists g \text{ such that } \\ v^n_g > nq_g \text{ and } v^n_h < nq_h, v^n_{gh} > v^n_g - nq_g, \forall h \neq g, v^n_{h_1h_2} \geq 1, \forall h_1, h_2 \neq g) = 0,$$

which proves that the second term on the right-hand side of (C.7) is also zero.

In conclusion, both terms on the right-hand side of (C.7) are null, so $\lim_{n\to\infty} \mathbb{P}^n(\Delta^n \in \mathscr{E}^n) = 0$. □

References

Abdulkadiroglu, Atila, Parag A. Pathak, and Alvin E. Roth (2005), "The New York City high school match." *American Economic Review*, 95, 364–367. [165]

Abdulkadiroglu, Atila and Tayfun Sonmez (1998), "Random serial dictatorship and the core from random endowments in house allocation problems." *Econometrica*, 66, 689–701. [165, 169]

Abdulkadiroglu, Atila and Tayfun Sonmez (2003), "Ordinal efficiency and dominated sets of assignments." *Journal of Economic Theory*, 112, 157–172. [168, 174]

Bogomolnaia, Anna and Hervé Moulin (2001), "A new solution of the random assignment problem." *Journal of Economic Theory*, 100, 295–328. [166, 168, 170]

Che, Yeon-Koo and Fuhito Kojima (2008), "Asymptotic equivalence of probabilistic serial and random priority mechanisms." Unpublished paper, Department of Economics, Columbia University. [167]

Hylland, Aanund and Richard Zeckhauser (1979), "The efficient allocation of individuals to positions." *Journal of Political Economy*, 87, 293–314. [165]

Immorlica, Nicole and Mohammad Mahdian (2005), “Marriage, honesty, and stability.” In *Proceedings of the Sixteenth Annual ACM–SIAM Symposium on Discrete Algorithms*, 53–62, Society for Industrial and Applied Mathematics, Philadelphia. [166, 174]

Katta, Akshay-Kumar and Jay Sethuraman (2006), “A solution to the random assignment problem on the full preference domain.” *Journal of Economic Theory*, 131, 231–250. [168]

Kojima, Fuhito and Mihai Manea (2006), “Strategy-proofness of the probabilistic serial mechanism in large random assignment problems.” Unpublished paper, Cowles Foundation, Yale University. [167, 177]

Manea, Mihai (2008a), “A constructive proof of the ordinal efficiency welfare theorem.” *Journal of Economic Theory*, 141, 276–281. [168]

Manea, Mihai (2008b), “Random serial dictatorship and ordinally efficient contracts.” *International Journal of Game Theory*, 36, 489–496. [168]

McLennan, Andrew (2002), “Ordinal efficiency and the polyhedral separating hyperplane theorem.” *Journal of Economic Theory*, 105, 435–449. [168]

Mitrinović, D. S. (1970), *Analytic Inequalities*. Springer-Verlag, New York. [184, 185]

Pathak, Parag A. (2008), “Lotteries in student assignment: The equivalence of queueing and a market-based approach.” Unpublished paper, Department of Economics, Massachusetts Institute of Technology. [167]

Pulleyblank, William R. (1995), “Matchings and extensions.” In *Handbook of Combinatorics, Volume 1* (Ronald L. Graham, Martin Grötschel, and László Lovász, eds.), 179–232, MIT Press, Cambridge, Massachusetts. [168]

Shapley, Lloyd and Herbert Scarf (1974), “On cores and indivisibility.” *Journal of Mathematical Economics*, 1, 23–37. [165]

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