Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2007, Article ID 13737, 9 pages
doi:10.1155/2007/13737

Research Article

Asymptotic Periodicity of a Higher-Order Difference Equation

Stevo Stević

Received 27 April 2007; Accepted 13 September 2007

We give a complete picture regarding the asymptotic periodicity of positive solutions of the following difference equation: $x_n = f(x_{n-p_1},\dots,x_{n-p_k},x_{n-q_1},\dots,x_{n-q_m})$, $n\in\mathbb{N}_0$, where p_i, $i\in\{1,\dots,k\}$, and q_j, $j\in\{1,\dots,m\}$, are natural numbers such that $p_1<p_2<\cdots<p_k$, $q_1<q_2<\cdots<q_m$ and $\gcd(p_1,\dots,p_k,q_1,\dots,q_m)=1$, the function $f\in C[(0,\infty)^{k+m},(\alpha,\infty)]$, $\alpha>0$, is increasing in the first k arguments and decreasing in other m arguments, there is a decreasing function $g\in C[(\alpha,\infty),(\alpha,\infty)]$ such that $g(g(x))=x$, $x\in(\alpha,\infty)$, $x=f(\underbrace{x,\dots,x}_{k},\underbrace{g(x),\dots,g(x)}_{m})$, $x\in(\alpha,\infty)$, $\lim_{x\to\alpha+}g(x)=+\infty$, and $\lim_{x\to+\infty}g(x)=\alpha$. It is proved that if all p_i, $i\in\{1,\dots,k\}$, are even and all q_j, $j\in\{1,\dots,m\}$ are odd, every positive solution of the equation converges to (not necessarily prime) a periodic solution of period two, otherwise, every positive solution of the equation converges to a unique positive equilibrium.

1. Introduction

Recently, there is a huge interest in studying nonlinear difference equations; see, for example, [1–29] and the references therein.

In [26], we proved the following theorem.

Theorem A. *Consider the following difference equation:*

$$x_n = 1+\frac{\sum_{i=1}^{k}\alpha_i x_{n-p_i}}{\sum_{j=1}^{m}\beta_j x_{n-q_j}},\quad n\in\mathbb{N}_0, \tag{1.1}$$

where α_i, $i \in \{1,\dots,k\}$, *and* β_j, $j \in \{1,\dots,m\}$, *are positive numbers such that* $\sum_{i=1}^{k}\alpha_i = \sum_{j=1}^{m}\beta_j = 1$, *and* p_i, $i \in \{1,\dots,k\}$, *and* q_j, $j \in \{1,\dots,m\}$, *are natural numbers such that* $p_1 < p_2 < \cdots < p_k$ *and* $q_1 < q_2 < \cdots < q_m$. *Assume that*

$$G := \gcd\left(p_1,\dots,p_k,q_1,\dots,q_m\right) = 1. \tag{1.2}$$

Then if all p_i, $i \in \{1,\dots,k\}$, *are even and all* q_j, $j \in \{1,\dots,m\}$, *are odd, every positive solution of (1.1) converges to a periodic solution of period two. Otherwise, every positive solution of (1.1) converges to a unique positive equilibrium.*

On the other hand, by the main result in [15], in [18], we proved the following result.

Theorem B. *Consider the difference equation*

$$x_{n+1} = F\left(x_n,x_{n-2},\dots,x_{n-2k}\right), \tag{1.3}$$

where $k \in \mathbb{N}$ *is fixed. If*

(a) $F \in C[(0,+\infty)^{k+1},(0,+\infty)]$ *is nonincreasing in each of its arguments,*

(b) $F(z_1,z_2,\dots,z_{k+1})$ *is strictly decreasing in the first argument* z_1,

(c) $g(g(x)) = x$ *for all* $x \in (0,+\infty)$, *where* $g(x) = F(x,x,\dots,x)$,

then every positive solution of (1.3) converges to (not necessarily prime) a period-two solution.

For closely related results to Theorem B, see [5, 7, 14, 16, 19] and the references therein.

These two theorems motivated us to investigate the behavior of positive solutions of the following difference equation:

$$x_n = f\left(x_{n-p_1},\dots,x_{n-p_k},x_{n-q_1},\dots,x_{n-q_m}\right), \quad n \in \mathbb{N}_0, \tag{1.4}$$

where p_i, $i \in \{1,\dots,k\}$, and q_j, $j \in \{1,\dots,m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$, and the function $f \in C[(0,\infty)^{k+m},(\alpha,\infty)]$, $\alpha > 0$, satisfies the following conditions:

(a) f is increasing in first k arguments and decreasing in other m arguments;

(b) there is a decreasing function $g \in C[(\alpha,\infty),(\alpha,\infty)]$ such that $g(g(x)) = x$, $x \in (\alpha,\infty)$;

(c)

$$x = f\left(\underbrace{x,\dots,x}_{k},\underbrace{g(x),\dots,g(x)}_{m}\right), \quad x \in (\alpha,\infty); \tag{1.5}$$

(d)

$$\lim_{x\to\alpha+} g(x) = +\infty, \qquad \lim_{x\to+\infty} g(x) = \alpha. \tag{1.6}$$

Note that if x is sufficiently close to α, then from (d) it follows that $x < g(x)$. From this and by (a) and (c), we have that

$$x = f\left(\underbrace{x,\dots,x}_{k},\underbrace{g(x),\dots,g(x)}_{m}\right) < f\left(\underbrace{x,\dots,x}_{k+m}\right). \tag{1.7}$$

On the other hand, if x is sufficiently large, from (d), it follows that $g(x) < x$. This, along with (a) and (c), yields

$$x = f\left(\underbrace{x,\dots,x}_{k},\underbrace{g(x),\dots,g(x)}_{m}\right) > f\left(\underbrace{x,\dots,x}_{k+m}\right). \tag{1.8}$$

Hence the equation $x = f(x,\dots,x)$ has a solution x^* on the interval (α,∞). In view of (c), it must be

$$x^* = f(x^*,\dots,x^*) = f\left(\underbrace{x^*,\dots,x^*}_{k},\underbrace{g(x^*),\dots,g(x^*)}_{m}\right). \tag{1.9}$$

This, and (a), imply that $g(x^*) = x^*$, which, along with (b), shows that x^* is a unique solution of the equation $g(x) = x$ on the interval (α,∞), and consequently, it is a unique solution of the equation $x = f(x,\dots,x)$ on (α,∞).

Here, we give a complete picture regarding the asymptotic stability of positive solutions of (1.4).

We may assume that

$$G := \gcd(p_1,\dots,p_k,q_1,\dots,q_m) = 1, \tag{1.10}$$

otherwise, (1.4) can be separated into the following G independent difference equations

$$x_l^{(t)} = f\left(x_{l-p_1/G}^{(t)},\dots,x_{l-p_k/G}^{(t)},x_{l-q_1/G}^{(t)},\dots,x_{l-q_m/G}^{(t)}\right), \quad l \in \mathbb{N}_0, \tag{1.11}$$

where $x_l^{(t)} = x_{Gl+t}$ and $t \in \{0,1,\dots,G-1\}$.

Remark 1.1. Note that by the definition of G, it follows that at least one of the numbers p_i/G, $i \in \{1,\dots,k\}$ and q_j/G, $j \in \{1,\dots,m\}$ is odd. This fact will be used in the proof of the main result of this paper, in Theorem 2.4.

Remark 1.2. Note also that some of p_i and q_j can be equal.

We also need the following result by Karakostas [10] (see also [11]).

Theorem C. *Let J be an interval of real numbers, $f \in C[J^l,J]$, and let $(x_n)_{n=-l}^{\infty}$ be a bounded solution of the difference equation*

$$x_n = f(x_{n-1},\dots,x_{n-l}), \quad n \in \mathbb{N}_0, \tag{1.12}$$

with $I = \liminf_{n\to\infty} x_n$, $S = \limsup_{n\to\infty} x_n$, *and with* $I, S \in J$. *Then there exist two solutions* $(I_n)_{n=-\infty}^{\infty}$ *and* $(S_n)_{n=-\infty}^{\infty}$ *of the difference equation*

$$x_n = f(x_{n-1}, \ldots, x_{n-l}), \tag{1.13}$$

which satisfy the equation for all $n \in \mathbb{Z}$, *with* $I_0 = I$, $S_0 = S$, $I_n, S_n \in [I, S]$ *for all* $n \in \mathbb{Z}$, *such that for every* $N \in \mathbb{Z}$, I_N *and* S_N *are limit points of* $(x_n)_{n=-l}^{\infty}$. *Furthermore, for every* $m \leq -l$, *there exist two subsequences* (x_{r_n}) *and* (x_{l_n}) *of the solution* $(x_n)_{n=-l}^{\infty}$ *such that the following are true:*

$$\lim_{n\to\infty} x_{r_n+N} = I_N, \quad \lim_{n\to\infty} x_{l_n+N} = S_N, \quad \text{for every } N \geq m. \tag{1.14}$$

The solutions $(I_n)_{n=-\infty}^{\infty}$ and $(S_n)_{n=-\infty}^{\infty}$ of (1.13) are called full-limiting solutions of (1.13) associated with the solution $(x_n)_{n=-l}^{\infty}$ of (1.12).

2. Main results

The first result in this section concerns the boundedness character of positive solutions of (1.4). Some other closely related results can be found, for example, in [2, 3, 8, 17, 20–24, 26, 27].

Theorem 2.1. *Every positive solution of (1.4) is bounded.*

Proof. Assume that $(x_n)_{n=-l}^{\infty}$ is a positive solution of (1.4). Then since $f : (0,\infty)^{k+m} \to (\alpha, \infty)$, we have that $x_n > \alpha$ for $n \geq 0$. From this and in view of condition (d), we have that there is a positive number l greater than α such that $l \leq x_i \leq g(l)$ for $i \in \{0, 1, \ldots, s-1\}$, where $s = \max\{p_k, q_m\}$. Employing condition (c) and (1.4), we obtain

$$\begin{aligned} l = f(l, \ldots, l, g(l), \ldots, g(l)) &\leq f(x_{s-p_1}, \ldots, x_{s-p_k}, x_{s-q_1}, \ldots, x_{s-q_m}) = x_s, \\ x_s = f(x_{s-p_1}, \ldots, x_{s-p_k}, x_{s-q_1}, \ldots, x_{s-q_m}) &\leq f(g(l), \ldots, g(l), l, \ldots, l) = g(l). \end{aligned} \tag{2.1}$$

By the induction, we obtain that $x_n \in [l, g(l)]$ for every $n \in \mathbb{N}_0$, finishing the proof of the theorem. □

Theorem 2.2. *Assume that* $(x_n)_{n=-l}^{\infty}$ *is a positive solution of (1.4) and let* $\liminf_{n\to\infty} x_n = I$ *and* $\limsup_{n\to\infty} x_n = S$. *Then* $I = g(S)$ *and* $S = g(I)$.

Proof. First, note that in view of Theorem 2.1, every positive solution (x_n) of (1.4) is bounded, which implies that there are finite $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$, moreover, we have that $\alpha < I$. By taking the limit inferior and limit superior in (1.4) and using condition (c), we obtain, respectively,

$$f(I, \ldots, I, S, \ldots, S) \leq I = f(I, \ldots, I, g(I), \ldots, g(I)), \tag{2.2}$$

$$f(S, \ldots, S, g(S), \ldots, g(S)) = S \leq f(S, \ldots, S, I, \ldots, I). \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$g(I) \leq S, \qquad I \leq g(S), \tag{2.4}$$

which, in view of condition (b), implies that

$$g(S) \le g(g(I)) = I, \qquad S = g(g(S)) \le g(I). \tag{2.5}$$

Hence

$$I = g(S), \qquad S = g(I), \tag{2.6}$$

as desired. □

Remark 2.3. Note that if all p_i are even and all q_j are odd, then for every $I > \alpha$, the sequence

$$(\dots, I, g(I), I, g(I), \dots) = (\dots, I, S, I, S, \dots) \tag{2.7}$$

is a period two solution of (1.4).

Before we formulate and prove the main result of this paper, we need the following notation. Let

$$\mathcal{P} = \{p_i \mid i = 1, \dots, k\}, \qquad \mathcal{Q} = \{q_j \mid j = 1, \dots, m\}. \tag{2.8}$$

Theorem 2.4. *Consider (1.4), where the function f satisfies conditions (a)–(d). Assume that*

$$G := \gcd(p_1, \dots, p_k, q_1, \dots, q_m) = 1. \tag{2.9}$$

Then if all p_i, $i \in \{1, \dots, k\}$, are even and all q_j, $j \in \{1, \dots, m\}$, are odd, every positive solution of (1.4) converges to (not necessarily prime) a periodic solution of period two. Otherwise, every positive solution of (1.4) converges to a unique positive equilibrium.

Proof. Let $(L_{-i})_{i\in\mathbb{Z}}$ be a full-limiting sequence of a solution $(x_n)_{n=-l}^{\infty}$ of (1.4) such that $L_0 = S$. Since $(L_{-i})_{i\in\mathbb{Z}}$ is a solution of (1.4) belonging to the interval $[I, S]$, by employing Theorems 2.1 and 2.2 and condition (c), we obtain

$$\begin{aligned} S = L_0 &= f(L_{-p_1}, \dots, L_{-p_k}, L_{-q_1}, \dots, L_{-q_m}) \\ &\le f(S, \dots, S, I, \dots, I) = f(S, \dots, S, g(S), \dots, g(S)) = S. \end{aligned} \tag{2.10}$$

From (2.10), it follows that $L_{-p_i} = S$ for every $i \in \{1, \dots, k\}$ and $L_{-q_j} = I$ for every $j \in \{1, \dots, m\}$.

If we assume further that $\mathcal{P} \cap \mathcal{Q} \neq \varnothing$, then we obtain $I = S$, from which the result follows in this case.

Now we assume that $\mathcal{P} \cap \mathcal{Q} = \varnothing$ and there is $p_{i_0} \in \mathcal{P}$, which is odd. Let $p_{i_0} = 2s + 1$ and let q_{j_0} be an arbitrary element of $\mathcal{Q}$. Then (1.4) can be written in the form

$$x_n = f(\dots, x_{n-(2s+1)}, \dots, x_{n-q_{j_0}}, \dots). \tag{2.11}$$

Let $(L_{-i})_{i\in\mathbb{Z}}$ be a full-limiting sequence of a solution (x_n) of (1.4) such that $L_0 = S = \limsup_{n\to\infty} x_n$. From

$$S = L_0 = f(\dots, L_{-(2s+1)}, \dots, L_{-q_{j_0}}, \dots), \tag{2.12}$$

similar to (2.10), we obtain

$$L_{-(2s+1)} = S, \qquad L_{-q_{j_0}} = I. \tag{2.13}$$

From (2.13), and since $(L_{-i})_{i\in\mathbb{Z}}$ is a solution of (2.11), it follows that

$$L_{-2(2s+1)} = S, \qquad L_{-2q_{j_0}} = S. \tag{2.14}$$

Indeed, since

$$\begin{aligned} S = L_{-(2s+1)} &= f\left(\dots, L_{-2(2s+1)}, \dots, L_{-q_{j_0}-(2s+1)}, \dots\right) \\ &\leq f(S,\dots,S,I,\dots,I) = f\left(S,\dots,S,g(S),\dots,g(S)\right) = S, \end{aligned} \tag{2.15}$$

we obtain the first equality in (2.14). On the other hand, from

$$\begin{aligned} I = L_{-q_{j_0}} &= f\left(\dots, L_{-q_{j_0}-(2s+1)}, \dots, L_{-2q_{j_0}}, \dots\right) \\ &\geq f(I,\dots,I,S,\dots,S) = I, \end{aligned} \tag{2.16}$$

the second equality in (2.14) follows.

By induction, we obtain

$$L_{-(2s+1)i} = S, \quad i \in \mathbb{N}, \tag{2.17}$$

$$L_{-q_{j_0}j} = \begin{cases} I, & j \text{ odd}, \\ S, & j \text{ even}. \end{cases} \tag{2.18}$$

If we take $i = q_{j_0}$ in (2.17) and $j = 2s+1$ in (2.18), we obtain $I = L_{-(2s+1)q_{j_0}} = S$, as desired.

Now assume that all $p_i \in P$ are even, and $\mathfrak{Q}$ has odd as well as even elements. Then (1.4) can be written in the form

$$x_n = f\left(x_{n-p_1}, \dots, x_{n-p_k}, \dots, x_{n-q_{j_0}}, \dots, x_{n-q_{j_1}}, \dots\right), \tag{2.19}$$

where $q_{j_0} = 2s$ and $q_{j_1} = 2t+1$.

Condition $G = 1$ implies that for each sufficiently large n, for example, $n \geq n_0$, there are nonnegative numbers $d_i \in \mathbb{N}_0$, $i \in \{1,\dots,k+m\}$, such that

$$\sum_{i=1}^{k} p_i d_i + \sum_{j=1}^{m} q_j d_{k+j} = n, \tag{2.20}$$

see, for example, [13]. From condition $G = 1$, by using (2.19), (2.20), and employing the procedure described above for getting formulae (2.17) and (2.18), we obtain that the subsequence $(L_{-i})_{i\geq n_0}$ of the full-limiting sequence $(L_i)_{i\in\mathbb{Z}}$ with $L_0 = S$ takes only values I and S.

If we replace n in (2.19) by $-n_0 - l$, $l \in \{0,1,\dots,p_1-1\}$, we obtain that $L_{-n_0-l} = L_{-n_0-l-p_1 i}$ for every $i \in \mathbb{N}$ and each $l \in \{0,1,\dots,p_1-1\}$, that is, $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with period p_1. Similarly, it can be proven that $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with periods $p_2,\dots,p_k$. The periodicity of $(L_{-i})_{i\in\mathbb{N}}$ with periods $2q_1,\dots,2q_m$ can be proved similar to (2.13), (2.14), and by using induction.

Since all $p_i \in \mathcal{P}$ are even and $G = 1$, we have that

$$\begin{aligned} 2 &\le \gcd\left(p_1,p_2,\dots,p_k,2q_1,\dots,2q_m\right) \\ &= 2\gcd\left(\frac{p_1}{2},\frac{p_2}{2},\dots,\frac{p_k}{2},q_1,\dots,q_m\right) \le 2G = 2, \end{aligned} \tag{2.21}$$

that is,

$$\gcd\left(p_1,p_2,\dots,p_k,2q_1,\dots,2q_m\right) = 2. \tag{2.22}$$

Hence the sequence $(L_{-i})_{i\in\mathbb{N}}$ is eventually periodic with period two. Since $(L_i)_{i\in\mathbb{Z}}$ is a solution of (1.4), we obtain that $(L_i)_{i\in\mathbb{Z}}$ is also periodic with period two. From this, since $L_0 = S$ and by Theorem 2.2, we have that

$$L_{2i} = S, \quad L_{2i-1} = I = g(S), \quad i \in \mathbb{Z}. \tag{2.23}$$

From (2.19), (2.23), and condition (c), we have that

$$f\left(\underbrace{S,\dots,S}_{k},I,\dots,I\right) = S = L_0 = f\left(\underbrace{S,\dots,S}_{k},\dots,S,\dots,I,\dots\right). \tag{2.24}$$

This and condition (a) imply that $S = I$.

If $\mathcal{P}$ contains only even elements while $\mathcal{Q}$ contains only odd elements, then from condition (c), we see that (1.4) has infinite prime two periodic solutions of the form $x,g(x),x,g(x),\dots$. Similar to (2.22), it can be proven that, in this case, the full-limiting sequence $(L_i)_{i\in\mathbb{Z}}$, $L_0 = S$ is periodic with period two and that

$$L_{2i} = S, \quad L_{2i-1} = I = g(S), \quad i \in \mathbb{Z}. \tag{2.25}$$

From (2.25) and condition (d), we have that for every $\varepsilon \in (0,S)$, there is a $k_0 \in \mathbb{Z}$ and $j \in \{1,2,\dots,[s/2]+1\}$ such that

$$S-\varepsilon < x_{k_0+2j}, \qquad x_{k_0+2j-1} < g(S-\varepsilon), \tag{2.26}$$

where $s = \max\{p_k,q_m\}$.

From (2.26), (1.4), and by conditions (b) and (c), we have that

$$\begin{aligned} x_{k_0+2[s/2]+3} &< f\left(g(S-\varepsilon),\dots,g(S-\varepsilon),S-\varepsilon,\dots,S-\varepsilon\right) = g(S-\varepsilon), \\ x_{k_0+2[s/2]+4} &> f\left(S-\varepsilon,\dots,S-\varepsilon,g(S-\varepsilon),\dots,g(S-\varepsilon)\right) = S-\varepsilon. \end{aligned} \tag{2.27}$$

By induction, we obtain

$$x_{k_0+2i+1} < g(S-\varepsilon), \qquad x_{k_0+2i} > S-\varepsilon, \tag{2.28}$$

for every $i \in \mathbb{N}$.

From (2.28) and the fact that $\lim_{\varepsilon\to 0} g(S-\varepsilon) = g(S) = I$, it follows that $\lim_{n\to\infty} x_{2n} = S$ and $\lim_{n\to\infty} x_{2n-1} = I$, or $\lim_{n\to\infty} x_{2n} = I$ and $\lim_{n\to\infty} x_{2n-1} = S$, finishing the proof of the theorem. □

Remark 2.5. If, in Theorem 2.4, all p_i, $i \in \{1,\dots,k\}$, are even and all q_j, $j \in \{1,\dots,m\}$, are odd, then the two periodic solutions to which the other solutions converge can be essentially different from each other in the sense that one of them cannot be transformed into another one by means of cyclic permutations.

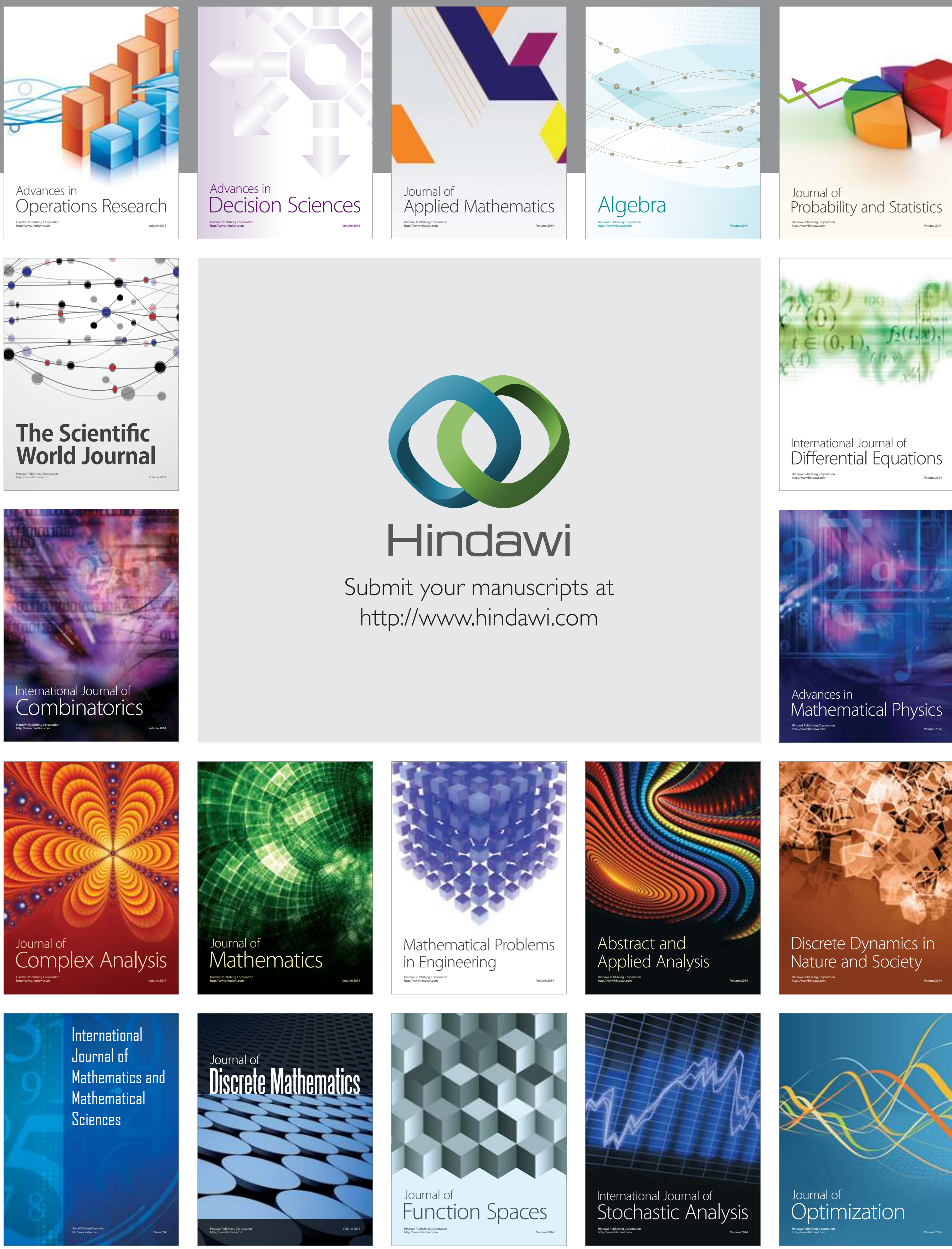
Remark 2.6. Note that Theorem 2.4 extends Theorem A as well as Theorem B (for the case when all arguments of the function F are decreasing).

References

[1] K. S. Berenhaut, J. D. Foley, and S. Stević, "Quantitative bounds for the recursive sequence $y_{n+1} = A + y_n/y_{n-k}$," *Applied Mathematics Letters*, vol. 19, no. 9, pp. 983–989, 2006.
[2] K. S. Berenhaut and S. Stević, "The behaviour of the positive solutions of the difference equation $x_n = A + (x_{n-2}/x_{n-1})^p$," *Journal of Difference Equations and Applications*, vol. 12, no. 9, pp. 909–918, 2006.
[3] L. Berg, "On the asymptotics of nonlinear difference equations," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 21, no. 4, pp. 1061–1074, 2002.
[4] L. Berg, "Inclusion theorems for non-linear difference equations with applications," *Journal of Difference Equations and Applications*, vol. 10, no. 4, pp. 399–408, 2004.
[5] J. Bibby, "Axiomatisations of the average and a further generalisation of monotonic sequences," *Glasgow Mathematical Journal*, vol. 15, pp. 63–65, 1974.
[6] D.-C. Chang and D.-M. Nhieu, "A difference equation arising from logistic population growth," *Applicable Analysis*, vol. 83, no. 6, pp. 579–598, 2004.
[7] E. T. Copson, "On a generalisation of monotonic sequences," *Proceedings of the Edinburgh Mathematical Society*, vol. 17, pp. 159–164, 1970/1971.
[8] R. DeVault, C. Kent, and W. Kosmala, "On the recursive sequence $x_{n+1} = p + x_{n-k}/x_n$," *Journal of Difference Equations and Applications*, vol. 9, no. 8, pp. 721–730, 2003.
[9] H. M. El-Owaidy, A. M. Ahmed, and M. S. Mousa, "On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + x_{n-1}^p/x_n^p$," *Journal of Applied Mathematics & Computing*, vol. 12, no. 1-2, pp. 31–37, 2003.
[10] G. Karakostas, "Convergence of a difference equation via the full limiting sequences method," *Differential Equations and Dynamical Systems*, vol. 1, no. 4, pp. 289–294, 1993.
[11] G. Karakostas, "Asymptotic 2-periodic difference equations with diagonally self-invertible responses," *Journal of Difference Equations and Applications*, vol. 6, no. 3, pp. 329–335, 2000.
[12] W. Kosmala and C. Teixeira, "More on the difference equation $y_{n+1} = (p + y_{n-1})/(qy_n + y_{n-1})$," *Applicable Analysis*, vol. 81, no. 1, pp. 143–151, 2002.
[13] I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers*, 2nd edition, John Wiley & Sons, New York, NY, USA, 1991.
[14] S. Stević, "A note on bounded sequences satisfying linear inequalities," *Indian Journal of Mathematics*, vol. 43, no. 2, pp. 223–230, 2001.
[15] S. Stević, "A generalization of the Copson's theorem concerning sequences which satisfy a linear inequality," *Indian Journal of Mathematics*, vol. 43, no. 3, pp. 277–282, 2001.
[16] S. Stević, "A global convergence result," *Indian Journal of Mathematics*, vol. 44, no. 3, pp. 361–368, 2002.
[17] S. Stević, "A note on the difference equation $x_{n+1} = \sum_{i=0}^{k} \alpha_i/x_{n-i}^{p_i}$," *Journal of Difference Equations and Applications*, vol. 8, no. 7, pp. 641–647, 2002.

[18] S. Stević, "A global convergence results with applications to periodic solutions," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 1, pp. 45–53, 2002.
[19] S. Stević, "Asymptotic behavior of a sequence defined by iteration with applications," *Colloquium Mathematicum*, vol. 93, no. 2, pp. 267–276, 2002.
[20] S. Stević, "On the recursive sequence $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_{n-j}$," *Taiwanese Journal of Mathematics*, vol. 7, no. 2, pp. 249–259, 2003.
[21] S. Stević, "On the recursive sequence $x_{n+1} = \alpha_n + x_{n-1}/x_n$. II," *Dynamics of Continuous, Discrete & Impulsive Systems. Series A*, vol. 10, no. 6, pp. 911–916, 2003.
[22] S. Stević, "Periodic character of a class of difference equation," *Journal of Difference Equations and Applications*, vol. 10, no. 6, pp. 615–619, 2004.
[23] S. Stević, "On the recursive sequence $x_{n+1} = \alpha + x_{n-1}^p/x_n^p$," *Journal of Applied Mathematics & Computing*, vol. 18, no. 1-2, pp. 229–234, 2005.
[24] S. Stević, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-k})/f(x_n, \ldots, x_{n-k+1})$," *Taiwanese Journal of Mathematics*, vol. 9, no. 4, pp. 583–593, 2005.
[25] S. Stević, "Asymptotic behavior of a class of nonlinear difference equations," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 47156, 10 pages, 2006.
[26] S. Stević, "On the recursive sequence $x_n = 1 + \sum_{i=1}^{k} \alpha_i x_{n-p_i} / \sum_{j=1}^{m} \beta_j x_{n-q_j}$," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 39404, 7 pages, 2007.
[27] T. Sun, H. Xi, and H. Wu, "On boundedness of the solutions of the difference equation $x_{n+1} = x_{n-1}/(p + x_n)$," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 20652, 7 pages, 2006.
[28] S.-E. Takahasi, Y. Miura, and T. Miura, "On convergence of a recursive sequence $x_{n+1} = f(x_{n-1}, x_n)$," *Taiwanese Journal of Mathematics*, vol. 10, no. 3, pp. 631–638, 2006.
[29] Q. Wang, F.-P. Zeng, G.-R. Zhang, and X.-H. Liu, "Dynamics of the difference equation $x_{n+1} = (\alpha + \beta_1 x_{n-1} + B_3 x_{n-3} + \cdots + B_{2k+1} x_{n-2k-1})/(A + B_0 x_n + B_2 x_{n-2} + \cdots + B_{2k} x_{n-2k})$," *Journal of Difference Equations and Applications*, vol. 12, no. 5, pp. 399–417, 2006.

Stevo Stević: Mathematical Institute of the Serbian Academy of Science,
Knez Mihailova 35/I, 11000 Beograd, Serbia
Email addresses: sstevic@ptt.yu; sstevo@matf.bg.ac.yu



Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization