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Asymptotic-preserving & well-balanced schemes for radiative transfer and the Rosseland approximation*

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Summary We are concerned with efficient numerical simulation of the radiative transfer equations. To this end, we follow the Well-Balanced approach's canvas and reformulate the relaxation term as a nonconservative product regularized by steady-state curves while keeping the velocity variable continuous. These steady-state equations are of Fredholm type. The resulting upwind schemes are proved to be stable under a reasonable parabolic CFL condition of the type $\Delta t \leq O(\Delta x^2)$ among other desirable properties. Some numerical results demonstrate the realizability and the efficiency of this process.

Key words Radiative transfer equation – Well-Balanced scheme – Nonconservative products.

Mathematics Subject Classification (1991): 82C70 – 65M06 – 35B25

1 Introduction

Radiative transfer problems are encountered in wide areas of application, like for instance asymptotics of Schrödinger equations with a random potential whose characteristic scale matches the one of the wave function, [4,13,15,36]. It is also used in climate evolution modeling, [38,40], in astrophysics, since the early works of Chandrasekhar, [12], or for neutron transport phenomena, [11,7,35]. In

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a three-dimensional nonaccretive setting with uniform optical thickness, it reads

$$\partial_t f + \boldsymbol{\xi} \cdot \nabla f = \frac{1}{4\pi} \int_{\mathbb{S}^2} f d\boldsymbol{\xi} - f, \qquad \mathbf{x} \in \mathbb{R}^3, t > 0. \tag{1}$$

The unknown is the nonegative specific radiation intensity $f(t, \mathbf{x}, \boldsymbol{\xi})$ which depends on time t, position $\mathbf{x}$ and velocity $\boldsymbol{\xi}$. It should also depend on its frequency usually denoted by $\nu > 0$; however since no emission source on the right-hand side of (1) will be taken into account before §4, we discard it. At thermal equilibrium, a canonical example for such a term would be Planck's blackbody radiation function, in standard notation

$$B_\nu(T) = \frac{2\hbar\nu^3}{c^2\left(e^{\frac{\hbar\nu}{kT}} - 1\right)},$$

with $T > 0$ the temperature, c the speed of light and $\hbar$, k standing respectively for Planck's and Boltzmann's constants. The one-dimensional Goldstein-Taylor model studied in [21] appears to match the so-called "two-stream approximation" of (1), see [38,40].

It is obvious that a direct numerical approach to (1) is very expensive because it requires the discretization of a seven-dimensional space. Hence it has been observed that in certain situations, it is sufficient to compute its diffusive (so-called Rosseland) approximation for which one gets rid of the velocity variable $\boldsymbol{\xi}$. The passage from one to the other is done via a rescaling of space and time, $t \to t/\varepsilon^2$, $\mathbf{x} \to \mathbf{x}/\varepsilon$; formal computations have been justified in *e.g.* [5,33].

The scope of this text is the efficient numerical simulation of the forthcoming one-dimensional Cauchy problem whose unknown is now a general nonegative kinetic density $f(t, x, \xi)$, $x \in \mathbb{R}$, $t > 0$,

$$\varepsilon \partial_t f + \xi \partial_x f = \frac{1}{\varepsilon}\left(\frac{1}{2}\int_{-1}^{1} f.d\xi - f\right), \qquad \xi \in [-1, 1], \tag{2}$$

together with an initial datum $0 \leq f_0(x, \xi)$. The parameter $\varepsilon \geq 0$ is the Knudsen number; for $\varepsilon \simeq 1$, we are in the so-called rarefied regime whereas $\varepsilon \to 0$ forces the energy density f to be fully scattered in an isotropic way. In this case, the asymptotic regime for the macroscopic density $\varrho(t, x) = \frac{1}{2}\int_{-1}^{1} f(t, x, \xi).d\xi$ is of the diffusive type,

$$\partial_t \varrho - \frac{1}{3}\partial_{xx}\varrho = 0, \qquad x \in \mathbb{R}, t > 0. \tag{3}$$

Several authors already proposed numerical treatments of (2), see *e.g.* [31,18,29,30,34] and [7,14] for an entropy closure approximation.

Recently, the so-called "Asymptotic-Preserving" (AP) schemes have been singled out because of their ability to remain stable as $\varepsilon \to 0$ within a fixed computational discretization and to be fully consistent with (3), see [26–28, 8].

The main drawback of these schemes at the time being lies in the difficulties one encounters when trying to establish rigorously their properties. This is one of the reasons why the authors decided to exploit the "Well-Balanced" approach after [23] (see also [19, 1, 6]) to treat this class of hyperbolic-parabolic relaxation problems. It has already been successful in the case of discrete velocity models, see [21, 22]. However, an extension towards problems involving a continuous distribution of the velocity variable has never been carried out before.

Therefore, as a first attempt, we propose here to focus on the simple model (2). Following the ideas from [19–22], we first recast this equation in the framework of nonconservative problems, [32, 1], while localizing its right-hand side on a discrete lattice by means of Dirac masses $\delta(.)$. More precisely, a positive parameter $h > 0$ being given, we plan to derive in §2 a Godunov type scheme based on the following singular equation inspired by (2) where $x_j = jh$,

$$\varepsilon\partial_t f^h + \xi\partial_x f^h = \frac{1}{\varepsilon}\sum_{j\in\mathbb{Z}} h\left(\frac{1}{2}\int_{-1}^{1} f^h.d\xi - f\right)\delta(x - x_{j-\frac{1}{2}}). \quad (4)$$

Here, $f^h(t, x, \xi)$ stands for an auxiliary unknown to be used as a building block in a Godunov scheme, [16]. Such a formulation is generally unstable for discontinuous $f^h(t, ., \xi)$; our treatment will follow the general theory of nonconservative products, [32]. We solve the Riemann problem for (4) in §2.1 in order to produce the numerical scheme which turns out to be endowed with the WB property, see (18), in §2.2. Along this presentation, we shall also consider briefly the nonlinear version of (2) namely,

$$\varepsilon\partial_t f + \xi\partial_x f = \frac{K(\varrho)}{\varepsilon}(\varrho - f), \quad \varrho = \frac{1}{2}\int_{-1}^{1} f.d\xi, \quad \xi \in [-1, 1], \quad (5)$$

with K a continuous function satisfying $K(\varrho) > 0$ if $\varrho > 0$. Its Rosseland approximation is given by the diffusion problem for $x \in \mathbb{R}$, $t > 0$:

$$\partial_t\varrho - \frac{1}{3}\partial_{xx}D(\varrho) = 0, \qquad D'(\varrho) = \frac{1}{K(\varrho)}. \quad (6)$$

Since this nonlinear case can be tackled in a quite similar way compared to (2), we also include it: see Remarks 1 and 3. Later, §3 is devoted to a complete convergence analysis of a simpler model, see

(34), in the diffusive limit $\varepsilon \to 0$. It is deduced from (18) by neglecting some terms of the order of ε, *cf.* §3.1. Finally, §4 is concerned with numerical experiments and some technical proofs (Theorem 1, Lemmas 5, 6, 7) are presented in an Appendix to improve readability.

2 Nonconservative reformulation and WB discretization

2.1 Steady-states and generalized jump relations

The Cauchy problem for (4) is unstable in a class of discontinuous functions, as explained in [32]. The first and basic step is therefore to give a precise mathematical meaning to the ambiguous products of the "Heaviside × Dirac" type appearing inside (4). This shall be done according to the theory presented in [1,19,22], that is to say by considering the steady-state equation related to (2). We call $\hat{f}(x,\xi)$ any solution with $\xi \neq 0$ of:

$$\varepsilon\xi\partial_x\hat{f} = \frac{1}{2}\int_{-1}^{1}\hat{f}.d\xi - \hat{f}, \qquad \xi \in [-1,1],\ x \in [0,h]. \tag{7}$$

Boundary conditions are to be specified later. We now split this equation into two for each positive u and negative v part of the kinetic density f. More precisely, we consider $u(t,x,\omega) = f(t,x,\omega)$ and $v(t,x,\omega) = f(t,x,-\omega)$ for $\omega \in [0,1]$. Thus the stationary relation (7) rewrites for $\hat{u}(x,\omega)$, $\hat{v}(x,\omega)$:

$$\left.\begin{aligned} \varepsilon\omega\partial_x\hat{u} &= \frac{1}{2}\int_0^1(\hat{u}+\hat{v}).d\omega - \hat{u},\\ \varepsilon\omega\partial_x\hat{v} &= \hat{v} - \frac{1}{2}\int_0^1(\hat{u}+\hat{v}).d\omega, \end{aligned}\right\} \qquad \omega \in]0,1],\ x \in [0,h]. \tag{8}$$

We complete this system with the following boundary conditions:

$$\forall\omega \in]0,1]; \qquad \hat{u}(0,\omega) = u_L(\omega), \qquad \hat{v}(h,\omega) = v_R(\omega). \tag{9}$$

The main objective of this paragraph is to derive a consistent approximation of the integral curves of (8)–(9) for all $\omega \in]0,1]$ and $h \in \mathbb{R}^+$. There is an important conservation law associated to (8), namely,

$$\partial_x\left(\int_0^1 \omega(\hat{u}-\hat{v}).d\omega\right) \equiv 0. \tag{10}$$

Hence we look for an approximation of (8) which could be solved exactly while keeping this property. From *e.g.* [24], we know that a simple Euler discretization of (8) won't preserve (10); however, the "mid-point rule" is more accurate.

Let us consider the Fredholm equations obtained by discretizing this way the system (8); the unknowns $\tilde{u}(\omega)$, $\tilde{v}(\omega)$ stand for an approximation of $\hat{u}(h,\omega)$, $\hat{v}(0,\omega)$ respectively for all $\omega \in [0,1]$,

$$\begin{aligned} \frac{2\varepsilon\omega}{h}(\tilde{u} - u_L)(\omega) &= \frac{1}{2}\int_0^1 (u_L + \tilde{v} + \tilde{u} + v_R).d\omega - (u_L + \tilde{u})(\omega), \\ \frac{2\varepsilon\omega}{h}(\tilde{v} - v_R)(\omega) &= \frac{1}{2}\int_0^1 (u_L + \tilde{v} + \tilde{u} + v_R).d\omega - (\tilde{v} + v_R)(\omega), \end{aligned} \tag{11}$$

In order to check consistency with (10), we integrate (11) in $\omega \in [0,1]$. What we have in mind is to establish that for the chosen discretization (11), there holds:

$$\int_0^1 \omega(u_L - \tilde{v})(\omega).d\omega = \int_0^1 \omega(\tilde{u} - v_R)(\omega).d\omega.$$

Hence we compute the value of an auxiliary quantity $\mathbf{I}$ for which a simplification occurs:

$$\begin{aligned} \mathbf{I} &= \frac{1}{2}\int_0^1 (u_L + \tilde{v} + \tilde{u} + v_R).d\omega - \int_0^1 (u_L + \tilde{u}).d\omega \\ &= \int_0^1 (\tilde{v} + v_R).d\omega - \frac{1}{2}\int_0^1 (u_L + \tilde{v} + \tilde{u} + v_R).d\omega \\ &= \frac{1}{2}\int_0^1 (\tilde{v} + v_R) - (u_L + \tilde{u}).d\omega. \end{aligned}$$

It remains to plug that into (11) and integrate in ω to end up with

$$\begin{aligned} \int_0^1 \omega(\tilde{u} - v_R)(\omega).d\omega &= \frac{h\mathbf{I}}{2\varepsilon} + \int_0^1 \omega(u_L - v_R).d\omega, \\ \int_0^1 \omega(u_L - \tilde{v})(\omega).d\omega &= \frac{h\mathbf{I}}{2\varepsilon} + \int_0^1 \omega(u_L - v_R).d\omega, \end{aligned} \tag{12}$$

which clearly is a numerical formulation of (10). Of course, if ε is replaced by a variable optical thickness $\varepsilon/\sigma(x)$ in (7), then (10) reads

$$\sigma(x)^{-1}\partial_x \left(\int_0^1 \omega(\hat{u} - \hat{v}).d\omega \right) \equiv 0,$$

and (11) should be amended accordingly; see §4.2.

Lemma 1 *Let $\hat{u}(x,\omega)$ and $\hat{v}(x,\omega)$ solve (8) with $u(0,\omega)=u_L(\omega)$ and $\hat{v}(h,\omega)=v_R(\omega)$; then their approximations produced by (11) read:*

$$\begin{aligned} \tilde{u}(\omega) &= u_L(\omega) - \frac{2h}{h+2\varepsilon\omega}\left(u_L(\omega) - \frac{1}{2}\int_0^1 \overline{u_L+v_R}(s).ds\right), \\ \tilde{v}(\omega) &= v_R(\omega) - \frac{2h}{h+2\varepsilon\omega}\left(v_R(\omega) - \frac{1}{2}\int_0^1 \overline{u_L+v_R}(s).ds\right), \end{aligned} \tag{13}$$

where we defined for $\omega\in[0,1]$, $\varepsilon>0$:

$$\int_0^1 \overline{u}(s).ds = \left(\int_0^1 a^\varepsilon(\omega).d\omega\right)^{-1}\int_0^1 a^\varepsilon(\omega)u(\omega).d\omega, \quad a^\varepsilon(\omega)=\frac{2\varepsilon\omega}{2\varepsilon\omega+h}.$$

Proof Let us set $u^*(\omega)=\tilde{u}(\omega)+u_L(\omega)$ and $v^*(\omega)=\tilde{v}(\omega)+v_R(\omega)$. Then from the equations (11) we obtain

$$\begin{cases} \dfrac{h+2\varepsilon\omega}{h}u^*(\omega) = \dfrac{1}{2}\displaystyle\int_0^1 (u^*+v^*)(\omega').d\omega' + \dfrac{4\varepsilon\omega}{h}u_L(\omega), \\ \dfrac{h+2\varepsilon\omega}{h}v^*(\omega) = \dfrac{1}{2}\displaystyle\int_0^1 (u^*+v^*)(\omega').d\omega' + \dfrac{4\varepsilon\omega}{h}v_R(\omega), \end{cases} \tag{14}$$

which gives at once the values of $u^*(\omega)$ and $v^*(\omega)$ in terms of their mean, $u_L(\omega)$ and $v_R(\omega)$, respectively. Hence, integrating on ω, and setting $a^\varepsilon(\omega)=\frac{2\varepsilon\omega}{2\varepsilon\omega+h}$ as announced, one obtains

$$\frac{1}{2}\int_0^1 (u^*+v^*)(\omega).d\omega = \left(\int_0^1 a^\varepsilon(\omega).d\omega\right)^{-1}\int_0^1 a^\varepsilon(\omega)(u_L+v_R)(\omega).d\omega. \tag{15}$$

Substituting into (14) leads easily to (13).

Remark 1 The nonlinear case (5) can be processed the same way because a straightforward modification of (11) still guarantees the conservation law (10). Thus one follows exactly the same lines to end up with formula (15), where now $a^\varepsilon_K(\omega)=\frac{hK(\varrho)}{hK(\varrho)+2\varepsilon\omega}$ inside which $\varrho=\frac{1}{2}\int_0^1\overline{u_L+v_R}(s).ds$. We can produce with a fixed point algorithm:

$$\begin{cases} \tilde{u}(\omega) = u_L(\omega) - \dfrac{2hK(\varrho)}{hK(\varrho)+2\varepsilon\omega}\left(u_L(\omega)-\varrho\right), \\ \tilde{v}(\omega) = v_R(\omega) - \dfrac{2hK(\varrho)}{hK(\varrho)+2\varepsilon\omega}\left(v_R(\omega)-\varrho\right). \end{cases} \tag{16}$$

Then we can reasonably choose to exploit the relations (13) in order to give a rigorous meaning to the measure source terms inside (4) within the theory of distributions. More precisely, let us fix some value $\omega \in]0,1]$ and consider the Riemann problem for (4) with initial states (u_L, v_L) for $x<0$, (u_R, v_R) for $x>0$; its solution is given by

$$\begin{cases} (u_L, v_L) & \text{for} \quad x < -\omega t, \\ (u_L, \tilde{v}) & \text{for} \quad -\omega t < x < 0, \\ (\tilde{u}, v_R) & \text{for} \quad 0 < x < \omega t, \\ (u_R, v_R) & \text{for} \quad x > \omega t. \end{cases} \tag{17}$$

At this level, one can notice the strong analogy between this Riemann structure (17) and the previous ones encountered within the study of discrete velocities models, see [21, 22].

2.2 An explicit WB Godunov scheme for rarefied regimes: $\varepsilon \gg h$

Formula (17) is enough to apply the ideas of Godunov [16] in order to derive a finite-difference scheme. To this end, we define a Cartesian computational grid with $h>0$, $\Delta t > 0$ standing respectively for the space and time steps. The initial data $u_0(x,\omega)$, $v_0(x,\omega)$ is discretized the following way:

$$\forall j \in \mathbb{Z}, \qquad u_{j,0}(\omega) = u_0(jh,\omega), \;\; v_{j,0}(\omega) = v_0(jh,\omega).$$

Hence for Δt small enough, one can update the numerical approximations just resolving Riemann problems for (4) whose solutions are given by (17)–(13) and using Stokes' theorem. The numerical process derived this way reads for $j \in \mathbb{Z}$, $n \in \mathbb{N}$:

$$\begin{cases} u_{j,n+1}(\omega) = u_{j,n}(\omega) - \dfrac{\omega \Delta t}{\varepsilon h}\left(u_{j,n}(\omega) - u_{j-1,n}(\omega)\right) \\ -\dfrac{2\omega\Delta t}{\varepsilon(h+2\varepsilon\omega)}\left(u_{j-1,n}(\omega)\Theta_\omega\left(\dfrac{h}{2\varepsilon}\right) - \dfrac{1}{2}\displaystyle\int_{h/2\varepsilon}^{1} \overline{u_{j-1,n}+v_{j,n}}(s).ds\right), \\ v_{j,n+1}(\omega) = v_{j,n}(\omega) + \dfrac{\omega\Delta t}{\varepsilon h}(v_{j+1,n}(\omega) - v_{j,n}(\omega)) \\ -\dfrac{2\omega\Delta t}{\varepsilon(h+2\varepsilon\omega)}\left(v_{j+1,n}(\omega)\Theta_\omega\left(\dfrac{h}{2\varepsilon}\right) - \dfrac{1}{2}\displaystyle\int_{h/2\varepsilon}^{1} \overline{u_{j,n}+v_{j+1,n}}(s).ds\right). \end{cases} \tag{18}$$

The interval of integration has been slightly modified for reasons which will become clear within the proof of Lemma 2. In (18) we used the following function:

$$\Theta_\omega(y) = 1 \quad \text{if } y \le \omega; \quad \Theta_\omega(y) = \frac{\omega}{y} \quad \text{if } y > \omega.$$

It is clear that it is nonnegative, continuous and bounded by one. Another elementary but fundamental observation is:

$$\frac{2\omega\Delta t}{\varepsilon(h+2\varepsilon\omega)} = \frac{\Delta t}{\varepsilon^2}a^\varepsilon(\omega); \quad 0 \le a^\varepsilon(\omega) = \frac{1}{1+h/2\varepsilon\omega}. \tag{19}$$

This scheme can be shown to be endowed with very strong properties in the so-called rarefied regime which corresponds to the case $h \ll \varepsilon$. Let us emphasize its consistency with the original problem (1); the upwinding of the source term ensures that steady states of the form (11) satisfying (10) will be preserved whatever the value of $h > 0$ under the CFL condition $\Delta t \le \varepsilon h$.

Lemma 2 *Assume the CFL condition $\Delta t \le \varepsilon h$ with $\varepsilon > 0$ and $0 \le f_0(x,\xi) \in L^\infty([-1,1];L^1(\mathbb{R}))$; then the numerical approximations produced by (18) are nonnegative and satisfy for all $n \in \mathbb{N}$,*

$$\sum_{j\in\mathbb{Z}} h \int_{h/2\varepsilon}^1 |u_{j,n}(\omega)| + |v_{j,n}(\omega)|.d\omega \le \int_{-1}^1 \|f_0(.,\xi)\|_{L^1(\mathbb{R})}.d\xi. \tag{20}$$

Proof We have in mind to derive convexity inequalities; the equation on $u_{j,n+1}$ in (18) implies:

$$\begin{aligned} |u_{j,n+1}(\omega)| \le |u_{j,n}(\omega)| &\left(1 - \frac{\omega\Delta t}{\varepsilon h}\right) \\ &+ |u_{j-1,n}(\omega)| \left(\frac{\omega\Delta t}{\varepsilon h} - \frac{2\omega\Delta t}{\varepsilon(2\varepsilon\omega+h)}\Theta_\omega\left(\frac{h}{2\varepsilon}\right)\right) \\ &+ \frac{\omega\Delta t}{\varepsilon(2\varepsilon\omega+h)}\int_{h/2\varepsilon}^1 \overline{|u_{j-1,n}| + |v_{j,n}|}(s).ds. \end{aligned} \tag{21}$$

We notice that under the prescribed CFL condition, the coefficient of $|u_{j,n}(\omega)|$ is nonnegative. Likewise, the nonnegativity of the coefficient of $|u_{j-1,n}(\omega)|$ follows from the definition of Θ_ω. The equation on $v_{j,n+1}(\omega)$ being treated the same way, it remains to integrate in $\omega \in [h/2\varepsilon, 1]$, to use (19) and to sum on $j \in \mathbb{Z}$ in order to derive

$$\begin{aligned} \sum_{j\in\mathbb{Z}} h \int_{h/2\varepsilon}^1 |u_{j,n+1}| + |v_{j,n+1}|.d\omega &\le \sum_{j\in\mathbb{Z}} h \int_{h/2\varepsilon}^1 |u_{j,n}| + |v_{j,n}|.d\omega \\ + \frac{\Delta t}{\varepsilon^2}\sum_{j\in\mathbb{Z}} h &\left\{\int_{h/2\varepsilon}^1 \left(a^\varepsilon(\omega) - \frac{2\varepsilon\omega}{h+2\varepsilon\omega}\right)(|u_{j,n}| + |v_{j,n}|).d\omega\right\}. \end{aligned}$$

Hence we obtain (20) from the definition of $a^\varepsilon(\omega)$ in Lemma 1. The consistency comes from (19) since $\frac{2\omega\Delta t}{\varepsilon(h+2\varepsilon\omega)} = \frac{\Delta t}{\varepsilon^2}\frac{1}{1+h/2\varepsilon\omega}$ when $\Theta_\omega = 1$. We notice further that $\Theta \neq 1$ on the interval $\omega \in [0, h/2\varepsilon]$ which Lebesgue measure shrinks to zero as $h \to 0$, $\varepsilon > 0$.

Other bounds in $L^p(\mathbb{R})$, $1 < p \leq +\infty$ can be derived the same way. In the nonlinear case (5), one has to exclude the critical interval $\omega < h/2\varepsilon\|K(\varrho_0)\|_{L^\infty(\mathbb{R})}$, $\varrho_0(x) = \frac{1}{2}\int_{-1}^{1} f_0(x,\xi).d\xi$. This is related to resonance problems for nonconservative balance laws, see [1, 2, 25] for instance, and also to the stiffness limit of the approximation (11). Other choices exist for Θ_ω; for instance, we may have taken the indicator function $\mathbf{1}_{h\leq 2\varepsilon\omega}$. The basic requirements are both to ensure nonnegativity for the incremental coefficients in (18) and consistency as h vanishes. In practice, the ω variable is discretized; the simplest way is then to exclude the velocities inside the critical interval.

For a given $\varepsilon > 0$ fixed, we define the piecewise constant functions,

$$u^h(t,x,\omega) = u_{j,n}(\omega), \quad v^h(t,x,\omega) = v_{j,n}(\omega), \qquad \omega \in [0,1], \tag{22}$$

for $x \in [(j-\frac{1}{2})\Delta x, (j+\frac{1}{2})\Delta x[$ and $t \in [n\Delta t, (n+1)\Delta t[$.

Theorem 1 *Assume the CFL condition $\Delta t \leq \varepsilon h$ with $\varepsilon > 0$ and $0 \leq f_0(x,\xi) \in L^\infty([-1,1]; L^1 \cap BV(\mathbb{R}))$; then, as $h \to 0$, u^h and v^h remain bounded in $L^\infty([-1,1]; BV_{loc}(\mathbb{R}^+_* \times \mathbb{R}))$ and converge towards the weak solution of (2), $f(t=0,.,.) = f_0$.*

The proof of this result has been forwarded to the Appendix.

Remark 2 Another way is to consider directly the system (8) while "freezing" the integral term in such a way (10) or its discrete version (12) can be preserved. Indeed, such an alternative approximation leads to an explicitly integrable system whose solutions read:

$$\begin{aligned}
\tilde{u}(\omega) &= u_L(\omega) - (1 - e^{-h/\varepsilon\omega})\left(u_L(\omega) - \frac{1}{2}\int_0^1 \widetilde{u_L + v_R}(s).ds\right), \\
\tilde{v}(\omega) &= v_R(\omega) - (1 - e^{-h/\varepsilon\omega})\left(v_R(\omega) - \frac{1}{2}\int_0^1 \widetilde{u_L + v_R}(s).ds\right).
\end{aligned} \tag{23}$$

The integrals are defined as in Lemma 1, but with a different function a which reads $\tilde{a}^\varepsilon(\omega) = \omega(1 - e^{-h/\varepsilon\omega})$. With these values, an analogue of Lemma 2 holds in the whole interval $\omega \in [0,1]$ since the modified scheme yields the following convexity inequality:

$$\begin{aligned}
|u_{j,n+1}(\omega)| \leq{} & |u_{j,n}(\omega)|\left(1 - \frac{\omega\Delta t}{\varepsilon h}\right) + |u_{j-1,n}(\omega)|\frac{\omega\Delta t}{\varepsilon h}e^{-h/\varepsilon\omega} \\
& + \frac{\omega\Delta t}{2\varepsilon h}(1 - e^{-h/\varepsilon\omega})\int_0^1 \left(\widetilde{|u_{j-1,n}|} + \widetilde{|v_{j,n}|}\right)(s).ds.
\end{aligned}$$

However, it is not completely clear how to extend such a discretization so as to handle the diffusive scaling efficiently.

In any case, both schemes (23) and (18) ask for a discretization of the velocity variable ω in order to update the kinetic densities $u_j^n(\omega)$, $v_j^n(\omega)$ and the average quantities have to be computed in an appropriate way (the usual choice is Gaussian quadrature rule, see *e.g.* [28]). In sharp contrast, we shall see that in the diffusive regime $\varepsilon \ll h$, it is possible to avoid this. Roughly speaking, this comes from the intuitive fact that if a scheme remains stable for small values of ε, then the values of $u_j^n(\omega)$, $v_j^n(\omega)$ must be very close to the Maxwellian distribution, which in the present case is independent of ω.

2.3 First considerations on the diffusive regime: $\varepsilon \ll h$

Following [21,22], we rewrite the values (13) splitting between a Maxwellian and a diffusive term. This can be done in several ways; the natural one reads

$$\begin{aligned}\tilde{u}(\omega) &= u_L(\omega) - \frac{h}{2\varepsilon\omega + h}\left(u_L(\omega) - \int_0^1 \overline{v_R}(s).ds\right)\\ &\quad -\frac{h}{2\varepsilon\omega + h}\left(u_L(\omega) - \int_0^1 \overline{u_L}(s).ds\right)\\ &= \int_0^1 \overline{v_R}(s).ds - \frac{2\varepsilon\omega}{2\varepsilon\omega + h}\left(\int_0^1 \overline{v_R}(s).ds - u_L(\omega)\right)\\ &\quad +\frac{h}{2\varepsilon\omega + h}\left(\int_0^1 \overline{u_L}(s).ds - u_L(\omega)\right),\end{aligned} \tag{24}$$

and leads to the following scheme if the last (Maxwellian) term is neglected: (compare with (18))

$$\begin{cases} u_{j,n+1}(\omega) + \dfrac{\omega\Delta t}{\varepsilon h}\left(u_{j,n+1}(\omega) - \displaystyle\int_0^1 \overline{v_{j,n+1}}(s).ds\right)\\ \qquad = u_{j,n}(\omega) - \dfrac{2\omega^2\Delta t}{h(h+2\varepsilon\omega)}\left(\displaystyle\int_0^1 \overline{v_{j,n}}(s).ds - u_{j-1,n}(\omega)\right),\\ v_{j,n+1}(\omega) + \dfrac{\omega\Delta t}{\varepsilon h}\left(v_{j,n+1}(\omega) - \displaystyle\int_0^1 \overline{u_{j,n+1}}(s).ds\right)\\ \qquad = v_{j,n}(\omega) - \dfrac{2\omega^2\Delta t}{h(h+2\varepsilon\omega)}\left(\displaystyle\int_0^1 \overline{u_{j,n}}(s).ds - v_{j+1,n}(\omega)\right). \end{cases} \tag{25}$$

Observe once again that because of the property (19), there holds

$$\int_0^1 \frac{\omega\Delta t}{\varepsilon h}\frac{h}{2\varepsilon\omega + h}\left(\int_0^1 \overline{u_L}(s).ds - u_L(\omega)\right).d\omega = 0,$$

which means that the neglected term is of zero average. We also comment on the fact that, taking advantage of the linear convection in

(4), we chosed as in [21,22] to treat implicitly the stiff terms divided by ε. This is not a drawback since the equations of (25) can be inverted analytically as we shall see in the sequel.

We introduce now a bit of notation as follows:

$$\begin{aligned}
\Gamma^u_{j,n}(\omega) &= u_{j,n}(\omega) - \tfrac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \left(\int_0^1 \overline{v_{j,n}}(s).ds - u_{j-1,n}(\omega) \right), \\
\Gamma^v_{j,n}(\omega) &= v_{j,n}(\omega) - \tfrac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \left(\int_0^1 \overline{u_{j,n}}(s).ds - v_{j+1,n}(\omega) \right).
\end{aligned} \tag{26}$$

We notice another important property of a^ε, namely,

$$\frac{a^\varepsilon(\omega)}{\int_0^1 a^\varepsilon(\omega').d\omega'} = 2\omega + o(\frac{\varepsilon}{h}). \tag{27}$$

Hence, up to an error of the order of ε, one can integrate (25) in ω and then is led to invert the resulting linear 2×2 system of equations which yields the analytic expression of the average values of $u_{j,n+1}(\omega)$ and $v_{j,n+1}(\omega)$:

$$\begin{cases}
u_{j,n+1}(\omega) + \dfrac{\Delta t}{2\varepsilon h} \left(2\omega u_{j,n+1}(\omega) - 2\omega \displaystyle\int_0^1 2\omega' v_{j,n+1}.d\omega' \right) = \Gamma^u_{j,n}(\omega), \\
v_{j,n+1}(\omega) + \dfrac{\Delta t}{2\varepsilon h} \left(2\omega v_{j,n+1}(\omega) - 2\omega \displaystyle\int_0^1 2\omega' u_{j,n+1}.d\omega' \right) = \Gamma^v_{j,n}(\omega).
\end{cases} \tag{28}$$

This system rewrites:

$$\begin{cases}
\displaystyle\int_0^1 2\omega u_{j,n+1}.d\omega - Z^\varepsilon \int_0^1 2\omega v_{j,n+1}.d\omega = \int_0^1 \frac{2\omega\varepsilon h}{\varepsilon h + \omega \Delta t} \Gamma^u_{j,n}(\omega).d\omega, \\
\displaystyle\int_0^1 2\omega v_{j,n+1}.d\omega - Z^\varepsilon \int_0^1 2\omega u_{j,n+1}.d\omega = \int_0^1 \frac{2\omega\varepsilon h}{\varepsilon h + \omega \Delta t} \Gamma^v_{j,n}(\omega).d\omega, \\
\qquad\qquad \text{with } Z^\varepsilon = \displaystyle\int_0^1 \frac{2\omega^2 \Delta t}{\varepsilon h + \omega \Delta t}.d\omega.
\end{cases} \tag{29}$$

At this point, one sees that the computation of the implicit averages asks for involved quantities on the right-hand side of the type $\int_0^1 \frac{2\omega\varepsilon h}{\varepsilon h + \omega \Delta t} u_{j,n}(\omega).d\omega$. However, in this diffusive regime, our main point is to compute $\varrho_{j,n}$, which satisfies the following simple equation:

$$\varrho_{j,n+1} = \frac{1}{2} \int_0^1 \Gamma^u_{j,n}(\omega) + \Gamma^v_{j,n}(\omega).d\omega, \qquad n \in \mathbb{N}. \tag{30}$$

Hence the expensive averages cancel when integrating and adding the scheme (28); it sounds therefore interesting to seek a simpler version of (29) being endowed with a correct asymptotic behaviour as $\varepsilon \to 0$. A natural way to get such a simplified scheme is to use simpler expressions in the left-hand side's brackets of formula (28).

2.4 The link with the Goldstein-Taylor model

A choice which gives this simplification makes us go from (13) to

$$\begin{aligned}\tilde{u}(\omega) &= v_R(\omega) + \frac{2\varepsilon\omega}{2\varepsilon\omega + h}\Big(u_L(\omega) - v_R(\omega)\Big) \\ &\quad + \frac{h}{h+2\varepsilon\omega}\left(\int_0^1 \overline{u_L + v_R}(s).ds - (u_L + v_R)(\omega)\right).\end{aligned}$$

One could choose to neglect the last Maxwellian term which is of zero average (always for the same reason) and this leads to the very simple family of schemes which are indexed by the ω variable:

$$\begin{cases} u_{j,n+1}(\omega) + \dfrac{\omega\Delta t}{\varepsilon h}\Big(u_{j,n+1}(\omega) - v_{j,n+1}(\omega)\Big) \\ \qquad\qquad = u_{j,n}(\omega) - \dfrac{2\omega^2\Delta t}{h(h+2\varepsilon\omega)}\Big(v_{j,n}(\omega) - u_{j-1,n}(\omega)\Big), \\ v_{j,n+1}(\omega) + \dfrac{\omega\Delta t}{\varepsilon h}\Big(v_{j,n+1}(\omega) - u_{j,n+1}(\omega)\Big) \\ \qquad\qquad = v_{j,n}(\omega) - \dfrac{2\omega^2\Delta t}{h(h+2\varepsilon\omega)}\Big(u_{j,n}(\omega) - v_{j+1,n}(\omega)\Big). \end{cases} \tag{31}$$

Of course, in the case $\omega \equiv 1$, one gets the "two-stream approximation" of the radiative transfer equation, that is to say the Goldstein-Taylor model, [17,37]. We notice also that since we neglected the Maxwellian terms inside the decomposition of $\tilde{u}(\omega)$, there is no coupling in $\omega \in [0,1]$ and what we obtain is but a collection of linear schemes of the type already studied in [21]. Thus we do not repeat all the analysis and we only state the final result:

Lemma 3 *Assume $0 \leq f_0 \in L^\infty([-1,1]; L^1(\mathbb{R}))$ is initial data for (2). Then under the CFL condition $2\Delta t \leq 3h^2$ there holds for all $t > 0$, $\omega \in [0,1]$ and $\varepsilon \to 0$:*

$$\sum_{j\in\mathbb{Z}} h\Big(|u_{j,n}(\omega)| + |v_{j,n}(\omega)|\Big) \leq \|u_0(.,\omega)\|_{L^1(\mathbb{R})} + \|v_0(.,\omega)\|_{L^1(\mathbb{R})}.$$

and the scheme (31) is positivity preserving. In the case $x \mapsto f_0(x,\omega) \in BV(\mathbb{R})$, one has also:

$$\sum_{j\in\mathbb{Z}} h|u_{j,n} - v_{j,n}|(\omega) \leq \varepsilon\Big(TV_x(u_0) + TV_x(v_0)\Big) + \|u_0(.,\omega) - v_0(.,\omega)\|_{L^1(\mathbb{R})}$$

where TV_x stands for the total variation in the space variable.

All the estimates are exactly the same than those in [21]. The only change comes from the consistency with (3) as $\varepsilon \to 0$ which is the AP property. Then one observes that each Maxwellian estimate gives that $(u_{j,n} - v_{j,n})(\omega)$ is of the order of ε in $L^1(\mathbb{R})$. Hence, summing the equations in (31) gives only:

$$\begin{aligned} u_{j,n+1}(\omega) + v_{j,n+1}(\omega) &= u_{j,n}(\omega) + v_{j,n}(\omega) \\ &+ \frac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \Big(v_{j+1,n}(\omega) - u_{j,n}(\omega) - v_{j,n}(\omega) + u_{j-1,n}(\omega) \Big). \end{aligned}$$

And this doesn't lead to (3) because there is still a dependence of $u_{j,n},\ v_{j,n}$ in ω; the relaxation estimate in Lemma 3 is too weak. As a fix, the term $\frac{h}{h+2\varepsilon\omega}\left(\int_0^1 \overline{v_R}(s).ds - v_R(\omega)\right)$ could be included inside (31), but the resulting scheme would be computationally expensive.

3 Convergence of a simple conservative AP scheme

3.1 Conservative Asymptotic-Preserving schemes

A third way to rewrite (13) is given by:

$$\begin{aligned} \tilde{u}(\omega) = \int_0^1 \overline{v_R}(s).ds &- \frac{2\varepsilon\omega}{h+2\varepsilon\omega} \int_0^1 \overline{v_R - u_L}(s).ds \\ &+ \frac{h-2\varepsilon\omega}{h+2\varepsilon\omega} \left(\int_0^1 \overline{u_L}(s).ds - u_L(\omega) \right). \end{aligned} \tag{32}$$

The main difference between (32) and (24) relies in the diffusive part, which is now entirely given by mean values. We notice that neglecting the last term in this expression is hopefully about to generate an error of order ε, the Maxwellian gap, on $\varrho_{j,n}$ because there holds (thanks again to (19)):

$$\begin{aligned} &\int_0^1 \frac{\Delta t\omega}{\varepsilon h}(1-2a^\varepsilon(\omega)) \left(\int_0^1 \overline{u_L}(s).ds - u_L(\omega) \right).d\omega = \\ -&\int_0^1 \frac{2\Delta t\omega^2}{h(h+2\varepsilon\omega)} \left(\int_0^1 \overline{u_L}(s).ds - u_L(\omega) \right).d\omega = O(\frac{\varepsilon\Delta t}{h^2}). \end{aligned}$$

Hence the important point is to ensure that the relaxation mechanism is kept strong enough despite these simplifications in order to maintain the errors on ϱ of order ε; so far we have moved from (13) to:

$$\begin{aligned} \tilde{u}(\omega) &= \int_0^1 \overline{v_R}(s).ds - \frac{2\varepsilon\omega}{2\varepsilon\omega+h} \int_0^1 \overline{v_R - u_L}(s).ds, \\ \tilde{v}(\omega) &= \int_0^1 \overline{u_L}(s).ds - \frac{2\varepsilon\omega}{2\varepsilon\omega+h} \int_0^1 \overline{u_L - v_R}(s).ds. \end{aligned}$$

Inserting these values $\tilde{u}(\omega)$, $\tilde{v}(\omega)$ leads to the following scheme

$$\begin{cases} u_{j,n+1}(\omega) + \dfrac{\Delta t}{2\varepsilon h}\left(2\omega u_{j,n+1}(\omega) - 2\omega \displaystyle\int_0^1 \overline{v_{j,n+1}}(s).ds\right) \\ \qquad = u_{j,n}(\omega) - \dfrac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \displaystyle\int_0^1 \overline{v_{j,n} - u_{j-1,n}}(s).ds, \\ v_{j,n+1}(\omega) + \dfrac{\Delta t}{2\varepsilon h}\left(2\omega v_{j,n+1}(\omega) - 2\omega \displaystyle\int_0^1 \overline{u_{j,n+1}}(s).ds\right) \\ \qquad = v_{j,n}(\omega) - \dfrac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \displaystyle\int_0^1 \overline{u_{j,n} - v_{j+1,n}}(s).ds. \end{cases} \tag{33}$$

We observe that this scheme differs from (28) mainly by the form of its right-hand side. However, it keeps on involving intricate implicit average quantities on the left-hand side. Thus we *choose* to replace them by simpler ones; namely, we define (compare with (26)):

$$\hat{\Gamma}^u_{j,n}(\omega) = u_{j,n}(\omega) - \frac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \int_0^1 v_{j,n} - u_{j-1,n}(\omega').d\omega',$$
$$\hat{\Gamma}^v_{j,n}(\omega) = v_{j,n}(\omega) - \frac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \int_0^1 u_{j,n} - v_{j+1,n}(\omega').d\omega'.$$

And we pass from (28) to the revised scheme where the diffusive terms are treated in a simple but conservative way:

$$\begin{cases} u_{j,n+1}(\omega) + \dfrac{\Delta t}{2\varepsilon h}\left(u_{j,n+1}(\omega) - \displaystyle\int_0^1 v_{j,n+1}(\omega').d\omega'\right) \\ \quad = u_{j,n}(\omega) - \dfrac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \displaystyle\int_0^1 v_{j,n}(\omega') - u_{j-1,n}(\omega').d\omega', \\ v_{j,n+1}(\omega) + \dfrac{\Delta t}{2\varepsilon h}\left(v_{j,n+1}(\omega) - \displaystyle\int_0^1 u_{j,n+1}(\omega').d\omega'\right) \\ \quad = v_{j,n}(\omega) - \dfrac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)} \displaystyle\int_0^1 u_{j,n}(\omega') - v_{j+1,n}(\omega').d\omega'. \end{cases} \tag{34}$$

This is an approximation which is valid for kinetic densities having weak dependence in ω since $\int_0^1 2\omega.d\omega = 1$ and which yields the following relation (compare with (30)) :

$$\varrho_{j,n+1} = \frac{1}{2} \int_0^1 \hat{\Gamma}^u_{j,n}(\omega) + \hat{\Gamma}^v_{j,n}(\omega).d\omega, \qquad n \in \mathbb{N}.$$

Finally we pass from (25)–(29) to a simpler linear system obtained by integrating (34) in ω:

$$\begin{cases} \displaystyle\int_0^1 u_{j,n+1}.d\omega - W^\varepsilon \int_0^1 v_{j,n+1}.d\omega = \frac{\varepsilon h}{\varepsilon h + \Delta t} \int_0^1 \hat{\Gamma}^u_{j,n}(\omega).d\omega, \\ \displaystyle\int_0^1 v_{j,n+1}.d\omega - W^\varepsilon \int_0^1 u_{j,n+1}.d\omega = \frac{\varepsilon h}{\varepsilon h + \Delta t} \int_0^1 \hat{\Gamma}^v_{j,n}(\omega).d\omega, \\ \qquad \text{with } W^\varepsilon = \dfrac{\Delta t}{2\varepsilon h + \Delta t} = 1 - \dfrac{2\varepsilon h}{2\varepsilon h + \Delta t}. \end{cases} \tag{35}$$

Moreover, it can be easily shown that (*cf.* Appendix)

$$|Z^\varepsilon - W^\varepsilon| = 2\varepsilon h \Delta t \left| \int_0^1 \frac{\omega(2\omega - 1).d\omega}{(2\varepsilon h + \Delta t)(\varepsilon h + \omega \Delta t)} \right| = O(\varepsilon / h), \tag{36}$$

assuming that $\Delta t = O(h^2)$ and both (29) and (35) are invertible. The main point of such a discretization is that it enforces the so-called "Asymptotic-Preserving" (AP) property. If one assumes that the data is locked on the Maxwellian distribution as $\varepsilon \to 0$, that is to say u and v independent of ω, then (34) reduces to an explicit discretization of (3) when integrated against $\omega \in [0, 1]$.

3.2 Positivity-preserving property

Apart from the constant velocity $\bar{\omega} = \frac{1}{2}$ on the left-hand side, (34) differs from (25) because of the presence of a monotone numerical flux-function on its right-hand side,

$$F(\omega; \mathcal{U}, \mathcal{V}) = \frac{2\omega^2}{h + 2\varepsilon\omega}(\mathcal{U} - \mathcal{V}), \qquad \partial_{\mathcal{U}} F \geq 0, \ \partial_{\mathcal{V}} F \leq 0, \tag{37}$$

and $F(\omega; 0, 0) = 0$ with the notations:

$$\mathcal{U} = \int_0^1 u(\omega').d\omega', \qquad \mathcal{V} = \int_0^1 v(\omega').d\omega'.$$

Remark 3 In the nonlinear case (5), one is led to define

$$\tilde{F}(\omega; \mathcal{U}, \mathcal{V}) = \frac{2\omega^2}{hK(\varrho) + 2\varepsilon\omega}(\mathcal{U} - \mathcal{V}), \qquad \varrho = \frac{1}{2}(\mathcal{U} + \mathcal{V}), \tag{38}$$

and the monotonicity is ensured if $2K(\varrho) \geq \varrho K'(\varrho)$.

We define also the piecewise constant functions:

$$\forall \omega \in [0,1], \qquad u^h(t,x,\omega) = u_{j,n}(\omega), \quad v^h(t,x,\omega) = v_{j,n}(\omega), \quad (39)$$

where $t \in [n\Delta t, (n+1)\Delta t[$, $x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$, $x_j = jh$. In this case, we are in position to establish results similar to those of [21]. We begin with the positivity-preserving property:

Lemma 4 *Assume $f_0(x,\xi)$ is nonnegative; then $2\Delta t \le h^2$ is enough to ensure that for (34), $u^h \ge 0$ and $v^h \ge 0$ with $0 \le \varepsilon < 3h/4$.*

Proof Taking (37) into account, the scheme (34) rewrites as follows:

$$u_{j,n+1}(\omega) + \frac{\Delta t}{2\varepsilon h}\left(u_{j,n+1}(\omega) - \mathcal{V}_{j,n+1}\right) = u_{j,n}(\omega) + \frac{\Delta t}{h} F(\omega; \mathcal{U}_{j-1,n}, \mathcal{V}_{j,n}),$$
$$v_{j,n+1}(\omega) + \frac{\Delta t}{2\varepsilon h}\left(v_{j,n+1}(\omega) - \mathcal{U}_{j,n+1}\right) = v_{j,n}(\omega) - \frac{\Delta t}{h} F(\omega; \mathcal{U}_{j,n}, \mathcal{V}_{j+1,n}).$$

Hence integrating it and adding yields (35) and the value of $\mathcal{U}_{j,n+1}$,

$$\mathcal{U}_{j,n+1} = \frac{\varepsilon h}{\Delta t + \varepsilon h}\left(\mathcal{U}_{j,n} + \frac{\Delta t}{h} K_\varepsilon (\mathcal{U}_{j-1,n} - \mathcal{V}_{j,n}) + \right.$$
$$\left. \frac{\Delta t}{2\varepsilon h}\Big(\mathcal{U}_{j,n} + \mathcal{V}_{j,n} + \frac{\Delta t}{h} K_\varepsilon (\mathcal{U}_{j-1,n} - \mathcal{V}_{j,n} - \mathcal{U}_{j,n} + \mathcal{V}_{j+1,n})\Big)\right),$$

together with the similar one of $\mathcal{V}_{j,n+1}$:

$$\mathcal{V}_{j,n+1} = \frac{\varepsilon h}{\Delta t + \varepsilon h}\left(\mathcal{V}_{j,n} - \frac{\Delta t}{h} K_\varepsilon (\mathcal{U}_{j,n} - \mathcal{V}_{j+1,n}) + \right.$$
$$\left. \frac{\Delta t}{2\varepsilon h}\Big(\mathcal{U}_{j,n} + \mathcal{V}_{j,n} + \frac{\Delta t}{h} K_\varepsilon (\mathcal{U}_{j-1,n} - \mathcal{V}_{j,n} - \mathcal{U}_{j,n} + \mathcal{V}_{j+1,n})\Big)\right),$$

We have used $K_\varepsilon = \int_0^1 \frac{2\omega^2}{h+2\varepsilon\omega}.d\omega \le 2/3h$. Now we insert all this inside the scheme (34) and rearrange terms. This leads to:

$$\left(1 + \frac{\Delta t}{2\varepsilon h}\right) u_{j,n+1}(\omega) = \frac{\Delta t}{\Delta t + \varepsilon h}\left(\mathcal{U}_{j,n} + \frac{\Delta t}{h} K_\varepsilon (\mathcal{U}_{j-1,n} - \mathcal{V}_{j,n}) \right.$$
$$+ \frac{\Delta t}{2\varepsilon h}\Big(\mathcal{U}_{j,n} + \mathcal{V}_{j,n}$$
$$+ \frac{\Delta t}{h} K_\varepsilon (\mathcal{U}_{j-1,n} - \mathcal{V}_{j,n} - \mathcal{U}_{j,n} + \mathcal{V}_{j+1,n})\Big)$$
$$\left. + u_{j,n}(\omega) + \frac{\Delta t}{h} F(\omega; \mathcal{U}_{j-1,n}, \mathcal{V}_{j,n})\right).$$

Inside this expression, we can split between terms in $O(1)$ and $O(1/\varepsilon)$; the expressions involving negative terms read:

$$\mathcal{U}_{j,n}\left(1 - \frac{\Delta t}{h} K_\varepsilon \Big(1 + \frac{\varepsilon h}{\Delta t}\Big)\right),$$

and

$$\mathcal{V}_{j,n}\left(\left(1+\frac{\varepsilon h}{\Delta t}\right)-\left(\frac{\Delta t}{h}K_\varepsilon+\frac{2\omega^2\varepsilon(\Delta t+\varepsilon)}{\Delta t(h+\varepsilon\omega)}\right)\right).$$

It is easy to check that each coefficient is positive under our assumptions. Computations for $v_{j,n+1}(\omega)$ are similar thus we are done.

Remark 4 From these computations, one can see that a simpler choice for F, for instance $F(\omega;u,v)=\frac{2\omega^2}{h+2\varepsilon\omega}(u(\omega)-v(\omega))$, couldn't keep such a property. Hence in §2.4, it wouldn't have been interesting to propose $\tilde{u}(\omega)=\int_0^1\overline{v_R}(s).ds+a^\varepsilon(\omega)(u_L-v_R)(\omega)$ just neglecting the other Maxwellian term (of zero average).

In practice, for $\varepsilon\simeq 0$, the terms of the order of ε become negligible and one can of course relax the CFL to $2\Delta t\le 3h^2$ like in [29].

3.3 $L^p(\mathbb{R})$ and $BV(\mathbb{R})$ regularity; control of the Maxwellian gap

As done before, we define the piecewise constant averages for $h\ge 0$:

$$\mathcal{U}^h(t,x)=\mathcal{U}_{j,n},\qquad \mathcal{V}^h(t,x)=\mathcal{V}_{j,n}, \tag{40}$$

where $t\in[n\Delta t,(n+1)\Delta t[, x\in[x_{j-\frac12},x_{j+\frac12}[$.

Lemma 5 *Assume $0\le f_0(x,\xi)\in L^\infty([-1,1];L^1\cap BV(\mathbb{R}))$; then if $2\Delta t\le h^2$ in (34), there hold for all $0\le\varepsilon<3h/4$, $t>0$:*

$$\|\mathcal{U}^h(t,.)\|_{L^1(\mathbb{R})}+\|\mathcal{V}^h(t,.)\|_{L^1(\mathbb{R})}\le\left\|\int_{-1}^1 f_0(.,\xi).d\xi\right\|_{L^1(\mathbb{R})}, \tag{41}$$

and

$$TV_x\left(\mathcal{U}^h(t,.)\right)+TV_x\left(\mathcal{V}^h(t,.)\right)\le TV_x\left(\int_{-1}^1 f_0(.,\xi).d\xi\right), \tag{42}$$

where TV_x stands for the total variation in the space variable.

The treatment of the nonlinear case (5), (38) is exactly the same; in particular, there is no need for any linearization of $\tilde{F}$ in $\mathcal{U}$ and $\mathcal{V}$ as in [22]. The only change is the explicit dependence of K_ε upon $\varrho_{j-\frac12,n}=\frac12(\mathcal{U}_{j-1,n}+\mathcal{V}_{j,n})$, see §4.3.

Lemma 6 *Assume $0\le f_0(x,\xi)\in L^\infty([-1,1];L^1\cap BV(\mathbb{R}))$; then if $2\Delta t\le h^2$ in (34), there holds for all $0\le\varepsilon<3h/4$, $t>0$:*

$$\begin{aligned}\|\mathcal{U}^h(t,.)-\mathcal{V}^h(t,.)\|_{L^1(\mathbb{R})}\le\|\mathcal{U}^h(0,.)-\mathcal{V}^h(0,.)\|_{L^1(\mathbb{R})}\\ +O(\varepsilon)TV_x\left(\int_{-1}^1 f_0(.,\xi).d\xi\right)\end{aligned} \tag{43}$$

where TV_x stands for the total variation in the space variable.

Hence (43) entails control on the distance from the Maxwellian distribution in the case the initial data is "well-prepared", that is to say: $\|\mathcal{U}^h(0,.) - \mathcal{V}^h(0,.)\|_{L^1(\mathbb{R})}$ of the order of ε. This kind of situation has already been encountered in [21,22].

Under this assumption and substracting the average in ω of the second equation in (34) to the first one, we can compute in a similar way for $\omega \in [0,1]$:

$$\Big(u_{j,n+1}(\omega) - \mathcal{V}_{j,n+1}\Big)\left(1 + \frac{\Delta t}{2\varepsilon h}\right) = u_{j,n}(\omega) - \mathcal{V}_{j,n} + \frac{\Delta t}{2\varepsilon h}\Big(\mathcal{U}_{j,n} - \mathcal{V}_{j,n}\Big)$$
$$+ \frac{2\omega^2 \Delta t}{h(h+2\varepsilon\omega)}\Big(\mathcal{U}_{j-1,n} - \mathcal{V}_{j,n}\Big) - \frac{\Delta t K_\varepsilon}{h}\Big(\mathcal{V}_{j+1,n} - \mathcal{U}_{j,n}\Big).$$

Hence an estimate which somehow gives credit to the simplifications made in §3.1 is deduced:

$$\sum_{j\in\mathbb{Z}} h|u_{j,n}(\omega) - \mathcal{V}_{j,n}| \leq \|\mathcal{U}^h(0,.) - \mathcal{V}^h(0,.)\|_{L^1(\mathbb{R})} +$$
$$O(\varepsilon) TV_x \left(\int_{-1}^{1} f_0(.,\xi).d\xi\right) + \prod_{i=1}^{n}\left(1 + \frac{\Delta t}{2\varepsilon h}\right)^{-1} \sum_{j\in\mathbb{Z}} h|u_{j,0}(\omega) - \mathcal{V}_{j,0}|.$$

3.4 Regularity in time and convergence

The basic time regularity result reads:

Lemma 7 *Assume* $0 \leq f_0(x,\xi) \in L^\infty([-1,1]; L^1 \cap BV(\mathbb{R}))$*; then if* $2\Delta t \leq h^2$*, there holds for all* $0 \leq \varepsilon \leq 3h/4$*,* $t \geq s \geq 0$*:*

$$\|\mathcal{U}^h(t,.) - \mathcal{U}^h(s,.)\|_{L^1(\mathbb{R})} + \|\mathcal{V}^h(t,.) - \mathcal{V}^h(s,.)\|_{L^1(\mathbb{R})} \leq$$
$$O\left(\tfrac{|t-s|}{\varepsilon h}\right) \|\mathcal{U}^h(s,.) - \mathcal{V}^h(s,.)\|_{L^1(\mathbb{R})} + O(h) TV_x \left(\textstyle\int_{-1}^{1} f_0(.,\xi).d\xi\right) \tag{44}$$

where TV_x *stands for the total variation in the space variable.*

From Lemmas 7 and 6 follows immediately an Hölder regularity for $\mathcal{U}^h$, $\mathcal{V}^h$ in the case the initial data is well-prepared and $h = O(\sqrt{\Delta t})$.

Theorem 2 *Assume* $0 \leq f_0(x,\xi) \in L^\infty([-1,1]; L^1 \cap BV(\mathbb{R}))$ *and the CFL condition* $2\Delta t \leq h^2$*; then if the initial datum is such that* $\|\mathcal{U}^h(0,.) - \mathcal{V}^h(0,.)\|_{L^1(\mathbb{R})} = O(\varepsilon)$*, the sequences* $\mathcal{U}^h$*,* $\mathcal{V}^h$ *are relatively compact in* $L^1_{loc}(\mathbb{R}^+_* \times \mathbb{R})$ *as* $h, \varepsilon \to 0$ *with* $\varepsilon < 3h/4$*. Moreover, the piecewise constant function* $\varrho^h = \frac{1}{2}(\mathcal{U}^h + \mathcal{V}^h)$ *converges towards the unique solution of (3) with the initial datum* $\varrho(0,.) = \frac{1}{2}\int_{-1}^{1} f_0(.,\xi).d\xi$*.*

Proof From Lemmas 5, 6, 7, one deduces strong L^1_{loc} compactness by means of Helly's theorem. Adding the two equations in (34), one sees that there holds with obvious notation:

$$\varrho_{j,n+1} = \varrho_{j,n} + \frac{\Delta t}{2h} K_\varepsilon(\varrho_{j+1,n} - 2\varrho_{j,n} + \varrho_{j-1,n}) + (R_\varepsilon)_{j,n}. \tag{45}$$

From previous estimates, one sees that $\|R_\varepsilon(n\Delta t, .)\|_{L^1(\mathbb{R})} = O(\varepsilon)$ and the consistency with the asymptotic equation (3) is ensured because $K_\varepsilon \to 2/3h$ as $\varepsilon \to 0$.

4 Numerical tests

4.1 The linear heat wave

This first numerical run consists in checking computationally the results established in §3.2–4, namely the approximation of (3) by (34) with a particular initial data $\varrho_0(x) = H(\frac{1}{2} - x)$ where $H(.)$ is the Heaviside function. The exact diffusion wave is given by:

$$\varrho(t,x) = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{x - \frac{1}{2}}{\sqrt{\frac{4}{3}t}}\right)\right].$$

On the top of Fig.1, we compare the $\varrho_{j,n}$ produced by (34) with this exact solution for $T = 0.07$ with $u_0 = v_0 = \varrho_0$, $\varepsilon = 2^{-6}$, $h = 0.02$ and $\Delta t = h^2$. On the bottom, we display the Maxwellian gap divided by ε for 20 decreasing values of ε together with the absolute error between the kinetic approximations and the analytical solution. The code involves the exact value of K_ε:

$$K_\varepsilon = \frac{1}{2\varepsilon} - \frac{h}{2\varepsilon^2} + \frac{h^2}{4\varepsilon^3}\ln\left(1 + \frac{2\varepsilon}{h}\right).$$

Looking at the error's decay, one notices a first regime for which the main factor is the value of ε (this corresponds to $\varepsilon \geq h$): the rate of decay is approximately of order one. This illustrates the fact that, in (45), the norm of R_ε is $O(\varepsilon)$ as stated in the proof of Theorem 2. In the regime for which $h > \varepsilon$, the terms of the order of h dominate and the error stalls.

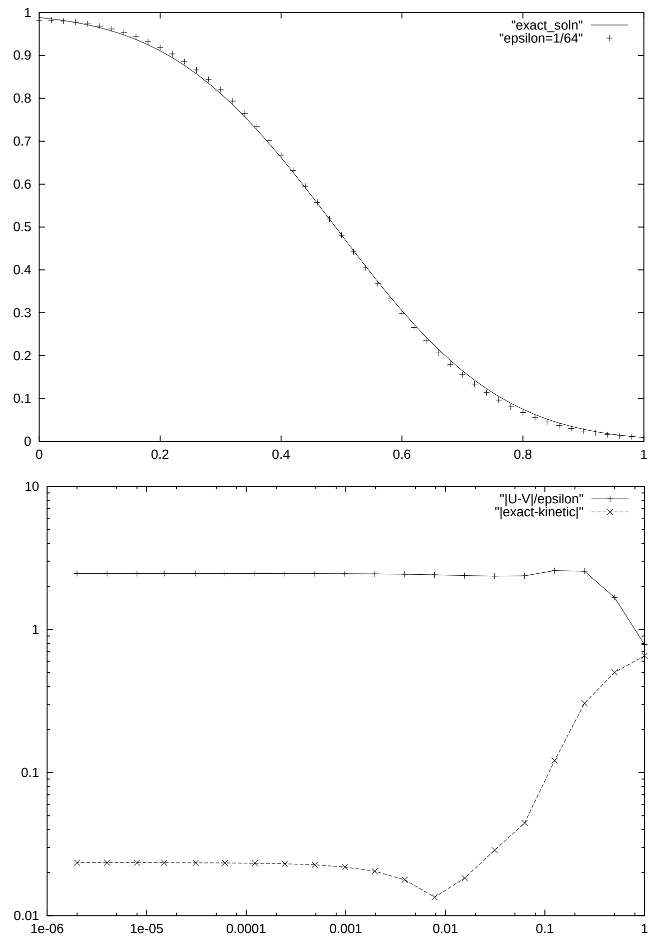

Fig. 1. Results for (34) on (2)–(3) with a Riemann initial datum.

4.2 A variable optical thickness

This second numerical test is devoted to a variant of (2); namely we want to consider a very simple production term and a variable optical thickness within a boundary-value problem as in [29],

$$\varepsilon\partial_t f + \xi\partial_x f = \frac{\sigma(x)}{\varepsilon}\left(\frac{1}{2}\int_{-1}^{1} f.d\xi - f\right) + \varepsilon, \qquad x \in [0,1], \tag{46}$$

with Maxwellian initial conditions $f_0 \equiv 0$ and $v(t, x = 1, \omega) = u(t, x = 0, \omega) = 0$ for all $\omega \in [0,1]$ and $t > 0$. The asymptotic behaviour of (46) as $\varepsilon \to 0$ is given by the slightly different diffusion

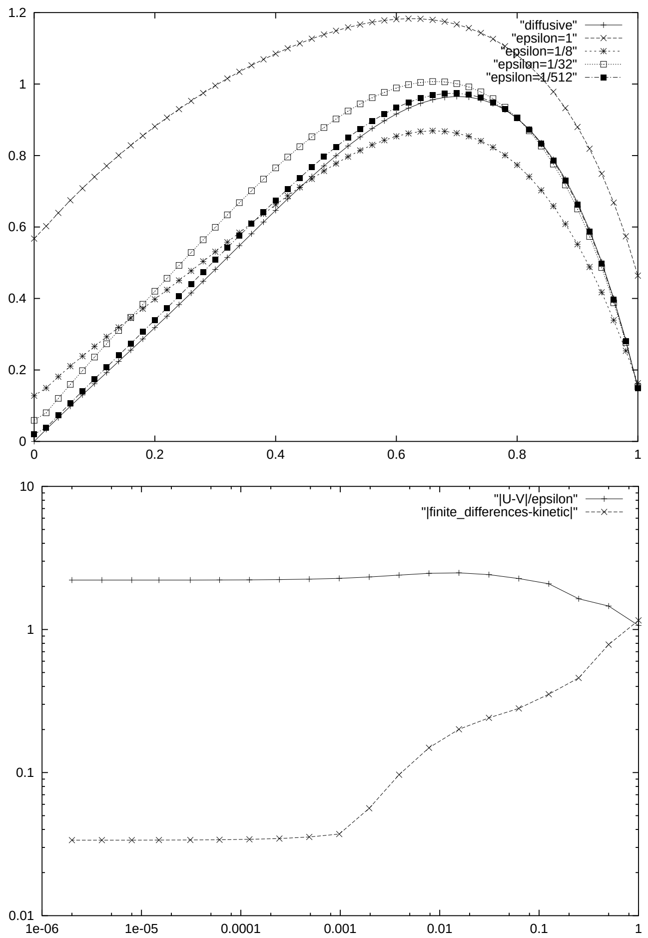

Fig. 2. Results in $T = 1.35$ for (46) with $f_0 \equiv 0$.

equation:

$$\partial_t \varrho - \frac{1}{3}\partial_x \left(\frac{\partial_x \varrho}{\sigma(x)} \right) = 1, \qquad \varrho(t,0) = \varrho(t,1) = 0,\ t > 0. \tag{47}$$

In our computations, we selected $\sigma(x) = 1 + 10x^2$ together with $h = 0.02$, $\Delta t = 0.0003$ and both schemes (18), in the setting proposed in Remark 2, and (34) in order to produce the results displayed in Fig.2. We used 20 values for ε between 1 and 2^{-20} and passed from (18) to (34) as soon as $\varepsilon < h\|\sigma\|_{L^\infty}$. We also computed directly the solution to (47) by a standard finite-difference scheme for comparison, indexed by "diffusive" on the top picture. On the bottom picture, we observe that the remaining terms R_ε separating this asymptotic solution from the kinetic one are of order $\varepsilon^{2/3}$ for moderate values. The Maxwellian gap keeps on being $O(\varepsilon)$ for every considered value ε. It turns out that despite the fact the following scheme remains stable for $\Delta t \leq \varepsilon h$,

$$\begin{cases} u_{j,n+1}(\omega) = u_{j,n}(\omega) + \Delta t - \dfrac{\omega \Delta t}{\varepsilon h}\Big\{ \Big(u_{j,n}(\omega) - u_{j-1,n}(\omega)\Big) \\ \quad + \Big(1 - e^{-h\sigma(x_{j-\frac{1}{2}})/\varepsilon\omega}\Big)\Big(u_{j-1,n}(\omega) - \dfrac{1}{2}\displaystyle\int_0^1 \widetilde{u_{j-1,n} + v_{j,n}}(s).ds\Big)\Big\}, \\ v_{j,n+1}(\omega) = v_{j,n}(\omega) + \Delta t + \dfrac{\omega \Delta t}{\varepsilon h}\Big\{ \Big(v_{j+1,n}(\omega) - v_{j,n}(\omega)\Big) \\ \quad - \Big(1 - e^{-h\sigma(x_{j+\frac{1}{2}})/\varepsilon\omega}\Big)\Big(v_{j+1,n}(\omega) - \dfrac{1}{2}\displaystyle\int_0^1 \widetilde{u_{j,n} + v_{j+1,n}}(s).ds\Big)\Big\}, \end{cases}$$

its performances decay strongly for $\varepsilon < 10h$ because of an excessive numerical dissipation, see top picture in Fig.2 for $\varepsilon = 0.125$. A 7-points Gaussian quadrature rule has been used to compute the modified velocity averages inside the aforementioned scheme.

4.3 The porous medium equation

At last, we simulated the so-called Barenblatt's solution of the porous medium equation corresponding to (6) with $K(\varrho) = 1/6\varrho$. We used the schemes (18)–(34) with the modified numerical fluxes (16)–(38) proposed in Remarks 1 and 3; the parameters were $h = 0.15$, $\Delta t = 0.01$ and an approximation for $K_\varepsilon(\varrho) \simeq 2\varrho\left(\frac{2-3\varepsilon}{h}\right)$. The picture on top of Fig.3 displays the numerical outcome at time $T = 3$ in the case $\varepsilon = 1$ and $\varepsilon = 2^{-9}$ respectively. One observes a good convergence despite the singularities on both sides of the solution. We switched from (18) to (34) for $\varepsilon < h/3$. This is noticeable if looking at the absolute errors' decay on the bottom picture. In the rarefied regime, the kinetic densities remain far from the Maxwellian distribution. When the diffusive scheme is used, a $O(\varepsilon^{2/3})$ decay is observed as in Figure 1 for moderate values of ε. For $\varepsilon \ll h$, the error stalls. Concerning the relaxation mechanism, one can observe that for $\varepsilon < 0.05$ the Maxwellian ratio is still of the order of ε since we took $f_0(t=0,x,\xi) = \varrho(t=0,x)$ for all ξ. The exact solution reads for $t \geq 0$:

$$\varrho(t,x) = \frac{1}{r(t)}\left(1 - \left(\frac{x}{r(t)}\right)^2\right)\mathbf{1}_{|x|\leq r(t)}, \qquad r(t) = (12(1+t))^{\frac{1}{3}}.$$

Its "two-stream approximation" is the model studied in [22], §4.1.

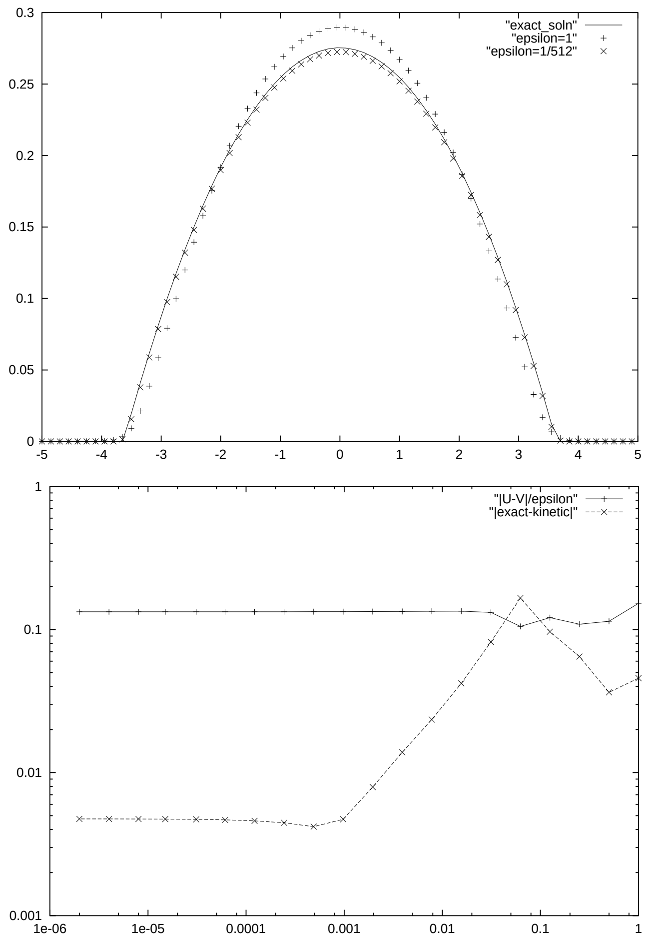

Fig. 3. Results for (18)–(34) on (5)–(6) with Barenblatt's solution.

5 Conclusion

In this paper, we applied the nowadays classical Well-Balanced approach to a simple model of radiative transfer equations (2) on the whole range of parameters $\varepsilon \in [0, 1]$. In contrast with [21], some new features appeared within the numerical processing of (4): first, the function $a^\varepsilon(\omega)$ which plays a key role when establishing the mass conservation property in the rarefied regime (Lemmas 1–2) and later, when splitting $\tilde{u}(\omega)$, $\tilde{v}(\omega)$ between Maxwellian and diffusive parts in the diffusive scaling. Second, it turned out that in order to preserve the properties (10)–(12), this Maxwellian term musts involve both $u_L(\omega)$ and $v_R(\omega)$. Hence it became necessary to "cut" some terms and make some assumptions inside this decomposition to produce a scheme like (34) which is as computationally expensive as the one in [21] and for which all the stability and asymptotic properties can be proved in the limit $\varepsilon \to 0$, see Lemmas 4–7.

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Appendix: Technical proofs

Proof of Theorem 1

As a consequence of Lax's equivalence theorem, we have to establish stability and consistency for (18) as $h \to 0$ with $\varepsilon > 0$ fixed under the CFL assumption $\Delta t \leq \varepsilon h$.

– Stability: We start from (21); summing on $j \in \mathbb{Z}$, we get for all $\omega \in [0,1]$:

$$\sum_{j\in\mathbb{Z}} h(|u_{j,n+1}| + |v_{j,n+1}|)(\omega) \leq \sum_{j\in\mathbb{Z}} h(|u_{j,n}| + |v_{j,n}|)(\omega) +$$
$$\frac{\Delta t a^\varepsilon(\omega)}{\varepsilon^2} \sum_{j\in\mathbb{Z}} h \Big\{ \int_{h/\varepsilon}^1 \overline{|u_{j,n}| + |v_{j,n}|}(s).ds - \Theta_\omega(\frac{h}{2\varepsilon})(|u_{j,n}| + |v_{j,n}|)(\omega) \Big\}.$$

We take the supremum over $\omega \in [0,1]$ on both sides. Since the integral term is a convex combination, we claim that the second part of this inequality's right-hand side is negative for h small enough. This yields:

$$\sup_{\omega\in[0,1]} \Big\{ \sum_{j\in\mathbb{Z}} h(|u_{j,n+1}|+|v_{j,n+1}|)(\omega) \Big\} \leq \sup_{\omega\in[0,1]} \Big\{ \sum_{j\in\mathbb{Z}} h(|u_{j,n}|+|v_{j,n}|)(\omega) \Big\}.$$

By linearity of (18), the same procedure can be applied to derive a bound on the total variation in the x variable. The piecewise constant functions (22) satisfy therefore for all time $t \in \mathbb{R}^+$:

$$u^h(t,.,.), v^h(t,.,.) \in L^\infty([0,1]; L^1 \cap BV(\mathbb{R})).$$

This allows to derive some regularity in the time variable; namely,

$$\sup_{\omega\in[0,1]} \Big\{ \sum_{j\in\mathbb{Z}} h(|u_{j,n+1}-u_{j,n}|+|v_{j,n+1}-v_{j,n}|)(\omega) \Big\} \leq \Delta t \sup_{\xi\in[-1,1]} TV_x(f_0(x,\xi)).$$

Hence $u^h, v^h \in L^\infty([0,1]; BV_{loc}(\mathbb{R}^+_* \times \mathbb{R}))$ uniformly in $h \geq 0$.

– Consistency: We rewrite the first equation of (18) as follows (the second one is treated exactly the same way) for any $\omega \in [0,1]$:

$$\frac{u_{j,n+1}(\omega) - u_{j,n}(\omega)}{\Delta t} + \frac{\omega}{\varepsilon} \frac{u_{j,n}(\omega) - u_{j-1,n}(\omega)}{h} =$$
$$\frac{a^\varepsilon(\omega)}{\varepsilon^2} \left(\frac{1}{2} \int_{h/\varepsilon}^{1} \overline{u_{j-1,n} + v_{j,n}}(s).ds - \Theta_\omega\left(\frac{h}{2\varepsilon}\right) u_{j-1,n}(\omega) \right).$$

The left part of this equation can be shown to be consistent in a very classical way. To carry out the other, one notices that by their very definition, $a^\varepsilon(\omega) \to 1$ (see Lemma 1) and $\Theta_\omega(h/2\varepsilon) \to 1$ as $h \to 0$. Thus the interval of integration is correct and the modified averages boil down to arithmetic ones in the limit. Moreover, by space regularity, the integral term converges to the right one.

Finally, u^h, v^h converge towards a weak solution of the system

$$\begin{cases} \partial_t u + \dfrac{\omega}{\varepsilon} \partial_x u = \dfrac{1}{\varepsilon^2} \Big(\displaystyle\int_0^1 (u+v)(\omega').d\omega' - u \Big), \\ \partial_t v - \dfrac{\omega}{\varepsilon} \partial_x v = \dfrac{1}{\varepsilon^2} \Big(\displaystyle\int_0^1 (u+v)(\omega').d\omega' - v \Big), \end{cases}$$

whose initial data is deduced from f_0. By uniqueness, all the sequence converges.

Proof of the equality (36)

The calculation goes as follows:

$$\begin{aligned} Z^\varepsilon - W^\varepsilon &= \Delta t \int_0^1 \frac{1}{2\varepsilon h + \Delta t} - \frac{2\omega^2}{\varepsilon h + \omega \Delta t}.d\omega \\ &= \Delta t \int_0^1 \frac{\varepsilon h + \omega \Delta t - 2\omega^2 (2\varepsilon h + \Delta t)}{(2\varepsilon h + \Delta t)(\varepsilon h + \omega \Delta t)} \\ &= \Delta t \Big\{ \int_0^1 \frac{(1-2\omega) 2\omega \varepsilon h}{(2\varepsilon h + \Delta t)(\varepsilon h + \omega \Delta t)}.d\omega + \int_0^1 \frac{1 - 2\omega}{2\varepsilon h + \Delta t}.d\omega \Big\}. \end{aligned}$$

It remains to observe that the last integral is zero and to take the modulus.

Proof of Lemma 5

Relying on the positivity of the incremental coefficients established in the proof of Lemma 4, we can take the moduli inside the schemes on $\mathcal{U}_{j,n}$, $\mathcal{V}_{j,n}$. This leads to:

$$\left(1+\frac{\Delta t}{\varepsilon h}\right)\Big(|\mathcal{U}_{j,n+1}|+|\mathcal{V}_{j,n+1}|\Big) \leq C_\varepsilon\Big(|\mathcal{U}_{j,n}|+|\mathcal{V}_{j,n}|\Big) \\ +D_\varepsilon\Big(|\mathcal{U}_{j-1,n}|+|\mathcal{V}_{j+1,n}|\Big).$$

We have:

$$C_\varepsilon = 1+\frac{\Delta t}{2\varepsilon h}\left(1-\frac{\Delta t}{h}K_\varepsilon\right)-\frac{\Delta t}{h}K_\varepsilon,$$
$$D_\varepsilon = \frac{\Delta t}{2\varepsilon h}\left(1+\frac{\Delta t}{h}K_\varepsilon\right)+\frac{\Delta t}{h}K_\varepsilon.$$

Hence one checks readily that $C_\varepsilon + D_\varepsilon = 1+\frac{\Delta t}{\varepsilon h}$. This is enough for (41). The linearity of the scheme ensures that (42) follows too.

Proof of Lemma 6

We compute the difference of the schemes on $\mathcal{U}_{j,n}$, $\mathcal{V}_{j,n}$:

$$\left(1+\frac{\Delta t}{\varepsilon h}\right)\Big(\mathcal{U}_{j,n+1}-\mathcal{V}_{j,n+1}\Big) = \left(1+\frac{2\Delta t K_\varepsilon}{h}\right)\Big(\mathcal{U}_{j,n}-\mathcal{V}_{j,n}\Big) \\ +\frac{\Delta t K_\varepsilon}{h}\Big(\mathcal{U}_{j-1,n}-\mathcal{U}_{j,n}\Big) \\ -\frac{\Delta t K_\varepsilon}{h}\Big(\mathcal{V}_{j+1,n}-\mathcal{V}_{j,n}\Big).$$

Taking the modulus, summing and taking advantage of the TVD property (*cf.* Lemma 5) leads to

$$\|\mathcal{U}^h(t,.)-\mathcal{V}^h(t,.)\|_{L^1(\mathbb{R})} \leq \alpha^n\|\mathcal{U}^h(0,.)-\mathcal{V}^h(0,.)\|_{L^1(\mathbb{R})} \\ +\tfrac{2\varepsilon}{3(1-\alpha)}TV_x\left(\textstyle\int_{-1}^1 f_0(.,\xi).d\xi\right),$$

for some convenient $n\in\mathbb{N}$ and

$$\alpha = \frac{1+\frac{2\Delta t K_\varepsilon}{h}}{1+\frac{\Delta t}{\varepsilon h}}.$$

Then, one notices that

$$\varepsilon < \frac{3h}{4} \qquad \Rightarrow \qquad \alpha < 1,$$

and (43) follows.

Proof of Lemma 7

We add the two equations in (34) and integrate in $\omega \in [0,1]$:

$$\begin{cases} \mathcal{U}_{j,n+1} - \mathcal{U}_{j,n} = -\dfrac{\Delta t}{2\varepsilon h}(\mathcal{U}_{j,n+1} - \mathcal{V}_{j,n+1}) - \dfrac{\Delta t K_\varepsilon}{h}(\mathcal{V}_{j,n} - \mathcal{U}_{j-1,n}) \\ \mathcal{V}_{j,n+1} - \mathcal{V}_{j,n} = \dfrac{\Delta t}{2\varepsilon h}(\mathcal{U}_{j,n+1} - \mathcal{V}_{j,n+1}) - \dfrac{\Delta t K_\varepsilon}{h}(\mathcal{U}_{j,n} - \mathcal{V}_{j+1,n}) \end{cases}$$

Adding and subtracting $\frac{\Delta t}{2\varepsilon h}(\mathcal{U}_{j,n} - \mathcal{V}_{j,n})$ in both members of the first equation leads to:

$$\left(1 + \frac{\Delta t}{2\varepsilon h}\right)(\mathcal{U}_{j,n+1} - \mathcal{U}_{j,n}) - \frac{\Delta t}{2\varepsilon h}(\mathcal{V}_{j,n+1} - \mathcal{V}_{j,n}) =$$
$$\left(\frac{\Delta t K_\varepsilon}{h} - \frac{\Delta t}{2\varepsilon h}\right)(\mathcal{U}_{j,n} - \mathcal{V}_{j,n}) - \frac{\Delta t K_\varepsilon}{h}(\mathcal{U}_{j,n} - \mathcal{U}_{j-1,n}).$$

Adding and subtracting the same quantity in the second equality yields a similar expression. At this level, one can take the modulus and add both equations to obtain:

$$|\mathcal{U}_{j,n+1} - \mathcal{U}_{j,n}| + |\mathcal{V}_{j,n+1} - \mathcal{V}_{j,n}| \leq \frac{\Delta t}{h}\left|\frac{1}{\varepsilon} - 2K_\varepsilon\right| |\mathcal{U}_{j,n} - \mathcal{V}_{j,n}|$$
$$+ \frac{\Delta t K_\varepsilon}{h}\Big\{|\mathcal{U}_{j,n} - \mathcal{U}_{j-1,n}| + |\mathcal{V}_{j+1,n} - \mathcal{V}_{j,n}|\Big\}.$$

All the terms appearing can be controlled by means of Lemmas 5 and 6 since $\Delta t K_\varepsilon / h = O(1)$; this yields the case where $t = (n+1)\Delta t$, $s = \Delta t$. Extension to the general case is straightforward.

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