Asymptotic Pricing of Commodity Derivatives using Stochastic Volatility Spot Models *

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It is well known that stochastic volatility is an essential feature of commodity spot prices. By using methods of singular perturbation theory, we obtain approximate but explicit closed form pricing equations for forward contracts and options on single- and two-name forward prices. The expansion methodology is based on a fast mean-reverting stochastic volatility driving factor, and leads to pricing results in terms of constant volatility prices, their Delta's and their Delta-Gamma's. The stochastic volatility corrections lead to efficient calibration and sensitivity calculations.

1. Introduction

A quick glance at any commodities price data will reveal the obvious fact that volatility is a stochastic quantity. A now classical and extremely popular model for incorporating this stochasticity of volatility is the Heston (1993) model, in which the instantaneous price variance follows a Cox, Ingersoll, and Ross (1985) (CIR) like process. Eydeland and Geman (1998) were among the first to utilize the Heston model in the context of energy derivatives. More recently, Richter and Sørensen (2006) introduce a stochastic convenience yield model with one underlying stochastic volatility factor in the same spirit of Heston. They make an extensive case study on soybean futures and options data and demonstrate that stochastic volatility is a significant factor. Since Heston inspired stochastic volatility models lead to affine structures, they appear natural; however, the resulting pricing equations are in terms of inverse Fourier transforms rather than explicitly in terms of elementary functions – or even special functions. This is not a substantial disadvantage when valuing only a few options; however, in a calibration and trading environment many contracts are involved and consistently calibrating all instruments to market prices would be difficult and time consuming. Furthermore, determining hedge ratios will require computations of the sensitivities of the price to various parameters – the so-called "Greeks" – which, if computed using Fourier methods, may result in further speed reduction. To circumvent these issues, we transport singular perturbation theory techniques first developed for equity derivatives (see Fouque, Papanicolaou, and Sircar, 2000a), and then for interest rate

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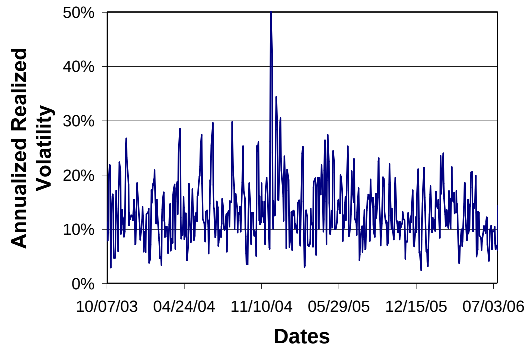

Figure 1. The annualized running five-day moving volatility of the NYMEX sweet crude oil spot price for the period 10/07/03 to 07/03/06.

derivatives in Cotton, Fouque, Papanicolaou, and Sircar (2004), into the context of commodities and commodities derivatives.

Asymptotic methods have three main advantages over traditional approaches: (i) they naturally lead to efficient calibration across a set of forward contracts; (ii) they lead to approximate, but *explicit*, closed form pricing equations for a wide class of contingent claims; and (iii) the resulting approximate prices are independent of the specific underlying volatility model. Notably, these prices are exact when the mean-reversion rate is large - serendipitously, this is precisely the manner in which the market prices seem to behave. In addition, many of the salient features of option prices – most strikingly the implied volatility smile or smirk – are captured by these methods. Fouque, Papanicolaou, and Sircar (2000a) were the first to introduce the use of asymptotic methods in the context of derivative pricing and, together with their collaborators, have written several articles on the application of these techniques to the equity and interest rate markets. To this date, none of these techniques have been applied to the commodities markets where a unique set of challenges arise.

To motivate the validity of asymptotic methods for commodities, we plot the running five-day realized volatility for the NYMEX sweet crude oil spot price for the period 10/07/03 to 07/03/06 in Figure 1 which clearly demonstrates the fast mean-reversion of volatility. We therefore model the underlying commodity spot price volatility as a function $\sigma_X(Z_t)$ of a fast-mean reverting hidden process Z_t. As is well known, commodities, unlike equities, tend to have strong mean-reversion effects in the prices themselves. Secondly, the long-run mean-reversion is not constant through time, rather it is stochastic. These and many other stylized empirical facts are well documented in, for example, Clewlow and Strickland (2000), Eydeland and Wolyniec (2003) and Geman (2005). Correctly accounting for such behavior together with stochastic volatility and using such a model to price derivatives is the main contribution of this article.

Hikspoors and Jaimungal (2007) introduced tractable two-factor mean-reverting models (with and without jumps) and priced forward and spread options on forward contracts. In this article, we successfully determine the asymptotic corrections for forward prices based on stochastic volatility extensions of the one- and two-factor mean-reverting diffusive spot price models. To this end, we quickly review the one- and two-factor spot price models, together with the resulting forward and option prices, in Section 2.1. The stochastic volatility extended one- and two-factor mean-reverting models are introduced in Section 2.2 and we illustrate that such a model does not provide closed form forward prices. Section 3 contains two of our main asymptotic expansion results: the forward prices for the stochastic volatility extended one- and two-factor mean-reverting models are shown to be well approximated by adjusted constant volatility results. By calibrating to existing forward prices, the volatility function $\sigma_X(z)$ is rendered irrelevant; instead, a new effective pseudo-parameter arises as a smoothed version of the stochastic volatility. This pseudo-parameter appears again in the pricing of contingent claims, allowing a consistent calibration between forward and options prices.

Given that the model is calibrated to forward prices, the next task is to determine the price corrections to contingent claims. Since typical single-name contingent claims are written on the forward prices, which we have already approximated, the asymptotic analysis relies on a consistent layering of approximations. In Section 4, these asymptotic price corrections to single-name contingent claims are explored. Interestingly, we demonstrate that the corrections depend solely on the Delta's and Delta-Gamma's of the option using the constant volatility model. Furthermore, once the free pseudo-parameter arising in the forward price approximation is calibrated to market prices, the option price corrections are uniquely determined. Section 5 contains the extension of these methods to contingent claims written on two forward prices. There are several subtle issues associated with the expansion; nonetheless, we pleasantly find that the resulting price corrections are once again in terms of the Delta's and Delta-Gamma's of the constant volatility price.

We close the paper with conclusions and some comments on ongoing and future work in Section 6.

2. Spot Price Models and Main Properties

This section first provides an overview of the standard one- and two-factor constant volatility models for energy spot price dynamics (for early uses of the one-factor models see Gibson and Schwartz, 1990; Cortazar and Schwartz, 1994). The forward prices, call and exchange option prices are also reviewed. Given these constant volatility models, the stochastic volatility (SV) extensions are then introduced and we briefly demonstrate that the SV extensions lack an affine structure.

We explain why and where asymptotic methods constitute a very useful set of tools in energy markets, as they already have been shown to be for their stocks and interest rate counterparts.

2.1. Constant Volatility Models

2.1.1. The One-Factor Model

For completeness, this section provides a quick review of a well known one-factor energy spot price model and its use in derivatives pricing. Let S_t denote the spot dynamics defined under the risk-neutral measure $\mathbb{Q}$. The standard model assumes

$$\begin{aligned} S_t &:= \exp\{g_t + X_t\}\ , &(1)\\ dX_t &= \beta\,(\phi - X_t)\,dt + \sigma_X\, dW_t^{(1)}\ , &(2) \end{aligned}$$

where σ_X is the constant volatility, g_t is a deterministic seasonality factor and $W^{(1)}$ is a $\mathbb{Q}$-Wiener process. An important traded commodity instrument is the futures contract with futures price $F_{t,T}$. In a no-arbitrage, deterministic interest rate, environment the futures and forward price coincides and the forward price must be given by $F_{t,T} := \mathbb{E}_t^{\mathbb{Q}}[S_T]$, where $\mathbb{E}_t^{\mathbb{Q}}[\mathcal{R}]$ represents the expectation of $\mathcal{R}$ conditional on the natural filtration $\mathcal{F}_t$ generated by the underlying Wiener process(es). The forward price process being a martingale, must satisfy the following PDE

$$\begin{cases} \mathcal{A}F(t,x) &= 0\,, \\ F(T,x) &= e^{g_T + x}\,, \end{cases} \tag{3}$$

where $\mathcal{A}$ is the infinitesimal generator of (t, X_t). Within the present context, a straightforward calculation provides the following result

$$F_{t,T} = \exp\left\{ g_T + \phi\left(1 - e^{-\beta(T-t)}\right) + \frac{\sigma_X^2}{2} h(t,T;2\beta) + e^{-\beta(T-t)}\left(\log(S_t) - g_t\right)\right\}. \tag{4}$$

Here, and in the sequel,

$$h(t,T;a) := (1 - e^{-a(T-t)})/a\ . \tag{5}$$

Turning to the valuation of European contingent claims, let $\varphi(F_{T_0,T})$ denote the terminal payoff at time T_0 of a European option written on a forward price. The no-arbitrage price Π_{t,T_0} is the discounted expectation under the risk-neutral measure $\mathbb{Q}$. Specifically,

$$\Pi_{t,T_0} := \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_s\,ds}\,\varphi\left(F_{T_0,T}\right)\right] = P(t,T_0)\,\mathbb{E}_t^{\mathbb{Q}}\left[\varphi\left(F_{T_0,T}\right)\right]\ . \tag{6}$$

Here, and in the remainder of this article, interest rates are deterministic, and we denote the T_0-maturity zero-coupon bond price contracted at time t by $P(t,T_0)$. Following the martingale techniques employed in Hikspoors and Jaimungal (2007) Section 3.4, the price $C_{t;T_0,T}$ at time t of a T_0-expiry call option with strike K written on the forward $F_{T_0,T}$ can be expressed in the following Black-Scholes like form:

$$C_{t;T_0,T} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_s\,ds}\left(F_{T_0,T} - K\right)_+\right] = P(t,T_0)\,\left[F_{t,T}\,\Phi(d^* + \sigma_{t;T_0}^*) - K\,\Phi(d^*)\right]\ . \tag{7}$$

Here, d^* and $\sigma_{t;T_0}^*$ are functions of the model parameters and time only, and $\Phi(\cdot)$ is the standard gaussian CDF. A similar result follows for forward exchange option prices:

$$\Pi_{t;T_0,T_1,T_2}^F = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_s\,ds}\left(F_{T_0,T_1}^{(1)} - \alpha F_{T_0,T_2}^{(2)}\right)_+\right] = P(t,T_0)\,\left[F_{t,T_1}^{(1)}\,\Phi(d + \sigma_{t;T_0}) - \alpha F_{t,T_2}^{(2)}\,\Phi(d)\right]\ . \tag{8}$$

The interested reader is referred to the original article for the precise form of the various coefficients.

2.1.2. The Two-Factor Model : *Mean-Reverting Long Run Mean*

Hikspoors and Jaimungal (2007) utilize a two-factor mean-reverting model, in which the long-run mean of the previous one-factor model is itself stochastic and mean-reverts to a second long-run mean. In that work, the authors study the valuation of forward contracts and exchange options and also include jumps into the spot price dynamics. In this article, we focus on the jump-free model; however, much of the results can be extended to the jump case with little additional complication.

In this two-factor model, the $\mathbb{Q}$-dynamics of the spot S_t is

$$\begin{aligned} S_t &= \exp\{g_t + X_t\}, & (9)\\ dX_t &= \beta\,(Y_t - X_t)\,dt + \sigma_X\,dW_t^{(1)}, & (10)\\ dY_t &= \alpha\,(\phi - Y_t)\,dt + \sigma_Y\,dW_t^{(2)}, & (11)\end{aligned}$$

with correlation structure,

$$d\left[W^{(1)}, W^{(2)}\right]_t = \rho_1\,dt\,. \tag{12}$$

Here, β controls the speed of mean-reversion of X_t to the stochastic long-run level Y_t; α controls the speed of mean-reversion of the long-run level Y_t to the target long-run mean ϕ; σ_X and σ_Y control the size of the fluctuations around these means. The forward price process can be shown to be

$$F_{t,T} = \exp\left\{g_T + R_{t,T} + G_{t,T} + e^{-\beta(T-t)}X_t + M_{t,T}\,Y_t\right\} \tag{13}$$

where the expressions for $M_{t,T}$, $G_{t,T}$ and $R_{t,T}$ are functions of time and the model parameters. Even within this more general setting, the call option price $C_{t;T_0,T}$ on a forward as well as the exchange option price $\Pi^F_{t;T_0,T_1,T_2}$ on forwards have similar forms to (7) and (8) respectively. More complicated expression for d^*, $\sigma^*_{t;T_0,T}$, d and $\sigma_{t;T_0,T}$ arise, yet they remain explicit functions only of the model parameters and time. The interested reader is once again referred to Hikspoors and Jaimungal (2007) for details.

2.2. Stochastic Volatility Extensions

2.2.1. The SV Extended One-Factor Model

In this section, the stochastic volatility (SV) extended one-factor model is explored in detail; in particular, the volatility σ_X is now assumed to be driven by a fast mean-reverting stochastic process. Explicitly, the spot is now modeled under the risk-neutral measure $\mathbb{Q}$ as

$$\begin{aligned} S_t &= \exp\{g_t + X_t\}, & (14)\\ dX_t &= \beta\,(\phi - X_t)\,dt + \sigma_X(Z_t)\,dW_t^{(1)}, & (15)\\ dZ_t &= \alpha\,(m - Z_t)\,dt + \sigma_Z\,dW_t^{(3)}, & (16)\end{aligned}$$

where $\sigma_X(\cdot)$ is a strictly positive smooth function bounded above and below by positive constants and with bounded derivatives. We also specify the following correlation structure

$$d\left[W^{(1)}, W^{(3)}\right]_t = \rho_2\,dt\,. \tag{17}$$

The smoothness and boundedness assumptions on the volatility function $\sigma_X(\cdot)$ may appear overly restrictive at first; however, as we later demonstrate, singular perturbation methods remarkably lead to pricing results that are completely independent of its detailed specification.

It is not possible to solve the system of SDEs (14)-(16) explicitly; nonetheless, we now explore its implications for forward prices. As usual, the forward price is $F(t,x,z) = \mathbb{E}^{\mathbb{Q}}_{t,x,z}[S_T]$. Equivalently, $F(t,x,z)$ can be characterized as the solution of the following PDE:

$$\begin{cases} \frac{\partial F}{\partial t} + \beta(\phi - x)\frac{\partial F}{\partial x} + \alpha(m-z)\frac{\partial F}{\partial z} + \frac{1}{2}\sigma_X^2(z)\frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\sigma_Z^2\frac{\partial^2 F}{\partial z^2} + \rho_2\sigma_Z\sigma_X(z)\frac{\partial^2 F}{\partial x \partial z} = 0 \\ F(T,x,z) = e^{g_T + x} \end{cases} \tag{18}$$

As we now show, a solution to (18) can be decomposed into two independent parts; one having a log-affine structure in x and the other being independent of x. First, let W_t be a $\mathbb{Q}$-Wiener process independent of $\left(W_t^{(1)}, W_t^{(3)}\right)$ and define the following

$$d\widetilde{Z}_t := \left(\alpha(m - \widetilde{Z}_t) + \rho_2\sigma_Z\sigma_X(\widetilde{Z}_t)e^{-\beta(T-t)}\right)dt + dW_t\,, \tag{19}$$

$$c(t,z) := \frac{1}{2}\sigma_X^2(z)e^{-2\beta(T-t)} + \beta\phi e^{-\beta(T-t)}\,, \tag{20}$$

$$M(t,z) := \mathbb{E}^{\mathbb{Q}}_{t,z}\left[\exp\left\{\int_t^T c(s,\widetilde{Z}_s)ds\right\}\right]. \tag{21}$$

Then, by smoothness and boundedness of $c(\cdot,\cdot)$ and of the coefficients of $d\widetilde{Z}_t$, $M(t,z)$ is finite and satisfies the following PDE (see Duffie, Pan, and Singleton (2000))

$$\begin{cases} \frac{\partial M}{\partial t} + \left(\alpha(m-z) + \rho_2\sigma_Z\sigma_X(z)e^{-\beta(T-t)}\right)\frac{\partial M}{\partial z} + \frac{1}{2}\sigma_Z^2\frac{\partial^2 M}{\partial z^2} + c(t,z)\,M = 0\,, \\ M(T,z) = 1\,. \end{cases} \tag{22}$$

By direct, tedious, computations $\exp\left\{g_T + e^{-\beta(T-t)}x\right\}M(t,z)$ is seen to satisfy the PDE (18); consequently, the forward price $F(t,x,z) = \exp\left\{g_T + e^{-\beta(T-t)}x\right\}M(t,z)$.

Given the form of $M(t,z)$, the forward prices clearly do not share the natural affine structure that other models often possess (e.g., compare with the constant volatility two-factor model (13)). It is also doubtful that an explicit (closed form) solution of the PDE (22) exists. Hence, this model appears to suffer from the deficiencies of Heston-like models which require either solving a PDE numerically or resorting to Fourier methods, rendering the models less useful for calibration purposes. Surprisingly, it is possible to partially overcome these difficulties if we accept to limit the range of applicability of our SV model to commodities having fast mean-reverting volatility ($\alpha \gg 1$). This is indeed the approach we pursue in the rest of this work.

2.2.2. The SV Extended Two-Factor Model

In this section, the stochastic volatility (SV) extended two-factor model is recorded for completeness. Starting with the two-factor model of Section 2.1.2, we make the volatility σ_X a function of a fast mean-reverting stochastic process – analogous to the SV extended one-factor model. The spot is now modeled under a $\mathbb{Q}$-measure as

$$dX_t = \beta\,(Y_t - X_t)\,dt + \sigma_X(Z_t)\,dW_t^{(1)}\,, \tag{23}$$

$$dY_t \;=\; \alpha_Y\,(\phi - Y_t)\,dt + \sigma_Y\,dW_t^{(2)}\,, \tag{24}$$

$$dZ_t \;=\; \alpha\,(m - Z_t)\,dt + \sigma_Z\,dW_t^{(3)}\,, \tag{25}$$

with correlation structure,

$$d\left[W^{(1)},W^{(2)}\right]_t = \rho_1\,dt\,,\quad d\left[W^{(1)},W^{(3)}\right]_t = \rho_2\,dt\,,\quad \text{and}\quad d\left[W^{(2)},W^{(3)}\right]_t = 0\,, \tag{26}$$

and restrictions on $\sigma_X(\cdot)$ parallel to the previous section.

Rather than repeating the analysis of the previous subsection, we instead point out that resulting forward prices are not of the affine form. Nevertheless, asymptotic methods will lead to approximate, but explicit, closed form forward and option prices.

3. Forward Price Approximation

It is well known that the invariant distribution of the volatility driving factor Z_t is Gaussian with a variance of $\nu^2 := \sigma_Z^2/2\alpha$. The asymptotic expansion revolves around assuming that $\alpha \gg 1$ and simultaneously holding the variance ν^2 of the invariant distribution finite and fixed. As such, our developments are primarily parameterized by the small parameter $\epsilon := \alpha^{-1}$. The ultimate goal of this section is to obtain a sound approximation (in a sense to be defined shortly) to the forward price, and in tandem eliminate the dependency of the approximate forward curve on the non-observable Z_t.

Such closed form forward price approximations will allow efficient statistical estimation of the model parameters, and lead to tractable pricing of derivatives written on these forward curves. We use the methodology originally applied in Fouque, Papanicolaou, and Sircar (2000a) and Cotton, Fouque, Papanicolaou, and Sircar (2004) for stock and IR options respectively. For detailed discussions on the fundamentals of these asymptotic techniques we refer to the monograph Fouque, Papanicolaou, and Sircar (2000b).

3.1. One-Factor Model + SV

In this section, we assume that the spot price dynamics is driven by the SV extended one-factor model in section 2.2.1. Recall that

$$F^\epsilon(t,x,z) \;:=\; \mathbb{E}^{\mathbb{Q}}_{t,x,z}\left[S_T\right]\,, \tag{27}$$

where the dependence on ϵ ($:= \alpha^{-1}$) is made explicit. Rewriting the PDE (18) as

$$\begin{cases} \mathcal{A}^\epsilon F^\epsilon = \left(\epsilon^{-1}\mathcal{A}_0 + \epsilon^{-\frac{1}{2}}\mathcal{A}_1 + \mathcal{A}_2\right) F^\epsilon(t,x,z) \;=\; 0\,, \\ \qquad\qquad\qquad\qquad F^\epsilon(T,x,z) \;=\; e^{g_T + x}\,, \end{cases} \tag{28}$$

with the three new operators defined as

$$\mathcal{A}_0 \;:=\; (m - z)\,\frac{\partial}{\partial z} + \nu^2\,\frac{\partial^2}{\partial z^2}\,, \tag{29}$$

$$\mathcal{A}_1 \;:=\; \sqrt{2}\rho_2\nu\sigma_X(z)\,\frac{\partial^2}{\partial x\partial z}\,, \tag{30}$$

$$\mathcal{A}_2 \;:=\; \frac{\partial}{\partial t} + \beta(\phi - x)\,\frac{\partial}{\partial x} + \frac{1}{2}\sigma_X^2(z)\,\frac{\partial^2}{\partial x^2}\,, \tag{31}$$

highlights the various scales of the individual operators. Note that $\mathcal{A}_0$ is the infinitesimal generator of a simple Vasicek (OU) process; $\mathcal{A}_2$ is the infinitesimal generator of the process (t, X_t); while the $\mathcal{A}_1$ operator accounts for the correlation between the log spot price X_t and the volatility driver Z_t processes.

Expanding F^ϵ in powers of $\sqrt{\epsilon}$

$$F^\epsilon = F^{(0)} + \sqrt{\epsilon}F^{(1)} + \epsilon F^{(2)} + \epsilon^{\frac{3}{2}}F^{(3)} + ... \tag{32}$$

where we impose the boundary conditions $F^{(0)}(T,x,z) := F^\epsilon(T,x,z) := e^{g_T+x}$ and $F^{(1)}(T,x,z) := 0$. We have explicitly assumed that the zeroth order term matches the payoff at maturity, while the first correction term vanishes at maturity. This terminal splitting is not required, however it is natural, leading to explicit closed form approximations, and allowing us to prove that the remaining corrections terms are $O(\epsilon)$.

Inserting this last expansion into the PDE (28) and collecting terms with like powers of $\sqrt{\epsilon}$ gives

$$\begin{aligned} 0 \;=\; & \frac{1}{\epsilon}\mathcal{A}_0 F^{(0)} + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{A}_1 F^{(0)} + \mathcal{A}_0 F^{(1)}\right) + \left(\mathcal{A}_2 F^{(0)} + \mathcal{A}_1 F^{(1)} + \mathcal{A}_0 F^{(2)}\right) \\ & + \sqrt{\epsilon}\left(\mathcal{A}_2 F^{(1)} + \mathcal{A}_1 F^{(2)} + \mathcal{A}_0 F^{(3)}\right) + \dots \,. \end{aligned} \tag{33}$$

From this last equation, the coefficients of the various powers of $\sqrt{\epsilon}$ must vanish individually. In the subsequent analysis we investigate these resulting equations and deduce from them the main properties of $F^{(i)}(t,x,z)$ for $i = 0, 1, 2$ and 3 explicitly.

- ϵ^{-1}–*Order Equation* : $\mathcal{A}_0 F^{(0)} = 0$
  This holds for all z; therefore $F^{(0)}$ must be independent of z: $F^{(0)} = F^{(0)}(t,x)$.
- $\epsilon^{-\frac{1}{2}}$–*Order Equation* : $\mathcal{A}_1 F^{(0)} + \mathcal{A}_0 F^{(1)} = 0$
  Since $F^{(0)}$ is independent of z, this implies $\mathcal{A}_0 F^{(1)} = 0$. This further implies $F^{(1)}$ is also independent of z; that is, $F^{(1)} = F^{(1)}(t,x)$.
- ϵ^0–*Order Equation* : $\mathcal{A}_2 F^{(0)} + \mathcal{A}_1 F^{(1)} + \mathcal{A}_0 F^{(2)} = 0$
  Since $F^{(1)}$ is independent of z, this implies the Poisson equation $\mathcal{A}_2 F^{(0)} + \mathcal{A}_0 F^{(2)} = 0$ and the resulting *centering equation* $\langle \mathcal{A}_2 F^{(0)} \rangle = 0$ is a necessary condition for a solution to exist. Here, and in the remainder of the article, the bracket notation $\langle f(z) \rangle$ denotes the expectation of $f(Z)$ where $Z \sim \mathbb{N}(m, \nu^2)$, the invariant distribution of the $\mathbb{Q}$-process Z_t, as defined in (16). Since $F^{(0)}$ is independent of z, the centering equation becomes $\langle \mathcal{A}_2 \rangle F^{(0)} = 0$. Remarkably, this is the PDE (3) satisfied by the forward price based on the one-factor spot model with constant volatility $\overline{\sigma}_X := \sqrt{\langle \sigma_X^2(z) \rangle}$. Enforcing the boundary condition $F^{(0)}(T,x) = \exp(g_T + x)$, implies that $F^{(0)}$ is the one-factor forward price (4) with constant volatility $\overline{\sigma}_X$.

  Up to this order, it is also possible to extract properties of $F^{(2)}$ which will prove useful in the subsequent analysis. Due to the centering equation $\langle \mathcal{A}_2 \rangle F^{(0)} = 0$, notice that

$$\mathcal{A}_2 F^{(0)} = (\mathcal{A}_2 - \langle \mathcal{A}_2 \rangle)\, F^{(0)} = \frac{1}{2}\left(\sigma_X^2(z) - \langle \sigma_X^2 \rangle\right) F_{xx}^{(0)} \,, \tag{34}$$

which allows the zero-order equation $\mathcal{A}_2 F^{(0)} + \mathcal{A}_0 F^{(2)} = 0$ to be rewritten as

$$F^{(2)} = -\frac{1}{2}\mathcal{A}_0^{-1}\left(\sigma_X^2(z) - \langle\sigma_X^2\rangle\right) F^{(0)}_{xx} = -\frac{1}{2}\left(\psi(z) + c(t,x)\right) F^{(0)}_{xx}\,, \tag{35}$$

where the function ψ is define as the solution of

$$\mathcal{A}_0\psi = \sigma_X^2 - \langle\sigma_X^2\rangle\,, \tag{36}$$

and $c(t,x)$ is an arbitrary constant of integration. A straightforward calculation also shows that

$$\psi' := \partial_z\psi = \frac{1}{\nu^2\Phi(z;m,\nu^2)}\int_{-\infty}^{z}\left(\sigma_X^2(u) - \langle\sigma_X^2\rangle\right)\Phi(u;m,\nu^2)\,du\,, \tag{37}$$

where $\Phi(\cdot;m,\nu^2)$ is the CDF of $\mathbb{N}(m,\nu^2)$, the invariant distribution of Z_t.

- $\epsilon^{\frac{1}{2}}$–*Order Equation* : $\mathcal{A}_2F^{(1)} + \mathcal{A}_1F^{(2)} + \mathcal{A}_0F^{(3)} = 0$
  This is a second Poisson equation, but now for $F^{(3)}$. Its centering equation is $\langle\mathcal{A}_2F^{(1)} + \mathcal{A}_1F^{(2)}\rangle = \langle\mathcal{A}_2\rangle F^{(1)} + \langle\mathcal{A}_1F^{(2)}\rangle = 0$ which is easily shown to transform into $\langle\mathcal{A}_2\rangle F^{(1)} = 2^{-\frac{1}{2}}\rho_2\nu\langle\sigma_X\psi'\rangle F^{(0)}_{xxx}$. Define $\widetilde{F}^{(1)} := \sqrt{\epsilon}F^{(1)}$ and $V := (\frac{\epsilon}{2})^{\frac{1}{2}}\rho_2\nu\langle\sigma_X\psi'\rangle$, the centering equation is then

$$\begin{cases} \langle\mathcal{A}_2\rangle\widetilde{F}^{(1)}(t,x) = V\,F^{(0)}_{xxx}\,, \\ \widetilde{F}^{(1)}(T,x) = 0\,. \end{cases} \tag{38}$$

  Equation (38) is the zero boundary version of the usual one-factor forward price PDE (3) with constant volatility $\overline{\sigma}_X$ and an additional source term of order $\sqrt{\epsilon}$. Using the previous result that $F^{(0)}$ has the form of the one-factor forward price (4), direct computations show that $\widetilde{F}^{(1)} = -V\,h(t,T;3\beta)\,F^{(0)}$ is a solution to equation (38).

Piecing together all of the above partial results, the price approximation based on the first two terms of the expansion (32) is succinctly written as

$$F^{\epsilon}(t,x,z) \simeq F^{(0)}(t,x) + \widetilde{F}^{(1)}(t,x) := (\,1 - V\,h(t,T;3\beta)\,)\,F^{(0)}(t,x)\,. \tag{39}$$

Intriguingly, the right hand side of (39) is independent of the unobservable Z_t process. This is an extremely convenient consequence of asymptotic derivative valuation results. It is also worth noting that for calibration purposes, the constant V can, and should, be used as a parameter in its own right. All of the details of the mapping from Z_t to the volatility process $(\sigma_X(Z_t))$ is averaged out and embedded in the constant V. Rather than specifying the "micro-structure" in the model, it is perfectly valid to specify the "macro-structure" in V as implied from futures price data.

We now state one of our main results on the validity of the approximation (39).

Theorem. 3.1 *For any fixed $(T,x,z) \in \mathbb{R}_+ \times \mathbb{R}^2$ and all $t \in [0,T]$, we have*

$$\left| F^\epsilon(t,x,z) - \left(F^{(0)}(t,x) + \widetilde{F}^{(1)}(t,x) \right) \right| = O(\epsilon)\,,$$

where the approximation $F^{(0)}(t,x) + \widetilde{F}^{(1)}(t,x)$ is defined in (39) and $F^{(0)}(t,x)$ as in (4) with σ_X replaced by $\sqrt{\langle \sigma_X^2(z) \rangle}$.

Proof. Define the function $\Upsilon^\epsilon(t,x,z)$ as the error terms of order ϵ^2 and higher. Explicitly,

$$\Upsilon^\epsilon := \left(F^{(0)} + \sqrt{\epsilon} F^{(1)} + \epsilon F^{(2)} + \epsilon^{\frac{3}{2}} F^{(3)} \right) - F^\epsilon\,. \tag{40}$$

We first aim at proving that $|\Upsilon^\epsilon| = O(\epsilon)$. Applying the infinitesimal generator $\mathcal{A}^\epsilon$ of (t, X_t, Z_t) on Υ^ϵ and canceling vanishing terms, based on our previous analysis of the $F^{(i)}$ functions, we find

$$\begin{aligned} \mathcal{A}^\epsilon\,\Upsilon^\epsilon &= \left(\epsilon^{-1}\mathcal{A}_0 + \epsilon^{-\frac{1}{2}}\mathcal{A}_1 + \mathcal{A}_2 \right) \left(F^{(0)} + \sqrt{\epsilon} F^{(1)} + \epsilon F^{(2)} + \epsilon^{\frac{3}{2}} F^{(3)} - F^\epsilon \right) \\ &= \epsilon \left(\mathcal{A}_2 F^{(2)} + \mathcal{A}_1 F^{(3)} + \sqrt{\epsilon}\mathcal{A}_2 F^{(3)} \right)\,. \end{aligned} \tag{41}$$

Now focus on each term from the right hand side of (41), paying attention to their growth properties as functions of x, z.

- $\mathcal{A}_2 F^{(2)}$*-Term:*
  Choosing the constant of integration in (35) to be zero, we have $F^{(2)} = -\frac{1}{2}\psi(z) F^{(0)}_{xx}$. In addition, since $\psi(z)$ satisfies the Poisson equation (36) and since its r.h.s. is bounded and satisfies the centering condition, then $\psi(z)$ grows at most linearly in $|z|$. Given the form of the forward price (4), it is clear that $F^{(0)}$ (and therefore $F^{(2)}$) is log-linear in x.

- $\mathcal{A}_1 F^{(3)}$ *and* $\mathcal{A}_2 F^{(3)}$*-Terms:*
  From the $\epsilon^{\frac{1}{2}}$-order analysis, $F^{(3)}$ satisfies the Poisson equation $\mathcal{A}_0 F^{(3)} + \mathcal{A}_2 F^{(1)} + \mathcal{A}_1 F^{(2)} = 0$ and the centering condition $\langle \mathcal{A}_2 F^{(1)} + \mathcal{A}_1 F^{(2)} \rangle = 0$. We then have, $\mathcal{A}_2 F^{(1)} + \mathcal{A}_1 F^{(2)} = \left(\mathcal{A}_2 F^{(1)} - \langle \mathcal{A}_2 F^{(1)} \rangle \right) + \left(\mathcal{A}_1 F^{(2)} - \langle \mathcal{A}_1 F^{(2)} \rangle \right)$. Consequently,

  $$F^{(3)} = -\sqrt{2}\rho_2 \nu \eta(z) F^{(0)}_{xxx} - \frac{1}{2}\zeta(z) F^{(1)}_{xx}\,, \tag{42}$$

  where $\eta(z)$ and $\zeta(z)$ are characterized by solutions of $\mathcal{A}_0 \eta = \sigma_X \psi' - \langle \sigma_X \psi' \rangle$ and $\mathcal{A}_0 \zeta = \sigma_X^2 - \langle \sigma_X^2 \rangle$, respectively, with both constants of integration set to zero. Both of these last two Poisson equations satisfy the centering equation and have bounded source terms, implying that $\eta(z), \zeta(z)$ are at most linearly growing in $|z|$ with bounded first derivatives. From these last properties of $\eta(z), \zeta(z)$ and the form of $F^{(3)}$ in (42) as well as the boundedness of $\sigma_X(z)$, we conclude that $\mathcal{A}_1 F^{(3)}$ and $\mathcal{A}_2 F^{(3)}$ are at most linearly growing in $|z|$ and log-linearly growing in x.

The above results allow us to bound the error term Υ^ϵ. Define $N := \mathcal{A}_2 F^{(2)} + \mathcal{A}_1 F^{(3)} + \sqrt{\epsilon}\mathcal{A}_2 F^{(3)}$ so that equation (41) becomes $\mathcal{A}^\epsilon \Upsilon^\epsilon = \epsilon N$. With this new terminology, the "Feynman-Kac" probabilistic representation of (41) can be expressed as (see Karatzas and Shreve (1991), section 5.7):

$$\Upsilon^\epsilon(t,x,z) = \epsilon\, \mathbb{E}^{\mathbb{Q}}_{t,x,z} \left[F^{(2)}(T,X_T,Z_T) + \sqrt{\epsilon} F^{(3)}(T,X_T,Z_T) - \int_t^T N(s,X_s,Z_s)\, ds \right] . \tag{43}$$

We have already demonstrated that $N(t,x,z)$, $F^{(2)}(T,x,z)$ and $F^{(3)}(T,x,z)$ are at most linearly bounded in $|z|$ and log-linearly growing in x. For the N function, this bound is uniform in $t \in [0,T]$. Furthermore, since $\sigma_X(\cdot)$ is bounded, a direct check (or see Lemma B.1 in Cotton, Fouque, Papanicolaou, and Sircar (2004)) shows that X_t has finite exponential moments. Similarly for the process Z_t, which implies a bound on its second moment (variance). Therefore, $|\Upsilon^\epsilon| = O(\epsilon)$, as previously claimed.

We make use of this last partial result and write

$$\left| F^\epsilon - (F^{(0)} + \widetilde{F}^{(1)}) \right| = \left| \epsilon F^{(2)} + \epsilon^{\frac{2}{3}} F^{(3)} - \Upsilon^\epsilon \right| \leq |\Upsilon^\epsilon| + \epsilon \left| F^{(2)} + \sqrt{\epsilon} F^{(3)} \right| , \tag{44}$$

which, by the properties of $F^{(2)}$ and $F^{(3)}$, completes the proof. □

We have succeeded in demonstrating that, when the mean-reversion rate is large, the forward prices in the SV extended one-factor model are well approximated by the constant volatility price with a small adjustment factor. The correction term is proportional to a parameter V which itself encapsulates the volatility function $\sigma_X(Z_t)$ information. However, from a calibration and pricing perspective, the detailed specification of this parameter in terms of the underlying volatility function is irrelevant, and instead it should be viewed as a free parameter in and of itself. There is one interesting limit to consider: the limit in which the correlation between the volatility factor Z_t and the log spot price process X_t is zero. In this limit, the correction term vanishes identically; however, the market will likely have a non-zero correlation between volatility and spot price returns. In fact, it is well known that for commodities there is an inverse leverage effect which drives volatility higher when spot prices rise.

We would like to make one last comment concerning the SV corrected forward price (39): the correction vanishes as $T \searrow t$ while it tends to $(1 - V/3\beta)$ as $T \to +\infty$. Specifically we have, $F^\epsilon(t,x,z) \overset{T\to+\infty}{\longrightarrow} \exp\{\phi + \ln(1 - V/3\beta) + \overline{\sigma}_X^2/4\beta\}$. Consequently, if one fixes the long-end of the log-forward curve and adjusts V, then V will control the mid-term of the forward curves. This is nice feature, because then, V can be viewed as an independent lever affecting the strength of the forward curve hump. To illustrate this point, in Figure 2 we plot sample forward curves with three choices of V. The diagram clearly shows that V affects the strength of the hump. Interestingly, regardless of the sign of V, in this specific example, the forward curve always becomes more humped.

3.2. Two-Factor Model + SV

In this section, we assume that the spot price dynamics is driven by the SV extended two-factor model of section 2.2.2 and look for an approximation to the implied forward prices. We omit the details of the calculations since the formal expansion procedure follows the same steps as in Section 3.1, with $\mathcal{A}_2$ containing additional terms due to the stochastic long-run mean.

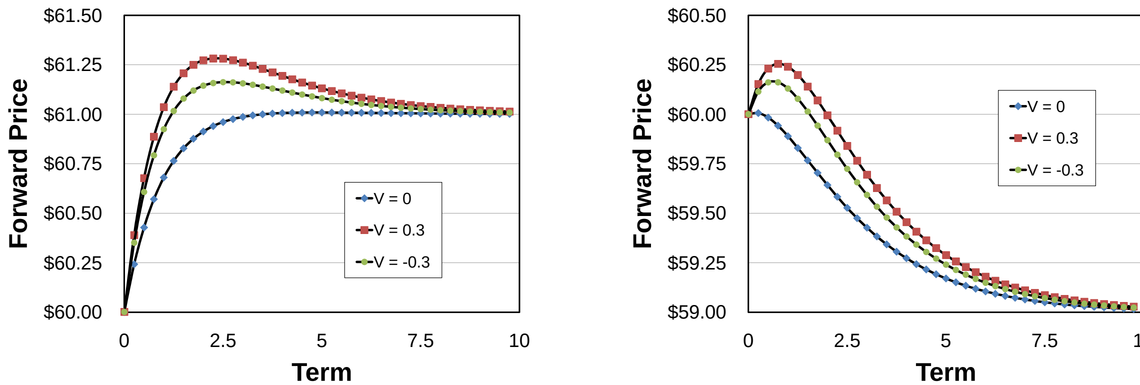

Figure 2. This diagram depicts typical forward curves implied the model for three choices of V. The long-run forward price is set at 61 in the left panel and 59 in the right panel. The spot is 60, $\beta = 0.4$ and $\overline{\sigma}_X = 0.2$.

Theorem. 3.2 *For any fixed* $(T, x, y, z) \in \mathbb{R}^+ \times \mathbb{R}^3$ *and all* $t \in [0, T]$, *we have*

$$F^{\epsilon}_{t,T} = \left(1 - V_1 h(t, T; 3\beta) - V_2 \frac{\beta}{\alpha_Y - \beta} \left[h(t, T; 3\beta) - h(t, T; \alpha_Y + 2\beta)\right]\right) F^{(0)}_{t,T} + O(\epsilon)\,, \tag{45}$$

where $F^{(0)}_{t,T}$ *is the two-factor forward curve (13) with constant volatility* σ_X *replaced by* $\sqrt{\langle \sigma^2_X(z) \rangle}$ *and the new parameters* $V_1 := \sqrt{\frac{\epsilon}{2}}\rho_2 \nu \langle \sigma_X \psi'_1 \rangle$ *and* $V_2 := \sqrt{2\epsilon}\rho_1 \rho_2 \nu \sigma_Y \langle \sigma_X \psi'_2 \rangle$.

From a calibration and pricing viewpoint, the detailed composition of V_1 and V_2 in terms of the initial parametrization is again irrelevant – they should now be considered as parameters in their own right. Furthermore, this approximation is, as in our previous forward approximation, independent of Z_t. This allows an easy calibration of the two-factor model to futures prices; see Hikspoors and Jaimungal (2007) and its reference for more details on these topics. Once again, these parameters can be viewed as levers to change strength and now also the shape of the forward-curve hump.

4. European Single-Name Options

Forward price determination is only the first stage of the analysis. For a model and method to be of any real use, it must lead to efficient valuation tools for single- and two-name option prices. In this section, we illustrate how the approximate forward prices from the previous section can be utilized to obtain approximate European single-name option prices. In Section 5, the issue of two-name contracts is addressed. Both single- and two-name approximations lead to closed form results which depend solely on constant volatility prices, Delta's and Delta-Gamma's.

4.1. Smooth Payoff Function

4.1.1. One-Factor Model + SV

Consider a smooth payoff function $\varphi(\cdot)$ with bounded derivatives and linear growth at infinity. Based on our SV extended one-factor spot price model of Section 2.2.1 we investigate the price $\Pi^\epsilon(t,x,z)$ at time t of the T_0-contingent claim $\varphi(F^\epsilon_{T_0,T})$ on the forward price $F^\epsilon_{T_0,T}$, that is

$$\Pi^\epsilon(t,x,z) \;=\; P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,x,z}\left[\varphi(F^\epsilon_{T_0,T})\right] . \tag{46}$$

To simplify notation we omit the explicit appearance of T_0 and T in the price function. To obtain an approximation scheme for (46), the previous asymptotic result $F^\epsilon_{T_0,T} = F^{(0)}_{T_0,T} + \widetilde{F}^{(1)}_{T_0,T} + O(\epsilon)$ from Theorem 3.1 will be used. To this end, consider a power expansion of the option payoff $\varphi(F^\epsilon_{T_0,T})$ around $\varphi(F^{(0)}_{T_0,T})$ – this is valid since we have made appropriate smoothness assumptions on $\varphi(\cdot)$

$$\varphi\left(F^\epsilon_{T_0,T}\right) = \varphi\left(F^{(0)}_{T_0,T}\right) - V h(T_0,T;3\beta) F^{(0)}_{T_0,T}\cdot\varphi'\left(F^{(0)}_{T_0,T}\right) + O(\epsilon)\,. \tag{47}$$

From (46) the price function Π^ϵ satisfies a similar PDE to the one F^ϵ satisfies (see (28)) with modified terminal conditions. Explicitly,

$$\begin{cases} \mathcal{A}^\epsilon\,\Pi^\epsilon = \left(\epsilon^{-1}\mathcal{A}_0 + \epsilon^{-\frac{1}{2}}\mathcal{A}_1 + \mathcal{A}_2^*\right)\Pi^\epsilon(t,x,z) \;=\; 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad \Pi^\epsilon(T_0,x,z) \;=\; 0\,, \end{cases} \tag{48}$$

where $\mathcal{A}_2^* := \mathcal{A}_2 - r(t)$, $r(t)$ is the short-rate and $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined in (29)-(31). Expanding Π^ϵ in powers of $\sqrt{\epsilon}$, as previously done with F^ϵ, we have

$$\Pi^\epsilon = \Pi^{(0)} + \sqrt{\epsilon}\Pi^{(1)} + \epsilon\Pi^{(2)} + \epsilon^{\frac{3}{2}}\Pi^{(3)} + \dots\,, \tag{49}$$

and plugging into (48) gives

$$\begin{aligned} 0 \;=\; & \frac{1}{\epsilon}\mathcal{A}_0\Pi^{(0)} + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{A}_1\Pi^{(0)} + \mathcal{A}_0\Pi^{(1)}\right) + \left(\mathcal{A}_2^*\Pi^{(0)} + \mathcal{A}_1\Pi^{(1)} + \mathcal{A}_0\Pi^{(2)}\right) \\ & +\sqrt{\epsilon}\left(\mathcal{A}_2^*\Pi^{(1)} + \mathcal{A}_1\Pi^{(2)} + \mathcal{A}_0\Pi^{(3)}\right) + \dots\,. \end{aligned} \tag{50}$$

An analysis of the various equations arising from (50) order-by-order in $\sqrt{\epsilon}$ – analogous to the study carried out in Section 3.1 and specifically for (33) – yields

$$\Pi^{(0)}(t,x) \;=\; P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,x}\left[\varphi\left(F^{(0)}_{T_0,T}(\overline{X}_{T_0})\right)\right]\,, \tag{51}$$

$$\begin{aligned} \widetilde{\Pi}^{(1)}(t,x) \;=\; & -V h(t,T_0;3\beta) P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,x}\left[F^{(0)}_{T_0,T}(\overline{X}_{T_0})\,\varphi'\left(F^{(0)}_{T_0,T}(\overline{X}_{T_0})\right)\right] \\ & -V\mathbb{E}^{\mathbb{Q}}_{t,x}\left[\int_t^{T_0} P(t,u)\,\Pi^{(0)}_{xxx}(u,\overline{X}_u)\,du\right]\,, \end{aligned} \tag{52}$$

$$\Pi^{(2)}(t,x,z) \;=\; -\frac{1}{2}\psi(z)\Pi^{(0)}_{xx}\,, \tag{53}$$

where $\widetilde{\Pi}^{(1)}(t,x) := \sqrt{\epsilon}\,\Pi^{(1)}$, $V = \left(\frac{\epsilon}{2}\right)^{\frac{1}{2}}\rho_2\nu\langle\sigma_X\psi'\rangle$ is the same parameter which arose in the analysis of the forward price approximation in Section 3.1, $\psi(z)$ is defined in (36), and the "smoothed" process $\overline{X}_t$ satisfies the SDE

$$d\overline{X}_u = \beta(\phi - \overline{X}_u)\,du + \overline{\sigma}_X\,dW^{(1)}_u\,, \qquad \overline{X}_t = X_t\,. \tag{54}$$

Here, $\overline{\sigma}_X^2 := \langle \sigma_X^2(z) \rangle$. Note that equation (52) is, due to its integral part, quite difficult to compute explicitly. It is, however, possible to transform $\widetilde{\Pi}^{(1)}$ into a much more tractable form. From the $\sqrt{\epsilon}$-order analysis, we find that $\widetilde{\Pi}^{(1)}$ satisfies the following PDE:

$$\begin{cases} \langle \mathcal{A}_2^* \rangle \widetilde{\Pi}^{(1)}(t,x) &= V\, \Pi^{(0)}_{xxx}\,, \\ \widetilde{\Pi}^{(1)}(T_0,x) &= -V h(T_0,T;3\beta) F^{(0)}_{T_0,T}\, \varphi'\left(F^{(0)}_{T_0,T}\right)\,. \end{cases} \tag{55}$$

Using the commutation relation

$$\langle \mathcal{A}_2^* \rangle \Pi^{(0)}_{xxx} = \left\{ \partial_x^3 \langle \mathcal{A}_2^* \rangle + \left[\partial_x^3 ; \langle \mathcal{A}_2^* \rangle \right] \right\} \Pi^{(0)}_{xxx} = 3\beta\, \Pi^{(0)}_{xxx} \tag{56}$$

where $[A;B] := AB - BA$, one can show that $G_1 := -V h(t,T_0;-3\beta)\Pi^{(0)}_{xxx}$ is a solution of (55) with zero boundary condition. Also, a specific solution (say G_2) to the homogeneous version of the PDE (55) provides a unique solution $G_1 + G_2$ to (55). Using Feynman-Kac with a source to obtain G_2, we conclude that

$$\begin{aligned} \widetilde{\Pi}^{(1)}(t,x) \;=\; & -V h(t,T_0;-3\beta)\Pi^{(0)}_{xxx} \\ & -V h(T_0,T;3\beta) P(t,T_0)\, \mathbb{E}^{\mathbb{Q}}_{t,x}\left[F^{(0)}_{T_0,T}(\overline{X}_{T_0})\, \varphi'\left(F^{(0)}_{T_0,T}(\overline{X}_{T_0}) \right) \right]\,. \end{aligned} \tag{57}$$

This last expression is now much simpler to compute for any reasonably well behaved payoff function. It is particularly interesting that the correction terms are dependent only on the zeroth order price, which themselves are determined in terms of the constant volatility model. Furthermore, the first term in the above correction explicitly depends on the Delta-Gamma of the constant vol option price. Contrastingly, the second term can be viewed as the price of a modified payoff assuming constant volatility. For example, if valuing a call option, then the second correction term is the price of an asset-or-nothing option. Finally, the parameter V which controls the impact of stochastic volatility is inherited from the forward price approximation (39).

We conclude this section by providing the conditions of validity of our price approximation in the following theorem.

Theorem. 4.1 *For any fixed $(T_0,T,x,z) \in \mathbb{R}_+^2 \times \mathbb{R}^2$ with $T_0 \le T$ and for all $t \in [0,T_0]$, we have*

$$\left| \Pi^{\epsilon}(t,x,z) - \left(\Pi^{(0)}(t,x) + \widetilde{\Pi}^{(1)}(t,x) \right) \right| = O(\epsilon)\,,$$

where the approximation $\Pi^{(0)}(t,x) + \widetilde{\Pi}^{(1)}(t,x)$ is defined in (51) and (57).

Proof. The proof follows along similar lines to the proof of Theorem 3.1. The one main complication is to demonstrate that x-derivatives of $\Pi^{(0)}$ and $\widetilde{\Pi}^{(1)}$ have at most exponential growth. This is achieved by appealing to the smoothness properties of $\varphi(\cdot)$ and Lebesgue's dominated convergence theorem, as similarly done in the more general situation of Section 5.1.1. We provide more details there. □

4.1.2. Two-Factor Model + SV

Based on our SV extended two-factor spot price model of Section 2.2.2, we seek an approximation to the price $\Pi^\epsilon(t,x,y,z)$ of a T_0-contingent claim with payoff $\varphi(F^\epsilon_{T_0,T})$, i.e.

$$\Pi^\epsilon(t,x,y,z) \;=\; P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,x,z}\left[\varphi(F^\epsilon_{T_0,T})\right] \,. \tag{58}$$

The forward approximation $F^\epsilon_{T_0,T} = F^{(0)}_{T_0,T} + \widetilde{F}^{(1)}_{T_0,T} + O(\epsilon)$ used in the expansion methodology is now the one from Theorem 3.2. The mathematical developments leading to the next theorem are very similar to those of Section 4.1.1; we therefore concentrate on the precise statement of the main result and omit the proof.

Theorem. 4.2 *For any fixed $(T_0,T,x,y,z)\in\mathbb{R}^2_+\times\mathbb{R}^3$ with $T_0\le T$ and for all $t\in[0,T_0]$, we have*

$$\left|\Pi^\epsilon(t,x,y,z) - \left(\Pi^{(0)}(t,x,y) + \widetilde{\Pi}^{(1)}(t,x,y)\right)\right| = O(\epsilon)\,,$$

where

$$\Pi^{(0)}(t,x,y) \;:=\; P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,x,y}\left[\varphi\left(F^{(0)}_{T_0,T}(\overline{X}_{T_0},Y_{T_0})\right)\right] \,, \tag{59}$$

with $F^{(0)}_{T_0,T}$ as in Theorem 3.2, the process X_t of (23) being replaced by its "smoothed version"

$$d\overline{X}_u = \beta(Y_u-\overline{X}_u)\,du + \overline{\sigma}_X\,dW^{(1)}_u\,, \qquad \overline{X}_t = X_t\,, \tag{60}$$

$\overline{\sigma}^2_X := \langle\sigma^2_X(z)\rangle$ *and*

$$\widetilde{\Pi}^{(1)}(t,x,y) \;:=\; l_1(t,T_0)\,\Pi^{(0)}_{xxx} + l_2(t,T_0)\,\Pi^{(0)}_{xxy} \tag{61}$$

$$\qquad + l(T_0,T)\,P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,x,y}\left[F^{(0)}_{T_0,T}(\overline{X}_{T_0},Y_{T_0})\,\varphi'\left(F^{(0)}_{T_0,T}(\overline{X}_{T_0},Y_{T_0})\right)\right] \,, \tag{62}$$

where,

$$l_1(t,T_0) \;:=\; -\frac{\beta\,V_2}{\beta-\alpha_Y}h(t,T_0;-2\beta-\alpha_Y) - \left[V_1 + \frac{3\,\beta^2\,V_2}{2\beta+\alpha_Y}\left(1-\frac{1}{\beta-\alpha_Y}\right)\right]h(t,T_0,-3\beta), \tag{63}$$

$$l_2(t,T_0) \;:=\; -V_2\,h(t,T_0;-2\beta-\alpha_Y)\,, \tag{64}$$

$$l(T_0,T) \;:=\; -V_1 h(T_0,T;3\beta) - V_2\frac{\beta}{\alpha_Y-\beta}\left[h(T_0,T;3\beta) - h(T_0,T;2\beta+\alpha_Y)\right] \,. \tag{65}$$

Furthermore, V_1 and V_2 are as in Theorem 3.2.

Once again, we find that the SV extended model option prices are written in terms of the constant volatility model prices with a smoothed volatility. The correction terms are again in terms of the various Delta's and Delta-Gamma's with coefficient proportional to the parameters V_i which themselves are inherited from the forward price approximation (45).

4.2. Nonsmooth Payoff: Calls and Puts

When the T_0-payoff function $\varphi(\cdot)$ is non-smooth, Theorem 4.1 and 4.2 can be generalized via a further approximation scheme. The main device is to approximate the non-smooth payoff function by a regularized version – in particular its discounted conditional expectation over a very small time – and then prove that the regularized option price well approximates the exact price. The required methodology is, due to the differentiability of our one-factor energy forward call/put option prices (7), a simplified version of the one originally developed for stock options in Fouque, Papanicolaou, Sircar, and Solna (2003). We therefore refer to that paper for further mathematical details.

For practical purposes, it suffices to know that the approximate prices developed in Theorems 4.1 and 4.2 are still valid for non-smooth call/put options as long as they are not used for extremely close to maturity option contracts. In practice, there would be no need for such a pricing methodology for small terms since it would be clear whether the option is in or out of the money.

5. European Two-Name Options

In this section, we pursue the approximations of options written on two correlated commodity forwards, where each commodity is driven by an SV extended one- or two-factor mean-reverting model. The analysis is more involved than previously; however, the end results inherit a similar structure to the single name case. In particular, the price is given in terms of the constant volatility model price with correction terms depending on the various Delta's and Delta-Gamma's. Interestingly, two new parameters arise in this case. These new parameters cannot be calibrated from forward prices, or options on the individual forward prices, instead they should be viewed as a flexibility lever allowing the trader to bias the prices (or equivalently the implied vol skew) upward or downward.

5.1. Smooth Payoff Function

Consider a smooth payoff function $\varphi(\cdot,\cdot)$ having bounded partial derivatives and a linear growth at infinity in each variable. Our main goal is to find a *well behaved* approximation to the option price Π^ϵ which as usual is written in terms of the discounted expectation under the risk-neutral measure

$$\Pi^{\vec{\epsilon}}(t,\vec{x},\vec{z}) = P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,\vec{x},\vec{z}}\left[\varphi\left(F^{\epsilon_1}_{T_0,T_1}, F^{\epsilon_2}_{T_0,T_2}\right)\right] . \tag{66}$$

Note that we allow the forward contracts to have different maturities, that is, we only require $T_0 \le T_1, T_2$. Most of the important steps in the derivation are explicitly provided for the SV extended one-factor model only, while the main Theorem for the two-factor model is simply stated.

5.1.1. One-Factor Model + SV

Here, the joint dynamics of the spot and forward price for the pair of commodities ($i = 1,2$) are assumed to satisfy the system of SDEs

$$S^{(i)}_t = \exp\left\{g^{(i)}_t + X^{(i)}_t\right\}, \tag{67}$$

$$F^{\epsilon_i}_{t,T} = \mathbb{E}^{\mathbb{Q}}_t\left[S^{(i)}_T\right] , \tag{68}$$

$$dX_t^{(i)} = \beta_i\left(\phi_i - X_t^{(i)}\right)dt + \sigma_{Xi}\left(Z_t^{(i)}\right)dW_t^{(1i)}\,, \tag{69}$$
$$dZ_t^{(i)} = \alpha_i\left(m_i - Y_t^{(i)}\right)dt + \sigma_{Zi}\,dW_t^{(3i)}\,, \tag{70}$$

with correlation structure $d\left[W^{(11)}, W^{(12)}\right]_t = \rho\,dt$, $d\left[W^{(1i)}, W^{(3i)}\right]_t = \rho_{2i}\,dt$ and all others zero. We also assume that the volatility functions $\sigma_{Xi}(\cdot)$ are again smooth, strictly positive and bounded functions with bounded derivatives. Also notice that the explicit dependence on the small parameter $\epsilon_i := 1/\alpha_i$ has been made. As before (see Section 3), the variance $\nu_i^2 := \sigma_{Zi}^2/2\alpha_i$ of the $Z_t^{(i)}$-invariant distributions are held fixed in the limit of small ϵ_i. We are now ready to develop an approximation to the price (66) which satisfies the PDE

$$\begin{cases} \mathcal{A}^{\vec{\epsilon}}\,\Pi^{\vec{\epsilon}} = \left(\epsilon_1^{-1}\mathcal{A}_0^{(1)} + \epsilon_2^{-1}\mathcal{A}_0^{(2)} + \epsilon_1^{-\frac{1}{2}}\mathcal{A}_1^{(1)} + \epsilon_2^{-\frac{1}{2}}\mathcal{A}_1^{(2)} + \mathcal{A}_2^*\right)\Pi^{\vec{\epsilon}} = 0\,, \\ \Pi^{\vec{\epsilon}}(T_0, \vec{x}, \vec{z}) = \varphi\left(F_{T_0,T_1}^{\epsilon_1}, F_{T_0,T_2}^{\epsilon_2}\right)\,, \end{cases} \tag{71}$$

where $\mathcal{A}^{\vec{\epsilon}}$ is the generator of $(t, M_t^{-1}, \vec{X}_t, \vec{Z}_t)$ with $M_t := \exp\{\int_0^t r_s\,ds\}$ the money market account and the operator

$$\begin{aligned} \mathcal{A}_2^* := \frac{\partial}{\partial t} + \beta_1(\phi_1 - x_1)\frac{\partial}{\partial x_1} + \beta_2(\phi_2 - x_2)\frac{\partial}{\partial x_2} + \frac{1}{2}\sigma_{X1}^2(z_1)\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}\sigma_{X2}^2(z_2)\frac{\partial^2}{\partial x_2^2} \\ + \rho\sigma_{X1}(z_1)\sigma_{X2}(z_2)\frac{\partial^2}{\partial x_1 \partial x_2} - r(t)\,. \end{aligned} \tag{72}$$

Expanding $\Pi^{\vec{\epsilon}}$ in powers of $\sqrt{\epsilon_1}$ and $\sqrt{\epsilon_2}$ we have

$$\begin{aligned} \Pi^{\vec{\epsilon}} = \Pi^{(0)} + \sqrt{\epsilon_1}\Pi^{(1,1)} + \sqrt{\epsilon_2}\Pi^{(1,2)} + \epsilon_1\Pi^{(2,1)} + \epsilon_2\Pi^{(2,2)} + \sqrt{\epsilon_1\epsilon_2}\Pi^{(2,3)} + \epsilon_1^{3/2}\Pi^{(3,1)} \\ + \epsilon_1\sqrt{\epsilon_2}\Pi^{(3,2)} + \sqrt{\epsilon_1}\epsilon_2\Pi^{(3,3)} + \epsilon_2^{3/2}\Pi^{(3,4)} + \ldots\,, \end{aligned} \tag{73}$$

with T_0-terminal condition

$$\begin{aligned} \varphi\left(F_{T_0,T_1}^{\epsilon_1}, F_{T_0,T_2}^{\epsilon_2}\right) = \varphi\left(F_{T_0,T_1}^{\epsilon_1(0)}, F_{T_0,T_2}^{\epsilon_2(0)}\right) &+ \widetilde{F}_{T_0,T_1}^{\epsilon_1(1)}\frac{\partial\varphi}{\partial F^{\epsilon_1}}\left(F_{T_0,T_1}^{\epsilon_1(0)}, F_{T_0,T_2}^{\epsilon_2(0)}\right) \\ &+ \widetilde{F}_{T_0,T_2}^{\epsilon_2(1)}\frac{\partial\varphi}{\partial F^{\epsilon_2}}\left(F_{T_0,T_1}^{\epsilon_2(0)}, F_{T_0,T_2}^{\epsilon_2(0)}\right) + O(\epsilon')\,. \end{aligned} \tag{74}$$

Here, and in the sequel, $\epsilon' := \max(\epsilon_1, \epsilon_2)$ and $F_{t,T}^{\epsilon_i(0)}$ ($F_{t,T}^{\epsilon_i(1)}$) is the first (second, resp.) order approximation of the forward price $F_{t,T}^{\epsilon_i}$, $i = 1, 2$ (see Section 3).

Now, collect terms of the equivalent orders arising from (71) on substitution of (73)-(74), as in the previous section. In the following, we emphasize the new aspects of the present (more general) asymptotic analysis and omit most of details. A study of the ϵ_1^{-1},ϵ_2^{-1},$\epsilon_1^{-1/2}$,$\epsilon_2^{-1/2}$, $\sqrt{\epsilon_1}/\epsilon_2$ and $\sqrt{\epsilon_2}/\epsilon_1$ - order equations results in $\Pi^{(0)}$ and $\Pi^{(1)}$ being independent of $\vec{z} := (z_1, z_2)$. Explicitly: $\Pi^{(0)} = \Pi^{(0)}(t, \vec{x})$ and $\Pi^{(1)} = \Pi^{(1)}(t, \vec{x})$.

- ϵ^0-*Order Equation*: $\mathcal{A}_0^{(1)}\Pi^{(2,1)} + \mathcal{A}_0^{(2)}\Pi^{(2,2)} + \mathcal{A}_2^*\Pi^{(0)} = 0$
  Any solution of the two Poisson equations

$$\mathcal{A}_0^{(1)}\Pi^{(2,1)} + \frac{1}{2}\mathcal{A}_2^*\Pi^{(0)} = 0\,, \qquad \text{and} \quad \mathcal{A}_0^{(2)}\Pi^{(2,2)} + \frac{1}{2}\mathcal{A}_2^*\Pi^{(0)} = 0\,. \tag{75}$$

is a solution of the ϵ^0-order PDE. Both Poisson equations have identical centering conditions $\langle \mathcal{A}_2^*\Pi^{(0)}\rangle = 0$, where $\langle f(\vec{Z}_\infty)\rangle$ is defined as the expectation of $f(\vec{Z}_\infty)$ with $\vec{Z}_\infty \sim \mathbb{N}(\vec{m}, \vec{\nu}^2)$, the invariant distribution of the process $\vec{Z}_t = (Z_t^{(1)}, Z_t^{(2)})$ defined in (70)[2]. The centering condition reduces to $\langle \mathcal{A}_2^*\Pi^{(0)}\rangle = \langle \mathcal{A}_2^*\rangle\Pi^{(0)} = 0$ and enforcing the b.c. (74) to zeroth order, implies that $\Pi^{(0)}(t,\vec{x})$ is the option price in the constant volatility one-factor model with $\overline{\sigma}_{Xi} := (\langle\sigma_{Xi}^2\rangle)^{1/2}$ and correlation $\overline{\rho} := \rho\langle\sigma_{X1}\rangle\langle\sigma_{X2}\rangle/(\langle\sigma_{X1}^2\rangle\langle\sigma_{X2}^2\rangle)^{1/2}$. The new correlation $\overline{\rho}$ is in $[-1,1]$ due to Hölder's inequality. Explicitly,

$$\Pi^{(0)}(t,\vec{x}) = P(t,T_0)\,\mathbb{E}_{t,\vec{x}}^{\mathbb{Q}}\left[\varphi\left\{F_{T_0,T_1}^{\epsilon_1(0)}\left(\overline{X}_{T_0}^{(1)}\right), F_{T_0,T_2}^{\epsilon_2(0)}\left(\overline{X}_{T_0}^{(2)}\right)\right\}\right]. \tag{76}$$

Here, the smoothed processes $\overline{X}_t^{(i)}$ are again defined by

$$d\overline{X}_u^{(i)} = \beta_i\left(\phi_i - \overline{X}_u^{(i)}\right)du + \overline{\sigma}_{Xi}\,d\overline{W}_u^{(i)}, \qquad \overline{X}_t^{(i)} = X_t^{(i)}, \tag{77}$$

with correlation $d[\overline{W}^{(1)}, \overline{W}^{(2)}] = \overline{\rho}$.

Using the above solution for $\Pi^{(0)}$ and following the arguments leading to equation (35), but starting with (75), we find

$$\Pi^{(2,1)} = -\frac{1}{4}\{\psi_1(z_1) + c_1(t,\vec{x},z_2)\}\,\Pi_{x_1x_1}^{(0)} + \frac{\rho}{2}\{\psi_{12}(\vec{z}) + c_{12}(t,\vec{x},z_2)\}\,\Pi_{x_1x_2}^{(0)}, \tag{78}$$

$$\Pi^{(2,2)} = -\frac{1}{4}\{\psi_2(z_2) + c_2(t,\vec{x},z_1)\}\,\Pi_{x_2x_2}^{(0)} + \frac{\rho}{2}\{\psi_{21}(\vec{z}) + c_{21}(t,\vec{x},z_1)\}\,\Pi_{x_1x_2}^{(0)}, \tag{79}$$

where the ψ_i's and ψ_{ij}'s are defined by

$$\begin{aligned} \mathcal{A}_0^{(1)}\psi_1 &= \sigma_{X1}^2 - \langle\sigma_{X1}^2\rangle, \quad \mathcal{A}_0^{(1)}\psi_{12} = \sigma_{X1}\sigma_{X2} - \langle\sigma_{X1}\sigma_{X2}\rangle, \\ \mathcal{A}_0^{(2)}\psi_2 &= \sigma_{X2}^2 - \langle\sigma_{X2}^2\rangle, \quad \mathcal{A}_0^{(2)}\psi_{21} = \sigma_{X1}\sigma_{X2} - \langle\sigma_{X1}\sigma_{X2}\rangle, \end{aligned} \tag{80}$$

with the c_i's and c_{ij}'s being their respective (arbitrary) constants of integration.

- $\sqrt{\epsilon_1/\epsilon_2}$, $\sqrt{\epsilon_2/\epsilon_1}$ *-Order Equations*: $\mathcal{A}_0^{(1)}\Pi^{(2,3)} = 0$ and $\mathcal{A}_0^{(2)}\Pi^{(2,3)} = 0$
These equations imply that $\Pi^{(2,3)} = \Pi^{(2,3)}(t,\vec{x})$ is independent of $\vec{z}$.

- $\sqrt{\epsilon_1}$*-Order Equation* : $\mathcal{A}_2^*\Pi^{(1,1)} + \mathcal{A}_1^{(1)}\Pi^{(2,1)} + \mathcal{A}_0^{(1)}\Pi^{(3,1)} + \mathcal{A}_0^{(2)}\Pi^{(3,3)} = 0$
Once again decoupling this PDE into two Poisson equations

$$\mathcal{A}_0^{(1)}\Pi^{(3,1)} + \frac{1}{2}\left(\mathcal{A}_2^*\Pi^{(1,1)} + \mathcal{A}_1^{(1)}\Pi^{(2,1)}\right) = 0, \tag{81}$$

$$\mathcal{A}_0^{(2)}\Pi^{(3,3)} + \frac{1}{2}\left(\mathcal{A}_2^*\Pi^{(1,1)} + \mathcal{A}_1^{(1)}\Pi^{(2,1)}\right) = 0, \tag{82}$$

leads to the centering condition $\langle \mathcal{A}_2^*\Pi^{(1,1)} + \mathcal{A}_1^{(1)}\Pi^{(2,1)}\rangle = 0$. Inserting the expression for $\Pi^{(2,1)}$ implies that

$$\left\{\begin{aligned} \langle\mathcal{A}_2^*\rangle\widetilde{\Pi}^{(1,1)} &= \frac{\sqrt{\epsilon_1}}{2\sqrt{2}}\rho_{21}\nu_1\langle\sigma_{X1}\psi_1'\rangle\Pi_{x_1x_1x_1}^{(0)} + \frac{\sqrt{\epsilon_1}}{\sqrt{2}}\rho\rho_{21}\nu_1\langle\sigma_{X1}\partial_{z_1}\psi_{12}\rangle\Pi_{x_1x_1x_2}^{(0)}, \\ \widetilde{\Pi}^{(1,1)}(T_0,\vec{X}_{T_0}) &= \widetilde{F}_{T_0,T_1}^{\epsilon_1(1)}\,\frac{\partial\varphi}{\partial F^{\epsilon_1}}\left(F_{T_0,T_1}^{\epsilon_1(0)}, F_{T_0,T_2}^{\epsilon_2(0)}\right). \end{aligned}\right. \tag{83}$$

[2]Since $Z_t^{(1)}$ and $Z_t^{(2)}$ are independent processes, they also have independent invariant distributions.

where $\widetilde{\Pi}^{(1,1)} := \sqrt{\epsilon_1}\,\Pi^{(1,1)}$ and its boundary condition being induced by (74). The commutation rules $[\langle\mathcal{A}_2^*\rangle;\partial_{x_1x_1x_1}] = 3\beta_1\partial_{x_1x_1x_1}$ and $[\langle\mathcal{A}_2^*\rangle;\partial_{x_1x_1x_2}] = (2\beta_1+\beta_2)\partial_{x_1x_1x_2}$, together with the fact that $\langle\mathcal{A}_2^*\rangle\Pi^{(0)} = 0$, allows one to write

$$\begin{aligned}\langle\mathcal{A}_2^*\rangle &\left(l_1(t)\Pi^{(0)}_{x_1x_1x_1} + l_2(t)\Pi^{(0)}_{x_1x_1x_2}\right)\\ &= \left(\partial_t l_1 + 3\beta_1 l_1\right)\Pi^{(0)}_{x_1x_1x_1} + \left(\partial_t l_2 + (2\beta_1+\beta_2) l_1\right)\Pi^{(0)}_{x_1x_1x_2}\,, \end{aligned} \tag{84}$$

for $l_1(t)$ and $l_2(t)$ arbitrary functions of time only. Matching the coefficients of the r.h.s. with coefficients in the r.h.s of the PDE (83), and solving the resulting ODEs for $l_{1,2}$ (with b.c. $l_1(T) = l_2(T) = 0$), allows us to solve the PDE (83) explicitly

$$\begin{aligned}\widetilde{\Pi}^{(1,1)}(t,\vec{x}) \;=\; & -\frac{V_1}{2}h(t,T_0;-3\beta_1)\Pi^{(0)}_{x_1x_1x_1} - V_{11}\,h(t,T_0;-2\beta_1-\beta_2)\Pi^{(0)}_{x_1x_1x_2}\\ & -V_1 h(T_0,T_1;3\beta_1)\,P(t,T_0)\\ & \times\mathbb{E}^{\mathbb{Q}}_{t,\vec{x}}\left[\widetilde{F}^{\epsilon_1(0)}_{T_0,T_1}\left(\overline{X}^{(1)}_{T_0}\right)\frac{\partial\varphi}{\partial F^{\epsilon_1}}\left\{F^{\epsilon_1(0)}_{T_0,T_1}\left(\overline{X}^{(1)}_{T_0}\right), F^{\epsilon_2(0)}_{T_0,T_2}\left(\overline{X}^{(2)}_{T_0}\right)\right\}\right]\,,\end{aligned} \tag{85}$$

with $V_1 := \sqrt{\frac{\epsilon_1}{2}}\rho_{21}\nu_1\langle\sigma_{X1}\psi_1'\rangle$ and $V_{11} := \sqrt{\frac{\epsilon_1}{2}}\rho\rho_{21}\nu_1\langle\sigma_{X1}\partial_{z_1}\psi_{12}\rangle$. Equation (85) depends solely on the Delta's and Delta-Gamma's of the constant volatility price and the constant volatility price of a modified payoff. These individual terms can be computed explicitly in many typical cases – such as Margrabe spread options.

- $\sqrt{\epsilon_2}$-*Order Equation* : $\mathcal{A}_2^*\Pi^{(1,2)} + \mathcal{A}_1^{(2)}\Pi^{(2,2)} + \mathcal{A}_0^{(1)}\Pi^{(3,2)} + \mathcal{A}_0^{(2)}\Pi^{(3,4)} = 0$
  Going through similar arguments as above, we find

$$\begin{aligned}\widetilde{\Pi}^{(1,2)}(t,\vec{x}) \;=\; & -\frac{V_2}{2}h(t,T_0;-3\beta_2)\Pi^{(0)}_{x_2x_2x_2} - V_{22}h(t,T_0;-\beta_1-2\beta_2)\Pi^{(0)}_{x_1x_2x_2}\\ & -V_2 h(T_0,T_2;3\beta_2)\,P(t,T_0)\\ & \times\mathbb{E}^{\mathbb{Q}}_{t,\vec{x}}\left[\widetilde{F}^{\epsilon_2(0)}_{T_0,T_2}\left(\overline{X}^{(2)}_{T_0}\right)\frac{\partial\varphi}{\partial F^{\epsilon_2}}\left\{F^{\epsilon_1(0)}_{T_0,T_1}\left(\overline{X}^{(1)}_{T_0}\right), F^{\epsilon_2(0)}_{T_0,T_2}\left(\overline{X}^{(2)}_{T_0}\right)\right\}\right]\,,\end{aligned} \tag{86}$$

with $\widetilde{\Pi}^{(1,2)} := \sqrt{\epsilon_2}\,\Pi^{(1,2)}$, $V_2 := \sqrt{\frac{\epsilon_2}{2}}\rho_{22}\nu_2\langle\sigma_{X2}\psi_2'\rangle$ and $V_{22} := \sqrt{\frac{\epsilon_2}{2}}\rho\rho_{22}\nu_2\langle\sigma_{X2}\partial_{z_2}\psi_{21}\rangle$.

We now aim at proving the main result of this section, which, according to our general expansion methodology (71)-(74) and its subsequent analysis, should take the form of

$$\Pi^{\vec{\epsilon}}(t,\vec{x},\vec{z}) \simeq \Pi^{(0)}(t,\vec{x}) + \widetilde{\Pi}^{(1,1)}(t,\vec{x}) + \widetilde{\Pi}^{(1,2)}(t,\vec{x})\,, \tag{87}$$

whenever the inverse mean-reversion parameters ϵ_1 and ϵ_2 are sufficiently small. The precise formulation of this approximation is the subject of our next Theorem.

Theorem. 5.1 *For any fixed* $(T_0,T_1,T_2,\vec{x},\vec{z}) \in \mathbb{R}^3_+\times\mathbb{R}^4$ *with* $T_0 \le T_1, T_2$ *and for all* $t\in[0,T_0]$*, we have*

$$\left|\Pi^{\vec{\epsilon}}(t,\vec{x},\vec{z}) - \left(\Pi^{(0)}(t,\vec{x}) + \widetilde{\Pi}^{(1,1)}(t,\vec{x}) + \widetilde{\Pi}^{(1,2)}(t,\vec{x})\right)\right| = O(\epsilon')\,, \tag{88}$$

where the terms $\Pi^{(0)}$,$\widetilde{\Pi}^{(1,1)}$*, and* $\widetilde{\Pi}^{(1,2)}$ *are defined in (76), (85), and (86). Finally,* $\epsilon' := \max\{\epsilon_1,\epsilon_2\}$.

Proof. First define the function $\Upsilon^{\vec{\epsilon}}(t,\vec{x},\vec{z})$ by

$$\begin{aligned}\Upsilon^{\vec{\epsilon}} \;&=\; \Big(\Pi^{(0)} + \widetilde{\Pi}^{(1,1)} + \widetilde{\Pi}^{(1,2)} + \epsilon_1\Pi^{(2,1)} + \epsilon_2\Pi^{(2,2)} + \epsilon_1^{\frac{3}{2}}\Pi^{(3,1)} + \epsilon_1\sqrt{\epsilon_2}\Pi^{(3,2)} \\ &\quad + \sqrt{\epsilon_1}\epsilon_2\Pi^{(3,3)} + \epsilon_2^{\frac{3}{2}}\Pi^{(3,4)}\Big) - \Pi^{\vec{\epsilon}} \qquad (89)\end{aligned}$$

Notice that the $\Pi^{(2,3)}$-term has purposefully been included in $\Upsilon^{\vec{\epsilon}}$ – this is a crucial splitting for the validity of the remaining analysis. The first step toward a proof of Therem 5.1 is once again to show that $\left|\Upsilon^{\vec{\epsilon}}\right| = O(\epsilon^{'})$. As similarly executed in Section 3, we study the properties of $\Upsilon^{\vec{\epsilon}}$ via its behavior when acted on by the generator $\mathcal{A}^{\vec{\epsilon}}$. From our previous analysis and the boundary condition (74), we have

$$\begin{cases} \begin{aligned} \mathcal{A}^{\vec{\epsilon}}\Upsilon^{\vec{\epsilon}} =\; & \epsilon_1\left(\mathcal{A}_2^*\Pi^{(2,1)} + \mathcal{A}_1^{(1)}\Pi^{(3,1)} + \mathcal{A}_1^{(2)}\Pi^{(3,2)}\right) \\ & +\epsilon_2\left(\mathcal{A}_2^*\Pi^{(2,2)} + \mathcal{A}_1^{(1)}\Pi^{(3,2)} + \mathcal{A}_1^{(2)}\Pi^{(3,4)}\right) \\ & +\sqrt{\epsilon_1\epsilon_2}\left(\mathcal{A}_1^{(1)}\Pi^{(3,2)} + \mathcal{A}_1^{(2)}\Pi^{(3,3)}\right) + \epsilon_1^{3/2}\mathcal{A}_2^*\Pi^{(3,1)} \\ & +\epsilon_1\sqrt{\epsilon_2}\mathcal{A}_2^*\Pi^{(3,2)} + \sqrt{\epsilon_1}\epsilon_2\mathcal{A}_2^*\Pi^{(3,3)} + \epsilon_2^{3/2}\mathcal{A}_2^*\Pi^{(3,4)}\,, \\[1ex] \Upsilon^{\vec{\epsilon}}(T_0,\vec{x},\vec{z}) =\; & \epsilon_1\Pi^{(2,1)}(T_0,\vec{x},\vec{z}) + \epsilon_2\Pi^{(2,2)}(T_0,\vec{x},\vec{z}) + \epsilon_1^{3/2}\Pi^{(3,1)}(T_0,\vec{x},\vec{z}) \\ & +\epsilon_1\sqrt{\epsilon_2}\,\Pi^{(3,2)}(T_0,\vec{x},\vec{z}) + \sqrt{\epsilon_1}\epsilon_2\Pi^{(3,3)}(T_0,\vec{x},\vec{z}) + \epsilon_2^{3/2}\Pi^{(3,4)}(T_0,\vec{x},\vec{z}) \\ & +O(\epsilon^{'})\,. \end{aligned} \end{cases} \qquad (90)$$

The detailed growth properties of the first two terms of (90) and related boundary conditions will now be examined. All other terms can be shown (with limitless patience!) to have similar bounds.

- $\mathcal{A}_2^*\Pi^{(2,1)}$-*Term:*
  Without loss of generality, we choose $c_1 = c_{12} = 0$ in (78). Then $\mathcal{A}_2^*\Pi^{(2,1)}$ involves multiplications of the terms ψ_1,ψ_{12} with up to the fourth order partial derivatives of $\Pi^{(0)}$ and the smooth (linear growth in $\vec{x}$ and bounded in $\vec{z}$) coefficients of $\mathcal{A}_2^*$. By the boundedness of the source terms in their defining Poisson equations (80), ψ_1 and ψ_{12} are at most linearly growing in their arguments and have bounded first derivative. From the Appendix, the partial derivatives of $\Pi^{(0)}$ are at most log-linearly bounded in $\vec{x}$. Aggregating these partial results, $\mathcal{A}_2^*\Pi^{(2,1)}$ has at most a linear growth in $\vec{z}$ and log-linear in $\vec{x}$. It also follows from these last arguments that $\Pi^{(2,1)}(T_0,\vec{x},\vec{z})$ share equivalent growth bounds.

- $\mathcal{A}_1^{(1)}\Pi^{(3,1)}$-*Term:*
  Since $\Pi^{(3,1)}$ solves the Poisson equation (81) with the corresponding centering equations, we can write

  $$\mathcal{A}_0^{(1)}\Pi^{(3,1)} = -\frac{1}{2}\left[\left(\mathcal{A}_2^* - \langle\mathcal{A}_2^*\rangle\right)\Pi^{(1,1)} + \left(\mathcal{A}_1^{(1)}\Pi^{(2,1)} - \langle\mathcal{A}_1^{(1)}\Pi^{(2,1)}\rangle\right)\right]\,. \qquad (91)$$

  Solving this yields,

  $$\Pi^{(3,1)} = -\frac{1}{4}\psi_1\Pi^{(1,1)}_{x_1x_1} - \frac{\rho}{2}\psi_{12}\Pi^{(1,1)}_{x_1x_2} + \frac{\rho_{21}\nu_1}{4\sqrt{2}}\xi_1\Pi^{(0)}_{x_1x_1x_1} + \frac{\rho\rho_{21}\nu_1}{2\sqrt{2}}\eta_1\Pi^{(0)}_{x_1x_1x_2}\,, \qquad (92)$$

where ξ_1, η_1 are solutions of (with constants of integration set to zero)

$$\mathcal{A}_0^{(1)}\xi_1 = \sigma_X^{(1)}\psi_1' - \langle \sigma_X^{(1)}\psi_1' \rangle \ , \quad \mathcal{A}_0^{(1)}\eta_1 = \sigma_X^{(1)}\partial_{z_1}\psi_{12} - \langle \sigma_X^{(1)}\partial_{z_1}\psi_{12} \rangle \, . \tag{93}$$

The source terms in (93) being bounded, ξ_1 and η_1 are at most linearly growing and can be chosen with bounded first derivatives. It follows that $\mathcal{A}_1^{(1)}\Pi^{(3,1)}$ is a linear combination of terms with at most linear growth in $\vec{z}$ multiplied by up to third order $\vec{x}$-derivatives of $\Pi^{(1,1)}$ or fifth order $\vec{x}$-derivatives of $\Pi^{(0)}$. By the Appendix's result on log-linear bounds of $\Pi^{(0)}$ and $\Pi^{(1,1)}$ under various orders of derivatives, we conclude that $\mathcal{A}_1^{(1)}\Pi^{(3,1)}$ is at most linearly growing in $\vec{z}$ and log-linearly growing in $\vec{x}$. It is now straightforward to see that $\Pi^{(3,1)}(T_0, \vec{x}, \vec{z})$ also shares these growth properties.

We remark that, with the use of similar techniques, the remaining terms from (90) can be shown to possess equivalent growth properties.

Letting the functions $M(t, \vec{x}, \vec{z})$ and $N(T_0, \vec{x}, \vec{z})$ denote the r.h.s. of the PDE (90) and its boundary condition, respectively, a probabilistic representation of the solution is

$$\Upsilon^{\vec{\epsilon}}(t, \vec{x}, \vec{z}) \;=\; \mathbb{E}^{\mathbb{Q}}_{t,\vec{x},\vec{z}}\left[P(t, T_0)\, N(T_0, \vec{X}_{T_0}, \vec{Z}_{T_0}) - \int_t^{T_0} P(t, u)\, M(u, \vec{X}_u, \vec{Z}_u)\, du\right] . \tag{94}$$

From Lemma B.1 in Cotton, Fouque, Papanicolaou, and Sircar (2004), or by direct computations, the processes $X_t^{(i)}$ and $Z_t^{(i)}$, $i \in \{1, 2\}$ have finite exponential moments, implying the finiteness of the first two moments of $Z_t^{(i)}$, $i \in \{1, 2\}$. Applying these considerations to (94) finally supplies us with the claimed assertion, that is $\left|\Upsilon^{\vec{\epsilon}}\right| = O(\epsilon')$.

We are now ready to conclude our proof:

$$\begin{aligned}
&\left|\Pi^{\vec{\epsilon}}(t, \vec{x}, \vec{z}) - \left(\Pi^{(0)}(t, \vec{x}) + \widetilde{\Pi}^{(1,1)}(t, \vec{x}) + \widetilde{\Pi}^{(1,2)}(t, \vec{x})\right)\right| \\
=\;& \left|\epsilon_1\Pi^{(2,1)} + \epsilon_2\Pi^{(2,2)} + \epsilon_1^{\frac{3}{2}}\Pi^{(3,1)} + \epsilon_1\sqrt{\epsilon_2}\Pi^{(3,2)} + \sqrt{\epsilon_1}\epsilon_2\Pi^{(3,3)} + \epsilon_2^{\frac{3}{2}}\Pi^{(3,4)} - \Upsilon^{\vec{\epsilon}}\right| && (95) \\
\le\;& \left|\Upsilon^{\vec{\epsilon}}\right| + \epsilon_1\left|\Pi^{(2,1)}\right| + \ldots + \epsilon_2\left|\sqrt{\epsilon_2}\Pi^{(3,4)}\right| && (96) \\
\le\;& \left|\Upsilon^{\vec{\epsilon}}\right| + \epsilon'\left|\Pi^{(2,1)}\right| + \ldots + \epsilon'\left|\sqrt{\epsilon_2}\Pi^{(3,4)}\right| && (97) \\
=\;& O(\epsilon') && (98)
\end{aligned}$$

□

It is noteworthy that the two parameters V_{11} and V_{22} arise only in the two-name case and are not induced by forward or single-name option prices. These parameters provide the trader with two additional degrees-of-freedom allowing a biasing of a two-name claim upward or downward relative to the single-name case. Equivalently, the two parameters may be used to tweak the implied volatility smile/skew. Furthermore, from the definitions of V_{ii} (see equations (85) and (86)), if the correlation between the two commodities is zero ($\rho = 0$) then $V_{ii} = 0$. Additionally, since each of these coefficients are proportional to the product of two correlations and the small parameter $\sqrt{\epsilon_i}$, they should in principle be very small.

5.1.2. Two-Factor Model + SV

This section's main goal is to find a *well behaved* approximation to the option price

$$\Pi^{\vec{\epsilon}}(t,\vec{x},\vec{y},\vec{z}) = P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,\vec{x},\vec{y},\vec{z}}\left[\varphi\left\{F^{\epsilon_1}_{T_0,T_1}, F^{\epsilon_2}_{T_0,T_2}\right\}\right], \tag{99}$$

where each of the forward curves $F^{\epsilon_i}_{T_0,T_i}$ are based on the SV extended two-factor spot price model of Section 2.2.2. We only state the main result without going through the proof, which follows along similar lines to the previous section.

Theorem. 5.2 *For any fixed* $(T_0,T_1,T_2,\vec{x},\vec{y},\vec{z}) \in \mathbb{R}^3_+ \times \mathbb{R}^6$ *with* $T_0 \le T_1, T_2$ *and for all* $t \in [0,T_0]$*, we have*

$$\left|\Pi^{\vec{\epsilon}}(t,\vec{x},\vec{y},\vec{z}) - \left(\Pi^{(0)}(t,\vec{x},\vec{y}) + \widetilde{\Pi}^{(1,1)}(t,\vec{x},\vec{y}) + \widetilde{\Pi}^{(1,2)}(t,\vec{x},\vec{y})\right)\right| = O(\epsilon'),$$

where $\epsilon' := Max\{\epsilon_1,\epsilon_2\}$,

$$\Pi^{(0)}(t,\vec{x},\vec{y}) := P(t,T_0)\,\mathbb{E}^{\mathbb{Q}}_{t,\vec{x},\vec{y}}\left[\varphi\left\{F^{\epsilon_1(0)}_{T_0,T_1}\left(\overline{X}^{(1)}_{T_0}, Y^{(1)}_{T_0}\right), F^{\epsilon_2(0)}_{T_0,T_2}\left(\overline{X}^{(2)}_{T_0}, Y^{(2)}_{T_0}\right)\right\}\right], \tag{100}$$

$F^{\epsilon_i(0)}_{t,T_i}$ *being as in (13) with* σ_{Xi} *replaced by* $\sqrt{\langle\sigma^2_{Xi}(z_i)\rangle}$ *and each* $\overline{X}^{(i)}_t$ *as defined in (77),*

$$\begin{aligned}
&\widetilde{\Pi}^{(1,1)}(t,\vec{x},\vec{y})\\
:=\;& l^{(1,1)}_1(t,T_0)\,\Pi^{(0)}_{x_1x_1x_1} + l^{(1,1)}_2(t,T_0)\,\Pi^{(0)}_{x_1x_1y_1} + l^{(1,1)}_3(t,T_0)\,\Pi^{(0)}_{x_1x_1x_2}\\
&+l^{(1,1)}(T_0,T_1)\,P(t,T_0)\\
&\times\mathbb{E}^{\mathbb{Q}}_{t,\vec{x},\vec{y}}\left[\widetilde{F}^{\epsilon_1(0)}_{T_0,T_1}\left(\overline{X}^{(1)}_{T_0}, Y^{(1)}_{T_0}\right)\frac{\partial\varphi}{\partial F^{\epsilon_1}}\left\{F^{\epsilon_1(0)}_{T_0,T_1}\left(\overline{X}^{(1)}_{T_0}, Y^{(1)}_{T_0}\right), F^{\epsilon_2(0)}_{T_0,T_2}\left(\overline{X}^{(2)}_{T_0}, Y^{(2)}_{T_0}\right)\right\}\right],
\end{aligned} \tag{101}$$

and its symmetric part

$$\begin{aligned}
&\widetilde{\Pi}^{(1,2)}(t,\vec{x},\vec{y})\\
:=\;& l^{(1,2)}_1(t,T_0)\,\Pi^{(0)}_{x_2x_2x_2} + l^{(1,2)}_2(t,T_0)\,\Pi^{(0)}_{x_2x_2y_2} + l^{(1,2)}_3(t,T_0)\,\Pi^{(0)}_{x_2x_2x_1}\\
&+l^{(1,2)}(T_0,T_2)\,P(t,T_0)\\
&\times\mathbb{E}^{\mathbb{Q}}_{t,\vec{x},\vec{y}}\left[\widetilde{F}^{\epsilon_2(0)}_{T_0,T_2}\left(\overline{X}^{(2)}_{T_0}, Y^{(2)}_{T_0}\right)\frac{\partial\varphi}{\partial F^{\epsilon_2}}\left\{F^{\epsilon_1(0)}_{T_0,T_1}\left(\overline{X}^{(1)}_{T_0}, Y^{(1)}_{T_0}\right), F^{\epsilon_2(0)}_{T_0,T_2}\left(\overline{X}^{(2)}_{T_0}, Y^{(2)}_{T_0}\right)\right\}\right],
\end{aligned} \tag{102}$$

where $V^{(i)}_1$ *and* $V^{(i)}_2$ *are (for each asset* $i \in \{1,2\}$*) given by the parameters value in there respective forward price approximation in Theorem 3.2. Furthermore, for each asset* $i \in \{1,2\}$*, the various "l-coefficients" are given by*

$$\begin{aligned}
l^{(1,i)}_1(t,T_0) &:= -\frac{\beta_i V^{(i)}_2}{2(\beta_i-\alpha_i)}h(t,T_0;-2\beta_i-\alpha_i)\\
&\quad -\left(\frac{V^{(i)}_1}{2} + \frac{3\beta_i^2 V^{(i)}_2}{4\beta_i+2\alpha_i}\left(1-\frac{1}{\beta_i-\alpha_i}\right)\right)h(t,T_0;-3\beta_i),
\end{aligned} \tag{103}$$

$$l^{(1,i)}_2(t,T_0) := -\frac{V^{(i)}_2}{2}h(t,T_0;-2\beta_i-\alpha_i), \tag{104}$$

$$l^{(1,i)}_3(t,T_0) := -V_{ii}h(t,T_0;-2\beta_i-\alpha_j)\;,\;(i,j)\in\{(1,2),(2,1)\}, \tag{105}$$

$$l^{(1,i)}(T_0,T_i) := -V^{(i)}_1 h(T_0,T_i;3\beta_i) - V^{(i)}_2\frac{\beta_i}{\alpha_{Yi}-\beta_i}\left[h(T_0,T_i;3\beta_i) - h(T_0,T_i;\alpha_{Yi}+2\beta_i)\right]. \tag{106}$$

Once more, these SV extended two-factor model two-name option prices depend on the constant volatility prices plus corrections depending on the Delta's and Delta-Gamma's of the option together with a modified payoff. The V_1 and V_2 coefficients are inherited from the forward price approximation, while the new parameters V_{11} and V_{22} arise as a result of the correlation between the spot prices.

5.2. Nonsmooth Payoff: Forward Spread

As similarly argued in Section 4.2, it is possible to extend the validity of our pricing results (Theorems 5.1 and 5.2) to non-smooth T_0-payoff functions $\varphi(\cdot,\cdot)$, such as forward spread option contracts. The arguments behind this assertion follows along the same lines as described in Section 4.2. In practice, our stated results apply for dates not extremely close to contract maturity. We again refer to Fouque, Papanicolaou, Sircar, and Solna (2003) for further technical explanations within the stock option context.

6. Conclusions and Future Work

This work focused on incorporating stochastic volatility into one- and two-factor mean-reverting commodities spot price models. Although stochastic volatility models have been studied in the context of commodities, none of the previous works produce closed form option prices and also only focus on single-name options. In contrast, we obtained explicit closed form pricing equations for single and two-name options on forward contracts.

By assuming the instantaneous stochastic volatility of the spot price process is driven by a hidden fast mean-reverting OU process, we were successful in obtaining explicit closed form derivative prices. Furthermore, we proved that these explicit approximations are correct up to order ϵ. This methodology produced forward and option prices which are independent of the specific mapping between the hidden process and the stochastic volatility. The key consequence is that forward prices can be written in terms of the constant volatility model where the effective constant volatility arises due to a smoothening of the hidden process driving the stochastic volatility. Although our results appear somewhat daunting at first, they are in fact quite simple in structure and, more importantly, they are *explicit*. The results can be used to compute the stochastic volatility corrections of many common commodity options such as calls, puts, and spreads. As Fouque, Papanicolaou, and Sircar (2000b) similarly found in equity derivatives, we find, using commutator methods, that these corrections are related to the delta's and delta-gamma's of the constant volatility prices and the price of a modified payoff.

Another very important aspect of this asymptotic approach is its ease of calibration. The arbitrary modeling specification is smoothed out and instead we find a new set of free parameters V, V_i and V_{ii} (see equations (39) & (45) and (85) & (86) and the arguments leading up to these results). Using a nonlinear least-squares minimization scheme, the first set of these parameters (V or V_1-V_2) can be calibrated directly from forward price data. The second set of parameters (V_{ii}) arise only in the two-name case when the payoff explicitly depends on both names. Consequently, this second class of parameters has two dual and equivalent interpretations: (i) if no market prices for two-name options exists, they are additional model inputs and provide the trader with the flexibility to tweak prices; or (ii) if at least two option prices on two-names exist, they are market determined parameters which can then be used to consistently price other options.

There are many directions left open for future work, the most obvious being applying the model to an extensive data set. It would be instructive to classify the set of commodities spot price data which are driven by a fast mean-reverting hidden OU process. Then, using the forward price approximations, we extract the market implied parameters and use these implied parameters to price single- and two-name options.

A more mathematically interesting direction to explore, and one we have already begun to, is to apply similar methods in the context of stochastic volatility HJM models for commodities data. Schwartz and Trolle (2007) are the first to introduce a second unspanned stochastic volatility component into the commodities framework using an HJM approach – previous work includes Cortazar and Schwartz (1994), Amin, Ng., and Pirrong (1995) and Miltersen and Schwartz (1998) who focus solely on the constant volatility cases – however, they resort to an affine Heston-like model and employ transform methods to compute forward and option prices as no closed form results exists. As commented earlier, this is not a huge disadvantage when pricing a few options; however, it becomes very computationally intensive when used as a calibration tool and/or for sensitivity analysis. Our preliminary work gives us confidence that we can indeed find closed form forward and option prices through an SV extension of the standard HJM models using the hidden fast mean-reverting OU process to drive stochastic volatility.

A. Appendix

Here we sketch a proof of some useful bounds on the growth of various partial derivatives of the first three terms in the price expansion (73). The main result is stated as follows.

Theorem. A.1 *Fix $p \in \mathbb{N}$. Then, for any $i, n \in \mathbb{N}$ s.t. $0 \le i \le n \le p$ and all $t \in [0, T_0)$, there exist constants C, C_1, C_2 such that*

$$\left|\frac{\partial^n \Pi^{(0)}}{\partial x_1^i \partial x_2^{n-i}}\right|, \left|\frac{\partial^n \Pi^{(1,1)}}{\partial x_1^i \partial x_2^{n-i}}\right|, \left|\frac{\partial^n \Pi^{(1,2)}}{\partial x_1^i \partial x_2^{n-i}}\right| \le C \exp\left(C_1 x_1 + C_2 x_2\right). \tag{107}$$

Proof. We know that for $t \le u \le T_0$

$$X_u^{(i)} = \phi_i + (x_i - \phi_i)\, e^{-\beta_i(u-t)} + \bar{\sigma}_{Xi} \int_t^u e^{-\beta_i(u-s)}\, dW_s^{(i)}, \tag{108}$$

with $d\left[W^{(1)}, W^{(2)}\right]_t = \rho\, dt$. That is $\vec{X}_u \sim N(\vec{m}, \bar{\sigma})$ with mean $m_i = \phi_i + (x_i - \phi_i)\, e^{-\beta_i(u-t)}$ and covariance matrix given by $\bar{\sigma}_{ii} = \langle\, \sigma_{Xi}^2 \rangle h(t, u; 2\beta_i)$ and $\bar{\sigma}_{12} = \bar{\sigma}_{21} = \rho\langle \sigma_{X1}\sigma_{X2}\rangle h(t, u; \beta_1 + \beta_2)$. Let $\bar{\Phi}'_{t,u}$ be the density of $N(\vec{m}, \bar{\sigma})$ and $\Phi'_{t,u}$ the one of $N(\vec{0}, \bar{\sigma})$. Also recall from (4) that we can write

$$\begin{aligned} F_{T_0,T_i}^{\epsilon_i(0)} &= \exp\left\{ g_{T_i}^{(i)} + \phi_i \left(1 - e^{-\beta_i(T_i - T_0)}\right) + \frac{\langle\, \sigma_{Xi}^2 \rangle}{2} h(T_0, T_i; 2\beta_i) + e^{-\beta_i(T_i - T_0)} X_{T_0}^{(i)} \right\} \\ &:= C^{(i)} \exp\left\{ D^{(i)} X_{T_0}^{(i)} \right\}, \end{aligned} \tag{109}$$

where we used the symbols $C^{(i)}$ and $D^{(i)}$ to simplify notations. Using these last formulas in (76) we can write

$$\Pi^{(0)}(t,\vec{x}) = P(t,T_0)\int_{\mathbb{R}^2} \varphi\left(C^{(1)}\, e^{D^{(1)}(m_1+y_1)}, C^{(2)}\, e^{D^{(2)}(m_2+y_2)}\right) \Phi'_{t,T_0}(y_1,y_2)\, d\vec{y}\,, \tag{110}$$

where the dependence on the $\vec{x}$-variable is imbedded in $\vec{m}$. Smoothness of φ and boundedness of its partial derivatives combined with the dominated convergence theorem (note that r.v. having the density $\Phi'(\cdot,\cdot)$ has finite exponential moments) imply the claimed bound on the derivatives of $\Pi^{(0)}$.

We also rewrite (85) as

$$\begin{aligned}\Pi^{(1,1)}(t,\vec{x}) \;\; = \;\; & -\int_{\mathbb{R}^2}\left(\frac{V_1}{2}h(t,T_0;-3\beta_1)\Pi^{(0)}_{x_1x_1x_1}(t,m_1+y_1,m_2+y_2)\right.\\ & \left. + \, V_{11}h(t,T_0;-2\beta_1-\beta_2)\Pi^{(0)}_{x_1x_1x_2}(t,m_1+y_1,m_2+y_2)\right)\Phi'_{t,T_0}(\vec{y})\, d\vec{y}\\ & -V_1 h(T_0,T_1;3\beta_1)\, P(t,T_0)\int_{\mathbb{R}^2} C^{(1)}\, e^{D^{(1)}(m_1+y_1)}\\ & \times\, \frac{\partial\varphi}{\partial F^{\epsilon_1}}\left(C^{(1)}\, e^{D^{(1)}(m_1+y_1)}, C^{(2)}\, e^{D^{(2)}(m_2+y_2)}\right)\Phi'_{t,T_0}(\vec{y})\, d\vec{y}\;, \end{aligned} \tag{111}$$

so that the log-linear bound on $\Pi^{(1,1)}$ follows from the same arguments as in $\Pi^{(0)}$ case and the previous bound on $\Pi^{(0)}$.

Finally, the log-linear bounds on the partial derivatives of $\Pi^{(1,2)}$ is found exactly as in the $\Pi^{(1,1)}$ case since the arguments are perfectly symmetric with respect to x_1 and x_2. □

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