Applied Probability Trust (1 September 2004)

ASYMPTOTIC PROBABILITIES OF AN EXCEEDANCE OVER RENEWAL THRESHOLDS WITH AN APPLICATION TO RISK THEORY

CHRISTIAN Y. ROBERT ,* *CNAM and CREST*

Abstract

Let $(Y_n, N_n)_{n\geq 1}$ be i.i.d. bivariate random variables such that N_n are positive with finite mean ν and Y_n have a common heavy-tailed distribution F. We consider the process $(Z_n)_{n\geq 1}$ defined by $Z_n = Y_n - \Sigma_{n-1}$, where $\Sigma_{n-1} = \sum_{k=1}^{n-1} N_k$. It is shown that the probability that the maximum $M = \max_{n\geq 1} Z_n$ exceeds x is approximately $\nu^{-1}\int_x^\infty \bar{F}(u)\,du$, as $x \to \infty$. Then we study the integrated tail of the maximum of a random walk with long-tailed increments and negative drift over the interval $[0, \sigma]$ defined by some stopping time σ, in case the randomly stopped sum is negative. Finally, an application to risk theory is considered.

Keywords: Renewal Process, Maximum of a Random Walk, Regenerative Process, Stopping Time, Heavy-tailed Distribution, Ruin Probability, Risk Theory

AMS 2000 Subject Classification: Primary 60G70
Secondary 60G50; 60G40; 60K30

1. Introduction

Throughout this paper, (Y, N) and $(Y_n, N_n)_{n\geq 1}$ are independent and identically distributed (i.i.d.) bivariate random variables (r.v.) such that N is positive with finite mean ν and Y has a heavy-tailed distribution F. We consider the process $(Z_n)_{n\geq 1}$ defined by $Z_n = Y_n - \Sigma_{n-1}$, where $\Sigma_{n-1} = \sum_{k=1}^{n-1} N_k$ ($\Sigma_0 = 0$), and study the probability that the maximum $M = \max_{n\geq 1} Z_n$ exceeds large x. It is also the probability that there exists n such that Y_n exceeds $x + \Sigma_{n-1}$, where $(\Sigma_{n-1})_{n\geq 1}$ is a sequence of renewal thresholds. Σ_{n-1} is independent of Y_n but can be a function of its past values. It is shown that this probability is approximately $\nu^{-1}\int_x^\infty \bar{F}(u)\,du$ for large x.

A possible application of this result is to risk theory. Let us consider the following risk model with renewal arrivals:

(i) The claim sizes $U_1, U_2, ...$ are i.i.d. positive r.v. with mean μ_U.

(ii) The claims happen at random times $s_1 < s_2 < ...$ such that $T_n = s_n - s_{n-1}$ are i.i.d. positive r.v. with mean μ_T, and are independent of $(U_n)_{n\geq 1}$.

(iii) The premium rate is assumed to be equal to 1, and $\mu_U < \mu_T$.

We define the claim surplus process (resp. the risk reserve process) by $S_t^c = \sum_{k=1}^{N_t} U_k - t$ (resp. $R_t = -S_t^c$), where $N_t = \max\{n \geq 0 : s_n \leq t\}$. Let u be the initial solvency margin which is met by capital provided by the shareholders. The classical probability $\psi(u)$ of ultimate ruin is the probability that the claim surplus

* Postal address: Centre de Recherche en Economie et Statistique, Timbre J320, 15 Boulevard Gabriel Peri, 92245 MALAKOFF Cedex, FRANCE

process ever exceeds the level u, $\mathbb{P}\left(\max_{t\geq 0} S_t^c > u\right)$. Since ruin can only occur at claim times, the ruin probability is equal to $\mathbb{P}\left(\max_{n\geq 0} S_n > u\right)$ where $(S_n)_{n\geq 1}$ is the random walk with increments $X_n = U_n - T_n$.

A reasonable modification of this model is that some dividends are paid out to the shareholders when the reserve process is sufficiently large. Dividend barrier models have a long history in risk theory (Bühlmann (1996)), but other situations can be considered. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function such that $0 \leq \varphi(x) \leq x$. The insurance company uses a stopping time σ_1 to decide when the reserves (just after a claim) $-S_{\sigma_1}$ are sufficiently large (necessarily positive) such that a part $\varphi(-S_{\sigma_1})$ is distributed to the shareholders and the other part $-S_{\sigma_1} - \varphi(-S_{\sigma_1})$ is kept to reinforce the solvency margin. At this time the reserves are reduced to 0. The same rule is then used to define a sequence of stopping times σ_k and a sequence of dividends $\varphi\left(S_{\sigma_{k-1}} - S_{\sigma_k}\right)$. Let us define the process $\{S_n^\varphi\}$ by

$$S_n^\varphi = S_n + \sum_{k\geq 1:\sigma_k<n} \varphi\left(S_{\sigma_{k-1}} - S_{\sigma_k}\right),$$

where $\sigma_0 = 0$ and $S_0 = 0$. The probability of ruin is thus equal to $\psi(u) = \mathbb{P}(M^\varphi > u)$, where $M^\varphi = \max\{S_n^\varphi : n \geq 1\}$. If we put $Y_n = \max_{0\leq k<\sigma_n-\sigma_{n-1}}\left(S_{k+\sigma_{n-1}} - S_{\sigma_{n-1}}\right)$ and $N_n = \rho\left(S_{\sigma_{n-1}} - S_{\sigma_n}\right)$ where $\rho(x) = x - \varphi(x)$, then it is easy to see that $\mathbb{P}(M^\varphi > u) = \mathbb{P}(M > u)$. According to our result, the probability of ruin is approximately

$$\psi(u) \sim \left(\mathbb{E}\rho(-S_{\sigma_1})\right)^{-1} \int_u^\infty \mathbb{P}(Y > x)\,dx, \text{ for large } u.$$

We shall not solve the problem of optimal dividend payment under a ruin constraint (see Bühlmann (1996) and references given there, as well as Gerber (1979) for such a problem).

The recent paper Foss and Zachary (2003) studies the tail behavior of the maximum of a random walk with long-tailed increments (X_n) and negative drift over the interval $[0, \sigma]$ defined by some stopping time σ. The authors show that

$$\mathbb{P}(Y > x) \sim \mathbb{E}\sigma\mathbb{P}(X > x), \text{ for large } x,$$

where X has the same distribution as the X_n. In case the randomly stopped sum is negative, we derive, as a corollary of our result, the similar equivalence

$$\int_u^\infty \mathbb{P}(Y > x)\,dx \sim \mathbb{E}\sigma \int_u^\infty \mathbb{P}(X > x)\,dx, \text{ for large } u.$$

Then the probability of ruin is approximately

$$\psi(u) \sim \left(\mathbb{E}\rho(-S_{\sigma_1})\right)^{-1} \mathbb{E}\sigma_1 \int_u^\infty \mathbb{P}(X > x)\,dx, \text{ for large } u.$$

In case no dividend is distributed, φ is equal to 0, and by using Wald's Identity (Chow et al. (1964)), $\mathbb{E}\rho(-S_{\sigma_1}) = -\mathbb{E}S_{\sigma_1} = -\mathbb{E}\sigma_1\mathbb{E}X$, which gives the classical asymptotics for the probability of ruin.

Section 2 presents the results of the paper. All the proofs are post-poned in a third section.

2. Results

We first state our main result, which is Theorem 2.1. We then give Corollary 2.1, Corollary 2.2 and Proposition 2.1 which set out the conditions for the application to risk theory.

Theorem 2.1. *Let us assume that $EN^2 < \infty$ and that*

$$\lim_{x\to\infty} \mathbb{P}(Y > x) \,/ \int_x^\infty \mathbb{P}(Y > u)\, du = 0, \tag{C.1}$$

then

$$\lim_{x\to\infty} \frac{\mathbb{P}(M > x)}{\int_x^\infty \mathbb{P}(Y > u)\, du} = \frac{1}{\mathbb{E}N}. \tag{1}$$

Remark 2.1. $(C.1)$ is not a classical condition to define heavy-tailed distributions. A more usual definition is that the distribution of Y is *long-tailed* (LT), i.e.

$$\lim_{x\to\infty} \frac{\mathbb{P}(Y > x-h)}{\mathbb{P}(Y > x)} = 1, \qquad \text{for all fixed } h > 0,$$

(see below for other definitions). If Y is LT, then it is easy to see that Condition $(C.1)$ holds.

Remark 2.2. Let $(S_n)_{n\geq 1}$ be a random walk with heavy-tailed increments X_n and negative mean. It is well-known that

$$\lim_{x\to\infty} \frac{\mathbb{P}(\max_{n\geq 1} S_n > x)}{\int_x^\infty \mathbb{P}(X > u)\, du} = -\frac{1}{\mathbb{E}X},$$

(see Veraverbecke (1977), Embrechts and Veraverbecke (1982), Korshunov (1997)). Theorem 2.1 seems to hold with $Y_n = X_n$ and $N_n = -X_n$ although $-X_n$ is not almost surely positive, but have a positive mean.

Now consider a regenerative process $\{V_n\}_{n\geq 0}$; there exists a zero-delayed renewal process with epochs $T_0 = 0 < T_1 < T_2 < ...$ such that the cycles $\left\{V_{n+T_{k-1}}\right\}_{0\leq n<T_k-T_{k-1}}$ are independent and have the same distribution. Regenerative processes have many important applications in queueing networks, storage processes, insurance or finance (see e.g. Asmussen (1987, 2000)). A basic example is the Lindley process and its cycles, that is, the time intervals separated by the instants where the process is equal to 0. We write $c_k = T_k - T_{k-1}$ for the cycle lengths and c is a r.v. which has the same distribution as the c_k. $\kappa = \mathbb{E}c$ is assumed to be finite. Let us define the maxima over cycles $M_{c_k} = \max\left\{V_{n+T_{k-1}} : 0 \leq n < T_k - T_{k-1}\right\}$.

Let us assume that there exist r.v. N_k such that $\left(\left\{V_{n+T_{k-1}}\right\}_{0\leq n<T_k-T_{k-1}}, N_k\right)_{k\geq 1}$ are i.i.d.. As a consequence of Theorem 2.1, we give the asymptotic tail behavior of $M = \max_{k\geq 1}(M_{c_k} - \Sigma_{k-1})$, i.e. the probability that the regenerative process exceeds high increasing thresholds which are defined by a renewal process and are constant on regenerative cycles.

Corollary 2.1. *Let us assume that $EN^2 < \infty$ and that*

$$\lim_{x\to\infty} \mathbb{P}(M_c > x) \,/ \int_x^\infty \mathbb{P}(M_c > u)\, du = 0, \tag{C.2}$$

then

$$\lim_{x\to\infty}\frac{\mathbb{P}(M>x)}{\int_x^\infty \mathbb{P}(M_c>u)\,du}=\frac{1}{\mathbb{E}N}. \tag{2}$$

The proof is omitted because it is a direct application of Theorem 2.1.

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables. By F_X we denote the common distribution of X_n and we assume that $\mu=\mathbb{E}X_n<0$ and $\mathbb{P}(X_1>0)>0$. We consider a stopping time σ with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$ where $\mathcal{F}_n=\sigma(X_1,...,X_n)$. We write $S_n=\sum_{i=1}^n X_i$ and $M_\sigma=\max\{S_n:n=0,1,...,\sigma-1\}$.

Before giving Corollary 2.2, we first provide some further definitions. For any distribution function H on $\mathbb{R}$ with finite mean we define the integrated distribution function H^s by $\overline{H^s}=\min\left(1,\int_x^\infty \overline{H}(u)\,du\right)$. A distribution function H on $\mathbb{R}_+$ is subexponential if and only if $H(x)>0$ for all x and $\lim_{x\to\infty}\overline{H^{*n}}(x)/\overline{H}(x)=n$, $n\geq 2$, where H^{*n} is the $n-$fold convolution of H with itself. (It is sufficient to verify the previous condition in the case $n=2$). More generally, a distribution function H on $\mathbb{R}$ is subexponential if and only if H^+ is subexponential, where $H^+=H\mathbb{I}_{\mathbb{R}_+}$ and $\mathbb{I}_{\mathbb{R}_+}$ is the indicator function of $\mathbb{R}_+$. A well-known result is that subexponential distributions are long-tailed. At last, a distribution H on $\mathbb{R}$ belongs to the class $\mathcal{S}^*$ if and only if $H(x)>0$ for all x and

$$\lim_{x\to\infty}\frac{\int_0^x \overline{H}(x-y)\,\overline{H}(y)\,dy}{\overline{H}(x)}=\int_0^\infty \overline{H}(y)\,dy. \tag{3}$$

It is also known that if $H\in\mathcal{S}^*$ then H and H^s are subexponential (see Klüppelberg (1988)).

Corollary 2.2. *(i) Suppose that $F_X\in\mathcal{S}^*$. Let σ be any stopping time such that $P(S_\sigma\leq 0)=1$ and $\mathbb{E}S_\sigma^2<\infty$. Then*

$$\lim_{x\to\infty}\frac{\int_x^\infty \mathbb{P}(M_\sigma>u)\,du}{\int_x^\infty \overline{F}_X(u)\,du}=\mathbb{E}\sigma. \tag{4}$$

(ii) Suppose that Equation (4) *holds for some stopping time σ such that $P(S_\sigma\leq 0)=1$ and $\mathbb{E}S_\sigma^2<\infty$, then F_X^s is subexponential.*

Remark 2.3. If we assume that $\mathbb{E}\left(X_1^+\right)^2<\infty$ and $E\sigma^2<\infty$, then $ES_\sigma^2<\infty$ (see Gut and Janson (1986) Theorem 3.1). Moreover if the stopping-time is the first passage time $\min\{n:S_n<c\}$ with c a nonpositive constant, then the condition $\mathbb{E}\left(X_1^+\right)^2<\infty$ implies that $E\sigma^2<\infty$ from Theorem 2.1 of Gut (1974), and then the condition $\mathbb{E}\left(X_1^+\right)^2<\infty$ is just needed.

Finally, we come back to the risk problem outlined in introduction. We found that, as in the classical risk model, the probability of ruin is equivalent to the integrated tail of X.

Proposition 2.1. *Suppose that $F_X\in\mathcal{S}^*$, then*

$$\lim_{x\to\infty}\frac{\psi(u)}{\int_u^\infty \overline{F}_X(x)\,dx}=\frac{\mathbb{E}\sigma}{\mathbb{E}\varphi(-S_\sigma)}.$$

3. Proofs

We first state a Lemma which is needed for the proof of Theorem 2.1.

Lemma 3.1. *Let* $\{\xi_n\}_{n\geq 1}$*be a sequence of i.i.d. r.v. with* $\mathbb{E}\xi_1 = -m < 0$ *and* $\mathbb{P}(\xi_1 > 0) > 0$*. Let us define* $\xi_1^+ = \max(0, \xi_1)$*, then*

$$E\left(\xi_1^+\right)^2 < \infty \Leftrightarrow \sum_{n\geq 1} \mathbb{P}\left(\sum_{i=1}^{n} \xi_i > 0\right) < \infty.$$

Proof. Consider the probability that n is the epoch of the first entry of the random walk with increments $\{\xi_n\}$ into $]-\infty;0[$, that is,

$$\tau_n^\xi = \mathbb{P}\left(\xi_1 \geq 0, ..., \sum_{i=1}^{n-1} \xi_i \geq 0, \sum_{i=1}^{n} \xi_i < 0\right).$$

$\left\{\tau_n^\xi\right\}$ is the distribution of the first descending ladder epoch τ_-^ξ. The finitness of the moments of τ_-^ξ are equivalent to the finitness of the moments of ξ_1^+ (see Gut (1974) Theorem 2.1): let $r \geq 1$, $\mathbb{E}\left|\xi_1^+\right|^r < \infty$ if and only if $\mathbb{E}\left(\tau_-^\xi\right)^r < \infty$.

Its probability generating function is given by $P^\xi(s) = \sum_{n\geq 1} \tau_n^\xi s^n$, $0 \leq s \leq 1$. Sparre-Anderson Theorem (see Feller (1971) Theorem XII.7.1) establishes that

$$\log \frac{1 - P^\xi(s)}{1 - s} = \sum_{n\geq 1} \frac{s^n}{n} \mathbb{P}\left(\sum_{i=1}^{n} \xi_i > 0\right), \qquad 0 \leq s < 1. \tag{5}$$

Since by assumption $\mathbb{E}\left|\xi_1^+\right| < \infty$, then $\mathbb{E}\tau_-^\xi < \infty$ and we can introduce the following probability generating function

$$Q^\xi(s) = \frac{1}{\mathbb{E}\tau_-^\xi} \frac{1 - P^\xi(s)}{(1 - s)}, \qquad 0 \leq s \leq 1. \tag{6}$$

of the random variable $I_{\tau_-^\xi}$ defined for $j \geq 0$ by $\mathbb{P}\left(I_{\tau_-^\xi} = j\right) = \sum_{k>j} \tau_j^\xi / \mathbb{E}\tau_-^\xi$ (see Feller (1970) p. 265). By differentiating (6), we have

$$\frac{\left(Q^\xi\right)'(s)}{Q^\xi(s)} = \sum_{n\geq 1} s^{n-1} \mathbb{P}\left(\sum_{i=1}^{n} \xi_i > 0\right), \qquad 0 \leq s < 1, \tag{7}$$

and then

$$\lim_{s\nearrow 1} \left(Q^\xi\right)'(s) < \infty \Leftrightarrow \sum_{n\geq 1} \mathbb{P}\left(\sum_{i=1}^{n} \xi_i > 0\right) < \infty \Leftrightarrow \mathbb{E}I_{\tau_-^\xi} < \infty \Leftrightarrow \mathbb{E}\left(\tau_-^\xi\right)^2 < \infty.$$

We conclude that $\mathbb{E}\left|\xi_1^+\right|^2 < \infty$ is equivalent to $\sum_{n\geq 1} \mathbb{P}\left(\sum_{i=1}^{n} \xi_i > 0\right) < \infty$. ■

Proof of Theorem 2.1. Let us denote $M^k = \max_{1\leq i\leq k}(Y_k - \Sigma_{k-1})$. Then

$$\mathbb{P}\left(M^k > x\right) = \mathbb{P}\left(\cup_{i=1}^k \{Y_i > x + \Sigma_{i-1}\}\right).$$

The proof of Theorem 2.1 is based on several steps.

Step 1: an upper bound for $\mathbb{P}(M > x)$. Since $\mathbb{P}\left(M^k > x\right)$ is bounded by $\sum_{i=1}^k \mathbb{P}(Y_i > x + \Sigma_{i-1})$, we focus our attention on the probability

$$\begin{aligned}
& \mathbb{P}(Y_i > x + \Sigma_{i-1}) \\
= \;& \mathbb{P}(Y_i > x + \Sigma_{i-1}, \Sigma_{i-1} < (i-1)\nu(1-\varepsilon)) + \mathbb{P}(Y_i > x + \Sigma_{i-1}, \Sigma_{i-1} \geq (i-1)\nu(1-\varepsilon)) \\
\leq \;& \mathbb{P}(Y_i > x, \Sigma_{i-1} < (i-1)\nu(1-\varepsilon)) + \mathbb{P}(Y_i \geq x + (i-1)\nu(1-\varepsilon)) \\
\leq \;& \mathbb{P}(Y_i > x)\,\mathbb{P}(\Sigma_{i-1} < (i-1)\nu(1-\varepsilon)) + \mathbb{P}(Y_i \geq x + (i-1)\nu(1-\varepsilon)).
\end{aligned}$$

It follows that

$$\mathbb{P}\left(M^k > x\right) \leq \overline{F}(x) \sum_{i=1}^k \mathbb{P}\left(\tilde{\Sigma}_{i-1}^u > 0\right) + \sum_{i=1}^k \overline{F}(x + (i-1)\nu(1-\varepsilon)),$$

where $\tilde{\Sigma}_{i-1}^u = (i-1)\nu(1-\varepsilon) - \Sigma_{i-1}$. According to Lemma 3.1 and since $\mathbb{E}\tilde{\Sigma}_1^u < 0$ and $\mathbb{E}(N)^2 < \infty$, we deduce that

$$\sum_{i=1}^k \mathbb{P}\left(\tilde{\Sigma}_{i-1}^u > 0\right) < \sum_{i=1}^\infty \mathbb{P}\left(\tilde{\Sigma}_{i-1}^u > 0\right) = K_1(\varepsilon) < \infty,$$

and

$$\mathbb{P}\left(M^k > x\right) \leq \overline{F}(x) K_1(\varepsilon) + \sum_{i=1}^k \overline{F}(x + (i-1)\nu(1-\varepsilon)).$$

Then let $k \to \infty$, $\mathbb{P}\left(M^k > x\right) \nearrow \mathbb{P}(M > x)$ and

$$\begin{aligned}
\mathbb{P}(M > x) &\leq \overline{F}(x) K_1(\varepsilon) + \sum_{i=1}^\infty \overline{F}(x + (i-1)\nu(1-\varepsilon)) \\
&\leq \frac{\int_x^\infty \overline{F}(u)\,du}{\nu(1-\varepsilon)} + \overline{F}(x)(K_1(\varepsilon) + 1).
\end{aligned}$$

Step 2: a lower bound for $\mathbb{P}(M > x)$. First let us note that

$$\mathbb{P}(M > x) \geq \sum_{i=1}^\infty \mathbb{P}(Y_i > x + \Sigma_{i-1}) - \sum_{1\leq i<j} \mathbb{P}(Y_i > x + \Sigma_{i-1}, Y_j > x + \Sigma_{j-1}).$$

On one hand, we have

$$\begin{aligned}
& \mathbb{P}\left(Y_i > x + \Sigma_{i-1}\right) \\
= \;& \int_0^\infty \mathbb{P}\left(Y_i > x + \sigma\right) f_{\Sigma_{i-1}}\left(\sigma\right) d\sigma \geq \int_0^{(i-1)\nu(1+\varepsilon)} \mathbb{P}\left(Y_i > x + \sigma\right) f_{\Sigma_{i-1}}\left(\sigma\right) d\sigma \\
\geq \;& \mathbb{P}\left(Y_i > x + (i-1)\,\nu\,(1+\varepsilon)\right) \int_0^{(i-1)\nu(1+\varepsilon)} f_{\Sigma_{i-1}}\left(\sigma\right) d\sigma \\
\geq \;& \mathbb{P}\left(Y_i > x + (i-1)\,\nu\,(1+\varepsilon)\right) \mathbb{P}\left(\Sigma_{i-1} \leq (i-1)\,\nu\,(1+\varepsilon)\right) \\
= \;& \mathbb{P}\left(Y_i > x + (i-1)\,\nu\,(1+\varepsilon)\right) - \mathbb{P}\left(Y_i > x + (i-1)\,\nu\,(1+\varepsilon)\right) \mathbb{P}\left(\Sigma_{i-1} > (i-1)\,\nu\,(1+\varepsilon)\right) \\
\geq \;& \mathbb{P}\left(Y_i > x + (i-1)\,\nu\,(1+\varepsilon)\right) - \mathbb{P}\left(Y_i > x\right) \mathbb{P}\left(\Sigma_{i-1} > (i-1)\,\nu\,(1+\varepsilon)\right).
\end{aligned}$$

It follows from Lemma 3.1 and $\mathbb{E}\left(N\right)^2 < \infty$ that $K_2\left(\varepsilon\right) = \sum_{i=1}^\infty \mathbb{P}\left(\tilde{\Sigma}_{i-1}^l > 0\right) < \infty$ where $\tilde{\Sigma}_{i-1}^l = \Sigma_{i-1} - (i-1)\,\nu\,(1+\varepsilon)$ since $\mathbb{E}\tilde{\Sigma}_1^l < 0$. Then

$$\begin{aligned}
\sum_{i=1}^\infty \mathbb{P}\left(Y_i > x + \Sigma_{i-1}\right) \;&\geq\; \sum_{i=1}^\infty \overline{F}\left(x + (i-1)\,\nu\,(1+\varepsilon)\right) - \overline{F}\left(x\right) K_2\left(\varepsilon\right) \\
&\geq\; \frac{\int_x^\infty \overline{F}\left(u\right) du}{\nu\left(1+\varepsilon\right)} - \overline{F}\left(x\right) K_2\left(\varepsilon\right).
\end{aligned}$$

On the other hand, let us denote $\Sigma_{i+1}^{j-1} = \sum_{k=i+1}^{j-1} N_k$ for $i < j$. We have

$$\begin{aligned}
& \mathbb{P}\left(Y_i > x + \Sigma_{i-1}, Y_j > x + \Sigma_{j-1}\right) \\
\leq \;& \mathbb{P}\left(Y_i > x + \Sigma_{i-1}, Y_j > x + \Sigma_{i+1}^{j-1}\right) \leq \mathbb{P}\left(Y_i > x + \Sigma_{i-1}\right) \mathbb{P}\left(Y_j > x + \Sigma_{i+1}^{j-1}\right) \\
= \;& \mathbb{P}\left(Y_i > x + \Sigma_{i-1}\right) \mathbb{P}\left(Y_j > x + \Sigma_{j-i-1}\right).
\end{aligned}$$

Analogously to Step 1

$$\sum_{1 \leq i < j} \mathbb{P}\left(Y_i > x + \Sigma_{i-1}, Y_j > x + \Sigma_{j-1}\right) \leq \left(\frac{\int_x^\infty \overline{F}\left(u\right) du}{\nu\left(1-\varepsilon\right)} + \overline{F}\left(x\right)\left(K_1\left(\varepsilon\right) + 1\right)\right)^2.$$

Thus a lower bound is given by

$$\mathbb{P}\left(M > x\right) \geq \frac{\int_x^\infty \overline{F}\left(u\right) du}{\nu\left(1+\varepsilon\right)} - \overline{F}\left(x\right) K_2\left(\varepsilon\right) - \left(\frac{\int_x^\infty \overline{F}\left(u\right) du}{\nu\left(1-\varepsilon\right)} + \overline{F}\left(x\right)\left(K_1\left(\varepsilon\right) + 1\right)\right)^2.$$

Step 3: Let us use Step 1 and 2 and let $x \to \infty$. Condition $(C.1)$ implies that

$$\frac{(1-\varepsilon)}{\mathbb{E}N} \leq \lim_{x \to \infty} \frac{\mathbb{P}\left(M > x\right)}{\int_x^\infty \mathbb{P}\left(Y > u\right) du} \leq \frac{(1+\varepsilon)}{\mathbb{E}N},$$

and then let $\varepsilon \to 0$ to complete the proof. ■

Proof of Corollary 2.2. (i) Let us define the sequence of stopping times $\{\sigma_k\}_{k \geq 0}$ such that $\sigma_0 = 0$, $\sigma_1 = \sigma$ and the cycles

$$\left\{S_{n+\sigma_{k-1}} - S_{\sigma_{k-1}}\right\}_{0 \leq n < \sigma_k - \sigma_{k-1}}$$

are independent and have the same distribution for $k=1,2,.....$ Let us write $V_{n+\sigma_{k-1}} = S_{n+\sigma_{k-1}} - S_{\sigma_{k-1}}$, for $0 \leq n < \sigma_k - \sigma_{k-1}$ and $N_k = S_{\sigma_{k-1}} - S_{\sigma_k}$. By the regenerative structure of the random walk, $\left\{ \left(\left\{ V_{n+\sigma_{k-1}} \right\}_{0 \leq n \leq \sigma_k - \sigma_{k-1}}, N_k \right), k \geq 1 \right\}$ are i.i.d.. Let us remark that

$$M = \max_{n \geq 1} S_n = \max_{k \geq 1} \max_{0 \leq n < \sigma_k - \sigma_{k-1}} \left(V_{n+\sigma_{k-1}} - \Sigma_{k-1} \right).$$

Since F_X^s is subexponential, Veraverbecke's theorem (Korshunov (1997)) establishes that

$$\lim_{x \to \infty} \frac{\mathbb{P}(M > x)}{\int_x^\infty \overline{F}_X(u)\, du} = \frac{1}{-\mu}.$$

Since $\mathbb{E}(N)^2 = \mathbb{E}(-S_\sigma)^2 < \infty$, to apply Corollary 2.1, it suffices to verify that

$$\lim_{x \to \infty} \mathbb{P}(M_\sigma > x) / \int_x^\infty \mathbb{P}(M_\sigma > u)\, du = 0.$$

Let us define the sequence of stopping times $\{\tau_k\}_{k \geq 0}$ by

$$\tau_0 = 0 \qquad \tau_k = \min \left\{ n : n > \tau_{k-1}, S_n \leq S_{\tau_{k-1}} \right\},$$

so that τ_k is the kth descending ladder time. Since $S_\sigma \leq 0$ *a.s.*, $\tau_1 \leq \sigma$ *a.s.*. Let $\mu(x) = \min \{n : S_n > x\}$,

$$\begin{aligned}
\mathbb{P}(M_\sigma > x) &= \sum_{k \geq 1} \mathbb{P}(\tau_k < \mu(x) \leq \tau_{k+1}, \sigma > \mu(x)) \\
&= \sum_{k \geq 1} \mathbb{P}(\sigma > \mu(x) > \tau_k)\, \mathbb{P}(\tau_k < \mu(x) \leq \tau_{k+1} | \sigma > \mu(x)) \\
&\leq \sum_{k \geq 1} \mathbb{P}(\sigma > \tau_k)\, \mathbb{P}(M_{\tau_k} > x) \\
&\leq \sum_{k \geq 1} \mathbb{P}(\sigma > k)\, \mathbb{P}(M_\tau > x) \leq \mathbb{E}\sigma \mathbb{P}(M_\tau > x).
\end{aligned}$$

Moreover, by using regenerative properties of the Lindley process

$$\begin{aligned}
\mathbb{P}(M_\tau > x) &\leq \mathbb{P}(\sharp \{n < \tau : S_n > x, S_{n+1} \leq x\} \geq 1) \\
&\leq \mathbb{E} \sum_{n=0}^{\tau - 1} \mathbb{I}\{S_n > x, S_{n+1} \leq x\} \\
&= \mathbb{E}\tau \int_0^\infty \pi(x + dt)\, \mathbb{P}(X \leq -t) \sim \mathbb{E}\tau \frac{\mu^-}{\mu} \mathbb{P}(X > x).
\end{aligned}$$

where $\pi(x) = P(M \leq x)$ and $\mu^- = \int_0^\infty F_X(-y)\, dy$. And remark that $\mathbb{P}(M_\sigma > x) \geq \mathbb{P}(X > x)$ to get

$$\frac{\mathbb{P}(M_\sigma > x)}{\int_x^\infty \mathbb{P}(M_\sigma > u)\, du} \leq \mathbb{E}\sigma \mathbb{E}\tau \frac{\mu^-}{\mu} \frac{\mathbb{P}(X > x)}{\int_x^\infty \mathbb{P}(X > u)\, du} \to 0,$$

as $x \to \infty$ since the distribution of X is long-tailed.

By Corollary 2.1, we have

$$\lim_{x\to\infty} \frac{\mathbb{P}(M > x)}{\int_x^\infty \mathbb{P}(M_\sigma > u)\, du} = -\frac{1}{\mathbb{E}S_\sigma}.$$

Moreover Wald's identity (see Chow et al. (1964)) gives that if either $\mathbb{E}|X_1| < \infty$ or $\mathbb{E}\sigma < \infty$, then $\mathbb{E}S_\sigma = \mu\mathbb{E}\sigma$. Using now Veraverbecke's theorem, we derive

$$\lim_{x\to\infty} \frac{\int_x^\infty \mathbb{P}(M_\sigma > u)\, du}{\int_x^\infty \overline{F}(u)\, du} = \mathbb{E}\sigma.$$

(ii) F^s is subexponential follows from the converse to Veraverbeke's theorem proved by Korshunov (1997). ■

Proof of Proposition 2.1. Let us consider the sequence of stopping times $\{\sigma_k\}_{k\geq 0}$. The cycles $\left\{S_{n+\sigma_{k-1}} - S_{\sigma_{k-1}}\right\}_{0\leq n<\sigma_k-\sigma_{k-1}}$ are independent and have the same distribution for $k = 1, 2, \ldots..$ Write $V_{n+\sigma_{k-1}} = S_{n+\sigma_{k-1}} - S_{\sigma_{k-1}}$, for $0 \leq n < \sigma_k - \sigma_{k-1}$ and $k \geq 1$, $Y_n = \max_{0\leq k<\sigma_n-\sigma_{n-1}} \left(S_{k+\sigma_{n-1}} - S_{\sigma_{n-1}}\right)$ and $N_n = \rho\left(S_{\sigma_{n-1}} - S_{\sigma_n}\right)$. By the regenerative structure of the random walk, $\left(\left\{V_{n+\sigma_{k-1}}\right\}_{0\leq n\leq\sigma_k-\sigma_{k-1}}, N_k\right)_{k\geq 1}$ are i.i.d.. Let us remark that

$$M^\varphi = \max_{n\geq 1} S_n^\varphi = \max_{k\geq 1} \max_{0\leq n<\sigma_k-\sigma_{k-1}} \left(V_{n+\sigma_{k-1}} - \Sigma_{k-1}\right).$$

According to Corollary 2.1

$$\lim_{x\to\infty} \frac{\mathbb{P}(M^\varphi > x)}{\int_x^\infty \mathbb{P}(Y > u)\, du} = \frac{1}{\mathbb{E}\rho(-S_{\sigma_1})},$$

and according to Corollary 2.2

$$\lim_{x\to\infty} \frac{\int_x^\infty \mathbb{P}(Y > u)\, du}{\int_x^\infty \overline{F}_X(u)\, du} = \mathbb{E}\sigma_1,$$

and then Proposition 2.1 is proved. ■

Acknowledgement

The author wishes to thank an anonymous referee for careful reading and detailed comments.

References

[1] Asmussen, S. (1987). *Applied Probability and queues*, John Wiley & Sons.

[2] Asmussen, S. (1998). Subexponential asymptotics for stochastic processes: Extremal behavior, stationary distributions and first passage probabilities. *Ann. Appl. Prob.*, **8**, 354–374.

[3] Asmussen, S. (2000). *Ruin Probabilities*, World Scientific.

[4] Bühlmann H. (1996). *Mathematical Methods of Risk Theory*, Springer, Berlin.

[5] Chow, Y.S., Robbins, H. and Teicher H. (1965). Moments of randomly stopped sums, *Ann. Math. Stat.*, **36,** 789–799.

[6] Embrechts, P. and Veraverbeke, N. (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims, *Ins. Math. Econom.*, **1,** 55–72.

[7] Feller, W. (1970). *An Introduction to Probability Theory and its Applications I*, Wiley, New-York, 3rd Edition.

[8] Feller, W. (1971). *An Introduction to Probability Theory and its Applications II*, Wiley, New-York, 2nd Edition.

[9] Foss, S. and Zachary, S. (2003). The maximum on a random time interval of a random walk with long-tailed increments and negative drift, *Ann. Appl. Prob.*, **13,** 37–53.

[10] Gerber H.U. (1979). *An Introduction to Mathematical Risk Theory*, S.S. Huebner Foundation Monographs, University of Pennsilvania.

[11] Gut, A. (1974). On the moments and limit distributions of some first passages times, *Ann. Prob.*, **2,** 277–306.

[12] Gut, A. and Janson, S. (1986). Converse results for existence of moments and uniform integrability for stopped random walks, *Ann. Prob.*, **14,** 1296–1317.

[13] Klüppelberg, C. (1988). Subexponential distributions and integrated tails, *J. Appl. Prob.*, **35,** 325–347.

[14] Korshunov, D.A. (1997). On distribution tail of the maximum of a random walk, *Stoch. Proc. Appl.*, **72,** 97–103.

[15] Veraverbeke, N. (1977). Asymptotic behavior of Wiener-Hopf factors of a random walk, *Stoch. Proc. Appl.*, **5,** 27–37.