

Asymptotic probability density of nonlinear phase noise

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The asymptotic probability density of nonlinear phase noise, often called the Gordon–Mollenauer effect, is derived analytically when the number of fiber spans is large. Nonlinear phase noise is the summation of infinitely many independently distributed noncentral χ^2 random variables with two degrees of freedom. The mean and the standard deviation of those random variables are both proportional to the square of the reciprocal of all odd natural numbers. Nonlinear phase noise can also be accurately modeled as the summation of a noncentral χ^2 random variable with two degrees of freedom and a Gaussian random variable. © 2003 Optical Society of America

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When optical amplifiers are used to compensate for fiber loss, the interaction of amplifier noise and the fiber Kerr effect causes phase noise, often called the Gordon–Mollenauer effect or nonlinear phase noise.¹ Nonlinear phase noise degrades both phase-shifted keying and differential phase-shifted keying signals,^{2–6} which have received renewed attention recently.^{7–9} Usually the performance of the system is estimated based on the variance of the nonlinear phase noise.^{1,4,5} However, the nonlinear phase noise is not Gaussian noise,^{3,6} and the variance is not sufficient to characterize the system. Knowledge of the probability density function (p.d.f.) is required for better understanding of the system and for evaluating the system's performance. In this Letter an analytical expression is provided of the asymptotic p.d.f. for nonlinear phase noise when the amplifier noise is modeled as a distributed process for a large number of fiber spans. First the characteristic functions are derived analytically as a simple expression, and the p.d.f. is the inverse Fourier transform of the corresponding characteristic function. The asymptotic p.d.f. can be accurately applied to systems that have more than 32 spans.

For an N -span fiber system, the overall nonlinear phase noise is¹

$$\phi_{\text{NL}} = \gamma L_{\text{eff}}(|A + n_1|^2 + |A + n_1 + n_2|^2 + \dots + |A + n_1 + \dots + n_N|^2), \quad (1)$$

where A is a real number that represents the amplitude of the transmitted signal; n_k , $k = 1, \dots, N$, are independent identically distributed complex zero-mean circular Gaussian random variables as the optical amplifier noise introduced into the system at the k th fiber span; γL_{eff} is the product of the fiber's nonlinear coefficient and the effective fiber length per span.

With large numbers of fiber spans the summation of Eq. (1) can be replaced by integration as

$$\phi_{\text{NL}} = \kappa \int_0^L |A + S(z)|^2 dz, \quad (2)$$

where $S(z)$ is a zero-mean complex-valued Wiener process or Brownian motion of $E[S(z_1)S^*(z_2)] = \sigma_s^2 \min(z_1, z_2)$ and $\kappa = N\gamma L_{\text{eff}}/L$ is the average

nonlinear coefficient per unit length. The variance of $\sigma_s^2 = N\sigma_{\text{ASE}}^2/L$ is the noise variance per unit length, where $E(|n_k|^2) = \sigma_{\text{ASE}}^2$, $k = 1, \dots, N$, is the noise variance per amplifier.

The p.d.f. is derived for the following normalized nonlinear phase noise:

$$\phi = \int_0^1 |\rho + b(t)|^2 dt, \quad (3)$$

where $b(t)$ is a complex Wiener process with an autocorrelation function

$$R_b(t, s) = E[b(s)b^*(t)] = \min(t, s). \quad (4)$$

Comparing the integrations of Eqs. (2) and (3), we scale the normalized phase noise of Eq. (3) by $\phi = L\sigma_s^2\phi_{\text{NL}}/\kappa$, $t = z/L$ is the normalized distance, $b(t) = S(tL)/\sigma_s/\sqrt{L}$ is the normalized amplifier noise, and $\rho = A/\sigma_s/\sqrt{L}$ is the normalized amplitude. The signal-to-noise ratio (SNR) is $\rho^2 = A^2/(L\sigma_s^2) = A^2/(N\sigma_{\text{ASE}}^2)$.

The classic paper by Cameron and Martin¹⁰ gave the characteristic function of the integration of the square of the Wiener process [Eq. (3)]. A brief derivation is provided here to simplify the model. The Wiener process of $b(t)$ can be expanded by use of the standard Karhunen–Loève expansion (Ref. 11, section 10-6)

$$b(t) = \sum_{k=1}^{\infty} \sigma_k x_k \psi_k(t), \quad (5)$$

where x_k are independent ideally distributed complex circular Gaussian random variables with zero mean and unity variance. The eigenvalues and eigenfunctions of σ_k^2 , $\psi_k(t)$, $0 \leq t \leq 1$ are (Ref. 11, p. 305)

$$\sigma_k = \frac{2}{(2k-1)\pi}, \quad \psi_k(t) = \sqrt{2} \sin\left[\frac{(2k-1)\pi}{2} t\right]. \quad (6)$$

Previous studies¹² produced the equivalent of the Karhunen–Loève transform of a finite number of random variables of Eq. (1) based on numerical calculation. Whereas the eigenvalues of the covariance matrix correspond approximately to σ_k^2 of Eqs. (6),

the eigenvectors always require numerical calculations.¹² The assumption of a distributed process of Eq. (2) can be used to derive both eigenvalues and eigenfunctions of Eqs. (6) analytically.

Using Eq. (5) with Eqs. (6) yields the normalized phase of Eq. (3) as

$$\phi = \rho^2 + 2\sqrt{2}\rho \sum_{k=1}^{\infty} \sigma_k^2 \Re(x_k) + \sum_{k=1}^{\infty} \sigma_k^2 |x_k|^2, \quad (7)$$

where $\Re(\cdot)$ denotes the real part of a complex number. Because $\sum_{k=1}^{\infty} \sigma_k^2 = 1/2$ (Ref. 13, section 0.234), we get

$$\phi = \sum_{k=1}^{\infty} \sigma_k^2 |\sqrt{2}\rho + x_k|^2. \quad (8)$$

The normalized nonlinear phase noise is the summation of infinitely many independently distributed noncentral χ^2 random variables with two degrees of freedom with noncentrality parameters of $2\sigma_k^2\rho^2$ and a variance parameter $\sigma_k^2/2$. The mean and the standard deviation of the random variables are both proportional to the square of the reciprocal of all odd natural numbers.

The characteristic function of $|\sqrt{2}\rho + x_k|^2$ is (Ref. 14, p. 44)

$$\Psi_{|\sqrt{2}\rho + x_k|^2}(j\nu) = \frac{1}{1 - j\nu} \exp\left(\frac{2j\nu\rho^2}{1 - j\nu}\right). \quad (9)$$

The characteristic function of the normalized phase ϕ of Eq. (3) is

$$\Psi_{\phi}(j\nu) = \prod_{k=1}^{\infty} \frac{1}{1 - j\nu\sigma_k^2} \exp\left(\frac{2j\nu\rho^2\sigma_k^2}{1 - j\nu\sigma_k^2}\right). \quad (10)$$

Using the expressions of Ref. 13, sections 1.431 and 1.421 enables the characteristic function of Eq. (10) to be simplified to^{3,10}

$$\Psi_{\phi}(j\nu) = \sec(\sqrt{j\nu}) \exp\left[\rho^2 \sqrt{j\nu} \tan(\sqrt{j\nu})\right]. \quad (11)$$

In Eq. (1), nonlinear phase noise is induced by self-phase modulation of the amplifier noise within a bandwidth matched to the signal. If the amplifier noise has a bandwidth that is m times larger than the signal bandwidth, the characteristic function of Eq. (11) becomes $\sec^m(\sqrt{j\nu}) \exp[\rho^2 \sqrt{j\nu} \tan(\sqrt{j\nu})]$.

The first eigenvalue of Eqs. (6) is much larger than other eigenvalues. The normalized phase of Eq. (7) is dominated by the noncentral χ^2 random variable that corresponds to the first eigenvalue because

$$\frac{\sigma_1^2}{\sigma_2^2 + \sigma_3^2 + \dots} = \frac{(2/\pi)^2}{1/2 - (2/\pi)^2} = 4.27, \quad (12)$$

$$\frac{\sigma_1^4}{\sigma_2^4 + \sigma_3^4 + \dots} = \frac{(2/\pi)^4}{1/6 - (2/\pi)^4} = 68.12. \quad (13)$$

The relationship $\sum_{k=1}^{\infty} \sigma_k^4 = 1/6$ is based on Ref. 12, section 0.234.

Besides the noncentral χ^2 random variable that corresponds to the largest eigenvalue of σ_1 , the other χ^2 random variables of $|\sqrt{2}\rho + x_k|^2$, $k > 1$, have more-or-less than same variance. From the central-limit theorem, the summation of many random variables with more-or-less the same variance

approaches a Gaussian random variable. The characteristic function of Eq. (10) can be accurately approximated by

$$\begin{aligned} \Psi_{\phi}(j\nu) \approx & \frac{1}{1 - 4j\nu/\pi^2} \exp\left(\frac{8j\nu\rho^2/\pi^2}{1 - 4j\nu/\pi^2}\right) \\ & \times \exp\left[j\nu(2\rho^2 + 1)\left(\frac{1}{2} - \frac{4}{\pi^2}\right)\right. \\ & \left. - \frac{1}{2}\nu^2(4\rho^2 + 1)\left(\frac{1}{6} - \frac{16}{\pi^4}\right)\right] \end{aligned} \quad (14)$$

as a summation of a noncentral χ^2 random variable with two degrees of freedom and a Gaussian random variable. Whereas the characteristic function of Eq. (11) is a simpler expression than approximation of relation (14) and can be derived easily,^{3,10} the physical meaning of relation (14) is more obvious.

One can calculate the p.d.f. of the normalized phase noise of Eq. (3) by taking the inverse Fourier transform of either the exact [Eq. (11)] or the approximate [relation (14)] characteristic functions. Figure 1 shows the p.d.f. of the normalized nonlinear phase noise for three SNRs ρ^2 ($=11, 18, 25$), which correspond to error probabilities of approximately 10^{-6} , 10^{-9} , and 10^{-12} , respectively, when amplifier noise is the only impairment. Figure 1 shows the p.d.f. that results when the exact [Eq. (11)] or the approximate [relation (14)] characteristic function is used and the Gaussian approximation with a mean and a variance of $m_{\phi} = \rho^2 + 1/2$ and $\sigma_{\phi}^2 = (4\rho^2 + 1)/6$, respectively. The exact and the approximate p.d.f.s overlap and cannot be distinguished from each other.

Figure 2 shows the cumulative tail probabilities as a function of the Q factor. The Q factor is defined as $Q = (\phi - m_{\phi})/\sigma_{\phi}$ and gives an error probability or tail probability of $1/2 \operatorname{erfc}(Q/\sqrt{2})$ for a Gaussian distribution, where $\operatorname{erfc}(\cdot)$ is the complementary error function. Figure 2 is plotted for $\rho^2 = 18$. From Fig. 2, the p.d.f.s calculated from the exact [Eq. (11)] or the approximate [relation (14)] characteristic function show no difference. The Gaussian approximation underestimates the cumulative tail probability for

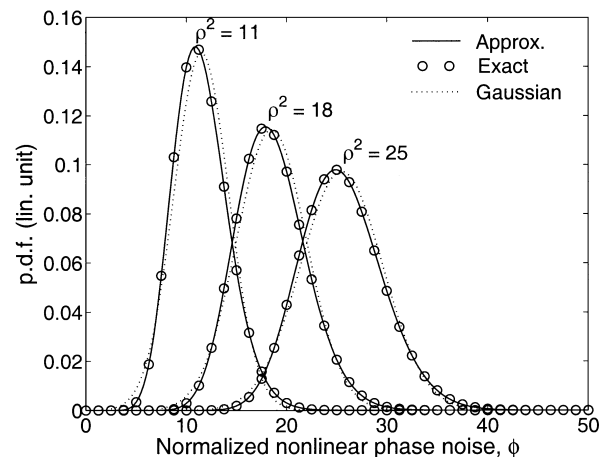


Fig. 1. p.d.f. of normalized nonlinear phase noise ϕ for SNRs ρ^2 of 11, 18, 25: lin., linear.

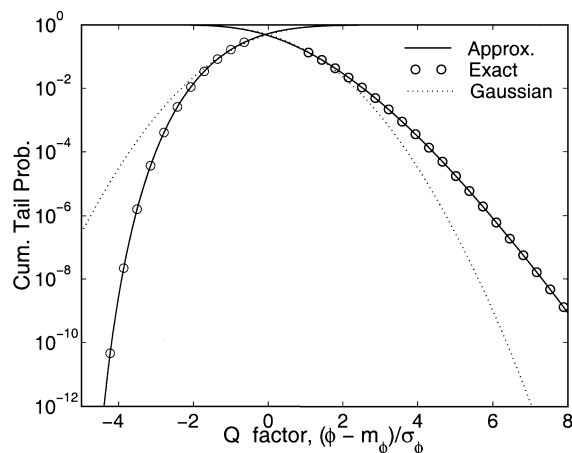


Fig. 2. Cumulative tail probabilities as functions of the Q factor.

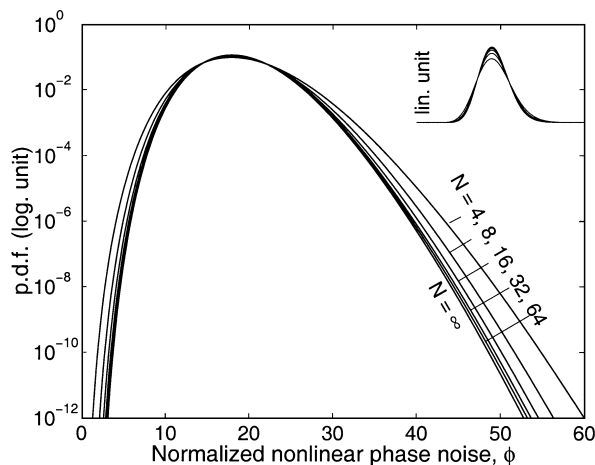


Fig. 3. Asymptotic p.d.f. of ϕ compared with the p.d.f.s of $N = 4, 8, 16, 32, 64$ fiber spans. The p.d.f. is shown in the inset on a linear scale.

$Q > 1$ but overestimates the cumulative tail probability for $Q < -1$.

The p.d.f. for a finite number of fiber spans was derived based on the orthogonalization of Eq. (1) by N independently distributed random variables.¹² Figure 3 shows a comparison of the p.d.f.s for $N = 4, 8, 16, 32, 64$ of fiber spans¹² with the distributed case of Eq. (11). Using the SNR $\rho^2 = 18$, I have plotted Fig. 3 on a logarithmic scale to show the differences in the tails. Figure 3 also provides an inset on a linear scale of the same p.d.f.s to show the difference near the mean. The asymptotic p.d.f. of Eq. (11) with distributed noise has the smallest spread in the tail of the p.d.f.s with N discrete noise sources. The asymptotic p.d.f. is highly accurate for $N \geq 32$ fiber spans.

The analysis here assumes dispersionless fiber. With fiber dispersion, if the nonlinear phase noise is confined to that induced by the amplifier noise that has a bandwidth matched to the signal, the analysis here should be a good approximation. Because they

have the same wavelength, both signal noise and amplifier noise propagate at the same speed.

The overall received phase noise of the signal is $\phi_r = \langle \Phi_{NL} \rangle \phi / (\rho^2 + 1/2) + \Theta_n$, where $\langle \Phi_{NL} \rangle$ is the mean nonlinear phase shift and Θ_n is the linear phase noise. Because of the interdependence of nonlinear and linear phase noise, it is difficult to derive the p.d.f. of the received phase, ϕ_r .

In summary, in this Letter the asymptotic p.d.f. of nonlinear phase noise when the number of fiber spans is large has been derived. A Gaussian approximation based solely on the variance cannot be used to predict the performance of the system accurately. The nonlinear phase noise can be modeled accurately as the summation of a noncentral χ^2 random variable with two degrees of freedom and a Gaussian random variable.

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