

ASYMPTOTIC PROPERTIES OF BROWNIAN MOTION DELAYED BY INVERSE SUBORDINATORS

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ABSTRACT. We study the asymptotic behaviour of the time-changed stochastic process ${}^fX(t) = B({}^fS(t))$, where B is a standard one-dimensional Brownian motion and fS is the (generalized) inverse of a subordinator, i.e. the first-passage time process corresponding to an increasing Lévy process with Laplace exponent f . This type of processes plays an important role in statistical physics in the modeling of anomalous subdiffusive dynamics. The main result of the paper is the proof of the mixing property for the sequence of stationary increments of a subdiffusion process. We also investigate various martingale properties, derive a generalized Feynman-Kac formula, the laws of large numbers and of the iterated logarithm for fX .

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. A subordinator $T^f(t)$, $t \geq 0$, is an increasing Lévy process, i.e. a stochastic process with stationary and independent increments whose sample paths are right-continuous with finite left limits. Being a Markov process, the law of T^f is uniquely characterized by the one-step transition functions. By definition, these are one-sided and infinitely divisible, and the Laplace transform of $T^f(t)$ can be written as (cf. [53])

$$\mathbb{E} \left(e^{-uT^f(t)} \right) = e^{-tf(u)},$$

where the characteristic (Laplace) exponent $f(u)$ is a Bernstein function; it is well known (cf. [54]) that all characteristic exponents are of the form

$$f(u) = \lambda u + \int_{(0, \infty)} (1 - e^{-ux}) \nu(dx),$$

where $\lambda \geq 0$ is the drift parameter and ν is a Lévy measure, i.e. a measure supported in $(0, \infty)$ such that $\int_{(0, \infty)} \min\{1, x\} \nu(dx) < \infty$. For simplicity, we will assume that $\lambda = 0$. In order to exclude the case of compound Poisson processes, we will further assume that $\nu(0, \infty) = \infty$; in particular, the sample paths $t \mapsto T_t^f$ are a.s. *strictly* increasing.

The first-passage time process of the subordinator T^f is the (generalized, right-continuous) inverse

$$(1) \quad {}^fS(t) = \inf\{\tau > 0 : T^f(\tau) > t\}, \quad t \geq 0.$$

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We will call ${}^fS(t)$ an *inverse subordinator*. The assumption $\nu(0, \infty) = \infty$ guarantees that the sample paths of T^f are a.s. strictly increasing, i.e. almost all paths $t \mapsto {}^fS(t)$ are continuous. Since the closure of the range of T^f has zero Lebesgue measure, the trajectories of fS are singular with respect to Lebesgue measure. Note also that T^f is a transient Lévy process and its potential measure satisfies $U([0, x]) = \mathbb{E}({}^fS(x))$. Moreover $U([0, x])/x$ is bounded as $x \rightarrow \infty$; see [5, p. 41].

In what follows we denote by $\hat{h}(u)$ the Laplace transform of a function $h(t)$, i.e.

$$\hat{h}(u) = \int_0^\infty e^{-ut} h(t) dt.$$

It is straightforward to verify that

$$(2) \quad \hat{s}(x, u) = \frac{f(u)}{u} e^{-xf(u)},$$

where $s(x, t)$ is the probability density function of ${}^fS(t)$ (existence of $s(y, t)$ under the assumption $\nu(0, \infty) = \infty$ was proved in [41] Theorem 3.1). In particular, we get for any independent, exponentially distributed random time $\tau \sim u e^{-ut} dt$

$$(3) \quad \begin{aligned} \mathbb{E} e^{\xi \cdot {}^fS(\tau)} &= \int_0^\infty \int_0^\infty e^{\xi x} s(x, t) u e^{-ut} dt dx \\ &= \int_0^\infty e^{\xi x} f(u) e^{-xf(u)} dx = \frac{f(u)}{f(u) - \xi}. \end{aligned}$$

Inverse subordinators have found many applications in probability theory. For their relationship with local times of some Markov processes, see [5]. Similarities between inverse subordinators and renewal processes were investigated in [7, 17, 27]. Applications of inverse subordinators in finance and physics are discussed in [57] and [22, 35, 56], respectively.

In this paper we study some sample path properties of the process

$$(4) \quad {}^fX(t) = B({}^fS(t)), \quad t \geq 0.$$

Here, B is a standard one-dimensional Brownian motion (Wiener process) and fS is an inverse subordinator; we assume that fS and B are independent. For every jump of the subordinator T^f there is a corresponding flat period of its inverse fS . These flat periods represent *trapping events* in which the test particle gets immobilized in a trap. Trapping slows down the overall dynamics of the diffusion process B , therefore ${}^fX(t) = B({}^fS(t))$ is called a *subdiffusion process*.

In Physics, subdiffusive dynamics is frequently described by fractional diffusion equations [44]. Using the relation $\mathbb{P}({}^fS(t) \leq s) = \mathbb{P}(T^f(s) \geq t)$ and the Fourier-Laplace transform in x and t , respectively, one sees that the probability density function $p(x, t)$ of the process ${}^fX(t)$ satisfies the following generalized diffusion equation

$$(5) \quad \frac{\partial p(x, t)}{\partial t} = \frac{1}{2} \Phi_t \frac{\partial^2}{\partial x^2} p(x, t)$$

(cf. [15, 29, 55]); this is also a special case of Theorem 2 below.

Here, Φ_t is an integro-differential operator given by

$$(6) \quad \Phi_t g(t) = \frac{d}{dt} \int_0^t M(t-s) g(s) ds$$

for sufficiently smooth functions g . The memory kernel $M(t)$ is defined via its Laplace transform

$$(7) \quad \hat{M}(u) = \int_0^\infty e^{-ut} M(t) dt = \frac{1}{f(u)}.$$

One can also write (5) with an operator Φ_t^{-1} on the left, i.e. a truly “fractional” time evolution. This operator is also a convolution operator, where the convolution kernel has the Laplace transform $f(u)/u$.

The most important case in applications is the case of α -stable waiting times, which corresponds to $f(u) = u^\alpha$ with $\alpha \in (0, 1)$ being the stability parameter. In this case, the operator Φ_t is the Riemann-Liouville fractional derivative ${}_0D_t^{1-\alpha}$ (cf. [52]),

$$\Phi_t g(t) = {}_0D_t^{1-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-y)^{\alpha-1} g(y) dy.$$

Then, (5) becomes the fractional diffusion equation [2, 35, 36, 43, 50]. The case of space-time fractional derivative operators can be found in [1]. Subdiffusion phenomena with α -stable waiting times have been empirically confirmed in a number of settings: Charge carrier transport in amorphous semiconductors, nuclear magnetic resonance, diffusion in percolative and porous systems, transport on fractal geometries, dynamics of a bead in a polymeric network, and protein conformational dynamics; see [44]. The corresponding process fX appears in a natural way as a scaling limit of uncoupled continuous-time random walk with heavy-tailed waiting times [3, 40]. Path properties of fX with $f(u) = u^\alpha$ have been investigated in [31, 39, 46].

Another important example is the tempered-stable case [11, 48], which corresponds to $f(u) = (u + \lambda)^\alpha - \lambda^\alpha$ with $\lambda > 0$ and $0 < \alpha < 1$. This type of anomalous dynamics has recently been investigated in [15, 37, 42]. Tempered stable distributions are particularly attractive in the modeling of transition from the initial subdiffusive character of motion to the standard diffusion for long times. In physics, applications of tempered stable distributions in the context of astrophysics and relaxation can be found in [56]. Modeling of the lipid granules dynamics with the use of tempered subdiffusion has recently been proposed in [20]. Note that the subordinate process $B(T^f(t))$, where $T^f(t)$ is the tempered stable subordinator, is the relativistic stable process with Fourier exponent equal to $(\xi^2 + m^{2/\beta})^{\beta/2} - m$ [25]. Here $\lambda = m^{2/\beta}$ and $\alpha = \beta/2$.

Another interesting example is the distributed order fractional diffusion equations [12, 24, 38], which correspond to the case $f(u) = \int_0^\infty (1 - e^{-ux}) \nu(dx)$ with $\nu(t, \infty) = \int_0^1 t^{-\beta} \mu(d\beta)$. Here, $\beta \in (0, 1)$ and μ is some distribution supported in $[0, 1]$. This type of Laplace exponent leads to the ultra-slow dynamics displayed by fX . The corresponding fractional Cauchy problem has recently been analyzed in [45].

Our paper is organized as follows: In the next section, we will investigate various properties of the subdiffusion process fX . In particular, we will describe its martingale properties, derive a generalized Feynman-Kac formula, and the laws of large numbers and of the iterated logarithm for fX . Our results extend those for the α -stable case in [31, 46]. In Section 3 we will construct a stationary sequence of increments of the appropriately modified process fX . We will show that this

sequence is ergodic and mixing. This is the main result of the paper; it can be applied to verify ergodicity and mixing of all subdiffusive complex systems modeled by the generalized diffusion equation (5).

2. ASYMPTOTIC BEHAVIOUR OF TRAJECTORIES

We begin with a result which establishes martingale properties of the subdiffusion process fX .

Theorem 1. (i) Both ${}^fX(t)$ and ${}^fY(t) := \exp\left\{{}^fX(t) - \frac{1}{2}{}^fS(t)\right\}$, $t \geq 0$, are martingales with respect to the probability measure \mathbb{P} .

(ii) Let $t \in [0, T]$. Then for each $\epsilon \geq 0$ the process ${}^fZ(t) = \exp\{{}^fX(t)\}$ is a martingale with respect to the probability measure \mathbb{P}_ϵ defined as

$$\mathbb{P}_\epsilon(A) = c_\epsilon \cdot \int_A \exp\left\{-\frac{1}{2}{}^fX(T) - \left(\epsilon + \frac{1}{8}\right){}^fS(T)\right\} d\mathbb{P}.$$

Here $A \in \mathcal{F}$ and $c_\epsilon^{-1} = \int_\Omega \exp\left\{-\frac{1}{2}{}^fX(T) - \left(\epsilon + \frac{1}{8}\right){}^fS(T)\right\} d\mathbb{P}$ is the normalizing constant.

Proof. (i) The proof of this part is similar to the proof [31, Theorem 2.1]. Note first that the quadratic variation of ${}^fX(t)$ satisfies $\langle {}^fX, {}^fX \rangle_t = \langle B, B \rangle_t = {}^fS(t)$. Set

$$(8) \quad \mathcal{F}_t = \bigcap_{u>t} \sigma(\{B(y) : 0 \leq y \leq u\}, \{{}^fS(t) : t \geq 0\}).$$

By definition, $\{\mathcal{F}_t\}$ is right-continuous, $(B(t), \mathcal{F}_t)_{t \geq 0}$ is a martingale, and for every fixed $t_0 > 0$ the random variable ${}^fS(t_0)$ is a stopping time with respect to $\{\mathcal{F}_t\}$. Thus $\{\mathcal{G}_t\}$ where $\mathcal{G}_t = \mathcal{F}_{{}^fS(t)}$ is a well-defined filtration. Let us introduce the sequence of $\{\mathcal{F}_t\}$ -stopping times

$$T_n = \inf\{u > 0 : |B(u)| = n\}.$$

By Doob's optional sampling theorem we have for $s < t$

$$\mathbb{E}(B(T_n \wedge {}^fS(t)) \mid \mathcal{G}_s) = B(T_n \wedge {}^fS(s)).$$

The right-hand side of the above equation converges to $B({}^fS(s))$, whereas the left-hand side converges to $\mathbb{E}(B({}^fS(t)) \mid \mathcal{G}_s)$ as $n \rightarrow \infty$. Therefore, fX is a martingale.

From [47, Proposition IV.3.4] we get that fY is a local martingale. To prove that it is also a martingale, we have to verify the integrability of the random variable $\sup_{0 \leq u \leq t} {}^fY(u)$. Using Doob's maximal inequality and the independence of B and fS , we get

$$\mathbb{E} \left[\left(\sup_{0 \leq u \leq t} \exp\{B({}^fS(u))\} \right)^2 \right] \leq 4\mathbb{E}[\exp\{2B({}^fS(t))\}] = 4\mathbb{E}(\exp\{2 \cdot {}^fS(t)\}).$$

Assume that the parameter u of τ in (3) is so large that $f(u) > 2$. Then we see that the expression on the right-hand side is finite. Thus, $\mathbb{E}[\sup_{0 \leq u \leq t} {}^fY(u)] < \infty$, and fY is a martingale.

(ii) Put

$$V(t) = \exp\left\{-\frac{1}{2}B(t) - \frac{1}{8}t\right\} \quad \text{and} \quad W(t) = \exp\{B(t)\}.$$

Then

$$V(t)W(t) = \exp\left\{\frac{1}{2}B(t) - \frac{1}{8}t\right\}.$$

Therefore, the product $V \cdot W$ is an $(\mathcal{F}_t, \mathbb{P})$ -martingale, and so is the stopped process $V(\cdot \wedge {}^fS(T))W(\cdot \wedge {}^fS(T))$. Since the bounded random variable $e^{-\epsilon {}^fS(T)}$ is \mathcal{F}_0 -measurable, it follows that $e^{-\epsilon {}^fS(T)}V(\cdot \wedge {}^fS(T))W(\cdot \wedge {}^fS(T))$ is an $(\mathcal{F}_t, \mathbb{P})$ -martingale. This in turn implies that $W(\cdot \wedge {}^fS(T))$ is an $(\mathcal{F}_t, \mathbb{P}_\epsilon)$ -martingale.

Denote by $\mathbb{E}^{\mathbb{P}_\epsilon}$ the expectation with respect to the measure \mathbb{P}_ϵ . Then, one shows similarly to (i) that

$$\mathbb{E}^{\mathbb{P}_\epsilon} \left(\sup_{t \geq 0} W(t \wedge {}^fS(T)) \right) = \mathbb{E}^{\mathbb{P}_\epsilon} \left(\sup_{t \leq {}^fS(T)} W(t) \right) < \infty.$$

Thus, $W(\cdot \wedge {}^fS(T))$ is a uniformly integrable $(\mathcal{F}_t, \mathbb{P}_\epsilon)$ -martingale. Consequently, there exists a random variable H such that $W(t \wedge {}^fS(T)) = \mathbb{E}^{\mathbb{P}_\epsilon}(H | \mathcal{F}_t)$. Moreover

$${}^fZ(t) = W({}^fS(t) \wedge {}^fS(T)) = \mathbb{E}^{\mathbb{P}_\epsilon}(H | \mathcal{F}_{{}^fS(t)}),$$

and this implies that fZ is an $(\mathcal{F}_{{}^fS(t)}, \mathbb{P}_\epsilon)$ -martingale. \square

The next result is a generalization of the Feynman-Kac formula. Consider the standard diffusion process V given by the following Itô stochastic differential equation:

$$\begin{aligned} dV(t) &= \mu(V(t)) dt + \sigma(V(t)) dB(t), \\ V(0) &= x. \end{aligned}$$

In order to ensure the existence and uniqueness of the solution we assume the usual (local) Lipschitz and (global) growth conditions on μ and σ . Then the infinitesimal generator of $V(t)$ is

$$\mathcal{A}\phi(x) = \mu(x) \frac{\partial}{\partial x} \phi(x) + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} \phi(x), \quad \phi \in C_c^2(\mathbb{R}).$$

Theorem 2. *Let g and h be continuous functions, such that g is bounded and $h \geq 0$. Then the function*

$$(9) \quad v(x, t) = \mathbb{E}^x \left(\exp \left\{ - \int_0^{{}^fS(t)} h(V(u)) du \right\} g(V({}^fS(t))) \right)$$

satisfies the generalized Feynman-Kac equation

$$(10) \quad \frac{\partial v(x, t)}{\partial t} = \Phi_t [\mathcal{A}v(x, t) - h(x)v(x, t)]$$

with the initial condition $v(x, 0) = g(x)$ and the operator Φ_t from (6).

Proof. Denote by $w(x, t)$ the function

$$w(x, t) = \mathbb{E}^x \left(\exp \left\{ - \int_0^t h(V(u)) du \right\} g(V(t)) \right)$$

and recall that $s(y, t)$ is the probability density of ${}^fS(t)$. With the above assumptions on g and h , $w(x, t)$ is in $C^{2,1}(\mathbb{R} \times \mathbb{R}_+)$.

Using the classical Feynman-Kac formula, we get that $w(x, t)$ satisfies in the Laplace space the equation

$$(11) \quad u\hat{w}(x, u) - w(x, 0) = \mathcal{A}\hat{w}(x, u) - h(x)\hat{w}(x, u).$$

Since fS and B are independent, we obtain

$$v(x, t) = \int_{\mathbb{R}} w(x, y) s(y, t) dy,$$

which by (2) implies that

$$(12) \quad \hat{v}(x, u) = \frac{f(u)}{u} \hat{w}(x, f(u)).$$

Replacing u by $f(u)$ in (11) we get

$$f(u) \hat{w}(x, f(u)) - w(x, 0) = \mathcal{A} \hat{w}(x, f(u)) - h(x) \hat{w}(x, f(u)).$$

Consequently, applying (12) and the fact that $w(x, 0) = v(x, 0)$, we conclude that $\hat{v}(x, u)$ satisfies

$$u \hat{v}(x, u) - v(x, 0) = \frac{u}{f(u)} [\mathcal{A} \hat{v}(x, u) - h(x) \hat{v}(x, u)].$$

Inverting the Laplace transform, we get that $v(x, t)$ satisfies (10). \square

We will now study the asymptotic behaviour of the trajectories of fX .

Theorem 3 (Law of large numbers). *The trajectories of ${}^fX(t)$ satisfy*

$$\lim_{t \rightarrow \infty} \frac{{}^fX(t)}{t} = 0 \quad a.s.$$

Proof. Fix $\epsilon > 0$ and define

$$A_n = \left\{ \sup_{2^n \leq t \leq 2^{n+1}} \left| \frac{{}^fX(t)}{t} \right| > \epsilon \right\}.$$

Using Markov's inequality we get that

$$\epsilon^2 \mathbb{P}(A_n) \leq \mathbb{E} \left[\left(\sup_{2^n \leq t \leq 2^{n+1}} \left| \frac{{}^fX(t)}{t} \right| \right)^2 \right] \leq \frac{\mathbb{E} \left[\left(\sup_{2^n \leq t \leq 2^{n+1}} |{}^fX(t)| \right)^2 \right]}{2^{2n}}.$$

Using the fact that ${}^fX(t)$ is a martingale, we obtain from Doob's maximal inequality the bound

$$\mathbb{E} \left[\left(\sup_{2^n \leq t \leq 2^{n+1}} |{}^fX(t)| \right)^2 \right] \leq 4 \mathbb{E} \left(|{}^fX(2^{n+1})|^2 \right) = 4 \mathbb{E} ({}^fS(2^{n+1})).$$

Since $\mathbb{E}({}^fS(x))/x = U([0, x])/x$ is bounded as $x \rightarrow \infty$, we get

$$\mathbb{E} ({}^fS(2^{n+1})) \leq c 2^{n+1}$$

for large n and some positive constant c . Thus

$$\mathbb{P}(A_n) \leq \frac{4c 2^{n+1}}{2^{2n} \epsilon^2}$$

for large enough n , and so

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty.$$

Finally, using the Borel-Cantelli lemma we obtain the desired result. \square

Now, we turn to the law of the iterated logarithm. It is known that for the inverse subordinator ${}^fS(t)$ we have (see [14])

$$\limsup_{t \rightarrow \infty} \frac{{}^fS(t)}{\phi(t)} = 1$$

for a certain function $\phi(t)$. Thus, by the celebrated law of the iterated logarithm for a Brownian motion B , we obtain the following bound for ${}^fX(t) = B({}^fS(t))$:

$$\limsup_{t \rightarrow \infty} \frac{{}^fX(t)}{\sqrt{2\phi(t) \log \log(\phi(t))}} \leq 1.$$

This result is, however, not sharp (cf. [31]): Just observe that the large increments of B do not coincide with those of fS . The next theorem determines precisely the asymptotic behaviour of the trajectories of fX for small and large times.

Theorem 4 (Law of the iterated logarithm). (i) *Assume that for some $\epsilon > 0$ the Laplace exponent of T^f satisfies $f(u) \geq u^\epsilon$ for sufficiently large u . Then, for $\gamma > 1$ there exists a constant $c > 0$ such that*

$$\limsup_{t \searrow 0} \frac{{}^fX(t)}{g(t)} = 1 \quad \text{and} \quad \liminf_{t \searrow 0} \frac{{}^fX(t)}{g(t)} = -1 \quad a.s.$$

(ii) *Assume that for some $\epsilon > 0$ the Laplace exponent of T^f satisfies $f(u) \leq u^\epsilon$ for sufficiently small u . Then, for $\gamma > 1$ there exist a constant $c > 0$ such that*

$$\limsup_{t \nearrow \infty} \frac{{}^fX(t)}{g(t)} = 1 \quad \text{and} \quad \liminf_{t \nearrow \infty} \frac{{}^fX(t)}{g(t)} = -1 \quad a.s.$$

Here, the function $g(t)$ is the inverse of

$$(13) \quad h(t) = \frac{c \log(|\log(t)|)}{\eta(\gamma t^{-1} \log(|\log(t)|))},$$

and $\eta(u) = f^{-1}(u^2/2)$.

Proof. In the proof we will apply the method used in [6] in the context of iterated Brownian motion.

(i) Put

$$B^*(t) = \sup_{s \in [0, t]} B(s)$$

and

$${}^fX^*(t) = \sup_{s \in [0, t]} {}^fX(s) = B^*({}^fS(t)).$$

Let us introduce

$$S_{1/2}(t) = \inf\{\tau : B^*(\tau) > t\}.$$

It is known [53] that $S_{1/2}(\tau)$ is the 1/2-stable subordinator with Laplace transform

$$\mathbb{E}\left(e^{-u S_{1/2}(\tau)}\right) = e^{-\tau \sqrt{2u}}.$$

Additionally, we have

$${}^fX^*(t) = \inf\{\tau : T^f(S_{1/2}(\tau)) > t\}.$$

The Laplace transform of the subordinator $T^f(S_{1/2}(t))$ is given by

$$\mathbb{E}\left(e^{-u T^f(S_{1/2}(t))}\right) = \mathbb{E}\left(e^{-S_{1/2}(t) f(u)}\right) = e^{-t \sqrt{2f(u)}}.$$

Using the assumptions on the Lévy exponent $f(u)$, we can apply [14, Theorem 1] to get

$$\liminf_{t \searrow 0} \frac{T^f(S_{1/2}(t))}{h(t)} = 1,$$

where $h(t)$ is defined in (13).

Since

$$B^*(fS(T^f(S_{1/2}(t)))) = t,$$

we obtain that

$$\mathbb{P}(B^*(fS(c_0 h(t))) \geq t \text{ infinitely often as } t \searrow 0) = \begin{cases} 1, & \text{if } c_0 > 1, \\ 0, & \text{if } c_0 < 1. \end{cases}$$

This in turn implies that

$$\limsup_{t \searrow 0} \frac{B^*(fS(t))}{g(t)} = 1,$$

where $g(t)$ is the inverse of $h(t)$. Clearly, B^* in the numerator can be replaced by B , and we obtain

$$\limsup_{t \searrow 0} \frac{fX(t)}{g(t)} = 1,$$

which proves the first half of part (i).

The result

$$\liminf_{t \searrow 0} \frac{fX(t)}{g(t)} = -1,$$

is a consequence of the symmetry of B .

The proof of part (ii) of the theorem is analogous. \square

Remark. In some particular cases it is possible to find the explicit form of the normalizing function g . For example in the α -stable case $g(t) \sim c_0 t^{\alpha/2} (\log |\log t|)^{(2-\alpha)/2}$ as $t \rightarrow 0^+$ and $t \rightarrow \infty$, where c_0 is an appropriate positive constant; cf. [31, 46].

3. ERGODIC PROPERTIES

Our standard reference for the ergodic theory of dynamical systems is [28]. For the convenience of our readers let us recall some basic facts.

Let (X, \mathcal{A}, ν, S) be a measure-preserving dynamical system: X is the phase space, \mathcal{A} is a σ -algebra on X , ν is a probability measure on X , and $S : X \rightarrow X$ is a measure-preserving transformation. A set $A \in \mathcal{A}$ is invariant if $S^{-1}(A) = A$. We say that (X, \mathcal{A}, ν, S) is *ergodic*, if every invariant set $A \in \mathcal{A}$ is trivial, i.e. if either $\nu(A) = 0$ or $\nu(X \setminus A) = 0$. We say that (X, \mathcal{A}, ν, S) is *mixing*, if for all $A, B \in \mathcal{A}$

$$\lim_{n \rightarrow \infty} \nu(A \cap S^{-n}B) = \nu(A)\nu(B).$$

Clearly, mixing is stronger than ergodicity.

Ergodicity and mixing have their origins in statistical physics. Intuitively, a system is ergodic if the phase space X cannot be divided into two regions such that a phase point starting in one region will always stay in that region. Thus, for ergodic transformations every phase point will visit the whole phase space. Another important property of ergodic systems is that their temporal and ensemble averages coincide, which follows from Birkhoff's ergodic theorem; cf. [28]. Mixing can be viewed as the asymptotic independence of the sets A and B under the transformation S .

In the context of stochastic processes, we use the following setup. Consider a real-valued stationary process $Y(n)$, $n \in \mathbb{N}$. In its canonical representation, $\{Y(n)\}$ can be identified with its law which is a probability measure \mathbb{P} on the space $\mathbb{R}^{\mathbb{N}}$. On this space we use the canonical σ -algebra \mathcal{B} generated by the cylinder sets, and we consider the standard shift transformation $S : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$. Stationarity of $\{Y(n)\}$

implies that the shift S is measure-preserving. Note that $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}, \mathbb{P}, S)$ is a typical object of study in the theory of dynamical systems. Therefore, ergodic properties of stationary stochastic processes can be studied in the framework of the theory of dynamical systems. A detailed analysis of ergodicity and mixing for the classes of infinitely divisible and fractional processes, can be found in [9, 10, 19, 49] and [32], respectively.

Here we will study ergodic properties of the anomalous diffusion process fX . However, one cannot verify ergodicity and mixing of fX in a straightforward manner, since neither fX nor its increments are stationary. Therefore, we will introduce a modification.

Let us assume that the first moment of T^f is finite: $\mu := \mathbb{E}(T^f(1)) < \infty$. Let

$$\tilde{T}^f(t) = T_0 + T^f(t)$$

denote the generalized subordinator whose initial distribution is given by

$$\mathbb{P}(T_0 \leq x) = \frac{1}{\mu} \int_0^x \int_y^\infty \nu(ds) dy = \frac{1}{\mu} \int_0^\infty \min(s, x) \nu(ds),$$

where T_0 is assumed to be independent of T^f . The corresponding inverse subordinator is

$$(14) \quad {}^f\tilde{S}(t) = \inf \{ \tau > 0 : \tilde{T}^f(\tau) > t \}.$$

With this choice of the initial distribution of T_0 , the process ${}^f\tilde{S}$ has stationary increments; cf. [27]. Consequently, the modification of fX defined as

$${}^f\tilde{X}(t) = B({}^f\tilde{S}(t)),$$

has stationary increments as well. We denote this stationary sequence by

$$(15) \quad {}^f\tilde{Y}(n) = {}^f\tilde{X}(n+1) - {}^f\tilde{X}(n), \quad n \in \mathbb{N}.$$

Now we are in position to verify its ergodicity and mixing properties.

Theorem 5. *The stationary sequence ${}^f\tilde{Y}$ defined in (15) is ergodic and mixing.*

Before we embark on the proof of the above theorem, we need two lemmas. The first shows that the increments of fS are asymptotically stationary. More precisely, the increments of fS converge in distribution to the increments of ${}^f\tilde{S}$.

Lemma 6. *Let $M \in \mathbb{N}$, $z_j \in \mathbb{R}_+$, $b_j \in \mathbb{R}_+$, $j = 1, \dots, M$, with $z_i < z_j$ for $i < j$. Then*

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ - \sum_{j=1}^M b_j ({}^fS(z_j + 1 + n) - {}^fS(z_j + n)) \right\} \right) \\ & \xrightarrow{n \rightarrow \infty} \mathbb{E} \left(\exp \left\{ - \sum_{j=1}^M b_j ({}^f\tilde{S}(z_j + 1) - {}^f\tilde{S}(z_j)) \right\} \right). \end{aligned}$$

Proof. Assume that $M = 1$. We will show that

$$(16) \quad \begin{aligned} & \mathbb{E} \left(\exp \left\{ - b_1 ({}^fS(z_1 + 1 + n) - {}^fS(z_1 + n)) \right\} \right) \\ & \xrightarrow{n \rightarrow \infty} \mathbb{E} \left(\exp \left\{ - b_1 ({}^f\tilde{S}(z_1 + 1) - {}^f\tilde{S}(z_1)) \right\} \right). \end{aligned}$$

Observe that ${}^f\tilde{S}(t) \leq {}^fS(t)$ and that, by (3) and the continuity of $t \mapsto {}^fS(t)$, ${}^fS(t)$ has exponential moments for any $t > 0$. Therefore, the moments determine the distribution of ${}^f\tilde{S}(t)$; cf. [13, Chapter VII, Section 3]. Consequently, the distribution of the random variable ${}^f\tilde{S}(z_1 + 1) - {}^f\tilde{S}(z_1)$ is also determined by its moments. This means that it is enough to prove the convergence of moments

$$(17) \quad \mathbb{E} \left[\left({}^fS(z_1 + 1 + n) - {}^fS(z_1 + n) \right)^k \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\left({}^f\tilde{S}(z_1 + 1) - {}^f\tilde{S}(z_1) \right)^k \right]$$

for every $k \in \mathbb{N}$.

Denote by $U(t) = \mathbb{E}({}^fS(t))$ the renewal function of fS and by $U(dx)$ the measure induced by U . We introduce another measure on $[z_1, z_1 + 1]$ defined as

$$U_n[z_1, x] = U(x + n) - U(n), \quad x \in [z_1, z_1 + 1].$$

Applying the renewal theorem we get that

$$U_n[z_1, x] \xrightarrow{n \rightarrow \infty} \frac{x}{\mu} \quad \text{pointwise,}$$

where $\mu = \mathbb{E}(T^f(1))$. This implies that

$$U_n \xrightarrow{n \rightarrow \infty} \frac{\lambda}{\mu} \quad \text{weakly,}$$

where λ is Lebesgue measure on $[z_1, z_1 + 1]$.

Now, using [27, Theorem 1], we obtain

$$\begin{aligned} & \mathbb{E} \left[\left({}^fS(z_1 + 1 + n) - {}^fS(z_1 + n) \right)^k \right] \\ &= k! \int_{z_1+n}^{z_1+1+n} \int_{x_1}^{z_1+1+n} \dots \int_{x_{k-1}}^{z_1+1+n} U(dx_k - x_{k-1}) \dots U(dx_2 - x_1) U(dx_1) \\ &= k! \int_{z_1}^{z_1+1} \int_{y_1}^{z_1+1} \dots \int_{y_{k-1}}^{z_1+1} U(dy_k - y_{k-1}) \dots U(dy_2 - y_1) U_n(dy_1), \end{aligned}$$

where the last equality was obtained by changing the variables $y_i = x_i - n$, $i = 1, \dots, k$. By the weak convergence of U_n to λ/μ and [27, Theorem 1], we get

$$\begin{aligned} & \mathbb{E} \left[\left({}^fS(z_1 + 1 + n) - {}^fS(z_1 + n) \right)^k \right] \\ & \xrightarrow{n \rightarrow \infty} k! \int_{z_1}^{z_1+1} \int_{y_1}^{z_1+1} \dots \int_{y_{k-1}}^{z_1+1} U(dy_k - y_{k-1}) \dots U(dy_2 - y_1) \frac{dy_1}{\mu} \\ &= \mathbb{E} \left[\left({}^f\tilde{S}(z_1 + 1) - {}^f\tilde{S}(z_1) \right)^k \right]. \end{aligned}$$

This ends the proof for $M = 1$.

The proof for arbitrary $M \in \mathbb{N}$ is similar. Note first that the random variable

$$\sum_{j=1}^M b_j ({}^f\tilde{S}(z_j + 1) - {}^f\tilde{S}(z_j))$$

is determined by its moments. So, it is enough to show

$$\mathbb{E} \left[\left(\sum_{j=1}^M b_j (fS(z_j+1+n) - fS(z_j+n)) \right)^k \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\left(\sum_{j=1}^M b_j (f\tilde{S}(z_j+1) - f\tilde{S}(z_j)) \right)^k \right]$$

for any $k \in \mathbb{N}$. As before, using the multinomial formula

$$\left(\sum_{j=1}^M x_j \right)^k = \sum_{\substack{k_1, \dots, k_M \geq 0 \\ k_1 + \dots + k_M = k}} \frac{k!}{k_1! \dots k_M!} x_1^{k_1} \dots x_M^{k_M},$$

one needs to show the convergence

$$(18) \quad \mathbb{E} \left[\prod_{j=1}^M (fS(z_j+1+n) - fS(z_j+n))^{k_j} \right] \xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\prod_{j=1}^M (f\tilde{S}(z_j+1) - f\tilde{S}(z_j))^{k_j} \right]$$

with $\sum k_j = k$. Applying [27, Theorem 1], we obtain

$$\mathbb{E} \left(\prod_{j=1}^M (fS(z_j+1+n) - fS(z_j+n))^{k_j} \right) = \prod_{i=1}^M k_i! \prod_{j=1}^k \int_C U(dx_j - x_{j-1}),$$

where $C = \{(x_0, \dots, x_k) : x_0 = 0, z_j + n < x_{k_0+\dots+k_{j-1}+1} < \dots < x_{k_0+\dots+k_j} \leq z_j + 1 + n, j = 1, \dots, M, k_0 = 0\}$. Finally, changing the variables $y_i = x_i - n$, $i = 1, \dots, k$, and using the weak convergence of U_n to λ/μ , we get (18). \square

We will use the following recursive relations for inverse subordinators from [23]:

$$(19) \quad \mathbb{E} \left(\exp \left\{ \sum_{j=1}^M \theta_j fS(\tau_j) \right\} \right) = \mathbb{E} \left(\exp \left\{ \sum_{j=2}^M \theta_j fS(\tau_j) \right\} \right) \\ + \frac{\theta_1}{\sum_{j=1}^M \theta_j} \int_0^{\tau_1} \mathbb{E} \left(\exp \left\{ \sum_{j=2}^M \theta_j fS(\tau_j - x) \right\} \right) d_x \mathbb{E} \left(\exp \left\{ fS(x) \sum_{j=1}^M \theta_j \right\} \right),$$

$$(20) \quad \mathbb{E} \left(\exp \left\{ \sum_{j=1}^M \theta_j f\tilde{S}(\tau_j) \right\} \right) = \mathbb{E} \left(\exp \left\{ \sum_{j=2}^M \theta_j f\tilde{S}(\tau_j) \right\} \right) \\ + \frac{\theta_1}{\sum_{j=1}^M \theta_j} \int_0^{\tau_1} \mathbb{E} \left(\exp \left\{ \sum_{j=2}^M \theta_j f\tilde{S}(\tau_j - x) \right\} \right) d_x \mathbb{E} \left(\exp \left\{ f\tilde{S}(x) \sum_{j=1}^M \theta_j \right\} \right),$$

where $M \geq 2$, $0 \leq \tau_1 \leq \dots \leq \tau_M$, $\theta_1, \dots, \theta_M \in \mathbb{R}$.

We will also need the following technical result.

Lemma 7. *Let $m, M \in \mathbb{N}$ with $1 \leq m \leq M$, $0 \leq z_1 \leq \dots \leq z_M$, $b_j \in \mathbb{R}$ for $j = 1, \dots, m$ and $b_j \in \mathbb{R}_+$ for $j = m+1, \dots, M$. Then*

$$(21) \quad \mathbb{E} \left(\exp \left\{ \sum_{j=1}^m b_j fS(z_j) - \sum_{j=m+1}^M b_j (fS(z_j+1+n) - fS(z_j+n)) \right\} \right) \\ \xrightarrow{n \rightarrow \infty} \mathbb{E} \left(\exp \left\{ \sum_{j=1}^m b_j fS(z_j) \right\} \right) \mathbb{E} \left(\exp \left\{ - \sum_{j=m+1}^M b_j (f\tilde{S}(z_j+1) - f\tilde{S}(z_j)) \right\} \right).$$

Proof. We will prove formula (21) by induction on the parameter M .

Step I. Let $M = 1$. Then $m = 1$ and the formula is trivially fulfilled.

Step II. Assume that the formula (21) holds for $M - 1$. We will show that it is also true for M . Applying (19) we have

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ \sum_{j=1}^m b_j fS(z_j) - \sum_{j=m+1}^M b_j (fS(z_j + 1 + n) - fS(z_j + n)) \right\} \right) \\ &= \mathbb{E} \left(\exp \left\{ \sum_{j=2}^m b_j fS(z_j) - \sum_{j=m+1}^M b_j (fS(z_j + 1 + n) - fS(z_j + n)) \right\} \right) \\ & \quad + \frac{b_1}{\sum_{j=1}^m b_j} \int_0^{z_1} \mathbb{E} \left(\exp \left\{ \sum_{j=2}^m b_j fS(z_j - x) \right. \right. \\ & \quad \left. \left. - \sum_{j=m+1}^M b_j (fS(z_j + 1 + n - x) - fS(z_j + n - x)) \right\} \right) d_x \mathbb{E} \left(\exp \left\{ fS(x) \sum_{j=1}^m b_j \right\} \right). \end{aligned}$$

Now, if $m > 1$ we apply the induction assumption, the dominated convergence theorem to get that the expression above converges to

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ \sum_{j=2}^m b_j fS(z_j) \right\} \right) \mathbb{E} \left(\exp \left\{ - \sum_{j=m+1}^M b_j (f\tilde{S}(z_j + 1) - f\tilde{S}(z_j)) \right\} \right) \\ & \quad + \mathbb{E} \left(\exp \left\{ - \sum_{j=m+1}^M b_j (f\tilde{S}(z_j + 1) - f\tilde{S}(z_j)) \right\} \right) \\ & \quad \times \frac{b_1}{\sum_{j=1}^m b_j} \int_0^{z_1} \mathbb{E} \left(\exp \left\{ \sum_{j=2}^m b_j fS(z_j - x) \right\} \right) d_x \mathbb{E} \left(\exp \left\{ fS(x) \sum_{j=1}^m b_j \right\} \right) \end{aligned}$$

and this is, by (19), equal to

$$\mathbb{E} \left(\exp \left\{ \sum_{j=1}^m b_j fS(z_j) \right\} \right) \mathbb{E} \left(\exp \left\{ - \sum_{j=m+1}^M b_j (f\tilde{S}(z_j + 1) - f\tilde{S}(z_j)) \right\} \right).$$

For $m = 1$ we apply Lemma 6 and use the same argumentation. This ends the proof. \square

Now we are ready to prove our main result.

Proof of Theorem 5. In order to prove that $f\tilde{Y}(n)$ defined in (15) is mixing, it is sufficient to show the following convergence of characteristic functions (cf. [34]):

$$\begin{aligned} & \mathbb{E} \left(\exp \left(i \sum_{j=1}^m a_j f\tilde{Y}(z_j) + i \sum_{j=m+1}^M a_j f\tilde{Y}(z_j + n) \right) \right) \\ & \xrightarrow{n \rightarrow \infty} \mathbb{E} \left(\exp \left(i \sum_{j=1}^m a_j f\tilde{Y}(z_j) \right) \right) \mathbb{E} \left(\exp \left(i \sum_{k=m+1}^M a_k f\tilde{Y}(z_k) \right) \right), \end{aligned}$$

for $m, M \in \mathbb{N}$, $1 \leq m \leq M$, $a_1, \dots, a_M \in \mathbb{R}$ and $0 \leq z_1 \leq \dots \leq z_M$. By the definition of ${}^f\tilde{Y}(t)$ we get that the above convergence is equivalent to

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ - \sum_{j=1}^m c_j ({}^f\tilde{S}(z_j + 1) - {}^f\tilde{S}(z_j)) - \sum_{j=m+1}^M c_j ({}^f\tilde{S}(z_j + 1 + n) - {}^f\tilde{S}(z_j + n)) \right\} \right) \\ & \xrightarrow{n \rightarrow \infty} \mathbb{E} \left(\exp \left\{ - \sum_{j=1}^m c_j ({}^f\tilde{S}(z_j + 1) - {}^f\tilde{S}(z_j)) \right\} \right) \\ & \quad \times \mathbb{E} \left(\exp \left\{ - \sum_{j=m+1}^M c_j ({}^f\tilde{S}(z_j + 1) - {}^f\tilde{S}(z_j)) \right\} \right), \end{aligned}$$

where $c_j = a_j^2/2$. We will prove the following more general assertion:

$$\begin{aligned} (22) \quad & \mathbb{E} \left(\exp \left\{ \sum_{j=1}^m b_j {}^f\tilde{S}(z_j) - \sum_{j=m+1}^M b_j ({}^f\tilde{S}(z_j + 1 + n) - {}^f\tilde{S}(z_j + n)) \right\} \right) \\ & \xrightarrow{n \rightarrow \infty} \mathbb{E} \left(\exp \left\{ \sum_{j=1}^m b_j {}^f\tilde{S}(z_j) \right\} \right) \mathbb{E} \left(\exp \left\{ - \sum_{j=m+1}^M b_j ({}^f\tilde{S}(z_j + 1) - {}^f\tilde{S}(z_j)) \right\} \right), \end{aligned}$$

where $m, M \in \mathbb{N}$ with $1 \leq m \leq M$, $0 \leq z_1 \leq \dots \leq z_M$, $b_j \in \mathbb{R}$ for $j = 1, \dots, m$ and $b_j \in \mathbb{R}_+$ for $j = m + 1, \dots, M$.

The proof follows the same lines as Lemma 7. To show (22) we will use induction on the parameter M :

Step I. For $M = 1$ also $m = 1$ and the convergence trivially holds.

Step II. Assume that the formula (22) holds for $M - 1$. We will show that it holds also for M . We apply (20) to get

$$\begin{aligned} & \mathbb{E} \left(\exp \left\{ \sum_{j=1}^m b_j {}^f\tilde{S}(z_j) - \sum_{j=m+1}^M b_j ({}^f\tilde{S}(z_j + 1 + n) - {}^f\tilde{S}(z_j + n)) \right\} \right) \\ & = \mathbb{E} \left(\exp \left\{ \sum_{j=2}^m b_j {}^f\tilde{S}(z_j) - \sum_{j=m+1}^M b_j ({}^f\tilde{S}(z_j + 1 + n) - {}^f\tilde{S}(z_j + n)) \right\} \right) \\ & \quad + \frac{b_1}{\sum_{j=1}^m b_j} \int_0^{z_1} \mathbb{E} \left(\exp \left\{ \sum_{j=2}^m b_j {}^fS(z_j - x) \right. \right. \\ & \quad \left. \left. - \sum_{j=m+1}^M b_j ({}^fS(z_j + 1 + n - x) - {}^fS(z_j + n - x)) \right\} \right) d_x \mathbb{E} \left(\exp \left\{ {}^f\tilde{S}(x) \sum_{j=1}^m b_j \right\} \right). \end{aligned}$$

Now, applying the induction assumption for the first summand, Lemma 7 for the expression inside the integral, and the dominated convergence theorem we obtain

that the above formula converges to

$$\begin{aligned} & \mathbb{E}\left(\exp\left\{\sum_{j=2}^m b_j {}^f\tilde{S}(z_j)\right\}\right) \mathbb{E}\left(\exp\left\{-\sum_{j=m+1}^M b_j ({}^f\tilde{S}(z_j+1) - {}^f\tilde{S}(z_j))\right\}\right) \\ & \quad + \mathbb{E}\left(\exp\left\{-\sum_{j=m+1}^M b_j ({}^f\tilde{S}(z_j+1) - {}^f\tilde{S}(z_j))\right\}\right) \\ & \quad \times \frac{b_1}{\sum_{j=1}^m b_j} \int_0^{z_1} \mathbb{E}\left(\exp\left\{\sum_{j=2}^m b_j {}^fS(z_j-x)\right\}\right) d_x \mathbb{E}\left(\exp\left\{{}^f\tilde{S}(x) \sum_{j=1}^m b_j\right\}\right), \end{aligned}$$

which is, by (20), the same as

$$\mathbb{E}\left(\exp\left\{\sum_{j=1}^m b_j {}^f\tilde{S}(z_j)\right\}\right) \mathbb{E}\left(\exp\left\{-\sum_{j=m+1}^M b_j ({}^f\tilde{S}(z_j+1) - {}^f\tilde{S}(z_j))\right\}\right).$$

This shows that ${}^f\tilde{Y}(n)$ is mixing and therefore also ergodic. \square

Remark. We point out that Theorem 5 holds only if $\mathbb{E}(T^f(1)) < \infty$. In particular this result can be applied in the case of tempered stable laws. This class of distributions is especially important in applications [20, 56], since it can model the so-called transient subdiffusion [37, 42]. The characteristic feature of transient subdiffusion processes is the transition from power-law to linear scaling of the variance, which is very different from the behavior of the standard Brownian diffusion.

For $\mathbb{E}(T^f(1)) = \infty$ one cannot construct the stationary-increment modification of fS as in (14). As a consequence, the so-called *weak ergodicity breaking* is observed; see [4].

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