# Asymptotic properties of fragmentation processes


submitted by

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Robert Knobloch

## Summary

Fragmentation processes describe phenomena of random splitting, with possibly infinite activity, according to certain rules that give rise to a close relation of these processes to branching processes and Lévy processes. In this thesis we study some asymptotic properties of fragmentation processes. More specifically, we prove certain strong laws of large numbers for self–similar fragmentations and we deal with the existence and uniqueness of solutions of the one–sided FKPP travelling wave equation for homogenous fragmentation processes. In addition to being concerned with standard fragmentation processes we also consider fragmentation processes with immigration, fragmentations stopped at a stopping line as well as killed fragmentation processes.

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# CONTENTS

# INTRODUCTION

This PhD thesis is devoted to the study of fragmentation processes and focuses mainly on themes regarding their asymptotic behaviour.

Fragmentation processes form a relatively new field of research within the theory of continuous–time stochastic processes and gained more and more popularity in recent years. By and large, the theory of fragmentation processes was developed within the last twenty years and much of this development is owed to Jean Bertoin and his former students in Paris as well as to David J. Aldous and Jim Pitman in Berkeley. The present exposition is concerned solely with the theoretical aspects of fragmentation processes. However, these mathematical objects also have applications with regard to real–world phenomena. The most prominent example of such an application in the literature deals with the fragmentation of blocks of minerals in the mining industry, and the reader interested in those aspects of fragmentations is referred to [BM05] as well as [FKM10].

A useful property of fragmentations is that they satisfy the Markov property. Moreover, the study of fragmentation processes benefits from their close relation to continuous–time branching random walks and general branching processes as well as from their intrinsic connection with the theory of Lévy processes. Fragmentation processes are mathematically challenging as there can be infinite activity in any finite time interval. That is to say, fragmentation processes are pure jump processes whose jump times may be dense in $[0, \infty)$. In this respect they differ decisively from classical branching processes. Even though all our results do in particular include the case of a jump structure where there are only finitely many jumps in any finite time interval, our focus is clearly on the case where the jump times are dense in $[0, \infty)$. The latter case is more interesting and challenging from a mathematical point of view.

This thesis is divided into two parts, preceded by an introductory chapter on fragmentation processes. That chapter aims at giving a fairly general introduction to the subject. In Part I and Part II we consider two problems which are different in nature and make use of different concepts and techniques. Both parts can be read independently of each other, but Chapter 1 is essential for the whole thesis. The first problem is to prove a strong law of large numbers for certain empirical measures associated with self–similar fragmentation processes, possibly with immigration. The motivation for this problem comes from similar strong laws in the literature on branching processes. Our approach requires us to deal with fragmentation processes stopped at a specific stopping line. The second problem deals with the FKPP equation in the context of fragmentation processes. More precisely, our goal is to study the existence and uniqueness of solutions to the one–sided FKPP travelling wave equation in the setting of fragmentation processes. To this end we develop a theory of killed fragmentation processes.

The topics and techniques of this dissertation are mainly related to mathematical stochastics and probability theory. In addition, there are various connections with several fields of mathematical analysis. There is a vast literature providing the necessary background on stochastics and measure theory. In this regard we refer for instance to [Bil95], [Kal01] and [Sch06] as well as [Kle08]. Concerning probabilistic aspects we also profited from [Dur91] and [Bre92]. In the present thesis we make extensive use of the theory of Lévy processes and in particular of subordinators. With regard to these processes our exposition is strongly influenced by [Ber96] and [Kyp06]. For a comprehensive treatise on fragmentation processes we refer to the monograph [Ber06] that covers many aspects in much more detail than our compilation in Chapter 1.

Let us now briefly describe the content of this dissertation.

Chapter 1 introduces the main concepts that are used in the subsequent chapters. That is to say, the first chapter prepares the ground for the more specialised considerations in Part I and Part II. More precisely, Chapter 1 introduces various kinds of self–similar fragmentation processes. Furthermore, the main concepts related to these classes of stochastic processes are developed. Our compilation in Chapter 1 is based on research papers by various authors. In this chapter we introduce three classes of fragmentation processes. We start by defining Lévy processes and related concepts that are of avail for our considerations later on. Subsequently, we introduce mass fragmentation processes. These are fragmentation processes where all the information about the fragments is given by their sizes. Then we introduce partition–valued fragmentation processes. The blocks of those fragmentations are subsets of $\mathbb{N}$. This class of fragmentation processes has the advantage over mass fragmentations that it has an intrinsic genealogical structure. Such a structure is also enjoyed by the third class of fragmentations that we consider, namely interval fragmentation processes. The blocks of interval fragmentations are open intervals in $(0, 1)$. A very useful fact, cf. Theorem 3 (ii) in [Ber01], is that there exists an intrinsic subordinator, that is a nondecreasing Lévy process, associated with the latter two classes of fragmentation processes. This subordinator is the object of Section 1.7. Moreover, self–similar fragmentation processes turn out to be time changed homogenous fragmentation processes, and the aforementioned subordinator is the Lévy process in the Lamperti representation, cf. [Lam72], of these positive self–similar Markov processes. Berestycki [Ber02a] and Bertoin [Ber02b] established one–to–one correspondences between any two of the classes of fragmentations considered here. Hence, we can always choose the class of fragmentations that is particularly useful in a specific situation. Similarly to the theory of branching processes there is an intrinsic additive martingale for fragmentation processes. This martingale is considered in Section 1.8. We conclude the introductory chapter by introducing the so–called tagged fragment as well as the spine decomposition. The latter is a popular tool for dealing with branching– and fragmentation processes.

Part I, which comprises Chapters 2 – 4, is devoted to strong laws of large numbers that are based on fragmentation processes. In particular, we extend in various aspects some results of [Ner81] and [Olo96] as well as [BM05].

Chapter 2 introduces stopping lines, which are extensions of the well known concept of stopping times. The main purpose of this chapter is to consider fragmentation processes stopped at a specific example of a stopping line. This specific stopping line at $t$ consists of the first times at which the blocks of the fragmentation process have mass less than $e^{-t}$. The blocks of the stopped fragmentation do not fragment any further after the stopping line. That is, the blocks of this stopped process at $t$ evolve as in the normal fragmentation until their size jumps to a value less than $e^{-t}$, and after this jump time this block remains unchanged. In particular this means that the stopped process at $t$ consists of blocks with size less than $e^{-t}$. Stopped fragmentation processes are essential for the considerations in Chapter 3 and Chapter 4. Section 2.4 deals with an intrinsic additive martingale in the context of stopped fragmentations. In Section 2.5 we consider a so–called many–to–one identity, which allows us to restrict our considerations to the behaviour of the tagged fragment in order to obtain some information about the behaviour of the whole fragmentation process.

In Chapter 3 we consider an empirical measure $\rho_t$ defined via the stopped fragmentation processes introduced in Chapter 2, and we consider the integral of bounded and measurable functions with respect to this measure. Our main theorem establishes a strong law of large numbers for this integral. In order to state this result more precisely, let $p^*$ be the so–called Malthusian parameter, let $(\lambda_{t,n})_{n\in\mathbb{N},t\in\mathbb{R}_0^+}$ be the stopped fragmentation process and let $f : \mathbb{R} \to \mathbb{R}_0^+$ be a bounded and measurable function. Under certain assumptions we show the almost sure convergence of

$$\int_{[0,1]} f \, \mathrm{d}\rho_t = \sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p^*} f\left(e^t \lambda_{t,n}\right), \tag{1}$$

as $t \to \infty$, to some limit that can be written as the product of a deterministic constant depending of $f$ and the almost sure martingale limit $\Lambda(p^*) := \lim_{t\to\infty} \Lambda_t(p^*)$. As a corollary we obtain the $\mathscr{L}^p$–convergence for some $p > 1$ of the integral in (1). Our method to prove the above result is to show that the conditional expectation of the right–hand side of (1), conditional on the natural filtration at $t$, is asymptotically a good approximation for that random variable itself and also for the proposed limiting random variable. This allows us to first prove the desired result along log–lattice times, that is for the integral in (1) considered at a discrete set of times, for homogenous fragmentations. By approximation arguments we then extend this to convergence along the real numbers. The extension to self–similar fragmentation processes follows from a time–change of the homogenous fragmentation. The relation of our main result to related results in the setting of Crump–Mode–Jagers processes are discussed in the final section of this chapter. Chapter 3 is based on the publication [HKK10] with Simon C. Harris and Andreas E. Kyprianou.

In Chapter 4 we extend the strong law of large numbers obtained in Chapter 3 to the situation of fragmentation processes with immigration. Fragmentation processes with immigration are more difficult to handle than standard fragmentation processes as there may be an infinite amount of immigrating particles. Our main result in this chapter basically boils down to showing that under certain assumptions

$$\sum_{k\in\mathbb{N}} v_k^{1+p_k^*} \sum_{n\in\mathbb{N}} \left[\lambda_{t,n}^{(k)}\right]^{1+p_k^*} f^{(k)}\left(e^t \lambda_{t,n}^{(k)}\right) \tag{2}$$

converges to some limit $\mathbb{P}$–a.s. as $t \to \infty$. Here $(v_j)_{j\in\mathcal{J}}$ is a summable decreasingly ordered sequence of nonnegative real numbers and the $\lambda^{(k)}$ are independent stopped mass fragmentation processes, each $f^{(k)}$ is a bounded and measurable function and $p_k^*$ is the Malthusian parameter associated with the respective fragmentation process. Although the series in (2) is some sort of an average of the integrals in (1), the result does not follow easily from Chapter 3 as in general neither the Dominated Convergence Theorem nor the Monotone Convergence Theorem is applicable to interchange the limit with the series in (2). Our approach relies on a martingale that appears in this setting with immigration.

Part II, consisting of Chapter 5 and Chapter 6, deals with the FKPP travelling wave equation in the setting of fragmentation processes. In order to obtain existence and uniqueness results for one–sided solutions of this integro–differential equation in Chapter 6 we consider killed fragmentation processes in Chapter 5.

Chapter 5 introduces a new class of fragmentation processes, namely those including a particular kind of killing. More precisely, a block is killed at the moment of its creation $t \in \mathbb{R}_0^+$ if the size of this block at time $t$ is less than $e^{-(x+ct)}$, where $c > 0$ and $x \in \mathbb{R}_0^+$. We say that the killed fragmentation process becomes extinct if there is a finite time $\zeta^x$ after which all its blocks are killed. The main results of this chapter are concerned with some additive and multiplicative martingales that appear in the setting of such killed fragmentations. Here we are interested in the extinction probability $\mathbb{P}(\zeta^x < \infty)$ as a function of $x$. Our goal is to derive various useful properties of this function and for this purpose we shall be concerned with the asymptotic behaviour of killed fragmentation processes. To begin with, we show that there exists some $c_{\bar{p}}$ such that for all $c > c_{\bar{p}}$ the map $x \mapsto \mathbb{P}(\zeta^x < \infty)$ is continuous and strictly monotonically decreasing on $\mathbb{R}_0^+$. The next theorem deals with a multiplicative stochastic process $(Z_t^{x,f})_{t\in\mathbb{R}_0^+}$. This result says the following: If $c > c_{\bar{p}}$, then there exists a unique monotone function $f : \mathbb{R}_0^+ \to [0, 1]$, given by $x \mapsto \mathbb{P}(\zeta^x < \infty)$, that satisfies $\lim_{x\to\infty} f(x) = 0$ and for which $Z^{x,f}$ is an $\mathscr{F}$–martingale for any $x \in \mathbb{R}_0^+$. We also establish the martingale property of some additive stochastic process. Moreover, if $c > c_{\bar{p}}$ then almost surely the corresponding martingale limit is strictly positive if and only if the killed fragmentation process becomes extinct. The last main result of Chapter 5 is concerned with the asymptotic speed of the largest fragment in the killed fragmentation. We show that in the setting with killing the asymptotic speed of the largest fragment, conditional on nonextinction, concurs with the one in the non–killed fragmentation process. Our methods of proof in this chapter are based on considering an intrinsic spectrally negative Lévy process that inherits the killing from the fragmentation process. Thus, we shall make extensive use of the theory of Lévy process.

Chapter 6 deals with the one–sided FKPP travelling wave equation in the setting of fragmentation processes. In the context of fragmentation processes the FKPP travelling wave equation is the following integro–differential equation:

$$cf'(x) + \int_{\mathcal{S}_1} \left( \prod_{n\in\mathbb{N}} f(x + \ln(s_n)) - f(x) \right) \nu(\mathrm{d}\mathbf{s}) = 0, \tag{3}$$

for certain $c > 0$ and all $x \in \mathbb{R}_0^+$, with boundary condition

$$\lim_{x \to \infty} f(x) = 0. \tag{4}$$

Here $\mathcal{S}_1$ consists of all decreasingly ordered sequences $(s_n)_{n\in\mathbb{N}}$ in $[0,1]$ that sum up to a value less than or equal to 1 and $\nu$ is the so–called dislocation measure, that is it is a measure on $\mathcal{S}_1$ that describes the jump structure of the fragmentation process. The probabilistic interpretation of this equation is similar to the interpretation of the classical FKPP travelling wave equation with regard to dyadic branching Brownian motion, see Section 6.2. In what follows we briefly describe the main results of Chapter 6. We show that for $c > c_{\bar{p}}$, where $c_{\bar{p}}$ is the positive constant introduced in Chapter 5, the following holds true. If $f$ belongs to a certain class of monotone and continuous functions, then for $x \in \mathbb{R}_0^+$ the process $(Z_t^{x,f})_{t\in\mathbb{R}_0^+}$, defined in Chapter 5, is a martingale under $\mathbb{P}$ if and only if $f$ solves (3). Moreover, any such function necessarily satisfies $f \in C^1(\mathbb{R}^+)$. This result is then used to prove the existence–and uniqueness result for travelling waves in the context of fragmentations. The latter says that if $c > c_{\bar{p}}$, then $\mathbb{P}(\zeta^{(\cdot)} < \infty)$ is the unique FKPP travelling wave with wave speed $c$, that is it solves the integro–differential equation (3) and satisfies the boundary condition (4). On the other hand, if $c \leq c_{\bar{p}}$, then there is no travelling wave at wave speed $c$. The approach in Chapter 6 is based on the results of Chapter 5.

# PRELIMINARIES

*Here we provide some general notation that is used without further mention troughout this dissertation.*

Throughout this thesis let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space.

We set $\mathbb{N} := \{1, 2, \dots\}$ and denote by $\mathbb{Q}$ and $\mathbb{R}$ the set of rational and real numbers respectively. Let us further adopt $\mathbb{R}^+ := (0, \infty)$ as well as $\mathbb{R}_0^+ := [0, \infty)$.

As usual, $a \wedge b$ and $a \vee b$ denote the minimum and maximum of $a, b \in \mathbb{R}$. Similarly, for the union and intersection of sets we use the symbols $\cup$ and $\cap$ respectively. For any sets $A$ and $B$ we adopt $A \setminus B := \{x \in A : x \notin B\}$. The symbol $\triangle$ denotes the symmetric difference between two sets $A$ and $B$. That is to say, $A \triangle B = [A \cup B] \setminus [A \cap B]$.

Let us further define some important measures that are used at various instances in our exposition. We denote the Lebesgue measure by $\mathrm{d}x$. Moreover, $\delta_x$ is the Dirac measure at $x$ and we use the symbol $\sharp$ for the counting measure on $\mathbb{N}$.

For any measure $\mu$ and every $p \geq 1$ we denote by $\mathscr{L}^p(\mu)$ the space of (equivalence classes of) all $p$–integrable functions with respect to $\mu$. As for the norms on these $\mathscr{L}^p$–spaces we denote for each $p \geq 1$ the $\mathscr{L}^p$–norm by $\| \cdot \|_p$, that is

$$\|f\|_p^p = \int f^p \mathrm{d}\mu$$

for all $f \in \mathscr{L}^p(\mu)$. Further, we use $C^n(E_1, E_2)$, $n \in \mathbb{N}$, to denote the space of $n$–times continuously differentiable functions from a topological space $E_1$ to a topological space $E_2$ as well as $C(E_1, E_2)$ for the space of continuous functions, and $C_b(E_1, E_2)$ for the space of bounded and continuous functions from $E_1$ to $E_2$. In this spirit we use $\mathrm{RCLL}(E_1, E_2)$ to denote the space of right–continuous functions from $E_1$ to $E_2$

with left–hand limits. For any $f : E_1 \to E_2$ and every $A \subseteq E_1$ we denoet by $f|_A$ the restriction of $f$ to $A$.

For any stochastic process $(X_t)_{t \in \mathbb{R}_0^+}$ and every random time $\tau$ we denote by $X_\tau$ the random variable defined by $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ for all $\omega \in \Omega$.

Let us emphasise that notions such as positivity and increasingness etc. are always used in the strict sense. In addition, we adopt the conventions that the empty product equals 1 as well as $\inf \emptyset := \infty$. We use the abbreviations DCT and MCT for the Dominated Convergence Theorem and the Monotone Convergence Theorem respectively. The indicator function is denoted by $\mathbb{1}$.

# CHAPTER 1

# FRAGMENTATION PROCESSES

*Here we give a brief introduction to the theory of fragmentation processes and related concepts. Our intention is to provide the basic notions that are used throughout the thesis.*

## 1.1 Introductory remarks

This chapter is devoted to the compilation of a couple of important definitions and results on fragmentation processes and Lévy processes that are used at various places in this dissertation. We stress that both Part I and Part II of the present thesis rely on the concepts developed in the present chapter.

We emphasise that our presentation in this introductory chapter is based on various research papers by several authors. The selection of results on fragmentation processes provided here is neither exhaustive nor do we provide proofs of most statements. In this regard let us mention that for the material covered in the present chapter we give references to the papers from where the respective results are taken. Hence, the interested reader can look up all the results in their original setting. Most results as well as additional background information can also be found in [Ber06].

The present chapter aims at introducing the various classes of fragmentation processes that we use in this dissertation and to show how they are related to each other. Furthermore, we present some general results and properties of fragmentation processes that are of avail in the subsequent more problem–specific chapters. In this respect let us point out that our choice which material to present here does on the one hand reflect our point of view regarding the important underlying issues of the theory on fragmentation processes and on the other hand it is motivated by the definitions and results needed for our considerations in Part I and Part II of this thesis.

## 1.2 Lévy processes

It may seem unusual to begin a chapter aiming at introducing fragmentation processes with a section on Lévy processes. However, let us point out that there are strong intrinsic connections of these two classes and we shall frequently exploit the close relation between fragmentations and Lévy processes in the present dissertation. It turns out that later on we shall resort to several notions and results related to Lévy processes and for this reason we start off with a brief compilation on various concepts regarding the theory of Lévy processes. Our exposition in this section is predominantly based on [Kyp06], but there are many other excellent monographs on Lévy processes such as [Ber96], [Sat99] and [App09] to name but three of them.

### 1.2.1 Basic notions

In this section we compile the fundamental definitions that are used for our considerations of Lévy processes.

**Definition 1.1** A *Lévy process* is a Markov process that has stationary and independent increments and whose paths are almost surely right–continuous with left–hand limits.

Note that any Lévy process $(X(t))_{t\in\mathbb{R}_0^+}$ satisfies $X(0) = 0$ $\mathbb{P}$–almost surely.

Lévy processes can be characterised by three entities describing the drift, the diffusion component and the jump structure respectively. More precisely, let us give the following definition based on the Lévy–Khintchine formula (see e.g. Theorem 1.3 in [Kyp06]) that is one of the fundamental results in the theory of Lévy processes.

**Definition 1.2** It follows from the Lévy–Khintchine formula that the law of any $\mathbb{R}$–valued Lévy process $X$ is characterised by the so–called *characteristic triple* $(a, \sigma, L_X)$, where $a \in \mathbb{R}$ describes the deterministic drift part of the Lévy process, $\sigma$ determines the diffusion component of the Lévy process and $L_X$ is a measure concentrated on $\mathbb{R} \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}} (1 \wedge x^2) L_X(\mathrm{d}x) < \infty$$

and characterises the jump structure of $X$. We call the measure $L_X$ the *Lévy measure* of $X$.

A very important subclass of Lévy processes are subordinators as given by the following definition:

**Definition 1.3** A *subordinator* is a Lévy process with nondecreasing trajectories.

In particular, the diffusion component is not present in a subordinator, that is the $\sigma$ in the characteristic triple introduced in Definition 1.2 equals 0. Moreover, we have the following lemma:

**Lemma 1.4 (Lemma 2.4 in [Kyp06])** *A Lévy process $X$ with characteristic triple $(a, \sigma, L_X)$ is a subordinator if and only if the drift $d = -(a + \int_{(0,1)} x L_X(dx))$ is positive, $\sigma = 0$ and $L_X(-\infty, 0) = 0$ as well as $\int_{(0,\infty)} (1 \wedge x) L_X(dx) < \infty$.*

A subordinator that plays a crucial role in the theory of fragmentation processes will be introduced in Section 1.7.

### 1.2.2 Spectrally negative Lévy processes

The reader who merely wants to become familiar with fragmentation processes may postpone or skip reading this section as it covers a more specialised topic that is only needed in Part II of this thesis. However, it is related to the above–defined concepts, and thus we decided to include this section here.

In Chapter 5 and Chapter 6 we shall make use of the well–developed theory of the following class of Lévy processes with one–sided jumps:

**Definition 1.5** A *spectrally negative Lévy process* is a Lévy process that has no upwards jumps and whose paths are not monotone.

Let us point out that in Chapter 5 we shall use a subordinator, see Definition 1.3, that appears frequently in the literature on fragmentations to define a spectrally negative Lévy process that will be be of avail for our considerations in Part II of the present dissertation. Let us further mention that a similar spectrally negative Lévy process, defined via the aforementioned subordinator, was considered in [Kre08] and [KR09].

For the time being, let $X$ be a spectrally negative Lévy process of bounded variation, that is without any diffusion component. Then $X$ is necessarily an increasing deterministic linear function minus a subordinator, that is $X(t) = ct - Y(t)$ for any $t \in \mathbb{R}_0^+$, where $c > 0$ is a constant and $Y$ is a pure jump subordinator.

Set

$$\tau_x^+ := \inf\{t \in \mathbb{R}_0^+ : X(t) > x\}$$

as well as

$$\tau_x^- := \inf\{t \in \mathbb{R}_0^+ : X(t) < -x\}$$

for all $x \in \mathbb{R}$. Note that $\tau_x^+$ and $\tau_x^-$ are stopping times, since $X$ is right–continuous.

**Definition 1.6 (cf. Theorem 8.1 in [Kyp06])** Let $X$ be a spectrally negative Lévy process with Laplace exponent $\Phi_X$. We call *scale function* of $X$ the unique monotonically increasing function $W_X : \mathbb{R} \to \mathbb{R}_0^+$ that is continuous on $\mathbb{R}_0^+$, equal to 0 on $(-\infty, 0)$ and whose Laplace transform satisfies

$$\int_{(0,\infty)} e^{-\beta x} W_X(x) \, \mathrm{d}x = \frac{1}{\Phi_X(\beta)}.$$

The following result on spectrally negative Lévy processes, taken from [Kyp06], will be used repeatedly in Chapter 5:

**Theorem 1.7 (Theorem 8.1 in [Kyp06])** *Let $X$ be a spectrally negative Lévy process with scale function $W_X$. Further, let $\Phi'_X$ denote the derivative of the Laplace exponent $\Phi_X$. Then we have*

$$\mathbb{P}_x(\tau_0^- < \infty) = \begin{cases} 1 - \Phi'_X(0+) W_X(x), & \Phi'_X(0+) > 0 \\ 1, & \Phi'_X(0+) \leq 0. \end{cases} \tag{1.1}$$

*for every $x \in \mathbb{R}_0^+$. Moreover,*

$$\mathbb{P}_x\left(\tau_0^- > \tau_y^+\right) = \frac{W_X(x)}{W_X(y)} \tag{1.2}$$

*holds true for all $x, y \in \mathbb{R}_0^+$ with $y \geq x$.*

Let $X$ be a spectrally negative Lévy process with scale function $W_X$. Notice that (1.2) implies in particular that for $X$ the point 0 is irregular for $(-\infty, 0)$, that is

$$\mathbb{P}_0(\tau_0^- = 0) = 0. \tag{1.3}$$

Indeed, by means of the continuity of $W_X$ on $\mathbb{R}_0^+$ there exists for every $\epsilon > 0$ some

$\delta > 0$ such that

$$\frac{W_X(0)}{W_X(\delta)} \geq 1 - \epsilon.$$

Since $\tau_\delta^+ > 0$ $\mathbb{P}$–a.s., this yields that

$$\mathbb{P}(\tau_0^- > 0) \geq \mathbb{P}\left(\tau_0^- > \tau_\delta^+\right) = \frac{W_X(0)}{W_X(\delta)} \geq 1 - \epsilon.$$

Letting $\epsilon \downarrow 0$ this shows that (1.3) holds true. In view of the shift invariance of Lévy processes the irregularity established in (1.3) immediately implies that for $X$ any $x \in \mathbb{R}_0^+$ is irregular for $(-\infty, x)$. Furthermore, (1.2) yields that

$$\lim_{x\to\infty} \mathbb{P}_x\left(\tau_0^- > \tau_{x+h}^+\right) = \lim_{x\to\infty} \frac{W_X(x)}{W_X(x+h)} = 1, \tag{1.4}$$

where for the last equality we have used that $W_X$ is monotone and bounded and thus has a limit as its argument tends to infinity. Since $\lim_{x\to\infty} \tau_x^+ = \infty$ $\mathbb{P}$–a.s., (1.4) results in

$$\lim_{x\to\infty} \mathbb{P}_x(\tau_0^- < \infty) = 0,$$

and thus we obtain as a consequence of (1.1) that

$$\lim_{x\to\infty} W_X(x) = \frac{1}{\psi_p'(0+)} \tag{1.5}$$

if $\psi_p'(0+) > 0$. Moreover, in [Kyp06] it is shown that if $X$ is of bounded variation, then there is a positive lower bound, as given by the following lemma, on the scale function of $X$.

**Lemma 1.8 (Lemma 8.6 in [Kyp06])** *Let $X$ be a spectrally negative Lévy process of bounded variation with drift $d \in \mathbb{R}^+$ and scale function $W_X$. Then we have that*

$$\inf_{x\in\mathbb{R}_0^+} W_X(x) = W_X(0) = d^{-1}.$$

## 1.3 Introduction to fragmentation processes

Fragmentation processes as considered in the present exposition are continuous–time Markov processes and are closely related to Lévy processes. In some sense Lévy processes can be seen as the continuous–time analogue of random walks and in a similar fashion fragmentation processes extend fragmentation chains to the continuous–time setting. Some of the mathematical roots of fragmentation processes lay with older families of spatial branching processes such as branching random walks and Crump–Mode–Jagers processes (also known as general branching processes). Such stochastic processes exemplify the phenomena of random splitting according to systematic rules and they may be seen as modelling the growth of special types of multi–particle systems.

The simplest example of a fragmentation process is the stick–breaking process, see Figure 1-1. More precisely, let us consider a stick of unit size and say that after an exponentially distributed time with some parameter $\alpha \in \mathbb{R}^+$ the stick breaks into two pieces of length $\beta$ and $1-\beta$ respectively, for some random $\beta : \Omega \to \mathbb{R}_0^+$. Then each of the resulting smaller sticks independently repeats the procedure and the process continues ad infinitum. The stochastic process $\lambda = (\lambda(t))_{t\in\mathbb{R}_0^+}$, consisting at each time $t \in \mathbb{R}_0^+$ of the decreasingly ordered set of the lengths $(\lambda_n(t))_{n\in\mathbb{N}}$ of the sub–sticks present at time $t$, constitutes a so–called mass fragmentation process. In general such processes can have a much more complicated structure. Firstly, the splitting does not need to be binary, that is the stick could break into a random, possibly infinite, number of pieces. Secondly, the time between two splittings does not need to be exponentially distributed with a finite parameter as the splitting times may be dense in $\mathbb{R}_0^+$. We give a rigorous definition of such a process in the following section.

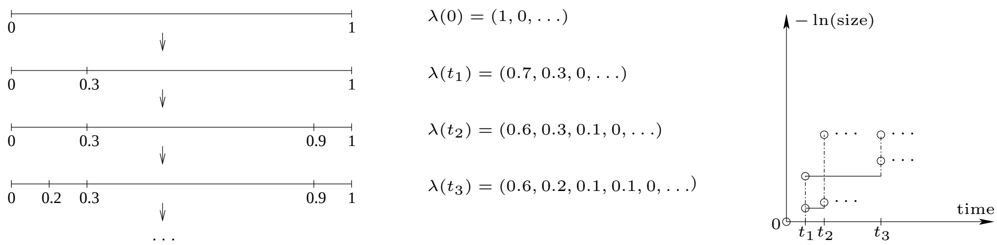



**Figure 1-1:** *Stick–breaking process* $(\lambda(t))_{t\in\mathbb{R}_0^+}$ *with jump times* $(t_n)_{n\in\mathbb{N}}$.

## 1.4 Mass fragmentation processes

Consider an infinite–dimensional vector space $\mathcal{S}_1$ of nonincreasing sequences in $[0, 1]$ given by

$$\mathcal{S}_1 := \left\{ \mathbf{s} := (s_n)_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} s_n \leq 1,\, 0 \leq s_j \leq s_i \leq 1 \,\forall\, i \leq j \right\}.$$

For any sequence $(x_n)_{n\in\mathbb{N}}$ of nonnegative real numbers we denote by $(x_n)^{\downarrow}_{n\in\mathbb{N}}$ the decreasing reordering of $(x_n)_{n\in\mathbb{N}}$, that is $(x_n)^{\downarrow}_{n\in\mathbb{N}} \in \mathcal{S}_1$ if and only if $\sum_{n\in\mathbb{N}} x_n \leq 1$. We consider $\mathcal{S}_1$ to be endowed with the uniform distance. That is to say, we work with the metric space $(\mathcal{S}_1, \rho_{\mathcal{S}_1})$, where the metric $\rho_{\mathcal{S}_1}$ on $\mathcal{S}_1$ is given by

$$\rho_{\mathcal{S}_1}(\mathbf{s}, \mathbf{u}) = \sup_{n \in \mathbb{N}} |s_n - u_n|$$

for all $\mathbf{s}, \mathbf{u} \in \mathcal{S}_1$. In what follows we consider continuity in probability of an $\mathcal{S}_1$–valued stochastic process with respect to the metric $\rho_{\mathcal{S}_1}$. That is to say, an $\mathcal{S}_1$–valued stochastic process $(\lambda(t))_{t \in \mathbb{R}_0^+}$ is continuous in probability if and only if for all $\epsilon > 0$ and any $u \in \mathbb{R}_0^+$ we have

$$\mathbb{P}\left(\rho_{\mathcal{S}_1}(\lambda(s), \lambda(u)) > \epsilon\right) \to 0$$

as $s \to u$.

Let us now give our first definition of fragmentation processes.

**Definition 1.9** We call an $\mathcal{S}_1$–valued Markov process $\lambda := (\lambda(t))_{t \in \mathbb{R}_0^+}$, continuous in probability, a *self–similar (standard) mass fragmentation process* with index $\alpha \in \mathbb{R}$ if

(i) $\lambda(0) = (1, 0, \ldots)$.

(ii) For any $s \in \mathbb{R}_0^+$, given that $\lambda(s) = (s_n)_{n \in \mathbb{N}}$, the process $((\lambda(s+t))_{t \in \mathbb{R}_0^+}$ has the same distribution as the process obtained by taking for any $t \in \mathbb{R}_0^+$ the components of $s_n \lambda^{(n)}(s_n^\alpha t)$ for all $n \in \mathbb{N}$, where the $\lambda^{(n)}$ are i.i.d. copies of $\lambda$, and ranking the resulting sequence in the decreasing order to obtain an element of $\mathcal{S}_1$.

If $\alpha = 0$ then the process is called *homogenous.*

In the above definition property (i) says that mass fragmentation processes start with exactly one fragment and this fragment has size 1. Property (ii) is called *fragmentation property* and is the analogue of the branching property in the theory of Markov branching processes. Observe that property (ii) says that for any $s, t \in \mathbb{R}_0^+$, given that $\lambda(s) = (s_n)_{n \in \mathbb{N}}$ we have

$$\lambda(s+t) \stackrel{\mathrm{d}}{=} \left(s_n \lambda_k^{(n)}(s_n^\alpha t)\right)_{k,n \in \mathbb{N}}^{\downarrow},$$

where the $\lambda^{(n)}$ are i.i.d. copies of $\lambda$. Here $\stackrel{\mathrm{d}}{=}$ means equality in distribution.

Note that the stick–breaking process in Figure 1-1 is an example of a homogenous mass fragmentation process. See Figure 1-2 for an illustration of a more sophisticated mass fragmentation process. Let us mention that an even more complicated example of a fragmentation process is depicted in Figure 5-1(a) in Chapter 5. However, in order to get a feeling for fragmentation processes we illustrate a relatively simple example in Figure 1-2.

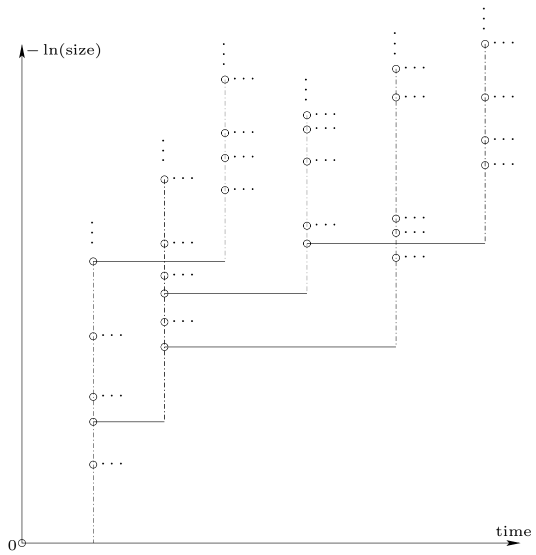



**Figure 1-2:** *Realisation of a standard (mass) fragmentation process with finite dislocation measure. In this illustration the term* size *refers to the values* $\lambda_n(t)$.

Let $(\mathbb{P}_x)_{x\in\mathbb{R}^+}$ denote the probabilities under which $\lambda$ is rescaled such that

$$\mathbb{P}_x(\lambda_1(0) = x) = 1. \tag{1.6}$$

Further, let $\mathbb{E}_x$ denote the expectation with respect to $\mathbb{P}_x$ and set $\mathbb{E} := \mathbb{E}_1$.

Similarly to the characterisation of Lévy processes in terms of the characteristic triple, cf. Definition 1.2, the following theorem, in the self–similar setting due to [Ber02b], provides us with a characterisation of self–similar standard mass fragmentation process in terms of a jump measure, a continuous drift and the index of self similarity. In this respect we also refer to Theorem 2 in [Ber01] as well as Corollary 3.1 in [Ber02a] for the homogenous case.

**Theorem 1.10 (Theorem 3 (ii) in [Ber02b])** *The distribution of any self–similar standard mass fragmentation process as given by Definition 1.9 is determined by*

- *a nonnegative rate of erosion,*
- *a measure* $\nu \not\equiv 0$ *on* $\mathcal{S}_1$ *that satisfies*

$$\nu(\{(1,0,\ldots)\}) = 0 \quad \textit{and} \quad \int_{\mathcal{S}_1}(1-s_1)\nu(d\mathbf{s}) < \infty, \tag{1.7}$$

- *the index of self–similarity* $\alpha \in \mathbb{R}$.

Let us mention that in [Ber01] and [Ber02b] Bertoin considered partition–valued fragmentation processes, a class of fragmentations with which we deal in the next section. However, it follows from a bijective correspondence between that class and the class of mass fragmentations that Theorem 1.10 can be stated for mass fragmentation processes. The above–mentioned bijective correspondence will be explained in more detail in Section 1.6.

Erosion means a continuous loss of mass, thus adding a continuous drift to the jumps of the fragmentation process. We emphasise that this phenomenon is not considered in the present thesis. That is to say, throughout this dissertation we assume without further mention the following hypothesis:

**Hypothesis 1.1** There is no erosion. That is, the erosion rate is zero.

Hypothesis 1.1 says that we consider fragmentation processes which change state only by a jump. Let us mention that in the literature on fragmentation processes erosion is usually not considered, and thus Hypothesis 1.1 does not pose an unusual restriction.

**Definition 1.11** The measure $\nu$ defined by (1.7) will be referred to as $\mathcal{S}_1$*–dislocation measure*. Further, we call the dislocation measure $\nu$, resp. the fragmentation process, *conservative* if $\nu(\sum_{n\in\mathbb{N}} s_n < 1) = 0$ and *dissipative* otherwise.

Throughout this dissertation we impose the following additional condition on $\nu$:

**Hypothesis 1.2** For all $a \in [0, 1]$ we have

$$\nu(\{(a, 0, \ldots)\}) = 0.$$

Hypothesis 1.2 ensures that the measure $\nu$ only charges those jumps of the fragmentation process, where a particle dislocates into at least two particles. The less restrictive assumption in (1.7) has the same meaning in the conservative case, which is often considered in the literature. Our slightly different requirement on $\nu$ stems from the consideration of the dissipative case, in which it would be possible that even a homogenous fragmentation process that satisfies (1.7) dies out in finite time. We remark that the sequence $(a, 0, \ldots)$ corresponds to the particle dislocating into exactly one particle with less mass, so rather than a proper dislocation this would form a discrete version of erosion.

**Remark 1.12** The dislocation measure $\nu$ specifies the rate at which blocks split. More precisely, a block of mass $x$ dislocates into a mass partition $x \cdot \mathbf{s} \in \mathcal{S}_1$, where $\mathbf{s} \in \mathcal{S}_1$, at rate $\nu(d\mathbf{s})$. Note that in the literature, cf. [Ber01], the measure $\nu$ is sometimes called Lévy measure. The motivation for that alternative name is that for mass fragmentation processes the measure $\nu$ plays the same role as the Lévy measure, see Definition 1.2, does for Lévy processes. Indeed, both measures are the jump measures of their respective processes in that both describe entirely the jump structure of a fragmentation process and a Lévy process respectively. In fact, that similar measures appear for both types of processes is not surprising as there is a very close relationship between fragmentation processes and Lévy processes. In Section 1.7 we will see that this similarity between the two classes of processes gives rise to an underlying Lévy process, more precisly a subordinator, which is used prevalently in this thesis and also in the literature on fragmentations. In the second part of this thesis we make extensive use of a spectrally negative Lévy process defined via a fragmentation process, which enables us to exploit the well–developed theory of spectrally negative Lévy processes. $\diamondsuit$

An important tool for dealing with homogenous fragmentation processes is their Poissonian structure as given by the following theorem.

**Theorem 1.13 (Section 3.1 in [Ber02a])** *For any homogenous mass fragmentation process $\lambda$ there exists an $\mathcal{S}_1 \times \mathbb{N}$–valued Poisson point process $(\Delta(t), k(t))_{t \in \mathbb{R}_0^+}$ with characteristic measure $\nu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$, such that $\lambda$ changes state at all times $t \in \mathbb{R}_0^+$ for which an atom $(\Delta(t), k(t))$ occurs in $(\mathcal{S}_1 \setminus \{(1, 0, \cdots)\}) \times \mathbb{N}$. At such a time $t \in \mathbb{R}_0^+$ the sequence $\lambda(t)$ is obtained from $\lambda(t-)$ by replacing its $k(t)$–th term, $\lambda_{k(t)}(t-) \in [0,1]$, with the sequence $\lambda_{k(t)}(t-) \cdot \Delta(t) \in \mathcal{S}_1$ and ranking the resulting sequence of all terms in decreasing order.*

Conversely, the above construction in terms of a Poisson point process for some $\mathcal{S}_1$–dislocation measure $\nu$ results in a homogenous mass fragmentation process.

**Remark 1.14** The mathematical approach to tackle problems involving self–similar fragmentation processes partly depends on whether the dislocation measure is finite or infinite. If $\nu$ is finite, then a block of size $x$ remains unchanged for an exponential period of time with parameter $\nu(\mathcal{S}_1) \in \mathbb{R}^+$. That is, in the homogenous case there is finite activity over finite time intervals in the underlying Poisson point process. By taking the negative logarithm of fragment sizes a fragmentation process with finite dislocation measure is closely related to continuous–time branching random walks and Crump–Mode–Jagers processes. If on the other hand $\nu(\mathcal{S}_1) = \infty$, then the jump times are dense in $\mathbb{R}_0^+$ and there is a countably infinite number of dislocations over any finite time horizon. Note that the denseness of the jump times does in particular imply that there is no first dislocation of the process and the infimum over all jump times is 0, although there is no dislocation at time 0. Fragmentation processes with an infinite dislocation measure are more interesting, both from a theoretical point of view and for applications as for instance in the mining industry. Moreover, in comparison to fragmentation processes with finite dislocation measure those processes are also mathematically more challenging. $\diamondsuit$

This thesis is primarily concerned with the situation of an infinite dislocation measure, but all our results hold true also in the finite activity case. Note that the illustration in Figure 1-2, as all illustrations in this thesis, only depicts a fragmentation process with finite dislocation measure, because a realisation of a fragmentation process with an infinite dislocation measure is much more difficult to visualise.

## 1.5 Fragmentation processes with a genealogical structure

One disadvantage of mass fragmentation processes is the lack of a genealogical structure. That is, in a mass fragmentation process it is difficult to define the notion of "ancestor" or "parent" of a given block. In this section we introduce two classes of fragmentation processes which avoid this problem and which are thus more applicable in many situations.

We shall assume without further mention that the analogues of Hypothesis 1.1 and Hypothesis 1.2 are also satisfied for the classes of processes that we are going to introduce in this section.

### 1.5.1 Partition–valued fragmentation processes

We denote by $\mathcal{P}$ the set of ordered partitions $\pi := (\pi_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$, ordered such that $\inf \pi_1 \leq \inf \pi_j$ for all $i \leq j \in \mathbb{N}$, with the convention $\inf \emptyset = \infty$. A partition of $\mathbb{N}$ is a sequence of blocks $\pi_n \subseteq \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \pi_n = \mathbb{N}$ and $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$. We equip $\mathcal{P}$ with the metric $\rho_{\mathcal{P}}$ on $\mathcal{P}$ defined as follows, cf. Section 2 of [Ber01]. For any two partitions $\pi_1, \pi_2$ of $\mathbb{N}$ we set

$$\rho_{\mathcal{P}}(\pi_1, \pi_2) = 2^{-N(\pi_1,\pi_2)},$$

where $N(\pi_1, \pi_2) := \sup(\{n \in \mathbb{N} : \pi_1|_{\{1,\dots,n\}} = \pi_2|_{\{1,\dots,n\}}\})$. Note that $\pi|_{\{1,\dots,n\}}$ denotes the restriction of a partition $\pi \in \mathcal{P}$ to the subset $\{1, \dots, n\} \subseteq \mathbb{N}$. We remark that the metric space $(\mathcal{P}, \rho_{\mathcal{P}})$ is compact.

The following definition provides us with a notion of "size" for the blocks of partitions in $\mathcal{P}$. This notion of asymptotic frequencies will be considered as the size of blocks in the context of partition–valued fragmentation processes as defined below.

**Definition 1.15** Let $\pi \in \mathcal{P}$ and $n \in \mathbb{N}$. Then we call

$$|\pi_n| := \limsup_{k\to\infty} \frac{\mathrm{card}(\pi_n \cap \{1, \dots, k\})}{k} \in [0, 1] \tag{1.8}$$

*asymptotic frequency* of the block $\pi_n$.

Let us adopt $|\pi| := (|\pi_n|)_{n\in\mathbb{N}}$ as well as $|\pi|^{\downarrow} := (|\pi_n|)^{\downarrow}_{n\in\mathbb{N}}$ for any $\pi \in \mathcal{P}$.

Our second definition of fragmentation processes reads as follows:

**Definition 1.16** We call a $\mathcal{P}$–valued Markov process $\Pi := (\Pi(t))_{t\in\mathbb{R}_0^+\cup\{\infty\}}$, continuous in probability, a *self–similar (standard) $\mathcal{P}$–fragmentation process* with index $\alpha \in \mathbb{R}$ if

(i) $\Pi(0) = (\mathbb{N}, \emptyset, \dots)$.

(ii) For any $s \in \mathbb{R}_0^+$, given that $\Pi(s) = (\pi_n)_{n\in\mathbb{N}}$, the process $((\Pi(s + t))_{t\in\mathbb{R}_0^+}$ has the same distribution as the process obtained by taking for any $t \in \mathbb{R}_0^+$ the components of $\pi_n \cap \Pi^{(n)}(|\pi_n|^\alpha t)$ for all $n \in \mathbb{N}$, where the $\Pi^{(n)}$ are i.i.d. copies of $\Pi$, and ordering the resulting sequence such that it is an element of $\mathcal{P}$.

If $\alpha = 0$ then the process is called *homogenous*. Further, we adopt $\Pi(\infty) := (\{n\})_{n\in\mathbb{N}}$.

In particular, a $\mathcal{P}$–fragmentation process starts with the trivial partition of $\mathbb{N}$, that is it starts with exactly one non–empty block that contains all natural numbers. As in the case of mass fragmentation processes, we call Property (ii) *fragmentation property.* The continuity in probability in Definition 1.16 is meant with respect to the metric $\rho_{\mathcal{P}}$.

For any $\pi \in \mathcal{P}$ let $\mathbb{P}_\pi$ denote the probability under which $\Pi$ is conditioned to start with the partition $\pi$, that is

$$\mathbb{P}_\pi(\Pi(0) = \pi) = 1.$$

In order to describe an interpretation of partition–valued fragmentation processes in terms of a Poisson point process, cf. Theorem 1.13 in the context of mass fragmentation processes, we first need to construct a dislocation measure on $\mathcal{P}$. For this purpose we shall resort to Kingman's paint–box as given by the following definition, cf. Chapter 2 in [Ber06].

**Definition 1.17** Let $\vartheta$ be an interval representation of some $\mathbf{s} \in \mathcal{S}$, that is $\vartheta$ is an open subset of $(0,1)$ such that the ranked sequence of the lengths of its interval components is given by $\mathbf{s}$. Let $(U_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of uniform random variables on $[0,1]$. We call *Kingman's paint–box* based on $\mathbf{s}$ the random partition[1] $\pi^{\mathbf{s}}$ of $\mathbb{N}$ induced by the following equivalence relation

$$i \overset{\pi^{\mathbf{s}}}{\sim} j \iff (U_i \text{ and } U_j \text{ belong to the same interval component of } \vartheta) \text{ or } (i=j).$$

Note that the alternative on the right–hand side is necessary, because the Lebesgue measure of $\vartheta$ may be less than one, and if $U_i$ does not belong to $\vartheta$ for some $i \in \mathbb{N}$, then $\{i\}$ is a singleton of $\pi^{\mathbf{s}}$.

The name "paint–box" stems from the following alternative description of the equivalence relation $\pi^{\mathbf{s}}$ described in Definition 1.17. Let us interpret the unit interval as a paint–box in which a different colour is assigned to each interval component of $\vartheta$. Every integer $i$ then receives the colour of the interval to which the random variable $U_i$ belongs, and $i$ does not receive any colour if $U_i$ is not in $\vartheta$. The equivalence classes are then given by the sets of indices with the same colour, where we adopt that indices with no colour form singletons.

**Definition 1.18** A random partition is call *exchangeable* if it is invariant under finite permutations.

**Remark 1.19** It is shown in [Ber06, Lemma 2.7] that Kingman's paint–box based on some $\mathbf{s} \in \mathcal{S}_1$ is independent of the choice of the interval representation of $\mathbf{s}$ and that it is an exchangeable random partition, cf. Definition 1.18. Moreover, in [Ber06, Proposition 2.8] Bertoin shows that $\mathbb{P}$–a.s. all blocks of Kingman's paint–box $\pi$ satisfy

$$|\pi_n| = \lim_{k\to\infty} \frac{\operatorname{card}(\pi_n \cap \{1,\ldots,k\})}{k}, \tag{1.9}$$

where $|\pi_n|$ is the asymptotic frequency, see Definition 1.15, of the block $\pi_n$. $\Diamond$

**Definition 1.20** For any $\mathcal{S}_1$–dislocation measure $\nu$ we call $\mathcal{P}$*–dislocation measure* the

[1] For an introduction to the theory of random partitions and related concepts as considered here, we refer to Section 2.3 in [Ber06].

measure $\mu_\nu$ on $\mathcal{P}$ defined by

$$\mu_\nu(\mathrm{d}\pi) = \int_{\mathcal{S}_1} \varrho_{\mathbf{s}}(\mathrm{d}\pi)\nu(\mathrm{d}\mathbf{s}) \tag{1.10}$$

for each $\pi \in \mathcal{P}$, where $\varrho_{\mathbf{s}}$ is the distribution of Kingman's paint–box based on $\mathbf{s}$.

The dislocation measure $\mu_\nu$ determines the distribution of the jumps of a partition–valued fragmentation process. We remark that the motivation for Definition 1.20 stems from Kingman's theory of random partitions, see Section 2.3.2 in [Ber06] and in particular [Ber06, Theorem 2.1] therein.

Similarly to the case of homogenous mass fragmentation processes, cf. Theorem 1.13, we have the following representation of homogenous $\mathcal{P}$–fragmentation process via Poisson point processes.

**Theorem 1.21 (Section 3.2 in [Ber01])** *For any homogenous $\mathcal{P}$–fragmentation $\Pi$ there exist an $\mathcal{S}_1$–dislocation measure $\nu$ and a $\mathcal{P} \times \mathbb{N}$–valued Poisson point process $(\pi(t), k(t))_{t \in \mathbb{R}_0^+}$ with characteristic measure $\mu_\nu \otimes \sharp$ such that $\Pi$ changes state at all times $t \in \mathbb{R}_0^+$ for which an atom $(\pi(t), k(t))$ occurs in $(\mathcal{P} \setminus (\mathbb{N}, \emptyset, \ldots)) \times \mathbb{N}$. At such a time $t \in \mathbb{R}_0^+$ the sequence $\Pi(t)$ is obtained from $\Pi(t-)$ by replacing its $k(t)$–th term, $\Pi_{k(t)}(t-) \subseteq \mathbb{N}$, with the sequence $\Pi_{k(t)}(t-) \cap \pi(t) \in \mathcal{P}$ and reordering the terms such that the resulting partition of $\mathbb{N}$ is an element of $\mathcal{P}$.*

It follows from (1.9) that for any $\mathcal{P}$–fragmentation process $\Pi$ and every $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$ the asymptotic frequency of $\Pi_n(t)$ satisfies

$$|\Pi_n(t)| = \lim_{k \to \infty} \frac{\mathrm{card}(\Pi_n(t) \cap \{1, \ldots, k\})}{k} \tag{1.11}$$

$\mathbb{P}$–almost surely. The following theorem, due to Bertoin, states that for homogenous partition–valued fragmentations an even stronger property holds true.

**Theorem 1.22 (Theorem 3 (i) in [Ber01])** *Homogenous partition–valued fragmentation processes are* nice. *That is to say, the asymptotic frequencies of any homogenous $\mathcal{P}$–fragmentation process $\Pi$ satisfy (1.11) $\mathbb{P}$–a.s. for every $n \in \mathbb{N}$ and all $t \in \mathbb{R}_0^+$ simultaneously.*

**Definition 1.23** We denote the random jump times of $\Pi$ by $(t_i)_{i \in \mathcal{I}}$, where $\mathcal{I}$ is a countably infinite index set. For any $t \in \mathbb{R}_0^+ \cup \{\infty\}$ let $B_n(t)$ denote the block in $\Pi(t)$ which contains the element $n \in \mathbb{N}$ and consider the process $B_n := (B_n(t))_{t \in \mathbb{R}_0^+ \cup \{\infty\}}$.

Notice that $B_n(\infty) = \{n\}$ for every $n \in \mathbb{N}$. Let us define $\mathcal{I}_n \subseteq \mathcal{I}$ such that the jump times of the process $B_n$ are $(t_i)_{i \in \mathcal{I}_n}$. In particular, note that $\mathcal{I}_1 = \{i \in \mathcal{I} : k(t_i) = 1\}$.

We shall make use of the fact, due to [Ber02b], that any self–similar $\mathcal{P}$–fragmentation process is a time–changed homogenous $\mathcal{P}$–fragmentation process. More precisely, let $\alpha \in \mathbb{R}$ and define

$$T_n^{(\alpha)}(t) := \inf \left\{ s \in \mathbb{R}_0^+ : \int_{(0,s)} |B_n(u)|^{-\alpha} \mathrm{d}u > t \right\}.$$

Let $\Pi^{(\alpha)}(t)$ be the random partition of $\mathbb{N}$ with the property that $i, j \in \mathbb{N}$ are in the same block of $\Pi^{(\alpha)}$ if and only if they are in the same block of $\Pi(T^{(\alpha)}(t))$. Bertoin proved the following result:

**Theorem 1.24 (Theorem 3 (i) in [Ber02b])** *If the process* $\Pi$ *is a self–similar* $\mathcal{P}$*–fragmentation with index* $\beta \in \mathbb{R}$*, then* $(\Pi^{(\alpha)}(t))_{t \in \mathbb{R}_0^+}$ *is a self–similar* $\mathcal{P}$*–fragmentation process with index* $\alpha + \beta$*.*

Let $\mathbf{s} = (s_n)_{n \in \mathbb{N}} \in \mathcal{S}_1$ and let $\rho_{\mathbf{s}}$ be Kingman's paint–box based on $\mathbf{s}$, see Definition 1.17. In addition, let $\pi^{\mathbf{s}}$ be a random partition with distribution $\rho_{\mathbf{s}}$. Proposition 2.8 in [Ber06] shows that $\mathbb{P}(|\pi^{\mathbf{s}}|^{\downarrow} = \mathbf{s}) = 1$. Moreover, $\mathbb{P}(|\pi_1^{\mathbf{s}}| = s_n) = s_n$ for all $n \in \mathbb{N}$. The latter means that the $[0,1]$–valued random variable $|\pi_1^{\mathbf{s}}|$ is a size–biased sample of $\mathbf{s}$ $\mathbb{P}$–almost surely. Let $\mu_\nu$ be given by (1.10). Resorting to Tonelli's theorem we then infer that

$$\begin{aligned} \int_{\mathcal{P}} g(|\pi|^{\downarrow}) f(|\pi_1|) \mu_\nu(\mathrm{d}\pi) &= \int_{\mathcal{S}_1} g(\mathbf{s}) \int_{\mathcal{P}} f(|\pi_1|) \rho_{\mathbf{s}}(\mathrm{d}\pi) \nu(\mathrm{d}\mathbf{s}) \\ &= \int_{\mathcal{S}_1} g(\mathbf{s}) \sum_{n \in \mathbb{N}} s_n f(s_n) \nu(\mathrm{d}\mathbf{s}) \qquad (1.12) \end{aligned}$$

holds for all nonnegative test functions $f : [0,1] \to \mathbb{R}_0^+$ with $f(0) = 0$ and $g : \mathcal{S}_1 \to \mathbb{R}_0^+$.

### 1.5.2 Interval fragmentation processes

A third kind of fragmentation processes that appears in the literature are so–called interval fragmentations. This kind of fragmentation processes was introduced by Bertoin [Ber02b] and was also considered by Basdevant [Bas06]. Our definition follows the lines of [Ber02b].

In order to define interval fragmentations we first need to define some more notation. The state space of the process will be the usual topology $\mathcal{T}_{(0,1)}$ on $(0,1)$. That is, $\mathcal{T}_{(0,1)}$

is the topology consisting of all unions of open intervals in $(0,1)$. Further, for any $U \in \mathcal{T}_{(0,1)}$ define a function $\chi_U : [0,1] \to [0,1]$ by

$$\chi_U(x) = \inf_{y \in U^{\complement}} |x-y|$$

for every $x \in [0,1]$, where $U^{\complement} := [0,1] \setminus U$. We endow $\mathcal{T}_{(0,1)}$ with the metric $\rho_{\mathcal{T}_{(0,1)}}$ defined by

$$\rho_{\mathcal{T}_{(0,1)}}(U,V) = \sup_{x \in [0,1]} |\chi_U(x) - \chi_V(x)|$$

for any $U, V \in \mathcal{T}_{(0,1)}$. Observe that $(\mathcal{T}_{(0,1)}, \rho_{\mathcal{T}_{(0,1)}})$ is a compact metric space. Further, note that for all $U, V \in \mathcal{T}_{(0,1)}$ the distance $\rho_{\mathcal{T}_{(0,1)}}(U,V)$ coincides with the Hausdorff distance between $U^{\complement}$ and $V^{\complement}$. Moreover, consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $[0,1]$ as well as $a, b \in [0,1]$. Then

$$\lim_{n\to\infty} \rho_{\mathcal{T}_{(0,1)}}((a_n,b_n),(a,b)) = 0 \iff \lim_{n\to\infty} \max\{|a-a_n|, |b-b_n|\} = 0$$

and

$$\lim_{n\to\infty} \rho_{\mathcal{T}_{(0,1)}}((a_n,b_n),\emptyset) = 0 \iff \lim_{n\to\infty} |a_n - b_n| = 0.$$

For further information regarding the metric space $(\mathcal{T}_{(0,1)}, \rho_{\mathcal{T}_{(0,1)}})$ we refer the reader to Section 2 in [Ber02b]. Let $\mathcal{T}_{0,1}$ be the topology induced by $\rho_{\mathcal{T}_{(0,1)}}$. We consider the measurable space $(\mathcal{T}_{(0,1)}, \mathscr{B}_{0,1})$, where $\mathscr{B}_{0,1}$ denotes the Borel–$\sigma$–algebra generated by $\mathcal{T}_{0,1}$, that is $\mathscr{B}_{0,1} := \sigma(\mathcal{T}_{0,1})$. Let $(p_t)_{t\in\mathbb{R}_0^+}$ be a set of probability measures on $(\mathcal{T}_{(0,1)}, \mathscr{B}_{0,1})$ such that the mapping $t \mapsto p_t$ is continuous. Further, let $\alpha \in \mathbb{R}$ and let $a, b \in [0,1]$ with $a < b$. We denote by $(\mathcal{T}_{(a,b)}, \mathcal{T}_{a,b})$ the topological subspace of $(\mathcal{T}_{(0,1)}, \mathcal{T}_{0,1})$ and we set $\mathscr{B}_{a,b} := \sigma(\mathcal{T}_{a,b})$. In addition, consider the map $g_{a,b} : \mathcal{T}_{(0,1)} \to \mathcal{T}_{(a,b)}$ by

$$g_{a,b}(U) = \{a + x(b-a) : x \in U\}.$$

for each $U \in \mathcal{T}_{(0,1)}$.

For any $t \in \mathbb{R}_0^+$ let us define a Markov kernel $p_t^{(\alpha)} : \mathcal{T}_{(0,1)} \times \mathscr{B}_{0,1} \to [0,1]$ as follows:

**Definition 1.25** Set $p_t^{(\alpha)}(\emptyset, \cdot) := \delta_\emptyset$, where $\delta_\emptyset$ denotes the Dirac point mass at $\emptyset$. For any non–empty interval $(a,b) \in \mathcal{T}_{(0,1)}$ set

$$p_t^{(\alpha)}((a,b), A) := p_s(g_{a,b}^{-1}(A))$$

for every $t \in \mathbb{R}_0^+$ $A \in \mathscr{B}_{a,b}$, where $s := t(b-a)^\alpha$. Note that $g_{a,b}^{-1}(U)$ denotes the preimage of $U$ under the function $g_{a,b}$. For any $A \in \mathscr{B}_{0,1} \setminus \mathscr{B}_{a,b}$ we set $p_t^{(\alpha)}((a,b), A) := 0$.

Now let $U \in \mathcal{T}_{(0,1)}$ and consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $[0,1]$ such that $U = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$. Further, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables $X_n$ with distribution $p_t^{(\alpha)}((a_n, b_n), \cdot)$. Then define $p_t^{(\alpha)}(U, \cdot)$ to be the distribution of $\bigcup_{n\in\mathbb{N}} X_n$.

We can now define interval fragmentation processes.

**Definition 1.26** We call a $\mathcal{T}_{(0,1)}$–valued Markov process $\mathfrak{I} := (\mathfrak{I}(t))_{t\in\mathbb{R}_0^+}$, continuous in probability, a *self–similar (standard) interval fragmentation process* with index $\alpha \in \mathbb{R}$ if

(i) $\mathfrak{I}(0) = (0, 1)$.

(ii) $\mathfrak{I}(t) \subseteq \mathfrak{I}(s)$ for all $s, t \in \mathbb{R}_0^+$ with $s \leq t$.

(iii) Denote the distribution of $\mathfrak{I}(t)$, $t \in \mathbb{R}_0^+$, by $p_t$. Then the transition semigroup of $\mathfrak{I}$ is determined by the Markov kernels $(p_t^{(\alpha)})_{t\in\mathbb{R}_0^+}$ provided by Definition 1.25.

If $\alpha = 0$ then the process is called *homogenous*.

Let us mention that the continuity in probability in Definition 1.26 is meant with respect to the metric $\rho_{\mathcal{T}_{(0,1)}}$. We further remark that similarly to the case of mass fragmentation processes, cf. Theorem 1.13, and $\mathbb{P}$–fragmentation processes, see Theorem 1.21, also homogenous interval fragmentation processes without erosion can be constructed via Poisson point processes.

## 1.6 Bijections between different classes of fragmentation processes

According to Proposition 2.6 in [Ber02a] the $\mathcal{S}_1$–valued process consisting of the reordered sequences of the asymptotic frequencies of a self–similar $\mathcal{P}$–fragmentation process with index $\alpha \in \mathbb{R}$ and $\mathbb{P}$–dislocation measure $\mu_\nu$ constitutes a self–similar mass fragmentation process with index $\alpha$ and $\mathcal{S}_1$–dislocation measure $\nu$. Moreover, in [Ber02a, Proposition 2.6] Berestycki also shows that the converse holds in the sense that for any self–similar mass fragmentation process $\lambda$ with index $\alpha \in \mathbb{R}$ and $\mathcal{S}_1$–dislocation measure $\nu$ there exists some self–similar $\mathcal{P}$–fragmentation process with index $\alpha$ and $\mathcal{P}$–dislocation measure $\mu_\nu$, whose asymptotic frequencies form a process having the same distribution as $\lambda$. That is, there exists a bijection between mass fragmentation processes and $\mathcal{P}$–fragmentation processes. Moreover, Section 3.2 in [Ber02b] shows that there is also a bijection between interval fragmentation processes and $\mathcal{P}$–fragmentation processes. Consequently, we have the following theorem:

**Theorem 1.27 ([Ber02a], [Ber02b])** *The three classes of fragmentations that we introduced in the previous sections are mutually in a one–to–one correspondence with each other.*

Therefore, without loss of generality we can always choose the representation that is most useful in a specific situation. In this regard, we remark that Figure 1-2 is an illustration of any kind of fragmentation processes as it is just concerned with the sizes of the blocks which always constitute a mass fragmentation process. Note that by size of a block we mean the asymptotic frequency for $\mathcal{P}$–fragmentation processes and the lengths of the interval components of open sets for interval fragmentations.

**Remark 1.28** It is shown in [Ber02a, Proposition 2.3] that self–similar standard mass fragmentation processes are Feller processes[2]. Moreover, the Feller property was established for self–similar $\mathbb{P}$–fragmentation processes in [Ber06, Lemma 3.13] and for self–similar interval fragmentation processes in [Ber02b, Lemma 4]. Hence, by Kinney's regularity theorem, see [Kal01, Theorem 17.15], these processes have a version which is almost surely right–continuous with limits from the left. We implicitly always assume that we are dealing with such a version when considering fragmentation processes. Consequently, in view of [Chu82, Theorem 1 in Section 2.3] or [Kal01, Theorem 17.17], self–similar fragmentation processes satisfy the strong Markov property. Note that here we have used that the state spaces of the fragmentation processes are in particular locally compact Polish spaces, so that the above–mentioned results in [Chu82] and [Kal01] are applicable. $\Diamond$

Throughout this thesis we consider a self–similar standard $\mathcal{P}$–fragmentation process $\Pi = (\Pi(t))_{t\in\mathbb{R}_0^+}$ without erosion. In addition, let $\lambda = (\lambda(t))_{t\in\mathbb{R}_0^+} := (|\Pi(t)|^{\downarrow})_{t\in\mathbb{R}_0^+}$ and $\mathfrak{I} := (\mathfrak{I}(t))_{t\in\mathbb{R}_0^+}$ be the corresponding mass fragmentation process and interval fragmentation process respectively, given by the aforementioned bijections, see Theorem 1.27, between these classes of fragmentation processes.

## 1.7 Subordinators associated with fragmentations

This section is devoted to a specific subordinator, that is a nondecreasing Lévy process, which appears in the context of fragmentation processes. This subordinator plays a

[2] For a definition of the Feller property for Markov processes see e.g. Section 2.2 in [Chu82].

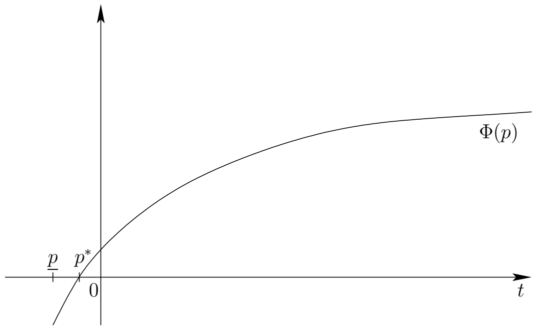



**Figure 1-3:** *Graph of the Laplace exponent $\Phi$ in the dissipative case with $\underline{p} < 0$, $\Phi(\underline{p}) > -\infty$ and $\Phi'(\underline{p}) < \infty$. Note that in this illustration there exists a $p^* \in (\underline{p}, 0)$ with $\Phi(p^*) = 0$.*

crucial role in our work.

**Definition 1.29** Let $\mathscr{F} := (\mathscr{F}_t)_{t \in \mathbb{R}_0^+}$ denote the filtration generated by the process $\Pi$ and note that $\lambda$ is adapted to $\mathscr{F}$. In addition, let $\mathscr{G} := (\mathscr{G}_t)_{t \in \mathbb{R}_0^+}$ be the sub–filtration generated by $\lambda$ and let $\mathscr{F}^1 := (\mathscr{F}_t^1)_{t \in \mathbb{R}_0^+}$ denote the filtration generated by $(\Pi_1(t))_{t \in \mathbb{R}_0^+}$.

Set

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{S}_1} \left| 1 - \sum_{n \in \mathbb{N}} s_n^{1+p} \right| \nu(\mathrm{d}\mathbf{s}) < \infty \right\} \in (-1, 0]. \tag{1.13}$$

It is well known that the function $\Phi : (\underline{p}, \infty) \to \mathbb{R}$, defined by

$$\Phi(p) = \int_{\mathcal{S}_1} \left( 1 - \sum_{n \in \mathbb{N}} s_n^{1+p} \right) \nu(\mathrm{d}\mathbf{s}) \tag{1.14}$$

for every $(\underline{p}, \infty)$, is monotonically increasing and concave. If $\underline{p} = 0$ in the conservative case, then we set $\Phi(\underline{p}) := 0$.

A typical graph of $\Phi$ is depicted in Figure 1-3. Note that this graph corresponds to the dissipative case. In the conservative case we always have that $\Phi$ passes through the origin, that is $\Phi(0) = 0$. Notice that the following three different possibilities for the behaviour of $\Phi$ at $\underline{p}$ can occur:

- $\Phi(\underline{p}) > -\infty$ and $\Phi'(\underline{p}+) < \infty$,
- $\Phi(\underline{p}) > -\infty$ and $\Phi'(\underline{p}+) = \infty$,
- $\Phi(\underline{p}) = -\infty$ and $\Phi'(\underline{p}+) = \infty$.

The illustration in Figure 1-3 depicts the first case.

**Definition 1.30 (cf. Lemma 1 in [Ber03])** Let $\bar{p}$ be the unique solution to

$$(1+p)\Phi'(p) = \Phi(p) \tag{1.15}$$

on $(\underline{p}, \infty)$, where $\Phi'$ denotes the derivative of $\Phi$.

Set

$$\zeta := \inf\{t \in \mathbb{R}_0^+ : |\Pi_1(t)| = 0\}$$

and note that $\zeta$ is an exponentially distributed random variable with parameter $\Phi(0)$.

The importance of the function $\Phi$, defined in (1.14), for fragmentation processes becomes clear in the following theorem that was proven in [Ber01] (see also Theorem 3.2 in [Ber06] in this regard).

**Theorem 1.31 (Theorem 3 (ii) in [Ber01])** *The process* $\xi := (\xi(t))_{t\in\mathbb{R}_0^+}$*, defined by*

$$\xi(t) := -\ln|\Pi_1(t)|\mathbb{1}_{\{t>\zeta\}} = -\ln|\Pi_1(t)|\mathbb{1}_{\{|\Pi_1(t)|>0\}}$$

*for any* $t \in \mathbb{R}_0^+$*, is a killed subordinator with killing time* $\zeta$*, Laplace exponent* $\Phi$ *and Lévy measure* $L_\xi$ *given by*

$$L_\xi(dx) = e^{-x} \sum_{n\in\mathbb{N}} \nu(-\ln(s_n) \in dx)$$

*for all* $x \in (0, \infty)$*.*

In particular, this theorem says that the subordinator $\xi$ is killed at rate $\Phi(0) \in \mathbb{R}_0^+$, with zero killing rate corresponding to the survival of $\xi$, and that

$$\Phi(p) = -\frac{1}{t}\ln\left(\mathbb{E}\left(e^{-p\xi(t)}\right)\right) = -\frac{1}{t}\ln\left(\mathbb{E}\left(|\Pi_1(t)|^p\mathbb{1}_{\{t<\zeta\}}\right)\right)$$

for every $p \in (\underline{p}, \infty)$.

The killed subordinator $\xi$ is an important tool in the theory of fragmentation processes and appears frequently in the literature on this subject. Moreover, resorting to this subordinator we shall construct a spectrally negative Lévy process, see Definition 1.5, in Chapter 5 that will enable us to make use of the results compiled in Section 1.2.2. Let us further point out that Theorem 1.24 shows that $\xi$ is the Lévy process in the *Lamperti representation*, cf. [Lam72], of the positive self–similar Markov process $\Pi$.

**Remark 1.32** We remark that by exchangeability of $\Pi$, cf. Definition 1.18, we also have that

$$\xi_n := (\xi_n(t))_{t\in\mathbb{R}_0^+} := \big(-\ln(|B_n(t)|)\mathbb{1}_{\{|B_n(t)|>0\}}\big)_{t\in\mathbb{R}_0^+}$$

as well as

$$\xi_U := (\xi_U(t))_{t\in\mathbb{R}_0^+} := \big(-\ln(|I_U(t)|)\mathbb{1}_{\{|B_U(t)|>0\}}\big)_{t\in\mathbb{R}_0^+}$$

are killed subordinators with Laplace exponent $\Phi$ for every $n \in \mathbb{N}$ and any uniformly distributed random variable $U$ on $[0,1]$. Notice that $\xi = \xi_1$, since $\Pi_1(t) = B_1(t)$ for all $t \in \mathbb{R}_0^+$. $\Diamond$

## 1.8 The intrinsic additive martingale for fragmentation processes

Throughout this section we assume that $\alpha = 0$, that is we consider a homogenous fragmentation process.

Let us start by considering the process $(e^{\Phi(p)t}|B_n(t)|^p\mathbb{1}_{\{|B_n(t)>0|\}})_{t\in\mathbb{R}_0^+}$ for $n \in \mathbb{N}$. Recall that

$$e^{\Phi(p)t}|B_n(t)|^p\mathbb{1}_{\{|B_n(t)>0|\}} = e^{\Phi(p)t-p\xi_n(t)}$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. This process with $n = 1$ was considered for instance in [BR03] and there it was used that it is a martingale with respect to the filtration $\mathscr{F}$. Let us briefly show that for any $n \in \mathbb{N}$ this process is indeed an $\mathscr{F}$–martingale. To this end, let $s, t \in \mathbb{R}_0^+$ and observe that the independent and identically distributed increments of the subordinator $\xi_n$ yield that

$$\begin{aligned}\mathbb{E}\left(e^{\Phi(p)(t+s)-p\xi_n(t+s)}\middle|\mathscr{F}_t\right) &= e^{\Phi(p)t-p\xi_n(t)}e^{\Phi(p)s}\mathbb{E}\left(e^{-p\xi_n(s)}\right)\\ &= e^{\Phi(p)t-p\xi_n(t)},\end{aligned}$$

where the final equality follows from $\Phi$ being the Laplace exponent of the killed subordinator $\xi_n$, see Theorem 1.31.

Later on we shall make use of another $\mathscr{F}$–martingale that in contrast to the above process is also $\mathscr{G}$–adapted. This martingale is given by the following lemma:

**Lemma 1.33** *The stochastic process $M(p) := (M_t(p))_{t\in\mathbb{R}_0^+}$, defined by*

$$M_t(p) := e^{\Phi(p)t}\sum_{n\in\mathbb{N}}\lambda_n^{1+p}(t)$$

*for all $t \in \mathbb{R}_0^+$ and $p \in (\underline{p}, \infty)$, is a martingale with respect to the filtration $\mathscr{F}$.*

The martingale property of the process $M(p)$ is well known, but since the proof is rather short we decided to present it here.

**Proof** Note first that for every Borel–measuable $A \subseteq [0,1]$ we have

$$\mathbb{P}(|B_1(t)| \in A) = \sum_{n \in \mathbb{N}} \mathbb{P}(|\Pi_n(t)| \in A)\mathbb{P}(B_1(t) = \Pi_n(t))$$

for all $t \in \mathbb{R}_0^+$. Since

$$\mathbb{P}(B_1(t) = \Pi_n(t)) = |\Pi_n(t)|,$$

it follows that

$$\mathbb{E}\left(\sum_{n \in \mathbb{N}} |\Pi_n(t)|^{1+p}\right) = \mathbb{E}\left(|B_1(t)|^p\right) = \mathbb{E}\left(|\Pi_1(t)|^p\right) = e^{-\Phi(p)t}, \tag{1.16}$$

where the final equality results from $(e^{t\Phi(t)}|\Pi_1(t)|^p \mathbb{1}_{\{t<\zeta\}})_{t \in \mathbb{R}_0^+}$ being a unit–mean martingale as mentioned above. Hence, we deduce from the fragmentation property that

$$\begin{aligned}
\mathbb{E}(M_{t+s}(p)|\mathscr{F}_t) &= \mathbb{E}\left( e^{\Phi(p)(t+s)} \sum_{n \in \mathbb{N}} |\Pi_n(t+s)|^{1+p} \middle| \mathscr{F}_t \right) \\
&= e^{\Phi(p)t} \sum_{n \in \mathbb{N}} |\Pi_n(t)|^{1+p} \mathbb{E}\left( e^{\Phi(p)s} \sum_{k \in \mathbb{N}} |\Pi_k(s)|^{1+p} \right) \\
&= e^{\Phi(p)t} \sum_{n \in \mathbb{N}} |\Pi_n(t)|^{1+p} \\
&= M_t(p),
\end{aligned}$$

which shows that $M(p)$ is a martingale. □

Let us mention that the martingale $M(p)$ appears frequently in the literature on fragmentation processes, see for example [Ber03], [BR03], [BM05], [HKK10] as well as [BHK10], and is often called *intrinsic additive martingale*. Moreover, similar additive martingales are also considered in the literature on branching processes, see for instance [Ner81] with regard to our considerations in Part I of this thesis and [Kyp04] with regard to Part II. In fact, $M(p)$ is the analogue of Biggins' classical additive martingale for branching random walks, see e.g. [Big92].

**Remark 1.34** By the martingale convergence theorem the nonnegative martingale

$M(p)$ has a $\mathbb{P}$–a.s. limit $M_\infty(p)$ for every $p > \underline{p}$. In [BR03, Theorem 1] (cf. also Theorem 4 in [BR05] for the conservative case) it is shown that $M_t(p) \to M_\infty(p)$ in $\mathscr{L}^1(\mathbb{P})$ as $t \to \infty$ for any $p \in (\underline{p}, \bar{p})$. Moreover, there it is also shown that $M_\infty(p) = 0$ $\mathbb{P}$–a.s for all $p \geq \bar{p}$. In [Ber03, Theorem 2] Bertoin showed that $M_\infty(p) > 0$ $\mathbb{P}$–a.s. if $\nu$ is conservative. $\Diamond$

Using the ideas of the proof of Theorem 2 in [Ber03] and adapting them to the dissipative case we obtain the corresponding results, to those in Remark 1.34, for dissipative fragmentation processes. The following lemma establishes the almost sure positivity of $M_\infty(p)$ for $p \in (\underline{p}, \bar{p})$. We remark that the uniform integrability of $M(p)$ follows from the forthcoming Proposition 3.5 in Chapter 3.

**Lemma 1.35** *Let $p \in (\underline{p}, \bar{p})$. Then we have that $M_\infty(p) > 0$ $\mathbb{P}$–almost surely.*

**Proof** Let $t \in \mathbb{R}^+$. Resorting to the fragmentation property of $\Pi$, we infer that

$$\mathbb{P}\left(M_\infty(p) = 0 | \mathscr{F}_t\right) = \prod_{n\in\mathbb{N}} \mathbb{P}_{\lambda_n(t)}(M_\infty(p) = 0)$$

$\mathbb{P}$–almost surely. Note that $\lambda_1(t) > 0$ $\mathbb{P}$–almost surely. Taking expectations we thus deduce that

$$\mathbb{P}\left(M_\infty(p) = 0\right) = \mathbb{E}\left(\prod_{n\in\mathbb{N}} \mathbb{P}_{\lambda_n(t)}(M_\infty(p) = 0)\right) \tag{1.17}$$

The homogeneity of $\Pi$ yields that

$$\mathbb{P}_x(M_\infty(p) = 0) = \mathbb{P}(xM_\infty(p) = 0) = \mathbb{P}(M_\infty(p) = 0)$$

for all $x > 0$. Note that $\mathbb{P}_0(M_\infty(p) = 0) = 1$. Hence, (1.17) results in

$$\mathbb{P}\left(M_\infty(p) = 0\right) = \mathbb{E}\left(\mathbb{P}(M_\infty(p) = 0)^{\mathrm{card}(\{n\in\mathbb{N}:\lambda_n(t)>0\})}\right),$$

that is

$$\mathbb{E}\left(\mathbb{P}\left(M_\infty(p) = 0\right) - \mathbb{P}(M_\infty(p) = 0)^{\mathrm{card}(\{n\in\mathbb{N}:\lambda_n(t)>0\})}\right) = 0. \tag{1.18}$$

Since, by Hypothesis 1.2, $\mathrm{card}(\{n \in \mathbb{N} : \lambda_n(t) > 0\}) > 1$ $\mathbb{P}$–a.s., we have that

$$\mathbb{P}\left(M_\infty(p) = 0\right) - \mathbb{P}(M_\infty(p) = 0)^{\mathrm{card}(\{n\in\mathbb{N}:\lambda_n(t)>0\})} \geq 0$$

$\mathbb{P}$–a.s., and thus we infer from (1.18) that

$$\mathbb{P}\left(M_\infty(p) = 0\right) = \mathbb{P}(M_\infty(p) = 0)^{\mathrm{card}(\{n\in\mathbb{N}:\lambda_n(t)>0\})}$$

$\mathbb{P}$–almost surely. Since $\mathrm{card}(\{n \in \mathbb{N} : \lambda_n(t) > 0\}) > 1$ $\mathbb{P}$–a.s., this implies that

$$\mathbb{P}(M_\infty(p) = 0) \in \{0, 1\}.$$

However, the uniform integrability of $M(p)$ thus yields that $\mathbb{P}(M_\infty(p) = 0) = 0$, because

$$\mathbb{E}(M_\infty(p)) = \mathbb{E}(M_0(p)) = 1 > 0.$$

□

## 1.9 Spine decomposition

The spine approach that we develop in this section is a tool that was successfully used with regard to various stochastic processes that possess a branching or fragmentation structure. For a detailed introduction to the spine method in the setting of branching diffusions we refer the reader to [HH09]. In the context of fragmentation processes we refer to [BR03] and [BR05].

In the present section we introduce a change of measure that is of avail for both Part I and Part II of this thesis. As in the previous section we consider a homogenous fragmentation process.

**Definition 1.36 (cf. Section 3.3 in [BR05])** We define for each $p \in (\underline{p}, \infty)$ a probability measure $\mathbb{P}^{(p)}$ on $\mathscr{F}_\infty := \bigcup_{t \in \mathbb{R}_0^+} \mathscr{F}_t$ by

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = e^{\Phi(p)t - p\xi(t)} \tag{1.19}$$

for all $t \in \mathbb{R}_0^+$.

The change of measure in Definition 1.36 is a so–called *Esscher transform*, cf. Section 3.3 in [Kyp06]. For any $p \in (\underline{p}, \infty)$ let $\mathbb{E}^{(p)}$ denote the expectation under $\mathbb{P}^{(p)}$. Theorem 3.9 in [Kyp06] shows that under the measure $\mathbb{P}^{(p)}$ the process $\xi$ is a subordinator with Laplace exponent $\Phi_p$ given by

$$\Phi_p(a) = \Phi(p + a) - \Phi(p)$$

for every $a \in \mathbb{R}_0^+$. Moreover, considering projections onto the sub–filtration $\mathscr{G}$ results in

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{G}_t} = M_t(p) \tag{1.20}$$

for any $p \in (\underline{p}, \infty)$ and $t \in \mathbb{R}_0^+$. Indeed, (1.20) holds true because we have

$$\mathbb{E}\left(e^{\Phi(p)t-p\xi(t)}\middle|\,\mathscr{G}_t\right) = e^{\Phi(p)t}\mathbb{E}\left(|B_1(t)|^p\middle|\,\mathscr{G}_t\right) \overset{(*)}{=} e^{\Phi(p)t}\mathbb{E}\left(\sum_{n\in\mathbb{N}}|\Pi_n(t)|^{1+p}\middle|\,\mathscr{G}_t\right) = M_t(p)$$

for all $p \in (\underline{p}, \infty)$ and $t \in \mathbb{R}_0^+$, where $(*)$ follows analogously to the first equality in (1.16).

**Remark 1.37** We remark that in view of Lemma 1.35 we have that restricted to the $\sigma$–algebra $\mathscr{G}_\infty := \bigcup_{t\in\mathbb{R}_0^+}\mathscr{G}_t$ the measures $\mathbb{P}^{(p)}$ and $\mathbb{P}$ are equivalent for any $p \in (\underline{p}, \bar{p})$. Moreover, since $M(p)$ is a uniformly integrable unit–mean martingale, we infer that $\mathbb{P}^{(p)}$ is a probability measure on $\mathscr{G}_\infty$. $\Diamond$

A similar change of measure has fruitfully been applied for branching processes in [LPP95] and [Lyo97]. In the light of these papers Bertoin and Rouault (cf. [BR03] and [BR05]) showed that under $\mathbb{P}^{(p)}$ the process $\Pi$ has the same distribution as the decreasingly ordered asymptotic frequencies of a $\mathcal{P}$–valued fragmentation process with a distinguished nested sequence of fragments. In the literature this sequence, from which all the other fragments descend, is often called the "spine" of the process. Bertoin and Rouault call the blocks in this distinguished sequence "tagged fragment" as one can imagine giving at each time of dislocation a tag to a uniformly chosen (among all fragments that exist at that time) fragment. This motivates the following definition:

**Definition 1.38** We call the stochastic process $(\Pi_1(t))_{t\in\mathbb{R}_0^+}$ the *spine* of $\Pi$ and for any $t \in \mathbb{R}_0^+$ we call $\Pi_1(t) = B_1(t)$, that is the block containing the element 1 at time $t$, the *tagged fragment*.

Note that by means of the exchangeability of $\Pi$, see Remark 1.19, we could also assume that the spine is $|B_n(t)|$ for any $n \in \mathbb{N}$ or $|B_U(t)|$ for any uniformly distributed random variable $U$ on $[0, 1]$.

The evolution of $\Pi$ under $\mathbb{P}^{(p)}$ differs from the evolution of $\Pi$ under $\mathbb{P}$ exactly at the behaviour of the spine, and all fragments that come off the spine evolve according to the behaviour of $\Pi$. More precisely, the evolution of $\Pi$ under $\mathbb{P}^{(p)}$ can be described by a Poisson point process on $\mathcal{P} \times \mathbb{N}$ with the following characteristic measure:

$$(\mu^{(p)} \otimes \sharp)|_{\mathcal{P}\times\{1\}} + (\mu \otimes \sharp)|_{\mathcal{P}\times\mathbb{N}\setminus\{1\}},$$

where the measure $\mu^{(p)}$ on $\mathcal{P}$ is given by

$$\mu^{(p)}(\mathrm{d}\pi) = |\pi_1|^p \mu(\mathrm{d}\pi) \tag{1.21}$$

for all $\pi \in \mathcal{P}$. Hence, we have the following *spine decomposition*:

$$|\Pi(t)| = (|\Pi_1(t)|, 0, \ldots) + \sum_{i \in \mathcal{I}_1 : t_i \leq t} \sum_{j \in \mathbb{N} \setminus \{1\}} \left|\Pi^{i,j}(t - t_i)\right| \tag{1.22}$$

$\mathbb{P}^{(p)}$–a.s., where the $\Pi^{i,j}$ are independent and satisfy

$$\mathbb{P}^{(p)}\left(|\Pi^{i,j}(u)| \in \cdot \,\middle|\, \mathscr{F}^1_{t_i}\right) = \mathbb{P}\left(x|\Pi(u)| \in \cdot\right)$$

with $x = |\Pi_1(t_i-) \cap \pi_j(t_i)|$. Moreover, the behaviour of the block $\Pi_1$ under $\mathbb{P}^{(p)}$ is determined by a Poisson point process with characteristic measure $\mu_\nu^{(p)}$.

# Part I

# Limit theorems for fragmentation processes

# CHAPTER 2

# STOPPED FRAGMENTATION PROCESSES

*This chapter is devoted to the study of a different class of fragmentations, namely those that are stopped at first passage below a given value. We derive results with regard to these processes that will be of avail in the subsequent chapters.*

## 2.1 Introduction

In the present chapter we consider stopping lines as a generalisation of the more common concept of stopping times. In the context of branching processes the concept of stopping lines was considered by various authors and in the setting of fragmentation processes it was introduced by Bertoin. The main purpose of our exposition in this chapter is to introduce fragmentation processes stopped at a specific example of such a stopping line. The concept of stopped fragmentation processes as considered in the present chapter plays a crucial role in Chapter 3 and Chapter 4. In this regard the present chapter aims at providing the necessary tools to which we shall resort predominantly in Chapter 3. The results in Section 2.4 deal with an intrinsic additive martingale for stopped fragmentations that is defined in a similar fashion to the intrinsic additive martingale for standard fragmentation processes as considered in Section 1.8. In Section 2.5 we shall derive a so–called many–to–one identity which allows us to reduce the consideration of the possibly infinitely many fragments in the fragmentation process at a given time or stopping line to the consideration of the behaviour of the spine of $\Pi$ that was defined in Definition 1.38.

Throughout this chapter we consider a homogenous fragmentation process $\Pi$ such that Hypothesis 1.1 and Hypothesis 1.2 hold. Further, let $B_n$, $n \in \mathbb{N}$, and $\lambda$ be defined as on page 16 and on page 20 respectively.

## 2.2 Stopping lines

Recall the filtration $\mathscr{F}^1$ defined in Definition 1.29 and note that $\mathscr{F}_t^1 = \sigma(B_1(t))$ for all $t \in \mathbb{R}_0^+$. With this in mind we define for any $n \in \mathbb{N} \setminus \{1\}$ a filtration $\mathscr{F}^n := (\mathscr{F}_t^n)_{t \in \mathbb{R}_0^+}$ by $\mathscr{F}_t^n := \sigma(B_n(t))$ for each $t \in \mathbb{R}_0^+$. A very useful concept for our considerations is the notion of stopping lines, cf. Definition 3.4 in [Ber06].

**Definition 2.1** A sequence $(L_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_0^+ \cup \{\infty\}$–valued random variables is called *stopping line* if

(i) $L_n$ is an $\mathscr{F}^n$–stopping time for every $n \in \mathbb{N}$.

(ii) $L_n = L_k$ for all $n \in \mathbb{N}$ and $k \in B_n(L_n)$.

Stopping lines were first considered in the theory of branching processes, see for example [Nev87], [Jag89] and [Cha91].

The fragmentation property of $\Pi$ extends to the situation where the deterministic times $s, t \in \mathbb{R}_0^+$ are replaced by stopping lines and is then called *extended fragmentation property*. More precisely, for any stopping line $L := (L_n)_{n\in\mathbb{N}}$ set

$$\mathcal{F}_L := \sigma(\{\Pi(L \wedge t) : t \in \mathbb{R}_0^+\}) = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathscr{F}_{L_n}^n\right).$$

Further, note that $\Pi(L) \in \mathcal{P}$ consists of all the blocks $\{B_n(L_n)) : n \in \mathbb{N}\}$. The extended fragmentation property then says that the conditional distribution, given $\mathcal{F}_L$, of the process $(\Pi(L+t))_{t\in\mathbb{R}_0^+}$ equals $\mathbb{P}_\pi(\Pi_t \in \cdot)$, where $\pi = \Pi(L)$.

The extended fragmentation property for fragmentation processes was established by Bertoin for $\mathcal{P}$–valued fragmentations in Lemma 3.14 in [Ber06] and for interval fragmentation processes (with the appropriate changes in Definition 2.1 and with an analogous definition of the extended fragmentation property) in Theorem 1 in [Ber02b].

We are mainly interested in a specific example of a stopping line, namely in the first passage times, defined by

$$\upsilon_{t,n} := \inf\left\{s \in \mathbb{R}_0^+ : |B_n(s)| < e^{-t}\right\} \tag{2.1}$$

for any $t \in \mathbb{R}_0^+$, when the asymptotic frequency of the block containing $n \in \mathbb{N}$ enters the interval $(0, e^{-t})$. Observe that $(\upsilon_{t,n})_{n\in\mathbb{N}}$ does indeed define a stopping line for any

$t \in \mathbb{R}_0^+$. In particular,

$$B_n(\upsilon_{t,n}) \cap B_k(\upsilon_{t,k}) \in \{\emptyset, B_n(\upsilon_{t,n})\}$$

for all $k, n \in \mathbb{N}$.

## 2.3 Stopped fragmentations

This section is devoted to introducing fragmentation processes stopped at the stopping line $(\upsilon_{t,n})_{n\in\mathbb{N}}$ that was defined in (2.1).

Our approach is to construct a stopped process $(\Pi_{t,n})_{n\in\mathbb{N},t\in\mathbb{R}_0^+}$ by describing the evolution of $(B_{t,n})_{t\in\mathbb{R}_0^+}$, the block in the stopped process that contains $n \in \mathbb{N}$. To this end, let $n \in \mathbb{N}$ as well as $t \in \mathbb{R}_0^+$ and set

$$B_{t,n}(s) := B_n(s \wedge \upsilon_{t,n})$$

for any $s \in \mathbb{R}_0^+$. The evolution $s \mapsto B_{t,k}(s)$ of distinct blocks is independent and happens according to the above description. Hence, at a given time $s \in \mathbb{R}_0^+$ only those blocks $B_{t,k}(s)$, $k \in \mathbb{N}$, still dislocate that are of size bigger than $e^{-t}$. This procedure describes $(B_{t,k}(s))_{s,t\in\mathbb{R}_0^+}$ for every $k \in \mathbb{N}$. We then define a process $(\Pi_{t,n})_{n\in\mathbb{N},t\in\mathbb{R}_0^+}$ by setting

$$\kappa_{1,s} := 1 \qquad \text{as well as} \qquad \kappa_{n,s} := \inf\left\{\mathbb{N} \setminus \left(\bigcup_{k=1}^{n-1} B_{t,k}(s)\right)\right\}$$

for any $n \in \mathbb{N} \setminus \{1\}$ and defining

$$\Pi_{t,n}(s) := B_{t,\kappa_{n,s}}(s)$$

for all $n \in \mathbb{N}$ and $s, t \in \mathbb{R}_0^+$. Notice that $\Pi_{t,n}(s) \in \mathcal{P}$ for all $s \in \mathbb{R}_0^+$. As with the non–stopped fragmentations it will be convenient to consider the $\mathcal{S}_1$–valued processes of the decreasingly ordered asymptotic frequencies, and consequently we adopt

$$\lambda_{t,n}(s) := |(\Pi_{t,k}(s))_{k\in\mathbb{N}}|_n^{\downarrow}$$

for every $n \in \mathbb{N}$ and $s, t \in \mathbb{R}_0^+$. Moreover, we will be interested in these stopped processes at the time at which they are stopped. In this regard, we set

$$B_{t,n} := B_{t,n}(\upsilon_{t,n}) = \lim_{s\to\infty} B_{t,n}(s)$$

as well as

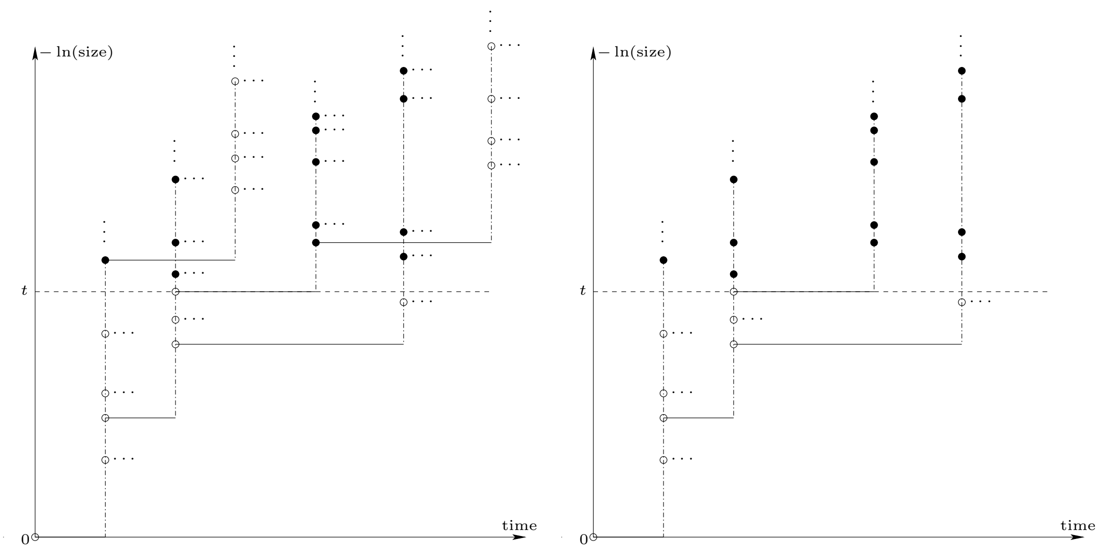
$$\Pi_{t,n} := \lim_{s\to\infty} \Pi_{t,n}(s) \qquad \text{and} \qquad \lambda_{t,n} := |(\Pi_{t,k})_{k\in\mathbb{N}}|_n^{\downarrow}$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. Note that the above limit exists as for sufficiently large $s \in \mathbb{R}_0^+$ the map $s \mapsto \Pi_{t,n}(s)$ is constant. Let us now define the stopped fragmentation process, see Figure 2-1, with which we shall be concerned in Chapter 3 and Chapter 4.

**Definition 2.2** The $\mathcal{S}_1$–valued stochastic process $\lambda^S := (\lambda_t)_{t\in\mathbb{R}_0^+}$ defined by

$$\lambda_t := (\lambda_{t,n})_{n\in\mathbb{N}}$$

for all $t \in \mathbb{R}_0^+$ is called *stopped fragmentation process.*




(a) Illustration of $\upsilon_{t,n}$. (b) Illustration of $\lambda^S$.

**Figure 2-1:** *Illustration (a) depicts the stopping line $\upsilon_{t,n}$ given by the first passage of the block sizes below $e^{-t}$ and (b) illustrates the stopped fragmentation process $\lambda^S$, stopped at $\upsilon_{t,n}$. The black dots indicate the blocks at the stopping line $\upsilon_{t,n}$, since their sizes are smaller than $e^{-t}$ and they result from the dislocation of blocks with size greater than or equal to $e^{-t}$.*

Let us denote by $\mathscr{H} := (\mathscr{H}_t)_{t\in\mathbb{R}_0^+}$ the filtration generated by the stopped $\mathcal{P}$–valued process $(\Pi_t)_{t\in\mathbb{R}_0^+}$, given by $\Pi_t = (\Pi_{t,n})_{n\in\mathbb{N}}$ for all $t \in \mathbb{R}_0^+$, that is

$$\mathscr{H}_t = \sigma(\{\Pi_{s,n} : n \in \mathbb{N}, s \in [0,t]\}) \tag{2.2}$$

for any $t \in \mathbb{R}_0^+$. Furthermore, we shall make use of

$$\sigma_{t,n} := \upsilon_{t,l_{t,n}} \tag{2.3}$$

for any $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$, where $l_{t,n} \in \mathbb{N}$ is chosen such that $\lambda_{t,n} = |B_{t,l_{t,n}}|$. That is to say, $\sigma_{t,n}$ is the time at which $\lambda_{t,n}$ is stopped. In addition, for any $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$ set

$$\sigma_t := \sup_{n\in\mathbb{N}} \sigma_{t,n} = \inf\left\{s \in \mathbb{R}_0^+ : \lambda_1(s) < e^{-t}\right\}. \tag{2.4}$$

## 2.4 The intrinsic additive martingale for stopped fragmentation processes

Recall the intrinsic additive martingale $M(p)$, $p \in (\underline{p}, \infty)$, that we considered in Section 1.8. In the present section we introduce an *intrinsic additive martingale* for stopped fragmentation processes. To this end, consider the processes $\Lambda(p) := (\Lambda_t(p))_{t\in\mathbb{R}_0^+}$, $p \in (\underline{p}, \infty)$, given by

$$\Lambda_t(p) := \sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p} e^{\Phi(p)\sigma_{t,n}}$$

for any $t \in \mathbb{R}_0^+$. The following lemma shows that $\Lambda(p)$ is a martingale with respect to the filtration $\mathscr{H}$ that was defined in (2.2).

**Lemma 2.3** *Assume that* $\Pi$ *is homogenous. Then the process* $\Lambda(p)$ *is a nonnegative, uniformly integrable* $\mathscr{H}$*–martingale for any* $p \in (\underline{p}, \bar{p})$.

**Proof** Fix an arbitrary $p \in (\underline{p}, \bar{p})$. For the time being, let $s, t \in \mathbb{R}_0^+$ and let $\mathcal{A}_t(s)$ denote the set of indices of fragments in $\lambda(s)$ whose size is greater than or equal to $e^{-t}$. Further, let $\mathcal{D}_t(s)$ denote the set of indices of fragments in $(\lambda_{t,n})_{n\in\mathbb{N}}$ which belong to $\lambda(u)$ for some $u \leq s$, that is which are either elements of $\lambda(u)$ or have descendants in $\lambda(u)$. Note that $\mathbb{E}(M_u(p)) = 1$ for all $u \in \mathbb{R}_0^+$. Resorting to the extended fragmentation property we thus obtain that

$$\begin{aligned}\mathbb{E}\left( M_s(p) \middle| \mathscr{H}_t\right) &= e^{\Phi(p)s} \sum_{n\in\mathcal{A}_t(s)} \lambda_n^{1+p}(s) + \sum_{n\in\mathcal{D}_t(s)} e^{\Phi(p)\sigma_{t,n}} \lambda_{t,n}^{1+p} \mathbb{E}\left( M^{(n)} \middle| \mathscr{H}_t\right) \\ &= e^{\Phi(p)s} \sum_{n\in\mathcal{A}_t(s)} \lambda_n^{1+p}(s) + \sum_{n\in\mathcal{D}_t(s)} e^{\Phi(p)\sigma_{t,n}} \lambda_{t,n}^{1+p} \end{aligned} \tag{2.5}$$

$\mathbb{P}$–a.s., where conditional on $\mathscr{H}_t$ the $M^{(n)}$ are independent and satisfy

$$\mathbb{E}\left(M^{(n)}\middle|\mathscr{H}_t\right) = f(s - \sigma_{t,n}) = 1$$

with $f(u) = \mathbb{E}\left(M_u(p)\right) = 1$ for any $u \in \mathbb{R}_0^+$.

An argument along the lines of the proof of Corollary 1.4 in [Ber06] yields that

$$\lim_{t\to\infty} -\frac{\ln(\lambda_1(t))}{t} = \Phi'(\bar{p}) \tag{2.6}$$

$\mathbb{P}$–almost surely. Since $t \mapsto \lambda_1(t)$ is right–continuous and $\sigma_t$ is its inverse, we infer that

$$\lim_{t\to\infty} \frac{\sigma_t}{t} = \frac{1}{\Phi'(\bar{p})} \tag{2.7}$$

$\mathbb{P}$–almost surely.

Let $t \in \mathbb{R}_0^+$. In view of (2.7) we have $\lim_{s\to\infty} \mathcal{A}_t(s) = \emptyset$ and $\lim_{s\to\infty} \mathcal{D}_t(s) = \mathbb{N}$ $\mathbb{P}$–almost surely. Consequently, (2.5) implies that

$$\lim_{s\to\infty} \mathbb{E}\left( M_s(p)\middle|\, \mathscr{H}_t\right) = \sum_{n\in\mathbb{N}} e^{\Phi(p)\sigma_{t,n}} \lambda_{t,n}^{1+p} = \Lambda_t(p) \tag{2.8}$$

$\mathbb{P}$–almost surely. Observe that

$$\mathbb{E}\left(\left|\mathbb{E}\left( M_s(p)\middle|\, \mathscr{H}_t\right) - \mathbb{E}\left( M_\infty(p)\middle|\, \mathscr{H}_t\right)\right|\right) \le \mathbb{E}\left(\left|M_s(p) - M_\infty(p)\right|\right) \to 0$$

as $s \to \infty$, since $M_\infty(p)$ is the $\mathscr{L}^1(\mathbb{P})$–limit of $M(p)$. Hence, $\mathbb{E}(M_\infty(p)|\mathscr{H}_t)$ is the $\mathscr{L}^1(\mathbb{P})$–limit of $(\mathbb{E}(M_s(p)|\mathscr{H}_t))_{s\in\mathbb{R}_0^+}$ as $s \to \infty$, and thus it follows from (2.8) that

$$\Lambda_t(p) = \mathbb{E}(M_\infty(p)|\mathscr{H}_t).$$

Therefore, $\Lambda(p)$ is a closed martingale, which proves the assertion. □

Let $p \in (\underline{p}, \bar{p})$ and set

$$\Lambda_\infty(p) := \limsup_{t\to\infty} \Lambda_t(p). \tag{2.9}$$

In view of the previous lemma we have $\Lambda_\infty(p) = \lim_{t\to\infty} \Lambda_t(p)$ $\mathbb{P}$–a.s. if $\Pi$ is homogenous. That is to say, for homogenous fragmentation processes the random variable $\Lambda_\infty(p)$ is the $\mathbb{P}$–a.s. limit of the nonnegative martingale $\Lambda(p)$. The following lemma is an analogue of Lemma 1.35 in that it shows that $\Lambda_\infty(p) > 0$ $\mathbb{P}$–almost surely.

**Lemma 2.4** *Assume that $\Pi$ is homogenous. Then we have*

$$\Lambda_\infty(p) > 0$$

*$\mathbb{P}$–a.s. for every $p \in (\underline{p}, \bar{p})$.*

**Proof** Resorting to the extended fragmentation property and the tower property of conditional expectations we obtain for any $s \in \mathbb{R}_0^+$ that

$$\begin{aligned}
\mathbb{P}\left(\lim_{t\to\infty}\sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p} e^{\Phi(p)\sigma_{t,n}} = 0\right) &= \mathbb{E}\left(\prod_{i\in\mathbb{N}} \mathbb{P}_{\lambda_{s,i}}\left(\lim_{t\to\infty}\sum_{j\in\mathbb{N}} \lambda_{t,j}^{1+p} e^{\Phi(p)\sigma_{t,j}} = 0\right)\middle| \mathscr{H}_s\right)\\
&= \mathbb{E}\left(\prod_{i\in\mathbb{N}} \mathbb{P}\left(\lim_{t\to\infty}\sum_{j\in\mathbb{N}} \lambda_j^{(i)} e^{\Phi(p)\sigma_j^{(i)}} = 0 \middle| \mathscr{H}_s\right)\right)\\
&= \mathbb{E}\left(\prod_{i\in\mathbb{N}} \mathbb{P}\left(\lim_{t\to\infty}\sum_{j\in\mathbb{N}} \lambda_{u,j}^{1+p} e^{\Phi(p)\sigma_{u,j}} = 0\right)\Bigg|_{u=t+\ln(\lambda_{s,i})}\right)
\end{aligned}$$

holds $\mathbb{P}$–a.s., where conditional on $\mathscr{H}_s$ the $\lambda_j^{(i)}$ are independent and satisfy

$$\mathbb{P}\left(\lambda_j^{(i)} \in \cdot \middle| \mathscr{H}_s\right) = \mathbb{P}\left(\lambda_{u,j}^{1+p} \in \cdot\right)\Big|_{u=t+\ln(\lambda_{s,i})}$$

and, given $\mathscr{H}_s$, also the $\sigma_j^{(i)}$ are independent and satisfy

$$\mathbb{P}\left(\sigma_j^{(i)} \in \cdot \middle| \mathscr{H}_s\right) = \mathbb{P}\left(\sigma_{u,j} \in \cdot\right)\big|_{u=t+\ln(\lambda_{s,i})}$$

$\mathbb{P}$–almost surely. Consequently, we deduce that

$$\mathbb{P}\left(\Lambda_\infty(p) = 0\right) = \prod_{i\in\mathbb{N}} \mathbb{P}\left(\Lambda_\infty(p) = 0\right). \tag{2.10}$$

Since $\Lambda_\infty(p)$ is the $\mathscr{L}^1(\mathbb{P})$–limit of $\Lambda(p)$ and $\mathbb{E}(\Lambda_t(p)) = 1$ for all $t \in \mathbb{R}_0^+$, cf. Lemma 2.3, we infer that $\mathbb{P}\left(\Lambda_\infty(p) = 0\right) < 1$. By (2.10) this implies that $\mathbb{P}\left(\Lambda_\infty(p) = 0\right) = 0$. □

**Remark 2.5** Assume there exists some $p^* \in [\underline{p}, 0]$ such that $\Phi(p^*) = 0$. Further, recall that according to Theorem 1.24 every self–similar fragmentation process is a time–changed homogenous one. Moreover, observe that $M(p^*)$ is just concerned with the sizes of the blocks of $\Pi$ but does not involve any time component. Therefore, a time change of $\Pi$ does not affect $M(p^*)$. Consequently, the statements of Lemma 2.3 and

Lemma 2.4 also hold for $p = p^*$ if $\Pi$ is self–similar with index of self–similarity $\alpha \neq 0$. In the light of (2.9) we thus have in particular that $\Lambda_\infty(p^*) = \lim_{t\to\infty} \Lambda_t(p^*)$ holds true $\mathbb{P}$–a.s. also in the self–similar setting.

A similar reasoning based on Theorem 1.24 will allow us to obtain the main results in Chapter 3 and Chapter 4 for self–similar fragmentation processes after proving these results in the homogenous setting. ◊

## 2.5 Many–to–one identities

In this section we develop a result that enables us to reduce the study of many fragments to that of a single fragment, viz the tagged fragment. For this reason this kind of result may be referred to as *many–to–one identity*. Such an identity first appeared in the literature on branching processes, see e.g. [BD75], [HW96] and [Har00]. For a version of a many–to–one identity in the context of fragmentation chains we refer to Lemma 5.1 in [HK08].

Recall that $B_n(t)$, $t \in \mathbb{R}_0^+$, denotes the block in $\Pi(t)$ which contains the element $n \in \mathbb{N}$. Furthermore, recall the stopped processes $(B_{t,n}(s))_{s\in\mathbb{R}_0^+}$ and $B_{t,n}$ that we defined in Section 2.2.

The many–to–one identity in our setting reads as follows:

**Lemma 2.6** *We have*

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}} |B_{u,n}(t)| f(\{|B_{u,n}(s)| : s \leq t\}) \mathbb{1}_{\{n=\min(B_{u,n}(t))\}}\right) = \mathbb{E}\left(f(\{|B_{u,1}(s)| : s \leq t\})\right)$$

*for every* $t, u \in \mathbb{R}_0^+ \cup \{\infty\}$ *and* $f : \mathrm{RCLL}([0,t],[0,1]) \to \mathbb{R}$*, where* $B_{u,n}(\infty)$ *is interpreted as* $B_{u,n}$ *and where we adopt* $B_{\infty,n}(s) := B_n(s)$ *for every* $s \in \mathbb{R}_0^+$*.*

Note that the indicator function that appears on the left–hand side above is needed in order to avoid counting a block multiple times. Using the indicator function ensures that to each block corresponds exactly one summand, namely the one associated with the least element of that block.

The identity provided by Lemma 2.6 will be used in different contexts in Chapter 3 and Chapter 5. Therefore, in Lemma 2.6 we give a fairly general version of a many–to–one identity in the context of fragmentation processes.

**Proof** Recall that in Section 1.6 we mentioned that for the $\mathbb{P}$–fragmentation process

$\Pi$ there exists a corresponding interval fragmentation $\mathfrak{I}$. Notice that the same holds true also for stopped fragmentations. Hence, for any $y \in (0,1)$ and $s \in \mathbb{R}_0^+$ let $\mathfrak{I}_{u,y}(s)$ be the interval at time $s$ in the interval representation of $(\Pi_{u,n}(s))_{n\in\mathbb{N}}$, $u \in \mathbb{R}_0^+ \cup \{\infty\}$, that contains $y$, where we adopt $\Pi_{\infty,n}(s) := \Pi_n(s)$. Further, fix $t \in \mathbb{R}_0^+$ as well as $u \in \mathbb{R}_0^+ \cup \{\infty\}$ and let $f : \mathrm{RCLL}([0,t],[0,1]) \to \mathbb{R}$. Then we have

$$\begin{aligned}
&\mathbb{E}\left(\sum_{n\in\mathbb{N}} |B_{u,n}(t)| f(\{|B_{u,n}(s)| : s \le t\}) \mathbb{1}_{\{n=\min(B_{u,n}(t))\}}\right) \\
&= \mathbb{E}\left(\int_{(0,1)} f(\{|\mathfrak{I}_{u,y}(s)| : s \le t\}) \, \mathrm{d}y\right) \\
&= \mathbb{E}\left(f(\{|\mathfrak{I}_{u,U}(s)| : s \le t\})\right),
\end{aligned}$$

where $U : \Omega \to (0,1)$ is a uniformly distributed random variable that is independent of $\Pi$. By means of the exchangeability of $\Pi$ the process $(-\ln(|\mathfrak{I}_{u,U}(t)|))_{t\in\mathbb{R}_0^+}$ has the same distribution under $\mathbb{P}$ as the stopped subordinator $(-\ln(|B_{u,1}(t)|))_{t\in\mathbb{R}_0^+}$, cf. Remark 1.32, and thus we have proven the assertion. □

Recall from (2.1) that

$$\upsilon_{t,1} = \inf\left\{s \in \mathbb{R}_0^+ : |B_1(s)| < e^{-t}\right\} = \inf\{s \in \mathbb{R}_0^+ : \xi(s) > t\}.$$

for all $t \in \mathbb{R}_0^+$. The following special case of a many–to–one identity, that follows easily from the identity in Lemma 2.6, will be used in Chapter 3.

**Corollary 2.7** *Let $f : [0,1] \to \mathbb{R}_0^+$ be a measurable function. Then*

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p^*} f(\lambda_{t,n})\right) = \mathbb{E}^{(p^*)}\left(f\left(e^{-\xi(\upsilon_{t,1})}\right)\right).$$

**Proof** By means of Lemma 2.6 we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p^*} f(\lambda_{t,n})\right) &= \mathbb{E}\left(\sum_{n\in\mathbb{N}} \lambda_{t,n} \left[\lambda_{t,n}^{p^*} f(\lambda_{t,n})\right]\right) \\
&= \mathbb{E}\left(|B_{t,1}|^{p^*} f(|B_{t,1}|)\right) \\
&= \mathbb{E}^{(p^*)}\left(f\left(e^{-\xi(\upsilon_{t,1})}\right)\right).
\end{aligned}$$

□

## 2.6 Concluding remarks

Mass fragmentation processes with a finite dislocation measure are akin to Crump–Mode–Jagers processes, where birth times correspond to the negative logarithm of the fragment sizes. In this respect we remark that in the context of Crump–Mode–Jagers processes the analogue of the stopped fragmentation process $\lambda^S$, see Definition 2.2, is often called *coming generation*. We refer to [Ner81] where this concept, in the setting of Crump–Mode–Jagers processes, was used in a context closely related to our considerations in Chapter 3.

Stopped fragmentation processes as introduced in this chapter are the main objects of our considerations in the following two chapters. Furthermore, they were also studied in a different context in [BM05]. More precisely, in [BM05] Bertoin and Martínez considered the energy that is needed to reduce a block of unit mass to fragments whose masses are smaller than a given value. Their result can be applied to the crushing of blocks of mineral in the mining industry. Hence, such stopped fragmentation processes are also interesting from a more applied point of view.

# CHAPTER 3

# STRONG LAW OF LARGE NUMBERS FOR FRAGMENTATION PROCESSES

*For self–similar fragmentation processes we show the almost sure convergence of an empirical measure associated with the stopping line corresponding to the first fragments of size smaller than $\eta \in (0,1]$.*

## 3.1 Introduction

In the spirit of a classical result for Crump–Mode–Jagers processes, cf. [Ner81, Theorem 5.4], we present a strong law of large numbers for self–similar fragmentation processes. In a more restrictive setting the limit theorem in question was also considered in [BM05, Corollary 2] with regard to $\mathscr{L}^2$–convergence. Here we are mainly interested in almost sure convergence. For an approach to deal with almost sure convergence in the case of a finite dislocation measure see [BM05, Corollary 1] in conjunction with [BM05, Remark 1 (b)]. Our goal in this chapter is to obtain the corresponding convergence result if the dislocation measure is infinite. In this regard we mention that all our results comprise the case of a finite dislocation measure, but more importantly we extend the aforementioned limit theorem to fragmentation processes with an infinite dislocation measure. Let us point out that such an extension is not straightforward. Indeed, we refer to Theorem 2 in [BM05] where a different limit theorem, in the sense of $\mathscr{L}^1$–convergence, is extended from the finite to the infinite activity setting. There Bertoin and Martínez make use of having shown the result for a finite dislocation measure and subsequently they use a discretisation method to infer the corresponding result for an infinite dislocation measure. Such a discretisation technique does not work for the problem under consideration in the present chapter and we do not resort to results that are already known in the finite activity case.

## 3.2 Set–up

In this chapter we consider a standard self–similar $\mathcal{P}$–fragmentation process $\Pi$ that satisfies Hypothesis 1.1 as well as Hypothesis 1.2. More specifically, we shall be mainly concerned with the corresponding stopped fragmentation process $\lambda^S$, see Definition 2.2, obtained by stopping the blocks of $\Pi$ at first passage below size $e^{-t}$ for any $t \in \mathbb{R}_0^+$.

Let $\underline{p}$ and $\Phi$ be given by (1.13) and (1.14) respectively.

**Definition 3.1** If there exists a $p^* \in [\underline{p}, 0]$ satisfying $\Phi(p^*) = 0$, then we call $p^*$ *Malthusian parameter.*

Recall from Definition 1.11 that we say the fragmentation process is dissipative if $\nu\left(\sum_{n\in\mathbb{N}} s_n < 1\right) > 0$. The following Hypothesis 3.1, commonly referred to as *Malthusian hypothesis*, provides us with the existence of a Malthusian parameter in the dissipative case.

**Hypothesis 3.1** If $\nu$ is dissipative, then there exists a $p^* \in (\underline{p}, 0)$ such that $\Phi(p^*) = 0$.

If $\nu$ is conservative, that is if $\nu\left(\sum_{n\in\mathbb{N}} s_n < 1\right) = 0$, then $\Phi(0) = 0$, and thus we set $p^* := 0$ in that case. Recall the definition of $\bar{p}$ in (1.15) and notice that it follows from Lemma 1 in [Ber03] that $p \geq \bar{p}$ if and only if $(1+p)\Phi'(p) \leq \Phi(p)$. Since $\Phi'(p) > 0$ for all $p \in (\underline{p}, \infty)$, we therefore have $p^* < \bar{p}$. Moreover, observe that Hypothesis 3.1 implies that $\underline{p} < 0$ and thus $\Phi'(0+) < \infty$ in the dissipative case. However, in the conservative case it is possible that $\underline{p} = 0$, in which case the expectation of the subordinator $\xi$ may be infinite. In order to guarantee that $\xi$ has finite expectation in the conservative case, we need the following hypothesis:

**Hypothesis 3.2** If $\underline{p} = 0$, then

$$\Phi'(0+) = \int_{\mathcal{S}_1} \left( \sum_{n\in\mathbb{N}} s_n \ln\left(s_n^{-1}\right) \right) \nu(\mathrm{ds}) < \infty.$$

In what follows, assume that Hypothesis 3.1 and Hypothesis 3.2 hold.

In order to state the main result of this chapter let us first introduce some notation. Let $\mathcal{B}^+$ denote the space of all bounded and measurable functions $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with

$f|_{[1,\infty)} \equiv 0$. In addition, set $\mathbf{1} :\equiv \mathbb{1}_{[1,\infty)}$. Moreover, for any $t \in \mathbb{R}_0^+$ consider the random measure $\rho_t$ on $[0,1]$ defined by

$$\rho_t := \sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p^*} \delta_{e^t \lambda_{t,n}},$$

where $\delta_x$ is the Dirac measure at $x \in [0,1]$. Our main result is concerned with the integral of test functions in $\mathcal{B}^+$ against the above–defined measure $\rho_t$. In this regard we define

$$\langle \rho_t, f \rangle := \int_{[0,1]} f \, \mathrm{d}\rho_t = \sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p^*} f\left(e^t \lambda_{t,n}\right) \tag{3.1}$$

for any $t \in \mathbb{R}_0^+$ and $f \in \mathcal{B}^+$. Notice that the time–parameter of the process $(\langle \rho_t, f \rangle)_{t\in\mathbb{R}_0^+}$ corresponds to the size rather than to the time of the fragmentation process $\Pi$. Define a measure $\rho$ on $[0,1]$ as follows:

$$\rho(\mathrm{d}t) = \frac{1}{\Phi'(p^*)} \left( \int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}} \mathbb{1}_{\{s_n<t\}} s_n^{1+p^*} \nu(\mathrm{d}\mathbf{s}) \right) \frac{\mathrm{d}t}{t},$$

where in the case $p^* = \underline{p} = 0$ we adopt $\Phi'(p^*) = \Phi'(0+)$. In the present chapter we are interested in the asymptotic behaviour of $\langle \rho_t, \cdot \rangle$ as $t$ tends to infinity. More precisely, our objective is to show that asymptotically as $t \to \infty$ the random function $t \mapsto \langle \rho_t, \cdot \rangle$ behaves $\mathbb{P}$–a.s. like the limit $\Lambda_\infty(p^*)$ of the nonnegative martingale $(\Lambda_t(p^*))_{t\in\mathbb{R}_0^+}$, up to a multiplicative function $\langle \rho, \cdot \rangle$ on $\mathcal{B}^+$ given by

$$\langle \rho, f \rangle = \int_{(0,1)} f \mathrm{d}\rho = \frac{1}{\Phi'(p^*)} \int_{(0,1)} f(t) \left( \int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}} \mathbb{1}_{\{s_n<t\}} s_n^{1+p^*} \nu(\mathrm{d}\mathbf{s}) \right) \frac{\mathrm{d}t}{t}$$

for all $f \in \mathcal{B}^+$.

## 3.3 Strong law of large numbers for fragmentation processes

Recall that we assume that $\Pi$ satisfies Hypothesis 1.1 and Hypothesis 1.2. Our main result in this chapter is the following strong law of large numbers for self–similar fragmentation processes:

**Theorem 3.2** *For any self–similar fragmentation process satisfying Hypothesis 3.1* *and Hypothesis 3.2 we have*

$$\lim_{t\to\infty} \langle \rho_t, f\rangle = \langle \rho, f\rangle \Lambda_\infty(p^*) \tag{3.2}$$

$\mathbb{P}$*–a.s. for any* $f \in \mathcal{B}^+$.

Theorem 3.2 follows a series of strong laws of large numbers that were obtained for different classes of branching processes. Related classical strong laws were considered in [AH76] and [AH77] for spatial branching processes. Nerman [Ner81] proved a more general strong law of large numbers in the context of Crump–Mode–Jagers processes. More recently, strong laws of large numbers in the spirit of Theorem 3.2 were obtained in [CS07], [CRW08] and [EHK10] for branching diffusions and in [EW06] as well as [Eng09] in the setting of superdiffusions. For related results, in the sense of $\mathscr{L}^2$–convergence, in the context of conservative fragmentation processes we refer to [BM05, Corollary 2] as well as [HK08], where the latter is concerned with conservative fragmentation chains.

It turns out that the DCT is applicable in order to get the corresponding result to Theorem 3.2 also in the sense of $\mathscr{L}^p$–convergence for some $p > 1$.

**Corollary 3.3** *For any self–similar fragmentation process satisfying Hypothesis 3.1 and Hypothesis 3.2 we have*

$$\lim_{t\to\infty} \langle \rho_t, f\rangle = \langle \rho, f\rangle \Lambda_\infty(p^*)$$

*in* $\mathscr{L}^p(\mathbb{P})$ *for any* $f \in \mathcal{B}^+$ *and all* $p \in [1, (1+\bar{p})(1+p^*)^{-1}]$.

## 3.4 Preliminary considerations

Recall the change of measure in (1.19) for $p^*$, that is

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p^*)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = e^{-p^*\xi(t)}.$$

Moreover, Theorem 3.9 in [Kyp06] also tells us that $L_\xi^{(p^*)}(\mathrm{d}t) = e^{-p^*t} L_\xi(\mathrm{d}t)$ for all $t \in \mathbb{R}^+$, where $L_\xi$ and $L_\xi^{(p^*)}$ are the Lévy measures associated with $\xi$ under $\mathbb{P}$ and $\mathbb{P}^{(p^*)}$ respectively. According to Theorem 1.31 the Lévy measure $L_\xi^{(p^*)}$ is related to the dislocation measure $\nu$ in the following way:

$$L_\xi^{(p^*)}(\mathrm{d}x) = e^{-(1+p^*)x} \sum_{n\in\mathbb{N}} \nu(-\ln(s_n) \in \mathrm{d}x) \tag{3.3}$$

for all $x \in \mathbb{R}^+$.

**Lemma 3.4** *Assume that Hypothesis 3.1 and Hypothesis 3.2 are satisfied. Then the limit* $\lim_{t\to\infty} \mathbb{E}(\langle \rho_t, f\rangle)$ *exists and satisfies*

$$\lim_{t\to\infty} \mathbb{E}(\langle \rho_t, f\rangle) = \langle \rho, f\rangle$$

*for all* $f \in \mathcal{B}^+$.

**Proof** Notice that

$$\mathbb{E}^{(p^*)}(\xi(1)) = \Phi'(p^*) < \infty$$

and recall from (2.1) that

$$\upsilon_{t,1} = \inf\left\{s \in \mathbb{R}_0^+ : |B_1(s)| < e^{-t}\right\} = \inf\{s \in \mathbb{R}^+ : \xi(s) > t\}.$$

for all $t \in \mathbb{R}_0^+$. By means of Theorem 1 in [BHS99] we thus infer that

$$\lim_{t\to\infty} \mathbb{P}^{(p^*)}(\xi(\upsilon_{t,1}) - t \in \mathrm{d}x) = \frac{1}{\Phi'(p^*)} L_\xi^{(p^*)}((x,\infty))\ \mathrm{d}x. \tag{3.4}$$

Observe that with the substitution $z := e^{-y}$ we infer from (3.3) that

$$\begin{aligned} L_\xi^{(p^*)}((x,\infty)) &= \int_{(x,\infty)} L_\xi^{(p^*)}(\mathrm{d}y) \\ &= \int_{(x,\infty)} e^{-(1+p^*)y} \sum_{n\in\mathbb{N}} \nu(-\ln(s_n) \in \mathrm{d}y) \\ &= \int_{(0,e^{-x})} z^{(1+p^*)} \sum_{n\in\mathbb{N}} \nu(s_n \in \mathrm{d}z) \end{aligned} \tag{3.5}$$

Since Corollary 2.7 shows that $\mathbb{E}(\langle \rho_t, f\rangle) = \mathbb{E}^{(p^*)} f\left(e^{-(\xi(\upsilon_{t,1})-t)}\right)$, the assertions of the lemma follow from the following calculation:

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}^{(p^*)} f\left(e^{-(\xi(\upsilon_{t,1})-t)}\right) &= \int_{\mathbb{R}_0^+} f\left(e^{-x}\right) \lim_{t\to\infty} \mathbb{P}^{(p^*)}(\xi(\upsilon_{t,1}) - t \in \mathrm{d}x) \\ &= \frac{1}{\Phi'(p^*)} \int_{\mathbb{R}_0^+} f\left(e^{-x}\right) L_\xi^{(p^*)}((x,\infty))\mathrm{d}x \end{aligned}$$

$$= \frac{1}{\Phi'(p^*)} \int_{\mathbb{R}_0^+} f\left(e^{-x}\right) \sum_{n\in\mathbb{N}} \int_{(0,e^{-x})} z^{(1+p^*)} \nu(s_n \in \mathrm{d}z)\mathrm{d}x$$

$$= \frac{1}{\Phi'(p^*)} \int_{\mathbb{R}_0^+} f\left(e^{-x}\right) \sum_{n\in\mathbb{N}} \int_{\mathcal{S}_1} s_n^{(1+p^*)} \mathbb{1}_{\{s_n<e^{-x}\}} \nu(\mathrm{d}\mathbf{s})\mathrm{d}x$$

$$= \frac{1}{\Phi'(p^*)} \int_{(0,1)} f\left(u\right) \sum_{n\in\mathbb{N}} \int_{\mathcal{S}_1} s_n^{(1+p^*)} \mathbb{1}_{\{s_n<u\}} \nu(\mathrm{d}\mathbf{s}) \frac{\mathrm{d}u}{u}$$

$$= \langle \rho, f \rangle$$

for every $f \in \mathcal{B}^+$. Note that in the above chain of equalities we applied the DCT for the first equality and the second equality follows from (3.4). The third equality is a consequence of (3.5) and for the penultimate equality we used the substitution $u := e^{-x}$. □

## 3.5 Proof of the strong law of large numbers for fragmentation processes

The goal of this section is to prove Theorem 3.2. Our method of proof is based on several auxiliary results that we shall develop below. Some of these auxiliary results are concerned with $\mathbb{E}(\langle \rho_{st}, f\rangle | \mathscr{H}_t)$, that is with the conditional expectation of the random variable under consideration in Theorem 3.2. Indeed, it turns out that asymptotically we can obtain good approximations of this conditional expectation with respect to both the left– and right–hand side of (3.2). These approximations then enable us to tackle the proof of Theorem 3.2.

In the present section we initially assume that $\Pi$ is homogenous. For that matter, let us point out that all the following auxiliary results are obtained in the setting of homogenous fragmentation processes. The generalisation to self–similar mass fragmentation processes with index of self–similarity $\alpha \neq 0$ will be made once we have proven Theorem 3.2 in the homogenous setting.

Let us start with the following proposition that provides us with $\mathscr{L}^p$–boundedness of the martingale $\Lambda(p)$, see Section 2.4, for some $p > 1$.

**Proposition 3.5** *Let $\tilde{p} \in (\underline{p}, 0)$. Then*

$$\sup_{t\in\mathbb{R}_0^+} \Lambda_t(\tilde{p}) \in \mathscr{L}^p(\mathbb{P})$$

*holds for all $p \in [1, (1+\bar{p})(1+\tilde{p})^{-1}]$.*

**Proof** The first part of the proof follows the lines of the proof of Theorem 2 in [Ber03]. However, the crucial argument in the proof of Theorem 2 in [Ber03] requires that the dislocation measure is conservative, and thus we have to develop a different argument in order to cater for the dissipative case.

Let $p \in (1,2)$. According to [Lép76] we have $\sup_{t\in\mathbb{R}_0^+} \mathbb{E}\left([M_t(\tilde{p})]^p\right) < \infty$ if

$$V_p(\tilde{p}) := \sum_{t\geq 0} |M_t(\tilde{p}) - M_{t-}(\tilde{p})|^p \in \mathscr{L}^1(\mathbb{P}), \tag{3.6}$$

where the above sum is taken over all $t \in (t_i)_{i\in\mathcal{I}}$, that is over all jump times of $\Pi$. For any such $t \in \mathbb{R}_0^+$ we have that

$$|M_t(\tilde{p}) - M_{t-}(\tilde{p})| = e^{\Phi(\tilde{p})t}\lambda_{k(t)}^{1+\tilde{p}} \left|1 - \sum_{n\in\mathbb{N}} \Delta_n^{1+\tilde{p}}(t)\right| .$$

By the compensation formula for Poisson point processes we have

$$\mathbb{E}\left(V_p(\tilde{p})\right) = c(\tilde{p},p) \int_{\mathbb{R}_0^+} e^{p\Phi(\tilde{p})t}\mathbb{E}\left(\sum_{n\in\mathbb{N}} \lambda_n^{p(1+\tilde{p})}(t)\right)\, \mathrm{d}t, \tag{3.7}$$

where

$$c(\tilde{p},p) := \int_{\mathcal{S}_1} \left|1 - \sum_{n\in\mathbb{N}} s_n^{1+\tilde{p}}\right|^p \nu(\mathrm{d}\mathbf{s}).$$

In order to apply the criterion (3.6) let us first show that $c(\tilde{p},p) < \infty$ for suitable $p > 1$. To this end, note that Jensen's inequality yields that

$$(u+v)^p \leq 2^{p-1}(u^p + v^p)$$

for all $u, v \in \mathbb{R}_0^+$. Hence, we have

$$\left|1 - \sum_{n\in\mathbb{N}} s_n^{1+\tilde{p}}\right|^p \leq \left[|1-s_1| + \sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+\tilde{p}}\right]^p \leq 2^{p-1}\,|1-s_1|^p + 2^{p-1}\left[\sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+\tilde{p}}\right]^p$$

Moreover, another application of Jensen's inequality yields that

$$\left[\sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+\tilde{p}}\right]^p = \left[\sum_{n\in\mathbb{N}\setminus\{1\}} s_n s_n^{\tilde{p}}\right]^p \leq \left[\sum_{n\in\mathbb{N}\setminus\{1\}} s_n\right]^{p-1} \sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+p\tilde{p}} \leq \sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+p\tilde{p}}.$$

Consequently,

$$c(\tilde{p},p) = \int_{\mathcal{S}_1} \left|1 - \sum_{n\in\mathbb{N}} s_n^{1+\tilde{p}}\right|^p \nu(\mathrm{d}\mathbf{s}) \leq \int_{\mathcal{S}_1} \left[2^{p-1}\left|1-s_1\right|^p + 2^{p-1} \sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+p\tilde{p}}\right] \nu(\mathrm{d}\mathbf{s}),$$

and thus it follows in view of (1.7) that

$$c(\tilde{p},p) < \infty \tag{3.8}$$

holds if

$$\int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+p\tilde{p}} \nu(\mathrm{d}\mathbf{s}) < \infty. \tag{3.9}$$

Let us now show that (3.9) holds true. To this end, observe first that

$$\begin{aligned} \sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+p\tilde{p}} &= \sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+p\tilde{p}} + s_1 - s_1 + 1 - 1 \\ &\leq \sum_{n\in\mathbb{N}\setminus\{1\}} s_n^{1+p\tilde{p}} + s_1^{1+p\tilde{p}} - s_1 + 1 - 1 \\ &\leq \left|\sum_{n\in\mathbb{N}} s_n^{1+p\tilde{p}} - 1\right| + \left|1 - s_1\right|. \end{aligned}$$

Hence, (1.7) and the definition of $\underline{p}$ imply that (3.9), and thus (3.8), holds true if $p\tilde{p} \in (\underline{p},\tilde{p})$. As a consequence we deduce that

$$[p \in (1, \underline{p}/\tilde{p})] \Longrightarrow [c(\tilde{p},p) < \infty]. \tag{3.10}$$

In order to deal with the integral in (3.7) notice that $\underline{p} < \tilde{p} < 0 < \bar{p}$ implies that

$$\frac{1+\tilde{p}}{\tilde{p}} < \frac{1+\bar{p}}{\underline{p}},$$

and thus for all $p \leq (1+\bar{p})(1+\tilde{p})^{-1}$ we have

$$p < \frac{\underline{p}}{\tilde{p}}. \tag{3.11}$$

Let $p \in (1, (1+\bar{p})(1+\tilde{p})^{-1}]$ and observe that

$$\tilde{p} < p(1+\tilde{p}) - 1 \leq \bar{p}. \tag{3.12}$$

Since $p \mapsto \Phi(p)/(1+p)$ is monotonically increasing on $(\underline{p}, \bar{p})$, cf. Lemma 1 in [Ber03], we infer from (3.12) that

$$\frac{\Phi(\tilde{p})}{1+\tilde{p}} < \frac{\Phi(p(1+\tilde{p})-1)}{p(1+\tilde{p})},$$

and consequently

$$p\Phi(\tilde{p}) < \Phi(p(1+\tilde{p})-1). \tag{3.13}$$

Note that $M(p)$ being a unit–mean martingale for any $p \in (\underline{p}, \infty)$, cf. Section 1.8, implies that $\mathbb{E}(M_t(p(1+\tilde{p})-1))$ for all $t \in \mathbb{R}_0^+$. Therefore,

$$\mathbb{E}\left(\sum_{n\in\mathbb{N}} \lambda_n^{p(1+\tilde{p})}(t)\right) = e^{-\Phi(p(1+\tilde{p})-1)t}.$$

for any $t \in \mathbb{R}_0^+$, and hence (3.13) results in

$$\int_{\mathbb{R}_0^+} e^{p\Phi(\tilde{p})t}\mathbb{E}\left(\sum_{n\in\mathbb{N}} \lambda_n^{p(1+\tilde{p})}(t)\right)\, \mathrm{d}t = \int_{\mathbb{R}_0^+} e^{(p\Phi(\tilde{p})-\Phi(p(1+\tilde{p})-1))t}\, \mathrm{d}t \in (0,\infty).$$

In view of (3.7), (3.10) and (3.11), we thus deduce that $\mathbb{E}(V_p(\tilde{p})) < \infty$. Consequently, the criterion (3.6) is satisfied and therefore we have

$$\lim_{t\to\infty} \mathbb{E}\left([M_t(\tilde{p})]^p\right) \le \sup_{t\in\mathbb{R}_0^+} \mathbb{E}\left([M_t(\tilde{p})]^p\right) < \infty.$$

Now Doob's maximal inequality, in conjunction with the MCT, yields that

$$\mathbb{E}\left(\left[\sup_{t\in\mathbb{R}_0^+} M_t(\tilde{p})\right]^p\right) = \lim_{s\to\infty} \mathbb{E}\left(\left[\sup_{t\in[0,s]} M_t(\tilde{p})\right]^p\right) \le \left(\frac{p}{p-1}\right)^p \lim_{s\to\infty} \mathbb{E}\left([M_s(\tilde{p})]^p\right) < \infty.$$

Resorting to Jensen's inequality for conditional expectations we then infer that

$$\begin{aligned}\sup_{t\in\mathbb{R}_0^+} \mathbb{E}\left([\Lambda_t(\tilde{p})]^p\right) &= \sup_{t\in\mathbb{R}_0^+} \mathbb{E}\left([\mathbb{E}\left(M_\infty(\tilde{p})|\mathscr{H}_t\right)]^p\right)\\ &\le \sup_{t\in\mathbb{R}_0^+} \mathbb{E}\left(\mathbb{E}\left(M_\infty^p(\tilde{p})|\mathscr{H}_t\right)\right)\\ &= \mathbb{E}\left(M_\infty^p(\tilde{p})\right),\end{aligned}$$

and another application of Doob's maximal inequality thus results in

$$\mathbb{E}\left(\left[\sup_{t\in\mathbb{R}_0^+} \Lambda_t(\tilde{p})\right]^p\right) \le \left(\frac{p}{p-1}\right)^p \lim_{t\to\infty} \mathbb{E}\left([\Lambda_t(\tilde{p})]^p\right) \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left(M_\infty^p(\tilde{p})\right) < \infty.$$

□

Notice that Proposition 3.5 implies that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}_0^+} \langle \rho_t, f\rangle^p\right) < \infty \tag{3.14}$$

for all $p \in [1, (1+\bar{p})(1+p^*)^{-1}]$.

Let us now establish an auxiliary result that will enable us to obtain a good asymptotic approximation, in an almost sure sense, of the right–hand side in (3.2) by the conditional expectation $\mathbb{E}(\langle \rho_{st}, f\rangle | \mathscr{H}_t)$. For this purpose, recall the definitions of $\sigma_{t,n}$ and $\sigma_t$ in (2.3) and (2.4) respectively. Furthermore, for any $t \in \mathbb{R}_0^+$ and $s > 1$ set

$$\mathcal{J}_{t,s} := \left\{n \in \mathbb{N} : \lambda_{t,n} \geq e^{-st}\right\} \quad \text{as well as} \quad \mathcal{J}_{t,s}^{\complement} := \left\{n \in \mathbb{N} : \lambda_{t,n} < e^{-st}\right\}.$$

**Lemma 3.6** *Assume that Hypothesis 3.1 and Hypothesis 3.2 are satisfied. Then there exists an* $s_0 \in (1, \infty)$ *such that*

$$\lim_{t\to\infty} \sum_{n\in\mathcal{J}_{t,s}^{\complement}} \lambda_{t,n}^{1+p^*} f\left(e^t \lambda_{t,n}\right) = 0$$

$\mathbb{P}$*–a.s. for all* $s \geq s_0$.

**Proof** We first prove the assertion in the conservative case (Part I), making use of Hypothesis 3.2, and then we use a different approach, which is based on Hypothesis 3.1, to prove the assertion in the dissipative case (Part II). Note that due the boundedness of $f$ is suffices to consider the case $f|_{[0,1]} \equiv 1$.

Part I Assume that $\nu$ is conservative, that is $p^* = 0$. In addition, fix some $s > 1$. Consider the corresponding interval fragmentation process $\mathfrak{I}$ and, motivated by the notation introduced in Section 1.7, let us define a stochastic process $\xi_u := (\xi_u(t))_{t\in\mathbb{R}_0^+}$, $u \in [0, 1]$, by

$$\xi_u(t) = -\ln(|\mathfrak{I}_u(t)|)$$

for any $t \in \mathbb{R}_0^+$. As in Section 1.7, we have that $\xi_u$, $u \in (0, 1)$, is a subordinator with Laplace exponent $\Phi$. Further, for any $u \in [0, 1)$ and $t \in [0, 1)$ set

$$\upsilon_{t,u} := \inf\{r \in \mathbb{R}_0^+ : |\mathfrak{I}_u(r)| < e^{-t}\} = \inf\{r \in \mathbb{R}_0^+ : \xi_u(r) > t\}.$$

Then we have

$$\sum_{n\in\mathcal{J}_{t,s}^{\complement}} \lambda_{t,n}^{1+p^*} = \int_{[0,1]} \mathbb{1}_{\{|\mathfrak{I}_u(\upsilon_{t,u})|<e^{-st}\}}\, \mathrm{d}u.$$

Moreover,

$$\left\{|\mathfrak{I}_u(\upsilon_{t,u})| < e^{-st}\right\} = \left\{\frac{\xi_u(\upsilon_{t,u})-t}{t} > s-1\right\}.$$

According to Hypothesis 3.2 the subordinator $\xi_u$ has finite mean, and thus the classical theory of subordinators, cf. [Ber99], yields that

$$\lim_{t\to\infty} \frac{\xi_u(\upsilon_{t,u})-t}{t} = 0$$

$\mathbb{P}$–almost surely. Hence, the DCT implies the assertion in the conservative case.

Part II Assume that $\nu$ is dissipative and, in view of Hypothesis 3.1, let $p \in (\underline{p}, p^*)$. Observe that

$$\sum_{n\in\mathcal{J}_{t,s}^{\complement}} \lambda_{t,n}^{1+p^*} \le e^{-(p^*-p)st-\Phi(p)\sigma_t} \sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p} e^{\Phi(p)\sigma_{t,n}} \tag{3.15}$$

holds true $\mathbb{P}$–a.s. for any $t \in \mathbb{R}_0^+$ and all $s \in (1,\infty)$. Recall that in (2.7) we showed that

$$\lim_{t\to\infty} \frac{\sigma_t}{t} = \frac{1}{\Phi'(\bar{p})}$$

$\mathbb{P}$–almost surely. Hence, it follows from (3.15) that

$$\sum_{n\in\mathcal{J}_{t,s}^{\complement}} \lambda_{t,n}^{1+p^*} \le \exp\left(-\left[(p^*-p)s + \Phi(p)\Phi'(\bar{p})^{-1}\right]t\right) \sum_{n\in\mathbb{N}} \lambda_{t,n}^{1+p} e^{\Phi(p)\sigma_{t,n}} \tag{3.16}$$

holds $\mathbb{P}$–a.s. for every $t \in \mathbb{R}_0^+$ and $s \in (1,\infty)$. Moreover, $(p^*-p)s + \Phi(p)\Phi'(\bar{p})^{-1} > 0$ for any

$$s > s_0 := \frac{\Phi(p)}{(p^*-p)\Phi'(\bar{p})}.$$

Hence, we infer from (3.16) and the martingale property of $\Lambda(p)$, cf. Lemma 2.3, that

$$\begin{aligned}
&\limsup_{t\to\infty} \sum_{n\in\mathcal{J}_{t,s}^{\complement}} \lambda_{t,n}^{1+p^*} \\
&\le \lim_{t\to\infty} \exp\left(-\left[(p^*-p)s + \Phi(p)\Phi'(\bar{p})^{-1}\right]t\right) \Lambda_\infty(p) \\
&= 0
\end{aligned}$$

$\mathbb{P}$–a.s. for all $s > s_0$. □

Recall the filtration $\mathscr{H} = (\mathscr{H}_t)_{t\in\mathbb{R}_0^+}$ given by (2.2).

**Lemma 3.7** *Assume that Hypothesis 3.1 and Hypothesis 3.2 are satisfied. Then there exists some $s_0 \in (1, \infty)$ such that*

$$\lim_{t\to\infty} \mathbb{E}(\langle \rho_{st}, f\rangle | \mathscr{H}_t) = \langle \rho, f\rangle \Lambda_\infty(p^*)$$

*$\mathbb{P}$–a.s. for all $s \geq s_0$.*

**Proof** For any $t \in \mathbb{R}_0^+$ and $s \in (1, \infty)$ the extended fragmentation property yields that

$$\begin{aligned}
&\mathbb{E}(\langle \rho_{2st}, f\rangle | \mathscr{H}_t) \\
&= \sum_{n\in\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*} \mathbb{E}\left( \sum_{k\in\mathbb{N}} \lambda_{t,s}^{(k)} f\left(e^{2st}\lambda_{t,n}\lambda_{t,s}^{(k)}\right) \middle| \mathscr{H}_t \right) \qquad (3.17) \\
&\quad + \sum_{n\in\mathcal{J}_{t,2s}\setminus\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*} \mathbb{E}\left( \sum_{k\in\mathbb{N}} \lambda_{t,s}^{(k)} f\left(e^{2st}\lambda_{t,n}\lambda_{t,s}^{(k)}\right) \middle| \mathscr{H}_t \right) + \sum_{n\in\mathcal{J}_{t,2s}^{\complement}} \lambda_{t,n}^{1+p^*} f\left(e^t\lambda_{t,n}\right)
\end{aligned}$$

$\mathbb{P}$–a.s., where conditional on $\mathscr{H}_t$ the $\lambda_{t,s}^{(k)}$ are independent and satisfy

$$\mathbb{E}\left(\lambda_{t,s}^{(k)} \in \cdot \middle| \mathscr{H}_t\right) = \mathbb{E}\left(\lambda_{u,k}^{1+p^*} \in \cdot\right)\Big|_{u=2st+\ln(\lambda_{t,n})}$$

$\mathbb{P}$–almost surely. Observe that

$$e^{2st}\lambda_{t,n}\lambda_{t,s}^{(k)} = e^{2st+\ln(\lambda_{t,n})}\lambda_{t,s}^{(k)},$$

and thus

$$\mathbb{E}\left(\lambda_{t,s}^{(k)} f\left(e^{2st}\lambda_{t,n}\lambda_{t,s}^{(k)}\right) \middle| \mathscr{H}_t\right) = \mathbb{E}\left(\lambda_{u,k}^{1+p^*} f\left(e^u\lambda_{u,k}\right)\right)\Big|_{u=2st+\ln(\lambda_{t,n})}.$$

$\mathbb{P}$–a.s. for all $k \in \mathbb{N}$. Therefore, (3.17) results in

$$\begin{aligned}
&\mathbb{E}(\langle \rho_{2st}, f\rangle | \mathscr{H}_t) \\
&= \sum_{n\in\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*}\, \mathbb{E}\left(\langle \rho_u, f\rangle\right)\big|_{u=2st+\ln(\lambda_{t,n})} \qquad (3.18) \\
&\quad + \sum_{n\in\mathcal{J}_{t,2s}\setminus\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*}\, \mathbb{E}\left(\langle \rho_u, f\rangle\right)\big|_{u=2st+\ln(\lambda_{t,n})} + \sum_{n\in\mathcal{J}_{t,2s}^{\complement}} \lambda_{t,n}^{1+p^*} f\left(e^t\lambda_{t,n}\right)
\end{aligned}$$

$\mathbb{P}$–almost surely. Let us first consider the first summand on the right–hand side of (3.18). Note that $2st + \ln(\lambda_{t,n}) \geq st$ for all $n \in \mathcal{J}_{t,s}$, $s \in (1, \infty)$ and $t \in \mathbb{R}_0^+$. Hence, Lemma 3.4 implies that

$$\mathbb{E}\left(\langle \rho_u, f \rangle\right)|_{u=2st+\ln(\lambda_{t,n})} \to \langle \rho, f \rangle$$

as $t \to \infty$ for every $s \in (1, \infty)$ and this convergence is uniform in $n \in \mathcal{J}_{t,s}$. We thus deduce from Lemma 3.6 that there exists some $s^* \in (1, \infty)$ such that

$$\begin{aligned} \lim_{t\to\infty} \sum_{n\in\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*} \, \mathbb{E}\left(\langle \rho_u, f \rangle\right)|_{u=2st+\ln(\lambda_{t,n})} &= \langle \rho, f \rangle \lim_{t\to\infty} \sum_{n\in\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*} \\ &= \langle \rho, f \rangle \Lambda_\infty(p^*) \end{aligned} \tag{3.19}$$

holds true $\mathbb{P}$–a.s. for any $s \geq s^*$. Moreover, according to Lemma 3.6 there exists some $s^{**} \in (1, \infty)$ such that

$$\lim_{t\to\infty} \sum_{n\in\mathcal{J}_{t,2s}\setminus\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*} \, \mathbb{E}\left(\langle \rho_u, f \rangle\right)|_{u=2st+\ln(\lambda_{t,n})} \leq \|f\|_\infty \lim_{t\to\infty} \sum_{n\in\mathcal{J}_{t,s}^{\complement}} \lambda_{t,n}^{1+p^*} = 0 \tag{3.20}$$

$\mathbb{P}$–a.s. for all $s \geq s^{**}$, where we have used that

$$\mathbb{E}\left(\langle \rho_u, f \rangle\right)|_{u=2st+\ln(\lambda_{t,n})} \leq \|f\|_\infty \, \mathbb{E}\left(\Lambda_u(p^*)\right)|_{u=2st+\ln(\lambda_{t,n})} = \|f\|_\infty.$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. By resorting to Lemma 3.6 once again we infer that there exists an $s^{***} \in (1, \infty)$ such that

$$\lim_{t\to\infty} \sum_{n\in\mathcal{J}_{t,2s}^{\complement}} \lambda_{t,n}^{1+p^*} f\left(e^t \lambda_{t,n}\right) = 0 \tag{3.21}$$

$\mathbb{P}$–a.s. for each $s \geq s^{***}$. Setting $s_0 := s^* \vee s^{**} \vee s^{***}$ we conclude by means of (3.18), (3.19), (3.20) and (3.21) that

$$\lim_{t\to\infty} \mathbb{E}(\langle \rho_{st}, f \rangle | \mathscr{H}_t) = \langle \rho, f \rangle \Lambda_\infty(p^*)$$

holds $\mathbb{P}$–a.s. for all $s \geq s_0$. □

The previous lemma shows that asymptotically, in the sense of almost sure convergence, $\mathbb{E}(\langle \rho_{st}, f \rangle | \mathscr{H}_t)$ is a good approximation of $\langle \rho, f \rangle \Lambda_\infty(p^*)$. If we could also show that almost surely $\mathbb{E}(\langle \rho_{st}, f \rangle | \mathscr{H}_t)$ is asymptotically a good approximation of $\langle \rho_{st}, f \rangle$, then the triangle inequality would prove Theorem 3.2. It turns out that we are not able to show this directly. However, what we can actually show is the following proposition which provides us with $\mathscr{L}^p$–convergence, for some $p > 1$, rather than the desired almost sure convergence. Moreover, it shows that the rate of convergence is exponentially fast and it turns out that this is enough for our purpose.

**Proposition 3.8** *Assume that Hypothesis 3.1 is satisfied. Then there exist some constants $\kappa, \gamma > 0$ and $p > 1$ such that*

$$\|\langle \rho_{st}, f\rangle - \mathbb{E}(\langle \rho_{st}, f\rangle | \mathscr{H}_t)\|_p \le \kappa e^{-\gamma t} \tag{3.22}$$

*holds for all $t \in \mathbb{R}_0^+$ and $s > 1$.*

**Proof** As in [Big92, Lemma 1] an application of Fatou's lemma (for conditional expectations) results in

$$\mathbb{E}\left( \left| \sum_{n\in\mathbb{N}} Z_n \right|^p \middle| \mathscr{H}_t \right) \le 2^p \sum_{n\in\mathbb{N}} \mathbb{E}\left( |Z_n|^p \,|\, \mathscr{H}_t \right) \tag{3.23}$$

holds for any $p \in [1, 2]$ and for all sequences $(Z_n)_{n\in\mathbb{N}}$ of independent random variables with $\mathbb{E}(Z_n|\mathscr{H}_t) = 0$. Moreover, according to Jensen's inequality we have

$$|u + v|^p \le 2^{p-1}(|u|^p + |v|^p) \tag{3.24}$$

for all $u, v \in \mathbb{R}$ and every $p \ge 1$.

Let $t \in \mathbb{R}_0^+$ as well as $s > 1$. By means of the extended fragmentation property we have

$$\begin{aligned}
\langle \rho_{st}, f\rangle - \mathbb{E}(\langle \rho_{st}, f\rangle | \mathscr{H}_t) &= \sum_{n\in\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*} \left( \langle \rho^{(n)}, f\rangle - \mathbb{E}(\langle \rho^{(n)}, f\rangle | \mathscr{H}_t) \right) \\
&\quad + \sum_{n\in\mathcal{J}_{t,s}^{\complement}} \lambda_{t,n}^{1+p^*} f\left(e^t \lambda_{t,n}\right) - \mathbb{E}\left( \sum_{n\in\mathcal{J}_{t,s}^{\complement}} \lambda_{t,n}^{1+p^*} f\left(e^t \lambda_{t,n}\right) \middle| \mathscr{H}_t \right) \\
&= \sum_{n\in\mathcal{J}_{t,s}} \lambda_{t,n}^{1+p^*} \left( \langle \rho^{(n)}, f\rangle - \mathbb{E}\left( \langle \rho^{(n)}, f\rangle \middle| \mathscr{H}_t \right) \right),
\end{aligned} \tag{3.25}$$

where conditional on $\mathscr{H}_t$ the $\langle \rho^{(n)}, f\rangle$ are independent and satisfy

$$\mathbb{P}\left( \langle \rho^{(n)}, f\rangle \in \cdot \middle| \mathscr{H}_t \right) = \mathbb{P}\left( \langle \rho_u, f\rangle \in \cdot \right)\big|_{u = st + \ln(\lambda_{t,n})}$$

$\mathbb{P}$–almost surely. Since

$$\begin{aligned}
&\mathbb{E}\left(\mathbb{1}_{\{n\in\mathcal{J}_{t,s}\}}\lambda_{t,n}^{1+p^*}\left(\langle\rho^{(n)},f\rangle-\mathbb{E}\left(\langle\rho^{(n)},f\rangle\middle|\,\mathscr{H}_t\right)\right)\middle|\,\mathscr{H}_t\right)\\
&=\mathbb{1}_{\{n\in\mathcal{J}_{t,s}\}}\lambda_{t,n}^{1+p^*}\mathbb{E}\left(\langle\rho^{(n)},f\rangle-\mathbb{E}\left(\langle\rho^{(n)},f\rangle\middle|\,\mathscr{H}_t\right)\middle|\,\mathscr{H}_t\right)\\
&=0,
\end{aligned}$$

we can apply (3.23) in order to deduce from (3.25) that

$$\begin{aligned}
&\mathbb{E}\left(\left|\langle\rho_{st},f\rangle-\mathbb{E}\left(\langle\rho_{st},f\rangle\right|\mathscr{H}_t\right)\right|^p\left|\,\mathscr{H}_t\right)\\
&\leq 2^p\sum_{n\in\mathcal{J}_{t,s}}\lambda_{t,n}^{p(1+p^*)}\mathbb{E}\left(\left|\langle\rho^{(n)},f\rangle-\mathbb{E}\left(\langle\rho^{(n)},f\rangle\middle|\,\mathscr{H}_t\right)\right|^p\middle|\,\mathscr{H}_t\right)\\
&\leq 2^{2p-1}\sum_{n\in\mathcal{J}_{t,s}}\lambda_{t,n}^{p(1+p^*)}\mathbb{E}\left(\langle\rho^{(n)},f\rangle^p+\mathbb{E}\left(\langle\rho^{(n)},f\rangle\middle|\,\mathscr{H}_t\right)^p\middle|\,\mathscr{H}_t\right) \qquad (3.26)\\
&\leq 2^{2p-1}\sum_{n\in\mathcal{J}_{t,s}}\lambda_{t,n}^{p(1+p^*)}\mathbb{E}\left(\langle\rho^{(n)},f\rangle^p+\mathbb{E}\left(\langle\rho^{(n)},f\rangle^p\middle|\,\mathscr{H}_t\right)\middle|\,\mathscr{H}_t\right)\\
&=2^{2p}\sum_{n\in\mathcal{J}_{t,s}}\lambda_{t,n}^{p(1+p^*)}\mathbb{E}\left(\langle\rho^{(n)},f\rangle^p\middle|\,\mathscr{H}_t\right)
\end{aligned}$$

holds true for any $p\in[1,2]$, where the $\langle\rho^{(n)},f\rangle$ are the same random variables that appear in (3.25). Notice that the first estimate in (3.26) results from (3.23) and (3.25), and the second estimate holds by means of (3.24). The third estimate is a consequence of Jensen's inequality for conditional expectations.

By taking the expectation on both sides in (3.26) we obtain

$$\begin{aligned}
\mathbb{E}\left(\left|\langle\rho_{st},f\rangle-\mathbb{E}\left(\langle\rho_{st},f\rangle\right|\mathscr{H}_t\right)\right|^p\right)&\leq 2^{2p}\mathbb{E}\left(\sum_{n\in\mathcal{J}_{t,s}}\lambda_{t,n}^{p(1+p^*)}\;\mathbb{E}\left(\langle\rho_u,f\rangle^p\right)\big|_{u=st+\ln(\lambda_{t,n})}\right)\\
&\leq 2^{2p}\sup_{u\in\mathbb{R}_0^+}\mathbb{E}\left(\langle\rho_u,f\rangle^p\right)\mathbb{E}\left(\sum_{n\in\mathcal{J}_{t,s}}\lambda_{t,n}^{p(1+p^*)}\right). \qquad (3.27)
\end{aligned}$$

Further, note that

$$\mathbb{E}\left(\sum_{n\in\mathcal{J}_{t,s}}\lambda_{t,n}^{p(1+p^*)}\right)\leq e^{-(p-1)(1+p^*)t}\mathbb{E}\left(\Lambda_t(p^*)\right)=e^{-(p-1)(1+p^*)t}, \qquad (3.28)$$

since $\Lambda(p^*)$ is a unit–mean martingale. In view of (3.14) let $p > 1$ be such that

$$K := \mathbb{E}\left(\sup_{u\in\mathbb{R}_0^+} \langle \rho_u, f\rangle^p\right) < \infty.$$

Then we infer from (3.27) and (3.28) that

$$\mathbb{E}\left(|\langle \rho_{st}, f\rangle - \mathbb{E}\left(\langle \rho_{st}, f\rangle | \mathscr{H}_t\right)|^p\right) \leq 2^{2p} K e^{-(p-1)(1+p^*)t}.$$

□

We are now in a position to prove Theorem 3.2. Let us emphasise that all the above results were obtained under the assumption that $\Pi$ is homogenous.

**Proof of Theorem 3.2**

The proof is divided into three parts. In the first part we establish the convergence along log–lattice times for homogenous fragmentation processes. The second part is devoted to extend that convergence from log–lattice times to convergence along the real numbers, still in the setting of homogenous fragmentations. Finally, in the third part we show that the result is also true for self–similar fragmentation processes with index of self–similarity $\alpha \neq 0$.

Part I As above we assume that $\Pi$ is a homogenous fragmentation process.

Let $f \in \mathcal{B}^+$ and in view of Proposition 3.8 let $\gamma, \kappa > 0$ as well as $p > 1$ be such that (3.22) holds for any $s > 1$. It then follows from the Chebyshev–Markov inequality that

$$\begin{aligned}\sum_{n\in\mathbb{N}} \mathbb{P}(|\langle \rho_{s\delta n}, f\rangle - \mathbb{E}(\langle \rho_{s\delta n}, X\rangle | \mathscr{H}_{\delta n})| > \varepsilon) &\leq \frac{1}{\varepsilon^p} \sum_{n\in\mathbb{N}} \|\langle \rho_{s\delta n}, f\rangle - \mathbb{E}(\langle \rho_{s\delta n}, f\rangle | \mathscr{H}_{\delta n})\|_p^p \\ &\leq \frac{\kappa^p}{\varepsilon^p} \sum_{n\in\mathbb{N}} e^{-p\gamma\delta n} \\ &< \infty\end{aligned}$$

holds for every $\delta, \varepsilon > 0$. Thus, we infer from the Borel–Cantelli lemma that

$$\lim_{n\to\infty} |\langle \rho_{s\delta n}, f\rangle - \mathbb{E}(\langle \rho_{s\delta n}, f\rangle | \mathscr{H}_{\delta n})| = 0 \tag{3.29}$$

holds $\mathbb{P}$–a.s. for all $\delta > 0$ and $s > 1$. In view of the triangle inequality, we thus deduce from (3.29) and Lemma 3.7 that there exists some $s_0 \in (1, \infty)$ such that

$$|\langle \rho_{s\delta' n}, f\rangle - \langle \rho, f\rangle \Lambda_\infty(p^*)|$$

$$\leq \quad |\langle \rho_{s\delta' n}, f\rangle - \mathbb{E}\left(\langle \rho_{s\delta' n}, f\rangle | \mathscr{H}_{\delta' n}\right)| + |\mathbb{E}\left(\langle \rho_{s\delta' n}, f\rangle | \mathscr{H}_{\delta' n}\right) - \langle \rho, f\rangle \Lambda_\infty(p^*)|$$

holds true $\mathbb{P}$–a.s. for all $n \in \mathbb{N}$, $\delta' > 0$ and any $s \geq s_0$. Let $\delta > 0$. Setting $\delta' := s^{-1}\delta$ in the above estimate we deduce that

$$\lim_{n\to\infty} \langle \rho_{\delta n}, f\rangle = \langle \rho, f\rangle \Lambda_\infty(p^*) \tag{3.30}$$

$\mathbb{P}$–almost surely.

Part II Let us now extend (3.30) to convergence along the real numbers. To this end, observe that for any $\delta > 0$ and all $t \in \mathbb{R}_0^+$ we have

$$\begin{aligned} \langle \rho_t, f\rangle &\geq \sum_{k \in \mathcal{J}^{\complement}_{n\delta, \frac{t}{n\delta}}} \lambda_{n\delta,k}^{1+p^*} f\left(e^t \lambda_{n\delta,k}\right) \\ &= \sum_{k\in\mathbb{N}} \lambda_{n\delta,k}^{1+p^*} f\left(e^t \lambda_{n\delta,k}\right) \\ &= \sum_{k\in\mathbb{N}} \lambda_{n\delta,k}^{1+p^*} f\left(e^{(t-n\delta)} e^{n\delta} \lambda_{n\delta,k}\right), \end{aligned} \tag{3.31}$$

where $n \in \mathbb{N}$ is chosen such that $t \in (n\delta, (n+1)\delta)$. Note that the penultimate equality results from $f \equiv 0$ on $[1, \infty)$. For the time being, let us assume that $f \in \mathcal{B}^+$ is continuous and has compact support, that is, by the Heine–Cantor theorem, $f$ is uniformly continuous. Therefore, for any $\epsilon > 0$ there exists some $\delta_\epsilon > 0$ such that $f(se^{(t-n\delta)}) \geq f(s) - \epsilon$ $\mathbb{P}$–a.s. for all $\delta \leq \delta_\epsilon$, $s \in \operatorname{supp}(f)$, $t \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$ with $t \in (n\delta, (n+1)\delta)$. Hence, under the above–mentioned assumptions on $f$ we deduce from (3.31) that

$$\langle \rho_t, f\rangle \geq \langle \rho_{n\delta}, f\rangle - \epsilon \langle \rho_{n\delta}, \mathbf{1}\rangle$$

holds for every $\delta \leq \delta_\epsilon$ and all $t \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$ with $t \in (n\delta, (n+1)\delta)$. Consequently, we infer from (3.30) that

$$\liminf_{t\to\infty} \langle \rho_t, f\rangle \geq \liminf_{n\to\infty} \left(\langle \rho_{n\delta}, f\rangle - \epsilon \langle \rho_{n\delta}, \mathbf{1}\rangle\right) = \left(\langle \rho, f\rangle - \epsilon\right) \Lambda_\infty(p^*)$$

$\mathbb{P}$–a.s. for all $\delta \leq \delta_\epsilon$. Letting $\epsilon \to 0$ we obtain

$$\liminf_{t\to\infty} \langle \rho_t, f\rangle \geq \langle \rho, f\rangle \Lambda_\infty(p^*) \tag{3.32}$$

$\mathbb{P}$–a.s. for any continuous $f \in \mathcal{B}^+$ with compact support.

Now assume that $f \in \mathcal{B}^+$ is continuous and let $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{B}^+$ be a sequence of continuous and compactly supported functions such that $f_k \uparrow f$ pointwise. Then we infer from (3.32) that

$$\liminf_{t\to\infty}\langle\rho_t, f\rangle \geq \lim_{k\to\infty}\liminf_{t\to\infty}\langle\rho_t, f_k\rangle \geq \lim_{k\to\infty}\langle\rho, f_k\rangle\Lambda_\infty(p^*) = \langle\rho, f\rangle\Lambda_\infty(p^*) \tag{3.33}$$

$\mathbb{P}$–a.s., where the final equality follows from the MCT and the continuity of $\langle\rho,\cdot\rangle$.

Let $A \subseteq \mathbb{R}_0^+$ be some open set. Then there exists an increasing sequence $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{B}^+$ of continuous functions such that $f_k \uparrow \mathbb{1}_A$ pointwise. Following the reasoning of (3.33) we obtain

$$\liminf_{t\to\infty}\langle\rho_t, \mathbb{1}_A\rangle \geq \langle\rho, \mathbb{1}_A\rangle\Lambda_\infty(p^*) \tag{3.34}$$

$\mathbb{P}$–almost surely. In view of $\langle\rho_t, \mathbf{1}\rangle = \Lambda_t(p^*)$ it follows from (3.34) that

$$\begin{aligned}\limsup_{t\to\infty}\langle\rho_t, \mathbb{1}_A\rangle &\leq \lim_{t\to\infty}\Lambda_t(p^*) - \liminf_{t\to\infty}\langle\rho_t, \mathbf{1} - \mathbb{1}_A\rangle \\ &\leq (1 - \langle\rho, \mathbf{1} - \mathbb{1}_A\rangle)\Lambda_\infty(p^*) \\ &= \langle\rho, \mathbb{1}_A\rangle\Lambda_\infty(p^*),\end{aligned} \tag{3.35}$$

$\mathbb{P}$–a.s., where we have used the linearity of $\langle\rho,\cdot\rangle$ and the fact that $\langle\rho, \mathbf{1}\rangle = 1$.

Next let $A \subseteq \mathbb{R}_0^+$ be some arbitrary set and let $(A_k)_{k\in\mathbb{N}}$ be a sequence of open sets with $A_k \downarrow A$ as $k \to \infty$. By means of (3.35) we then have

$$\limsup_{t\to\infty}\langle\rho_t, \mathbb{1}_A\rangle \leq \lim_{k\to\infty}\limsup_{t\to\infty}\langle\rho_t, \mathbb{1}_{A_k}\rangle \leq \lim_{k\to\infty}\langle\rho, \mathbb{1}_{A_k}\rangle\Lambda_\infty(p^*) = \langle\rho, \mathbb{1}_A\rangle\Lambda_\infty(p^*)$$

$\mathbb{P}$–almost surely. Similarly to the argument in (3.35) we thus deduce that

$$\begin{aligned}\liminf_{t\to\infty}\langle\rho_t, \mathbb{1}_A\rangle &\geq \lim_{t\to\infty}\Lambda_t(p^*) - \limsup_{t\to\infty}\langle\rho_t, \mathbf{1} - \mathbb{1}_A\rangle \\ &\geq (1 - \langle\rho, \mathbf{1} - \mathbb{1}_A\rangle)\Lambda_\infty(p^*) \\ &= \langle\rho, \mathbb{1}_A\rangle\Lambda_\infty(p^*)\end{aligned} \tag{3.36}$$

$\mathbb{P}$–almost surely.

In view of the well–known measure theoretical fact that every nonnegative measurable function can be approximated from below by an increasing sequence of nonnegative step functions, drop any additional assumptions on $f \in \mathcal{B}^+$ and let $(f_k)_{k\in\mathbb{N}}$ be a sequence of nonnegative linear combinations of indicator functions such that $f_k \uparrow f$ as $k \to \infty$. Resorting to (3.36) we then infer analogously to (3.33) that

$$\liminf_{t\to\infty}\langle\rho_t, f\rangle \geq \langle\rho, f\rangle\Lambda_\infty(p^*) \tag{3.37}$$

$\mathbb{P}$–almost surely. Consequently, the same reasoning as in (3.35) results in

$$\begin{aligned}\limsup_{t\to\infty}\langle\rho_t,f\rangle &\leq \|f\|_\infty \lim_{t\to\infty}\Lambda_t(p^*)-\liminf_{t\to\infty}\langle\rho_t,\|f\|_\infty-f\rangle\\ &\leq (\|f\|_\infty-\langle\rho,\|f\|_\infty-f\rangle)\,\Lambda_\infty(p^*)\\ &= \langle\rho,f\rangle\Lambda_\infty(p^*)\end{aligned}$$

$\mathbb{P}$–almost surely. Combining this estimate with (3.37) we conclude that

$$\lim_{t\to\infty}\langle\rho_t,f\rangle=\langle\rho,f\rangle\Lambda_\infty(p^*) \tag{3.38}$$

$\mathbb{P}$–almost surely.

Part III So far we have assumed that the fragmentation process $\Pi$ is homogenous. It remains to show the assertion for self–similar fragmentation processes with index $\alpha\neq 0$. Recall that Theorem 1.24 shows that any self–similar fragmentation process is a time–changed homogenous fragmentation process. Furthermore, observe that the definition of $\langle\rho_t,\cdot\rangle$ is only concerned with stopping lines associated with fragment sizes and with the size of the blocks at these times. The times themselves do not influence the values of $\langle\rho_t,\cdot\rangle$. In other words, a time–change of $\Pi$ does not affect the stopped process $\lambda^S$ and thus it does not change $\langle\rho_t,\cdot\rangle$ as this depends on $\Pi$ only via $\lambda_t$, cf. Definition 2.2. In view of (3.38) this proves Theorem 3.2. □

In the light of Theorem 3.2 and (3.14) the proof of Corollary 3.3 is a straightforward application of the DCT.

**Proof of Corollary 3.3** Resorting to Theorem 3.2 and (3.14) we infer from the DCT that

$$\langle\rho_t,f\rangle\to\langle\rho,f\rangle\Lambda_\infty(p^*)$$

in $\mathscr{L}^p(\mathbb{P})$ for any $f\in\mathcal{B}^+$ and all $p\in[1,(1+\bar{p})(1+p^*)^{-1}]$. □

## 3.6 Concluding remarks

As mentioned earlier, in the literature there are several strong laws of large numbers, for various classes of branching processes, that are similar in spirit to Theorem 3.2. Here we are going to provide some more detail in this regard. In [Ner81, Theorem 5.4] Nerman obtained strong and weak laws of large numbers, in the setting of Crump–Mode–Jagers processes, that do not directly look like the convergence in Theorem 3.2. The connection with our result can be seen in the case of a finite dislocation measure. Indeed, exploiting the fact that Crump–Mode–Jagers processes can be seen as fragmentation processes with finite dislocation measure [Ner81, Theorem 5.4] essentially proves Theorem 3.2 in the situation of a finite activity fragmentation. In [BM05, Corollary 1] Bertoin and Martínez made this claim rigorous for obtaining $\mathscr{L}^1$–convergence, but the same approach works for almost sure convergence, cf. [BM05, Remark 1 (b)]. Moreover, in [BM05, Corollary 2] they considered the problem for fragmentation processes with infinite dislocation measure, where Nerman's results are not applicable. More precisely, in [BM05, Corollary 2] Bertoin and Martínez proved $\mathscr{L}^2$–convergence for conservative fragmentation processes. Note that in Theorem 3.2 we show almost sure convergence in the dissipative setting.

In [HK08] the convergence result of Theorem 3.2 is considered from a different point of view. In fact, motivated by an application to mathematical statistics Hoffmann and Krell use the convergence of the integral with respect to the empirical measure $\rho_t$ in order to estimate the Lévy measure of the subordinator $\xi_U$, where $U$ is a uniformly distributed random variable on $[0, 1]$, cf. Remark 1.32. More specifically, in [HK08, Theorem 3.1] they prove $\mathscr{L}^2$–convergence of the object under consideration in Theorem 3.2 for conservative fragmentation chains. In their setting [HK08, Theorem 3.1] extends [BM05, Corollary 2], under some additional assumptions, in that it not only shows convergence but also establishes the rate of convergence.

# CHAPTER 4

# STRONG LAW OF LARGE NUMBERS FOR FRAGMENTATIONS WITH IMMIGRATION

*In this chapter we extend Theorem 3.2, to the setting of fragmentation processes with immigration.*

## 4.1 Fragmentation processes with immigration

Set

$$\mathcal{S} := \left\{ \mathbf{s} := (s_n)_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}} s_n < \infty,\, 0 \le s_j \le s_i < \infty\, \forall\, i \le j \right\}.$$

On $\mathcal{S}$ we define the binary operator $+$ as the decreasingly ordered concatenation of two sequences in $\mathcal{S}$. The corresponding iterated operator is denoted by $\sum$.

**Definition 4.1** Let $\mathbf{u} := (u_n)_{n\in\mathbb{N}} : \Omega \to \mathcal{S}$. Then we call *self–similar mass fragmentation process starting from* $\mathbf{u}$ with index $\alpha_{\mathbf{u}} := (\alpha_n)_{n\in\mathbb{N}}$, $\alpha_n \in \mathbb{R}$, the $\mathcal{S}$–valued Markov process $\lambda^{\mathbf{u}} := (\lambda^{\mathbf{u}}(t))_{t\in\mathbb{R}_0^+}$, defined by

$$\lambda^{\mathbf{u}}(t) := \sum_{n\in\mathbb{N}} u_n \lambda^{(n)}(u_n^{\alpha_n} t)$$

for all $t \in \mathbb{R}_0^+$, where the $\lambda^{(n)}$ are independent self–similar standard mass fragmentation processes with index $\alpha_n$ as given by Definition 1.9, where we assume that the $\lambda^{(n)}$ are also independent of $\mathbf{u}$.

Now we can define fragmentation processes with immigration.

**Definition 4.2** Let $\mathbf{u} : \Omega \to \mathcal{S}$ and consider a random measure $N$ on $\mathcal{S} \otimes \mathbb{R}_0^+$ with atoms $(\mathbf{s}(t_i), t_i)_{i \in \mathcal{I}}$, where the index set $\mathcal{I}$ is at most countably infinite. Furthermore, consider some sequences $(\alpha_n)_{n \in \mathbb{N}}$ and $(\alpha_{i,j})_{i \in \mathcal{I}, j \in \mathbb{N}}$ in $\mathbb{R}$. For $i \in \mathcal{I}$ let $\lambda^{\mathbf{u}} := (\lambda^{\mathbf{u}}(t))_{t \in \mathbb{R}_0^+}$ and $\lambda^{\mathbf{s}(t_i)} := (\lambda^{\mathbf{s}(t_i)}(t))_{t \in \mathbb{R}_0^+}$ be given by

$$\lambda^{\mathbf{u}}(t) = \sum_{n \in \mathbb{N}} u_n \lambda^{(n)} \left(u_n^{\alpha_n} t\right) \quad \text{as well as} \quad \lambda^{\mathbf{s}(t_i)}(t) = \sum_{j \in \mathbb{N}} \mathbf{s}(t_i)_j \lambda^{(i,j)} \left(\mathbf{s}(t_i)_j^{\alpha_{i,j}} t\right),$$

where the $\lambda^{(n)}$ and $\lambda^{\mathbf{s}(t_i)}$ are independent self–similar standard mass fragmentation processes with index $\alpha_n$ and $\alpha_{i,j}$ respectively. Note that $\lambda^{\mathbf{u}}$ and $\lambda^{\mathbf{s}(t_i)}$, $i \in \mathcal{I}$, are self–similar mass fragmentation processes starting from $\mathbf{u}$ and $\mathbf{s}(t_i)$ respectively, and assume that conditionally on $\mathbf{u}$ and $(\mathbf{s}(t_i), t_i)_{i \in \mathcal{I}}$ these processes are independent. Then we call the $\mathcal{S}$–valued process $\lambda^I := (\lambda^I(t))_{t \in \mathbb{R}_0^+}$, defined by

$$\lambda^I(t) := \lambda^{\mathbf{u}}(t) + \sum_{i \in \mathcal{I} : t_i \leq t} \lambda^{\mathbf{s}(t_i)}(t - t_i)$$

for all $t \in \mathbb{R}_0^+$, a *self–similar mass fragmentation process with immigration* starting from $\mathbf{u}$.

## 4.2 Set–up

For all $i \in \mathcal{I}$ and $j, n \in \mathbb{N}$ let $\Pi^{(n)}$ and $\Pi^{(i,j)}$ be independent self–similar standard $\mathcal{P}$–fragmentation processes rescaled such that $|\Pi_1^{(n)}(0)| = u_n$ and $|\Pi_1^{(i,j)}(0)| = \mathbf{s}(t_i)_j$ $\mathbb{P}$–almost surely. Further, we assume that the evolution of $\Pi^{(n)}$ [resp. $\Pi^{(i,j)}$] is independent of the starting value $u_n$ [resp. $\mathbf{s}(t_i)_j$]. In view of the previous chapter we assume that $\Pi^{(n)}$ and $\Pi^{(i,j)}$ satisfy Hypothesis 1.1 and Hypothesis 1.2 as well as Hypothesis 3.1 and Hypothesis 3.2. We denote the Malthusian parameter associated with $\Pi^{(n)}$ [resp. $\Pi^{(i,j)}$] by $p_n^*$ [resp. $p_{i,j}^*$]. In addition, let $\lambda^{(n)}$ [resp. $\lambda^{(i,j)}$] be the mass fragmentation obtained from the asymptotic frequencies of $\Pi^{(n)}$ [resp. $\Pi^{(i,j)}$], cf. Section 1.6, and let $\lambda^I$ be the the corresponding fragmentation process with immigration, see Definition 4.2.

**Definition 4.3** Bearing in mind the above set–up we call *stopped fragmentation process with immigration* the $\mathcal{S}$–valued stochastic process $(\lambda_t^I)_{t \in \mathbb{R}_0^+}$ defined by

$$\lambda_t^I := \left(\lambda_{t,m}^I\right)_{m \in \mathbb{N}} = \sum_{n \in \mathbb{N}} \lambda_t^{(n)} + \sum_{i \in \mathcal{I}} \sum_{j \in \mathbb{N}} \lambda_t^{(i,j)}.$$

For any $i,j,n$ set $\tilde{\lambda}^{(n)} := u_n^{-1}\lambda^{(n)}$ as well as $\tilde{\lambda}^{(i,j)} := \mathbf{s}(t_i)_j^{-1}\lambda(i,j)$ and observe that

$$\lambda_t^{(n)} = \tilde{\lambda}^{(n)}_{t+\ln(u_n)} \qquad \text{and} \qquad \lambda_t^{(i,j)} = \tilde{\lambda}^{(i,j)}_{t+\ln(\mathbf{s}(t_i)_j)} . \tag{4.1}$$

In the context of fragmentation processes with immigration, the objects considered in Chapter 3 have analogous notions that we now introduce. Recall that we denote the space of all bounded and measurable function $f : \mathbb{R} \to \mathbb{R}_0^+$ with $f|_{[1,\infty)} \equiv 0$ by $\mathcal{B}^+$. Let $f^{(n)}, f^{(i,j)} \in \mathcal{B}^+$ for all $i \in \mathcal{I}$ and $j,n \in \mathbb{N}$, and let $f^I := (f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{B}^+$ such that for every $n \in \mathbb{N}$ there exists some $k \in \mathbb{N}$ such that $f_k = f^{(n)}$ and for any $i \in \mathcal{I}$ and $j \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $f_k = f^{(i,j)}$. In addition, let $\eta : \mathbb{R}_0^+ \times \mathbb{N} \to \mathbb{N} \cup (\mathcal{I} \times \mathbb{N})$ be a random function that assigns to each $(t,m) \in \mathbb{R}_0^+ \times \mathbb{N}$ the index $n \in \mathbb{N}$ or $(i,j) \in \mathcal{I} \times \mathbb{N}$ such that $\lambda^I_{t,m}$ corresponds to $\lambda^{(n)}_{t,k}$ or $\lambda^{(i,j)}_{t,k}$ for some $k \in \mathbb{N}$. In the setting with immigration the analogue of $\langle \rho_t, \cdot \rangle$, which was introduced in (3.1) in the context of standard fragmentation processes, is defined as follows:

$$\langle \rho_t^I, f^I \rangle := \sum_{m\in\mathbb{N}} \left[\lambda^I_{t,m}\right]^{1+p^*_{\eta(t,m)}} f^{(\eta(t,m))}\left(e^t \lambda^I_{t,m}\right) .$$

By means of (4.1) we then have

$$\begin{aligned} \langle \rho_t^I, f^I \rangle &= \sum_{k\in\mathbb{N}} \left( \sum_{n\in\mathbb{N}} \left[\lambda^{(n)}_{t,k}\right]^{1+p^*_n} f^{(n)}\left(e^t\lambda^{(n)}_{t,k}\right) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathbb{N}} \left[\lambda^{(i,j)}_{t,k}\right]^{1+p^*_{i,j}} f^{(i,j)}\left(e^t\lambda^{(i,j)}_{t,k}\right)\right) \\ &= \sum_{n\in\mathbb{N}} u_n^{1+p^*_n} \langle \rho_t^n, f^{(n)}\rangle + \sum_{i\in\mathcal{I}}\sum_{j\in\mathbb{N}} \mathbf{s}(t_i)_j^{1+p^*_{i,j}} \left\langle \rho_t^{i,j}, f^{(i,j)} \right\rangle , \end{aligned} \tag{4.2}$$

where

$$\langle \rho_t^n, f^{(n)}\rangle = \sum_{k\in\mathbb{N}} \left[\tilde{\lambda}^{(n)}_{t+\ln(u_n),k}\right]^{1+p^*_n} f^{(n)}\left(e^{t+\ln(u_n)}\tilde{\lambda}^{(n)}_{t+\ln(u_n),k}\right) \tag{4.3}$$

as well as

$$\langle \rho_t^{i,j}, f^{(i,j)}\rangle = \sum_{k\in\mathbb{N}} \left[\tilde{\lambda}^{(i,j)}_{t+\ln(\mathbf{s}(t_i)_j),k}\right]^{1+p^*_{i,j}} f^{(i,j)}\left(e^{t+\ln(\mathbf{s}(t_i)_j)}\tilde{\lambda}^{(i,j)}_{t+\ln(\mathbf{s}(t_i)_j),k}\right) .$$

for any $t \in \mathbb{R}_0^+$. Furthermore, set

$$\Lambda^I_\infty(f^I) := \sum_{n\in\mathbb{N}} u_n^{1+p^*_n} \langle \rho^n, f^{(n)}\rangle \Lambda^{(n)}_\infty(p^*_n) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathbb{N}} \mathbf{s}(t_i)_j^{1+p^*_{i,j}} \langle \rho^{i,j}, f^{(i,j)}\rangle \Lambda^{(i,j)}_\infty(p^*_{i,j}),$$

where

$$\langle \rho^n, f^{(n)} \rangle := \lim_{t\to\infty} \mathbb{E}(\langle \rho^n_t, f^{(n)} \rangle) \qquad \text{and} \qquad \langle \rho^{i,j}, f^{(i,j)} \rangle := \lim_{t\to\infty} \mathbb{E}(\langle \rho^{i,j}_t, f^{(i,j)} \rangle)$$

as well as

$$\Lambda^{(n)}_\infty(p^*_n) := \lim_{t\to\infty} \langle \rho^n_t, \mathbf{1} \rangle \qquad \text{and} \qquad \Lambda^{(i,j)}_\infty(p^*_{i,j}) := \lim_{t\to\infty} \langle \rho^{i,j}_t, \mathbf{1} \rangle$$

for all $i \in \mathcal{I}$ and $j, n \in \mathbb{N}$.

## 4.3 Strong law of large numbers for fragmentation processes with immigration

Here we resort to the set–up established in the previous section. For the main result of this chapter we need the following condition:

$$\sum_{n\in\mathbb{N}} u_n^{1+p^*_n} \in \mathscr{L}^1(\mathbb{P}) \quad \text{and} \quad \sum_{i\in\mathcal{I}} \sum_{j\in\mathbb{N}} \mathbf{s}(t_i)_j^{1+p^*_{i,j}} \in \mathscr{L}^1(\mathbb{P}) \tag{4.4}$$

$\mathbb{P}$–almost surely.

Similar to Crump–Mode–Jagers processes (cf. [Olo96]), it is possible to lift the strong law for fragmentation processes in Theorem 3.2 up to fragmentation processes with immigration. More precisely, our main result in this chapter is the following theorem:

**Theorem 4.4** *Assume that for each $n \in \mathbb{N}$ and $(i, j) \in \mathcal{I} \times \mathbb{N}$ the processes $\lambda^{(n)}$ and $\lambda^{(i,j)}$ satisfy Hypothesis 3.1 and Hypothesis 3.2. If in addition (4.4) holds, then*

$$\lim_{t\to\infty} \langle \rho^I_t, f^I \rangle = \Lambda^I_\infty(f^I) \tag{4.5}$$

*$\mathbb{P}$–a.s. for all $f^{(n)}, f^{(i,j)} \in \mathcal{B}^+_1 := \{f \in \mathcal{B}^+ : \|f\|_\infty \le 1\}$.*

In view of Theorem 3.2 this says that the limit of the series in (4.2) as $t \to \infty$ is the same as taking the limit inside the series. Note that this is not an obvious result, since in general neither the DCT nor the MCT is applicable in this situation.

## 4.4 Proof of the strong law of large numbers for fragmentations with immigration

In order to tackle the proof of Theorem 4.4 we first need to develop some auxiliary lemmas. We remark that the method of proof is based on ideas of [AH77] and [Olo96].

Let $\mathcal{J}$ be some index set which is at most countably infinite and consider a random sequence $\mathbf{v} := (v_j)_{j\in\mathcal{J}} \in \mathcal{S}$.

In view of (4.4) we assume that

$$\sum_{j\in\mathcal{J}} \mathbb{E}\left(v_j^{1+p_j^*}\right) < \infty \tag{4.6}$$

Let $\Pi^{(j)}$, $j \in \mathcal{J}$, be independent homogenous standard $\mathcal{P}$–fragmentation processes rescaled such that $|\Pi^{(j)}| = v_j$ $\mathbb{P}$–almost surely. In addition, assume that the evolution of $\Pi^{(j)}$ is independent of $v_j$ and that $\Pi^{(j)}$ satisfies Hypothesis 3.2, Hypothesis 1.1, Hypothesis 3.1 as well as Hypothesis 1.2.

Bearing in mind (4.2) set

$$\langle \rho_t^{\mathbf{v}}, \cdot\rangle := \sum_{j\in\mathcal{J}_t} v_j^{1+p_j^*} \left\langle \rho_t^j, \cdot \right\rangle,$$

where the $\langle \rho_t^j, \cdot\rangle$ are defined analogously to (4.3) with respect to $\Pi^{(j)}$. Further, for every $t \in \mathbb{R}_0^+$ set

$$\mathcal{J}_t := \{j \in \mathcal{J} : v_j \geq e^{-t}\} \qquad \text{as well as} \qquad \mathcal{J}_t^{\complement} := \{j \in \mathcal{J} : v_j < e^{-t}\}. \tag{4.7}$$

For every $j \in \mathcal{J}$ and $t \in \mathbb{R}_0^+$ let $(\lambda_t^{(j)})_{t\in\mathbb{R}_0^+}$ be the stopped fragmentation process, see Definition 2.2, associated with $\Pi^{(j)}$. In addition, let $(\mathscr{H}_t^{(j)})_{t\in\mathbb{R}_0^+}$, $j \in \mathcal{J}$, be the filtration generated by the stopped process $(\lambda_t^{(j)})_{t\in\mathbb{R}_0^+}$

$$\mathscr{H}_t^{(j)} = \sigma\left(\{\lambda_s^{(j)} : s \leq t\}\right)$$

and note that $v_j$ is $\mathscr{H}_t^{(j)}$–measurable for each $t \in \mathbb{R}_0^+$. Furthermore, consider the filtration $\mathscr{H}^{\mathcal{J}} := (\mathscr{H}_t^{\mathcal{J}})_{t\in\mathbb{R}_0^+}$ given by

$$\mathscr{H}_t^{\mathcal{J}} := \sigma\left(\bigcup_{j\in\mathcal{J}_t} \mathscr{H}_t^{(j)}\right)$$

for any $t \in \mathbb{R}_0^+$.

According to Lemma 2.3 we have that $\Lambda^{(j)}(p_j^*) := (\langle \rho_t^j, \mathbf{1} \rangle)_{t \in \mathbb{R}_0^+}$, $j \in \mathcal{J}$, is a nonnegative uniformly integrable $\mathscr{H}^{(j)}$–martingale with unit mean and with $\mathbb{P}$–a.s. limit

$$\Lambda_\infty^{(j)}(p_j^*) := \lim_{t \to \infty} \langle \rho_t^j, \mathbf{1} \rangle. \tag{4.8}$$

**Lemma 4.5** *There exists a $\Lambda_\infty^{\mathcal{J}} \in \mathscr{L}^1(\mathbb{P})$ such that $\langle \rho_t^{\mathbf{v}}, \mathbf{1} \rangle \to \Lambda_\infty^{\mathcal{J}}$ $\mathbb{P}$–a.s. as $t \to \infty$.*

**Proof** By means of the MCT and independence we infer from (4.6) that

$$\sup_{t \in \mathbb{R}_0^+} \mathbb{E}(\langle \rho_t^{\mathbf{v}}, \mathbf{1} \rangle) = \sup_{t \in \mathbb{R}_0^+} \sum_{j \in \mathcal{J}} \mathbb{E}\left(v_j^{1+p_j^*}\right) \mathbb{E}\left(\langle \rho_t^j, \mathbf{1} \rangle\right) = \sum_{j \in \mathcal{J}} \mathbb{E}\left(v_j^{1+p_j^*}\right) < \infty. \tag{4.9}$$

Moreover, the MCT for conditional expectations in conjunction with the martingale property of $(\langle \rho_t^j, \mathbf{1} \rangle)_{t \in \mathbb{R}_0^+}$ for any $j \in \mathcal{J}$ yields that

$$\begin{aligned}
\mathbb{E}\left(\langle \rho_{t+s}^{\mathbf{v}}, \mathbf{1} \rangle \middle| \mathscr{H}_t^{\mathcal{J}}\right) &= \sum_{j \in \mathcal{J}} v_j^{1+p_j^*} \mathbb{E}\left(\left\langle \rho_{t+s}^j, \mathbf{1} \right\rangle \middle| \mathscr{H}_t^{(j)}\right) \\
&\geq \sum_{j \in \mathcal{J}_t} v_j^{1+p_j^*} \mathbb{E}\left(\left\langle \rho_{t+s}^j, \mathbf{1} \right\rangle \middle| \mathscr{H}_t^{(j)}\right) \\
&= \sum_{j \in \mathcal{J}_t} v_j^{1+p_j^*} \left\langle \rho_t^j, \mathbf{1} \right\rangle \\
&= \langle \rho_t^{\mathbf{v}}, \mathbf{1} \rangle
\end{aligned}$$

$\mathbb{P}$–a.s. for all $s, t \in \mathbb{R}_0^+$, which shows that under $\mathbb{P}$ the process $(\langle \rho_t^{\mathbf{v}}, \mathbf{1} \rangle)_{t \in \mathbb{R}_0^+}$ is a nonnegative $\mathscr{H}^{\mathcal{J}}$–submartingale. Note that here we have used the independence of $(\Pi^{(j)})_{j \in \mathcal{J}}$. In view of (4.9) the submartingale convergence theorem thus ensures that $\mathbb{P}$–a.s. there exists a $\Lambda_\infty^{\mathcal{J}} \in \mathscr{L}^1(\mathbb{P})$ such that $\langle \rho_t^{\mathbf{v}}, \mathbf{1} \rangle \to \Lambda_\infty^{\mathcal{J}}$ $\mathbb{P}$–a.s. as $t \to \infty$. □

The previous lemma can be strengthened in the sense that the obtained limiting random variable can be described explicitly. This claim is the statement of the following proposition.

**Proposition 4.6** *We have*

$$\langle \rho_t^{\mathbf{v}}, \mathbf{1} \rangle \to \sum_{j \in \mathcal{J}} v_j^{1+p_j^*} \Lambda_\infty^{(j)}(p_j^*)$$

*$\mathbb{P}$–a.s. as $t \to \infty$.*

**Proof** As a consequence of the MCT and (4.6) we obtain that

$$\mathbb{E}\left(\sum_{j\in\mathcal{J}} v_j^{1+p_j^*}\Lambda_\infty^{(j)}(p_j^*)\right) = \sum_{j\in\mathcal{J}}\mathbb{E}\left(v_j^{1+p_j^*}\right)\mathbb{E}\left(\Lambda_\infty^{(j)}(p_j^*)\right) = \sum_{j\in\mathcal{J}}\mathbb{E}\left(v_j^{1+p_j^*}\right) < \infty, \quad (4.10)$$

where we have used independence and the fact that the unit–mean martingale $\Lambda^{(j)}(p_j^*)$ is uniformly integrable. Further, let $\Lambda_\infty^{\mathcal{J}}$ be given by Lemma 4.5 and recall the definition of $\mathcal{J}_t$ as well as $\mathcal{J}_t^{\complement}$ in (4.7). Observe that for any $0 \le s \le t$ we have

$$\begin{aligned} &\Lambda_\infty^{\mathcal{J}} - \sum_{j\in\mathcal{J}} v_j^{1+p_j^*}\Lambda_\infty^{(j)}(p_j^*) \\ &= \Lambda_\infty^{\mathcal{J}} - \langle \rho_t^{\mathbf{v}}, \mathbf{1}\rangle + \sum_{j\in\mathcal{J}_s} v_j^{1+p_j^*}\left(\left\langle \rho_t^j, \mathbf{1}\right\rangle - \Lambda_\infty^{(j)}(p_j^*)\right) \qquad (4.11) \\ &\quad + \sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*}\left\langle \rho_t^j, \mathbf{1}\right\rangle - \sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*}\Lambda_\infty^{(j)}(p_j^*). \end{aligned}$$

According to Lemma 4.5 we have that $\Lambda_\infty^{\mathcal{J}} - \langle \rho_t^{\mathbf{v}}, \mathbf{1}\rangle \to 0$ $\mathbb{P}$–a.s. as $t \to \infty$. Furthermore, by means of (4.6) the third term converges to zero $\mathbb{P}$–a.s. as $t \to \infty$. Indeed, notice that the sum in the third term has only finitely many summands as infinitely many $j \in \mathcal{J}$ with $v_j \ge e^{-s}$ would contradict (4.6). Hence, we can take the limit inside the sum. Moreover, resorting to (4.10) we obtain that the last term above tends to zero $\mathbb{P}$–a.s. as $s \to \infty$. Let us now consider the fourth term, and note that $\lim_{t\to\infty}\sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*}\langle \rho_t^j, \mathbf{1}\rangle$ exists $\mathbb{P}$–a.s., since, according to Lemma 4.5, $\lim_{t\to\infty}\langle \rho_t^{\mathbf{v}}, \mathbf{1}\rangle$ exists $\mathbb{P}$–a.s. and also $\lim_{t\to\infty}\sum_{j\in\mathcal{J}_s} v_j^{1+p_j^*}\langle \rho_t^j, \mathbf{1}\rangle$ exists because the sum is taken over only finitely many summands. Since

$$s \mapsto \lim_{t\to\infty}\sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*}\left\langle \rho_t^j, \mathbf{1}\right\rangle$$

is monotonically decreasing in $s$, we infer that the limit as $s \to \infty$ exists $\mathbb{P}$–a.s., and thus we deduce from Fatou's lemma that

$$\begin{aligned} \mathbb{E}\left(\lim_{s\to\infty}\lim_{t\to\infty}\sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*}\left\langle \rho_t^j, \mathbf{1}\right\rangle\right) &\le \liminf_{s\to\infty}\liminf_{t\to\infty}\sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*}\mathbb{E}\left(\left\langle \rho_t^j, \mathbf{1}\right\rangle\right) \\ &= \lim_{s\to\infty}\sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*} = 0 \end{aligned}$$

$\mathbb{P}$–a.s., since $\sum_{j\in\mathcal{J}} v_j^{1+p_j^*} < \infty$ $\mathbb{P}$–almost surely. Consequently, as

$$\lim_{s\to\infty}\lim_{t\to\infty}\sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*}\left\langle \rho_t^j, \mathbf{1}\right\rangle \geq 0,$$

this implies that

$$\lim_{s\to\infty}\lim_{t\to\infty}\sum_{j\in\mathcal{J}_s^{\complement}} v_j^{1+p_j^*}\left\langle \rho_t^j, \mathbf{1}\right\rangle = 0.$$

Hence, all the terms in (4.11) converge to 0 as first $t\to\infty$ and then $s\to\infty$, which proves the assertion. □

We are now in a position to prove Theorem 4.4.

**Proof of Theorem 4.4**

Observe that Theorem 3.2 in conjunction with Fatou's lemma yields that

$$\liminf_{t\to\infty}\langle \rho_t^{\mathbf{v}}, f^{\mathbf{v}}\rangle \geq \sum_{j\in\mathcal{J}} v_j^{1+p_j^*}\langle \rho^j, f^{(j)}\rangle \Lambda_\infty^{(j)}(p_j^*) \tag{4.12}$$

$\mathbb{P}$–a.s., where $\langle \rho^j, f^{(j)}\rangle := \lim_{t\to\infty}\mathbb{E}(\langle \rho_t^j, f^{(j)}\rangle)$. As a consequence of the additivity of $\langle \rho^j, \cdot\rangle$ and $\langle \rho^j, \mathbf{1}\rangle = 1$ for all $j\in\mathcal{J}$, we infer from Proposition 4.6 and (4.12) that

$$\begin{aligned}
\limsup_{t\to\infty}\langle \rho_t^{\mathbf{v}}, f^{\mathbf{v}}\rangle &\leq \lim_{t\to\infty}\langle \rho_t^{\mathbf{v}}, \mathbf{1}\rangle - \liminf_{t\to\infty}\langle \rho_t^{\mathbf{v}}, \mathbf{1}-f^{\mathbf{v}}\rangle \\
&\leq \sum_{j\in\mathcal{J}} v_j^{1+p_j^*}\langle \rho^j, \mathbf{1}\rangle \Lambda_\infty^{(j)}(p_j^*) - \sum_{j\in\mathcal{J}} v_j^{1+p_j^*}\langle \rho^j, \mathbf{1}-f^{(j)}\rangle \Lambda_\infty^{(j)}(p_j^*) \\
&= \sum_{j\in\mathcal{J}} v_j^{1+p_j^*}\langle \rho^j, f^{(j)}\rangle \Lambda_\infty^{(j)}(p_j^*)
\end{aligned}$$

$\mathbb{P}$–a.s., which combined with (4.12) proves that

$$\lim_{t\to\infty}\langle \rho_t^{\mathbf{v}}, f^{\mathbf{v}}\rangle = \sum_{j\in\mathcal{J}} v_j^{1+p_j^*}\langle \rho^j, f^{(j)}\rangle \Lambda_\infty^{(j)}(p_j^*) \tag{4.13}$$

$\mathbb{P}$–almost surely.

Recall the setup and notations developed in the introduction of this chapter and assume that (4.4) is satisfied. In view of (4.4) we can replace $\mathbf{v}$ in (4.13) by $\mathbf{u}$ and $(\mathbf{s}(t_i))_{i\in\mathcal{I}}$, respectively, and thus we deduce that

$$\lim_{t\to\infty}\langle \rho_t^I, f^I\rangle = \lim_{t\to\infty}\langle \rho_t^{\mathbf{u}}, f^{\mathbf{u}}\rangle + \lim_{t\to\infty}\left\langle \rho_t^{(\mathbf{s}(t_i))_{i\in\mathcal{I}}}, f^{(\mathbf{s}(t_i))_{i\in\mathcal{I}}}\right\rangle$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{N}} u_n^{1+p^*}\langle \rho^n, f^{(n)}\rangle \Lambda_\infty^{(n)}(p^*) + \sum_{i\in\mathcal{I}}\sum_{j\in\mathbb{N}} \mathbf{s}(t_i)_j^{1+p^*}\langle \rho^{i,j}, f^{(i,j)}\rangle \Lambda_\infty^{(i,j)}(p^*) \\
&= \Lambda_\infty^I(f^I) \qquad (4.14)
\end{aligned}
$$

holds true $\mathbb{P}$–almost surely. Note that, due to the countability of $\mathcal{I}$, above we can indeed replace $\mathbf{v}$ by $(\mathbf{s}(t_i))_{i\in\mathcal{I}}$, since we can interpret $(\mathbf{s}(t_i))_{i\in\mathcal{I}}$ as $\sum_{i\in\mathcal{I}}\mathbf{s}(t_i)$, the decreasingly ordered concatenation of the sequences $\mathbf{s}(t_i)\in\mathcal{S}$, $i\in\mathcal{I}$, and (4.4) ensures that $\sum_{i\in\mathcal{I}}\mathbf{s}(t_i)\in\mathcal{S}$. This argument works, because $\langle \rho_t^I, f^I\rangle$ is only concerned with the sizes of the immigrating particles but not with the times of immigration.

The extension of (4.14) from homogenous to self–similar fragmentation processes follows by the same reasoning as described in Part III of the proof of Theorem 3.2. □

**Remark 4.7** Assume that the $\Pi^{(j)}$, $j\in\mathcal{J}$, are identically distributed. Set $p^* := p^*_{j_0}$, with $j_0 = \min J$, and notice that $p^* = p^*_j$ for all $j\in\mathcal{J}$. Then (4.13) can be proven without resorting to Lemma 4.5 and Proposition 4.6. Indeed, by means of Fatou's lemma in conjunction with Doob's maximal inequality and Proposition 3.5 we infer that there exists some $p>1$ such that

$$
\mathbb{E}\left(\sup_{t\in\mathbb{R}_0^+}\langle \rho_t^j, \mathbf{1}\rangle^p\right) \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left(\Lambda_\infty^{(j)}(p_j^*)^p\right) < \infty \qquad (4.15)
$$

for every $j\in\mathcal{J}$. Set

$$
\mathbb{E}\left(\sup_{t\in\mathbb{R}_0^+}\Lambda_t(p^*)\right) := \mathbb{E}\left(\sup_{t\in\mathbb{R}_0^+}\langle \rho_t^{j_0}, \mathbf{1}\rangle\right) < \infty \qquad (4.16)
$$

with $j_0 = \min J$, where the finiteness follows from (4.15). Since the $\lambda^{(j)}$, $j\in\mathcal{J}$, are identically distributed we have

$$
\mathbb{E}\left(\sup_{t\in\mathbb{R}_0^+}\langle \rho_t^j, \mathbf{1}\rangle\right) = \mathbb{E}\left(\sup_{t\in\mathbb{R}_0^+}\Lambda_t(p^*)\right) \qquad (4.17)
$$

for all $j\in\mathcal{J}$. Therefore, resorting to the DCT we obtain that

$$
\begin{aligned}
\lim_{t\to\infty}\langle \rho_t^{\mathbf{v}}, \mathbf{1}\rangle &= \lim_{t\to\infty}\sum_{j\in\mathcal{J}} v_j^{1+p^*}\langle \rho_t^j, \mathbf{1}\rangle \\
&= \sum_{j\in\mathcal{J}} v_j^{1+p^*}\lim_{t\to\infty}\langle \rho_t^j, \mathbf{1}\rangle \qquad (4.18)
\end{aligned}
$$

$$= \sum_{j \in \mathcal{J}} v_j^{1+p^*} \Lambda_\infty^{(j)}(p_j^*)$$

holds $\mathbb{P}$–a.s., which proves the statement of Proposition 4.6 in this special situation. Note that in (4.18) we can indeed resort to the DCT, since an application of the MCT yields that

$$\begin{aligned} \mathbb{E}\left( \sup_{t \in \mathbb{R}_0^+} \sum_{j \in \mathcal{J}} v_j^{1+p^*} \langle \rho_t^j, \mathbf{1} \rangle \right) &\leq \sum_{j \in \mathcal{J}} \mathbb{E}\left( v_j^{1+p^*} \right) \mathbb{E}\left( \sup_{t \in \mathbb{R}_0^+} \langle \rho_t^j, \mathbf{1} \rangle \right) \\ &= \mathbb{E}\left( \sup_{t \in \mathbb{R}_0^+} \Lambda_t(p^*) \right) \sum_{j \in \mathcal{J}} \mathbb{E}\left( v_j^{1+p^*} \right) \\ &< \infty. \end{aligned} \tag{4.19}$$

Let us mention that the first estimate in (4.19) results from the MCT and the assumption that $\mathbf{v}$ is independent of $\Pi^{(j)}$ for any $j \in \mathcal{J}$. Moreover, the equality in (4.19) follows from (4.17) and the finiteness is a consequence of (4.6) and (4.16).

## 4.5 Example

The aim of this section is to consider an example of a homogenous mass fragmentation process for which we can give an alternative proof that (4.5) holds. This example is based on the spine decomposition introduced in Section 1.9.

Throughout this section fix some $p \in (\underline{p}, \infty)$. Let $\nu$ be an $\mathcal{S}_1$–dislocation measure and let the measure $\mu_\nu$ on $\mathcal{P}$ be given by (1.10). In addition, let $p \in (\underline{p}, \infty)$ and consider the measure $\mu_\nu^{(p)}$ on $\mathcal{P}$ given by $\mu_\nu^{(p)}(d\pi) = |\pi_1|^p \mu_\nu(d\pi)$. Let $(\pi(t))_{t \in \mathbb{R}_0^+}$ be a Poisson point process on $\mathcal{P}$ with characteristic measure $\mu_\nu^{(p)}$ and let $(t_i)_{i \in \mathcal{I}_1}$, where $\mathcal{I}_1$ is an at most countable index set, be the times for which this process takes a value in $\mathcal{P} \setminus \{(\mathbb{N}, \emptyset, \ldots)\}$. Furthermore, let $\Pi^{(p)}$ be a standard homogenous $\mathcal{P}$–fragmentation process under $\mathbb{P}$ with dislocation measure $\mu_\nu^{(p)}$ and such that the Poisson point process on $\mathcal{P}$ underlying $(\Pi_1^{(p)}(t))_{t \in \mathbb{R}_0^+}$ coincides with $(\pi(t))_{t \in \mathbb{R}_0^+}$. In addition, set

$$\Delta(t) := e^{\ln(|\Pi_1^{(p)}(t-)|)} \left| (\pi_n(t))_{n \in \mathbb{N} \setminus \{1\}} \right|^{\downarrow}$$

for any $t \in \mathbb{R}_0^+$. Notice that $(\Delta(t))_{t \in \mathbb{R}_0^+}$ is a Poisson point process on $\mathcal{S}_1$ whose atoms in $\mathcal{S}_1 \setminus \{(0, \ldots)\}$ are given by $(\Delta(t_i))_{i \in \mathcal{I}_1}$. Let $\lambda^{\Delta(t_i)}$, $i \in \mathcal{I}_1$, be mutually independent fragmentation processes, each starting from $\Delta(t_i)$, with dislocation measure $\nu$. Consider the fragmentation process with immigration $\lambda^I := (\lambda^I(t))_{t\in\mathbb{R}_0^+}$ defined by

$$\lambda^I(t) = \sum_{i\in\mathcal{I}_1:t_i\le t} \lambda^{\Delta(t_i)}(t-t_i) \tag{4.20}$$

for all $t \in \mathbb{R}_0^+$. Observe that this process starts from $\mathbf{u} = (0,\ldots)$, that is to say $\lambda^I$ is a pure immigration process.

**Lemma 4.8** *Let $f \in \mathcal{B}^+$. If $\nu$ satisfies Hypothesis 3.1 and Hypothesis 3.2, then the process $\lambda^I$ constructed in (4.20) satisfies (4.5), where $f^{(i,j)} :\equiv f$ for all $i \in \mathcal{I}_1$ and $j \in \mathbb{N}$.*

**Proof** Let $\Pi$ be a standard homogenous $\mathcal{P}$–fragmentation process under $\mathbb{P}$ with dislocation measure $\mu_\nu$. Further, recall the change of measure in (1.19), that is

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = e^{\Phi(p)t-p\xi(t)}.$$

In view of (1.22) in Section 1.9, consider the following spine decomposition:

$$|\Pi(t)| = (|\Pi_1(t)|, 0, \ldots) + \sum_{i\in\mathcal{I}_1:t_i\le t}\ \sum_{j\in\mathbb{N}\setminus\{1\}} \left|\Pi^{(i,j)}(t-t_i)\right|$$

$\mathbb{P}^{(p)}$–a.s., where the $\Pi^{(i,j)}$ are independent and satisfy

$$\mathbb{P}^{(p)}\left(|\Pi^{(i,j)}(u)| \in \cdot \,\middle|\, \mathscr{F}^1_{t_i}\right) = \mathbb{P}^{(p)}\left(x_{i,j}|\Pi(u)| \in \cdot\right)$$

$\mathbb{P}^{(p)}$–a.s. with $x_{i,j} = |\Pi_1(t_i-) \cap \pi_j(t_i)|$. Moreover, the behaviour of the block $\Pi_1$ is determined by a Poisson point process with characteristic measure $\mu_\nu^{(p)}$, cf. (1.21).

Recall the construction of $\lambda^I$ in (4.20) and observe that

$$\lambda(t) = |\Pi_1(t)| + \lambda^I(t)$$

$\mathbb{P}^{(p)}$–almost surely. That is to say, under $\mathbb{P}^{(p)}$ we can interpret the immigrating particles of $\lambda^I(t)$ as those particles that result from the fragmentation of the spine $\Pi_1$ at the jump times $(t_i)_{i\in\mathcal{I}_1}$ except for the tagged fragments $\Pi_1(t_i)$, $i \in \mathcal{I}_1$, themselves.

Recall the definition of $\upsilon_{t,1}$ in (2.1). Using notations introduced in Chapter 3 and Section 4.1 we infer from Theorem 3.2 that

$$\lim_{t\to\infty}\langle\rho_t^I, f^I\rangle = \lim_{t\to\infty}\langle\rho_t, f\rangle - \lim_{t\to\infty}\left[|\Pi_1(\upsilon_{t,1})|^{1+p^*} f\left(e^t|\Pi_1(\upsilon_{t,1})|\right)\right]$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \langle \rho_t, f\rangle \\
&= \langle \rho, f\rangle \Lambda(p^*) \qquad (4.21)\\
&= \Lambda^I_\infty(f^I)
\end{aligned}
$$

$\mathbb{P}^{(p)}$–almost surely. Note that in order to apply Theorem 3.2 we have used that $\mathbb{P}^{(p)}$ and $\mathbb{P}$ are equivalent measures on $\mathscr{F}_\infty$, cf. Remark 1.37, to deduce that the convergence in Theorem 3.2 holds true $\mathbb{P}^{(p)}$–almost surely. Since the event $\{\lim_{t\to\infty}\langle \rho^I_t, f^I\rangle = \Lambda^I_\infty(f^I)\}$ is $\mathscr{F}_\infty$–measurable, we conclude in view of (4.21) and resorting again to the fact that $\mathbb{P}^{(p)}$ and $\mathbb{P}$ are equivalent measures on $\mathscr{F}_\infty$ that

$$\lim_{t\to\infty}\langle \rho^I_t, f^I\rangle = \Lambda^I_\infty(f^I)$$

$\mathbb{P}$–almost surely. □

We remark that it follows from [BR03, Lemma 2] that (4.4) is satisfied for the process $\lambda^I$ given by (4.20), and thus the statement of Lemma 4.8 also follows from Theorem 4.4.

Let us now assume that the dislocation measure $\nu$ is conservative, cf. Definition 1.11, and let us finish this chapter by having a closer look at the characteristic measure under $\mathbb{P}^{(p)}$ of the Poisson random measure $N$ that describes the immigration structure of (4.20). Note that $N$ is a random measure on $\mathcal{S}_1\otimes\mathbb{R}_0^+$ with atoms $(|(\pi_j(t_i))_{j\in\mathbb{N}\setminus\{1\}}|^\downarrow)_{i\in\mathcal{I}_1}$ in $\mathcal{S}_1\setminus\{(0,\ldots)\}$. The first thing to mention is that under $\mathbb{P}^{(p)}$ the intensity of $N$ is of the form $I\otimes \mathrm{d}t$, where, for the time being, $I$ is a $\sigma$–finite measure on $\mathcal{S}_1$ and $\mathrm{d}t$ denotes the Lebesgue measure on $\mathbb{R}_0^+$. Further, recall that the Poisson point process on $\mathcal{P}$ with atoms $(\pi(t_i))_{i\in\mathcal{I}_1}$ in $\mathcal{P}\setminus(\mathbb{N},\emptyset,\ldots)$ has characteristic measure $\mu_\nu^{(p)}$. Hence, since $\nu$ is conservative, the measure $I$ is the projection of $\mu_\nu^{(p)}$ on $\mathcal{S}_1$ and in view of (1.12) we thus infer that

$$\int_{\mathcal{S}_1} g(\mathbf{s})I(\mathrm{d}\mathbf{s}) = \int_{\mathcal{P}} g(|\pi|^\downarrow)\mu_\nu^{(p)}(\mathrm{d}\pi) = \int_{\mathcal{P}} g(|\pi|^\downarrow)|\pi_1|^p\mu_\nu(\mathrm{d}\pi) = \int_{\mathcal{S}_1} g(\mathbf{s})\sum_{n\in\mathbb{N}} s_n^{1+p}\nu(\mathrm{d}\mathbf{s})$$

holds for any nonnegative test function $g:\mathcal{S}_1\to\mathbb{R}_0^+$, which results in

$$I(\mathrm{d}\mathbf{s}) = \sum_{n\in\mathbb{N}} s_n^{1+p}\nu(\mathrm{d}\mathbf{s})$$

for all $\mathbf{s}\in\mathcal{S}_1$.

## 4.6 Concluding remarks

Fragmentation processes with immigration were introduced in [Haa05]. There Haas was interested in a stationary distribution of such processes. The definition of the immigration process in [Haa05] is similar to our definition, cf. Definition 4.2. One difference is that in [Haa05] Haas requires the Poisson random measure $N$, that describes the immigration, to be of the form $I \cdot \mathrm{d}t$, where $I$ is a $\sigma$–finite measure on $\mathcal{S}$ and $\mathrm{d}t$ is the Lebesgue measure on $\mathbb{R}_0^+$. In Definition 4.2 we allow the Poisson random measure to be of the form $I \otimes \mathrm{d}t$, that is the distribution of the immigrating particles can depend on the time at which the particles are immigrating.

The main theorem of this chapter is a natural extension of Theorem 3.2. However, the difficulty arises that in general neither the DCT nor the MCT can be applied in order to deduce the statement of Theorem 4.4 directly from Theorem 3.2. A similar issue lies at the heart of [Olo96, Theorem 4.2], where Olofsson extends an $\mathscr{L}^1$–convergence limit theorem, obtained for Crump–Mode–Jagers processes in [Ner81], to such processes with immigration. However, his techniques do not yield almost sure convergence. Theorem 4.4 is also similar to more classical results on branching processes. In particular we refer to [AH77] for a result on supercritical immigration–branching pocessers that is in the spirit of Theorem 4.4.

# Part II

# Killed fragmentation processes

CHAPTER 5

# MARTINGALES ASSOCIATED WITH KILLED FRAGMENTATION PROCESSES

*This chapter is devoted to the study of fragmentation processes that exhibit a specific kind of killing. Our goal is to derive various properties regarding these killed fragmentation processes.*

## 5.1 Introduction

In this chapter we introduce a new class of fragmentation processes. Recall that in Chapter 1 we considered various types of self–similar standard fragmentation processes. In Chapter 2 and Chapter 3 we were concerned with so–called stopped fragmentation processes and in Chapter 4 this was extended to fragmentations with immigration. The goal of the present chapter is to introduce a certain kind of killing in the context of homogenous fragmentation processes. Here we kill blocks when they are sufficiently small relative to their time of creation. This description will be made rigorous below.

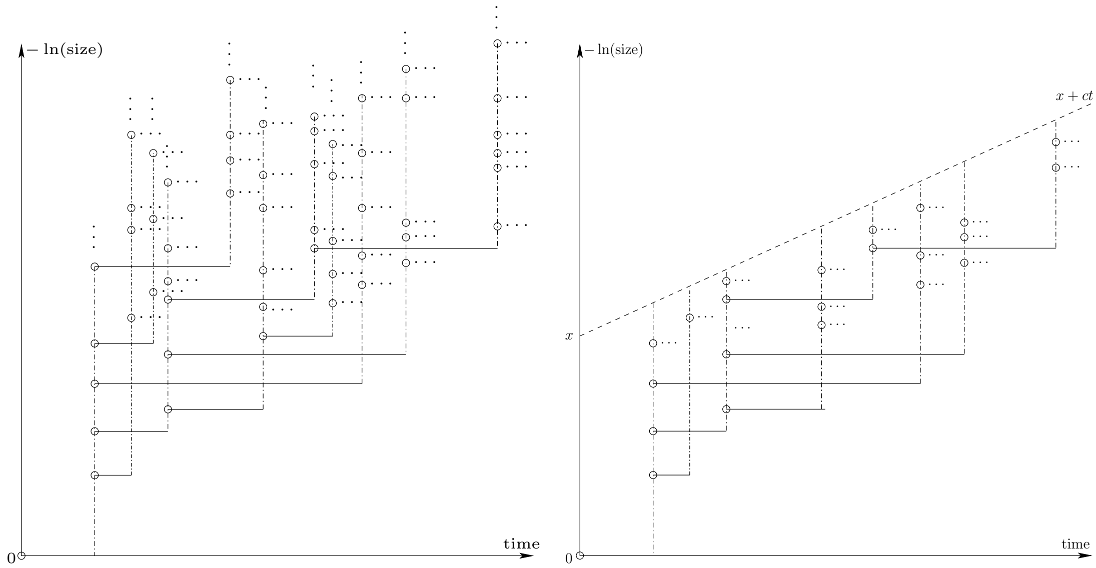
Throughout this chapter we consider a homogenous fragmentation process $\Pi$ with $B_n$, $n \in \mathbb{N}$, and $\lambda$ being defined as on page 16 and on page 20 respectively. Furthermore, we assume that Hypothesis 1.1 and Hypothesis 1.2 hold.

## 5.2 Killed fragmentation processes

Let $c > 0$, $x \in \mathbb{R}$ and introduce killing of $\Pi$ upon hitting the space–time barrier

$$\left\{(y,t) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : y < e^{-(x+ct)}\right\}.$$

That is, a block $\Pi_n(t)$ is killed at the moment of its creation $t \in \mathbb{R}_0^+$ if $|\Pi_n(t)| < e^{-(x+ct)}$, see Figure 5-1. A block that is killed is set to be $\emptyset$.




(a) Realisation of a fragmentation process without killing.

(b) Realisation of the killed fragmentation process corresponding to (a).

**Figure 5-1:** *Realisation of a fragmentation process with finite dislocation measure without killing, cf. (a), and with killing, cf. (b).*

**Definition 5.1** The resulting killed process, denoted by $\Pi^x := (\Pi_n^x)_{n\in\mathbb{N}}$, is called *killed fragmentation process* and $(\emptyset, \ldots)$ can be interpreted as a *cemetery state* for $\Pi^x$.

Notice that $\Pi^x$ is not necessarily $\mathcal{P}$–valued, since the killing may result in the union of it blocks being a strict subset of $\mathbb{N}$. That is to say, $\Pi^x$ takes values in the set of ordered partitions of subsets of $\mathbb{N}$.

**Remark 5.2** Let us emphasise that the killed fragmentation process $\Pi^x$ depends on the constant $c > 0$. In order to keep the notation as simple as possible we do not include the parameter $c$ in the notation of this chapter, because this constant does not change within results or proofs. However, the reader should keep in mind that all notions related to killed fragmentation processes depend on $c$. $\lozenge$

**Definition 5.3** For any $x \in \mathbb{R}_0^+$ we refer to the space–time barrier with initial value $x$ as *$x$–killing line.*

Clearly, the jump times of the killed fragmentation process $\Pi^x$ are a subset of the jump times of $\Pi$ as every block still alive evolves synchronously to the unkilled version until it gets killed. In this regard we adopt the following definition:

**Definition 5.4** Let $\mathcal{I}^x \subseteq \mathcal{I}$ be the index set of the jump times pertaining to $\Pi^x$. That is, the jump times of the killed process $\Pi^x$ are given by $(t_i)_{i\in\mathcal{I}^x}$. Moreover, let $\mathcal{I}_1^x := \{i \in \mathcal{I}^x : k(t_i) = 1\}$. Thus, $(t_i)_{i\in\mathcal{I}_1^x}$ are the jump times of the block containing 1 in the killed fragmentation process $\Pi^x$.

For each $n \in \mathbb{N}$ the block of $\Pi^x$ containing $n$ has a killing time that may be finite or infinite. In this chapter we shall answer the question whether it is possible that the supremum over all the respective individual killing times is finite.

**Definition 5.5** We say that $\Pi^x$ becomes *extinct* if there exists some finite time after which all its blocks are killed.

In the course of our analysis of killed fragmentation processes it turns out that there is a critical drift for the killing line such that for all smaller drifts (including the critical drift) the killed process becomes extinct $\mathbb{P}$–a.s. and for all larger drifts the extinction probability is less than 1. The forthcoming Theorem 5.11 shows that the critical drift is given by the following definition. Recall the definition of $\bar{p}$ in Definition 1.30.

**Definition 5.6** Set

$$c_{\bar{p}} := \Phi'(\bar{p}), \tag{5.1}$$

where $\Phi'$ is the derivative of $\Phi$.

## 5.3 An associated spectrally negative Lévy process

In this section we introduce a spectrally negative Lévy process, that is a Lévy process with no upwards jumps and non–monotone paths, cf. Definition 1.5. This process is closely related to the subordinator introduced in Section 1.7. The spectrally negative Lévy process considered here is of bounded variation, and thus it enables us to make use of the well–established theory for this class of Lévy processes, see Section 1.2.2.

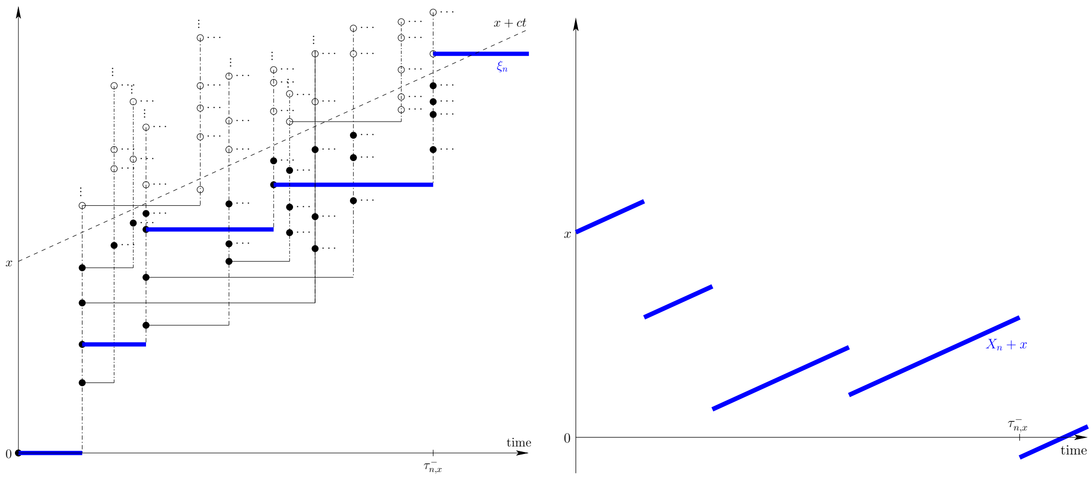
Recall that $B_n(t)$ denotes the block in $\Pi(t)$ which contains the element $n \in \mathbb{N}$, see Figure 5-2(a), and recall that under $\mathbb{P}$ the process $\xi_n = (-\ln(|B_n(t)|)\mathbb{1}_{\{|B_n(t)|>0\}})_{t\in\mathbb{R}_0^+}$ is a killed subordinator.

**Definition 5.7** For every $n \in \mathbb{N}$ let the process $X_n := (X_n(t))_{t\in\mathbb{R}_0^+}$ be defined by

$$X_n(t) := ct - \xi_n(t)$$

for all $t \in \mathbb{R}_0^+$.

Notice that under $\mathbb{P}$ the process $X_n$ is a spectrally negative Lévy process of bounded variation. Moreover, the jump times of $X_n$ are exactly $(t_i)_{i \in \mathcal{I}_n}$, that is $X_n$ jumps exactly when the subordinator $\xi_n$ jumps. Observe that under $\mathbb{P}_{e^x}$, cf. (1.6), the process $X_n(t)$ is shifted by $x \in \mathbb{R}_0^+$, that is $X_n(0) = x$ $\mathbb{P}_{e^x}$–almost surely. Hence, $X_n(t)$ under $\mathbb{P}_{e^x}$ is equivalent to $x + X_n(t)$ under $\mathbb{P}$. The two processes $\xi_n$ and $x + X_n$ under $\mathbb{P}$ are illustrated in Figure 5-2.




(a) Illustration of $\xi_n$ under $\mathbb{P}$. (b) Illustration of $x + X_n$ under $\mathbb{P}$.

**Figure 5-2:** *Illustration of the shifted spectrally negative Lévy process $x+X_n$, cf. (b), associated with a fragmentation process with finite dislocation measure. The randomness of $X_n(t)$ is entirely determined by the subordinator $\xi_n$, cf. (a). Note that the drift of $X_n$ is the constant c, that is it coincides with the slope of the dashed line in (a).*

For any $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$ consider the following $\mathscr{F}$–stopping times:

$$\tau_{n,x}^+ := \inf\{t \in \mathbb{R}_0^+ : X_n(t) > x\} \qquad \text{as well as} \qquad \tau_{n,x}^- := \inf\{t \in \mathbb{R}_0^+ : X_n(t) < -x\}.$$

Let $\psi$ be the Laplace exponent of $X_1$ under $\mathbb{P}$, that is

$$\begin{aligned}
\psi(\beta) &= \ln\left(\mathbb{E}\left(e^{\beta X_1}\right)\right) \\
&= \ln\left(\mathbb{E}\left(e^{\beta c + \beta \ln(|B_1(1)|)} \mathbb{1}_{\{|B_1(1)|>0\}}\right)\right) \\
&= \ln\left(e^{\beta c}\right) + \ln\left(\mathbb{E}\left(e^{-\beta \xi(1)}\right)\right) \\
&= \beta c - \Phi(\beta)
\end{aligned}$$

holds for all $\beta \in \mathbb{R}_0^+$.

For the time being, let $p \in (\underline{p}, \infty)$ and recall the change of measure in (1.19) given by

$$\left.\frac{\mathrm{d}\mathbb{P}^{(p)}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = e^{\Phi(p)t - p\xi(t)} = e^{pX_1(t) - \psi(p)t}.$$

Corollary 3.10 in [Kyp06] shows that under the measure $\mathbb{P}^{(p)}$ the process $X_1$ is a spectrally negative Lévy process with Laplace exponent $\psi_p$ that satisfies

$$\psi_p(\beta) = \psi(p+\beta) - \psi(p)$$

for any $\beta \in \mathbb{R}_0^+$. Let $W$ and $W_p$ be the scale functions, see Definition 1.6, of the spectrally negative Lévy process $X_1$ under $\mathbb{P}$ and $\mathbb{P}^{(p)}$ respectively. Further, let $\psi_p'$ denote the derivative of $\psi_p$ and recall from (1.1) and (1.2) that

$$\mathbb{P}^{(p)}(\tau_{n,x}^- < \infty) = \mathbb{P}_{e^x}^{(p)}(\tau_{n,0}^- < \infty) = \begin{cases} 1 - \psi_p'(0+)W_p(x), & \psi_p'(0+) > 0 \\ 1, & \psi_p'(0+) \leq 0. \end{cases} \tag{5.2}$$

as well as

$$\mathbb{P}\left(\tau_{n,x}^- > \tau_{n,y}^+\right) = \mathbb{P}_{e^x}\left(\tau_{n,0}^- > \tau_{n,x+y}^+\right) = \frac{W(x)}{W(x+y)} \tag{5.3}$$

for all $x, y \in \mathbb{R}_0^+$.

**Remark 5.8** Let $p \in (\underline{p}, \bar{p})$ and let $\Phi_p'$ denote the derivative of $\Phi_p$. In view of (5.2) we are interested in the situation that $\psi_p'(0+) > 0$. In this regard we remark that

$$\psi_p'(0+) = \mathbb{E}^{(p)}(X_1(1)) = c + \mathbb{E}^{(p)}(\ln(|B_1(1)|)) \tag{5.4}$$

is positive if and only if

$$c > \mathbb{E}^{(p)}(-\ln(|B_1(1)|)) = \Phi_p'(0+) = \Phi'(p),$$

where in the light of Corollary 3.10 in [Kyp06] the final equality follows from

$$\Phi_p(\beta) = \Phi(p+\beta) - \Phi(p)$$

for all $\beta \in \mathbb{R}_0^+$. However, since $\Phi$ is concave we have in particular that $\Phi'(\bar{p}) \leq \Phi'(p)$, and thus $\psi_p'(0+)$ may be nonpositive for $c \geq c_{\bar{p}} = \Phi'(\bar{p})$. For a given $c \geq c_{\bar{p}}$ we thus frequently choose some $p \in (\underline{p}, \bar{p})$ such that $c > \Phi'(p)$ in order to have $\psi_p'(0+) > 0$.

Note that such a choice is possible as $c > \Phi'(\bar{p})$ and $\Phi'$ is continuous. $\Diamond$

In what follows, the killed version of $X_n$, killed upon hitting $(-\infty, -x)$ for a given $x \in \mathbb{R}_0^+$, plays a crucial role.

**Definition 5.9** Let $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$. We define a killed and shifted spectrally negative Lévy process $X_n^x := (X_n^x(t))_{t \in \mathbb{R}_0^+}$ as follows: For $t \in \mathbb{R}_0^+ \cup \{\infty\}$ set

$$X_n^x(t) := (X_n(t) + x)\mathbb{1}_{\{\tau_{n,x}^- > t\}} = (x + ct + \ln(|B_n(t)|))\,\mathbb{1}_{\{\tau_{n,x}^- > t\}}.$$

The killed process $X_n^x(t)$ defined above is depicted in Figure 5-3.

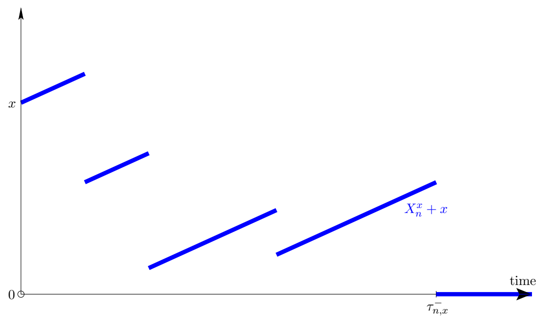



**Figure 5-3:** *Illustration of $X_n^x + x$, killed upon hitting $(-\infty, 0)$, under $\mathbb{P}$. Observe that the graph of $X_n^x$ coincides with the one of $X_n + x$, cf. Figure 5-2(b), up to the time $\tau_{n,x}^-$, which is the hitting time of the negative half–line $(-\infty, 0)$ by the process $X_n + x$. After the killing time $\tau_{n,x}^-$ the killed process $X_n^x$ remains forever in the so–called cemetery state $\{0\}$.*

For the time being, fix some $x \in \mathbb{R}_0^+$. Notice that $X_n^x$, $n \in \mathbb{N}$, is killed upon hitting the open interval $(-\infty, 0)$. Further, note that the killing time of $X_n^x$ is $\tau_{n,x}^-$ and that this is also the killing time of the block of $\Pi^x$ containing $n \in \mathbb{N}$. Hence, the extinction time of $\Pi^x$, which is the time after which all the blocks of $\Pi^x$ are killed, is given by

$$\zeta^x := \sup_{n \in \mathbb{N}} \tau_{n,x}^-. \tag{5.5}$$

Let us point out that $\zeta^x$ is not necessarily finite. In particular, the killed fragmentation process $\Pi^x$ becomes *extinct* if and only if $\zeta^x < \infty$. If $\zeta^x = \infty$, then we say that $\Pi^x$ *survives*. Note that the irregularity with respect to $X_1^0$ of 0 for $(-\infty, 0)$, see (1.3), implies that

$$\mathbb{P}(\zeta^0 = 0) \leq \mathbb{P}(\tau_{1,0}^- = 0) = 0.$$

This means the probability that $\Pi^x$ dies out instantaneously is 0.

For any $c, t \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$ define

$$\lambda_n^x(t) := \left[ (|\Pi_k^x(t)|)_{k\in\mathbb{N}}^{\downarrow} \right]_n .$$

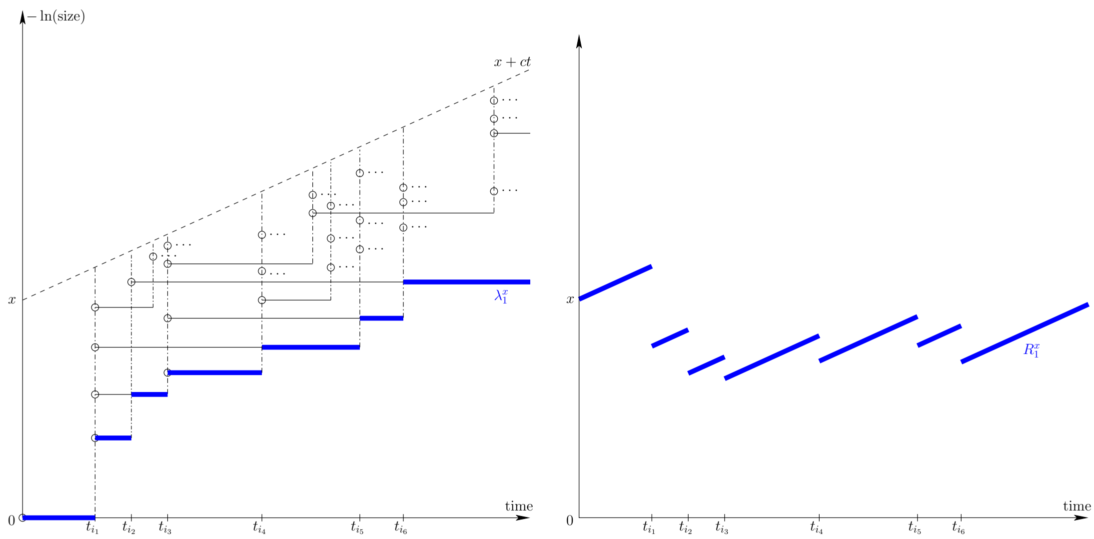
That is to say, $\lambda_n^x(t)$ denotes the size of the $n^{\text{th}}$–largest block alive in the killed fragmentation process $\Pi^x$ at time $t \in \mathbb{R}_0^+$. In particular, notice that

$$\lambda_1^x(t) = \sup_{k\in\mathbb{N}} (|\Pi_k^x(t)|)$$

is the size of the largest block alive at time $t$ and $\lambda_1^x(t) = 0$ for all $t \geq \zeta^x$. In addition, for every $t \in \mathbb{R}_0^+$ we let $R_1^x(t)$ denote the largest value that any of the killed spectrally negative Lévy processes attains at time $t$, that is we set

$$R_1^x(t) := \sup_{k\in\mathbb{N}} X_k^x(t) = x + ct + \ln(\lambda_1^x(t)) .$$

We refer to Figure 5-4 for an illustration of $\lambda_1^x(t)$ and $R_1^x(t)$ in the finite activity case.




(a) Illustration of $\lambda_1^x$. (b) Illustration of $R_1^x$.

**Figure 5-4:** *Illustration of the largest fragment at time $t \in \mathbb{R}_0^+$ in the killed fragmentation process, cf. (a), and the largest value that any of the processes $X_n$, $n \in \mathbb{N}$, attains at time $t$, cf. (b). Here the times $t_{i_k}$, $k \in \mathbb{N}$, form a subset of the jump times $(t_i)_{i\in\mathcal{I}^x}$. More precisely, the sequence $(t_{i_k})_{k\in\mathbb{N}}$ consists of all those jump times of $\Pi^x$ at which the currently largest block dislocates. Observe that the drift of the process $R_1^x$ is the constant $c$, that is it coincides with the slope of the $x$–killing line at which the process $X_n^x$ is killed. Further, note that in this illustration $R_1^x$ remains positive as $\lambda_1^x$ remains below the killing line.*

**Definition 5.10** For every $t \in \mathbb{R}_0^+ \cup \{\infty\}$ set

$$\mathcal{N}_t^x := \left\{n \in \mathbb{N} : \left[t < \tau_{n,x}^-\right] \wedge \left[\exists\, k \in \mathbb{N} : n = \min \Pi_k^x(t)\right]\right\}.$$

The above definition says that $\mathcal{N}_t^x$ consists of all the indices of blocks $B_n$ that are not yet killed by time $t$. Let us remark that the first condition "$t < \tau_{n,x}^-$" in the definition of $\mathcal{N}_t^x$ ensures that $B_n$, the block containing $n \in \mathbb{N}$, is still alive at time $t$ and the second condition "$\exists\, k \in \mathbb{N} : n = \min(\Pi_k^x(t))$" is used to avoid considering the same block multiple times. That is, for a block $B_n(t)$ that is alive at time $t \in \mathbb{R}_0^+$ only its least element is an element of $\mathcal{N}_t^x$. Without the second condition all elements of $B_n(t)$ would be in $\mathcal{N}_t^x$.

## 5.4 Main results on killed fragmentation processes

Our goal in the present chapter is to use the shifted and killed spectrally negative Lévy process $X_n^x$, that was defined in Definition 5.9, to obtain results which are related to certain additive and multiplicative martingales. The main results in this chapter make use of the extinction probability $\mathbb{P}(\zeta^x < \infty)$ of the killed fragmentation process $\Pi^x$. To begin with, the following theorem establishes some properties of the extinction probability that will be useful later on. Recall the constant $c_{\bar{p}}$ that we defined in Definition 5.6.

**Theorem 5.11** *For all $c \le c_{\bar{p}}$ we have $\mathbb{P}(\zeta^x < \infty) = 1$ for any $x \in \mathbb{R}_0^+$. If $c > c_{\bar{p}}$, then $x \mapsto \mathbb{P}(\zeta^x < \infty)$ is a continuous and strictly monotonically decreasing $(0,1)$–valued function on $\mathbb{R}_0^+$.*

We shall prove this theorem in Section 5.5.

Note that as a consequence of Theorem 5.11 we obtain that the conditional probability $\mathbb{P}(\cdot|\zeta^x = \infty)$ is well defined if $c > c_{\bar{p}}$. This turns out to be useful in what follows. The continuity established in Theorem 5.11 will be needed in Chapter 6.

For any function $f : \mathbb{R} \to [0,1]$ and $x \in \mathbb{R}_0^+$ let $Z^{x,f} := (Z_t^{x,f})_{t \in \mathbb{R}_0^+}$ be given by

$$Z_t^{x,f} = \prod_{n \in \mathcal{N}_t^x} f\left(X_n^x(t)\right)$$

for every $t \in \mathbb{R}_0^+$. The question for which functions $f$ the process $Z^{x,f}$ is a *product martingale* is partially answered by the following theorem that will be proven in

Section 5.6.

**Theorem 5.12** *Let $c > c_{\bar{p}}$ and let $f : \mathbb{R} \to [0,1]$ be a monotone function. Then the following two statements are equivalent*

- *For any $x \in \mathbb{R}_0^+$ the process $Z^{x,f}$ is a martingale with respect to the filtration $\mathscr{F}$ and*
$$\lim_{x\to\infty} f(x) = 0.$$
- *For all $x \in \mathbb{R}_0^+$ :*
$$f(x) = \mathbb{P}\left(\zeta^x < \infty\right).$$

Let us consider the process $M^x(p) := (M_t^x(p))_{t\in\mathbb{R}_0^+}$, $p \in (\underline{p}, \infty)$, defined by

$$M_t^x(p) := \sum_{n\in\mathcal{N}_t^x} W_p\left(X_n^x(t)\right) e^{\Phi(p)t} |B_n(t)|^{1+p}. \tag{5.6}$$

The process $M^x(p)$ is defined in the spirit of similar stochastic processes for branching processes and non–killed fragmentations. In this respect note its similarity with the intrinsic additive martingales $M(p)$ and $\Lambda(p)$ that we introduced in Section 1.8 and Section 2.4 respectively. The following theorem states in particular that for certain values of $c$ and $p$ the process $M^x(p)$ is an *intrinsic additive martingale* in the setting of killed fragmentation processes.

**Theorem 5.13** *Let $c > c_{\bar{p}}$ and let $p \in (\underline{p}, \bar{p})$ be such that $c > \Phi'(p)$. Then the process $M^x(p)$ is a nonnegative $\mathscr{F}$–martingale with $\mathbb{P}$–a.s. limit $M_\infty^x(p)$. Moreover, this martingale limit satisfies*

$$\mathbb{P}\left(\{M_\infty^x(p) = 0\} \triangle \{\zeta^x < \infty\}\right) = 0, \tag{5.7}$$

*where $\triangle$ denotes the symmetric difference.*

The proof of Theorem 5.13 will be provided in Section 5.7 and relies on Theorem 5.12.

The final main result of this chapter is concerned with the asymptotic speed of the largest fragment $\lambda_1^x(t)$ in the killed fragmentation conditional on the event of survival of the killed fragmentation process.

**Theorem 5.14** *Let $c > c_{\bar{p}}$ and $x \in \mathbb{R}_0^+$. Then we have*

$$\lim_{t\to\infty} \frac{-\ln(\lambda_1^x(t))}{t} = c_{\bar{p}}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–*almost surely.*

Since the killing of blocks in $\Pi^x$ results in having less blocks that may constitute the largest fragment at a given time, one may expect that the killing increases the asymptotic speed of the largest fragment. However, comparing Theorem 5.14 with (2.6) shows that the asymptotic speed of the largest fragment in the killed fragmentation is exactly the same as the asymptotic speed of the largest fragment in the non–killed fragmentation process. We shall prove Theorem 5.14 in Section 5.8.

The above results show that the map $x \mapsto \mathbb{P}(\zeta^x < \infty)$ plays a crucial role in this chapter. For this reason we adopt the following definition:

**Definition 5.15** We define a function $\varphi : \mathbb{R} \to [0, 1]$ by

$$\varphi(x) = \mathbb{P}(\zeta^x < \infty) \tag{5.8}$$

for all $x \in \mathbb{R}$.

Let us point out that $\varphi$, as the extinction probability on the right–hand side of (5.8), depends on the drift $c > 0$ of the $x$–killing line. In this regard see Remark 5.2.

## 5.5 Properties of the extinction probability

In this section we prove various properties of $\varphi$ in separate lemmas, which combined constitute the proof of Theorem 5.11.

Let us first deal with the easier, but less interesting, case of drifts, that is with $c \in (0, c_{\bar{p}}]$.

**Lemma 5.16** *Let $c \in (0, c_{\bar{p}}]$. Then $\mathbb{P}(\zeta^x < \infty) = 1$ for all $x \in \mathbb{R}_0^+$.*

**Proof** Let $p \geq \bar{p}$ be such that

$$c = c_p := \frac{\Phi(p)}{1 + p}. \tag{5.9}$$

In view of (1.15) the constant $c_p$ in (5.9) concurs for $p = \bar{p}$ with $c_{\bar{p}}$ as defined in Definition 5.6. Note that such a $p \geq \bar{p}$ satisfying (5.9) does indeed exist, since according to Lemma 1 in [Ber03] the mapping $p \mapsto \Phi(p)/(1+p)$ is continuous and decreasing to 0 as $p \to \infty$. Recall that $M_t(p)$ denotes the martingale for fragmentation processes without killing, see Section 1.8. That is,

$$M_t(p) = e^{\Phi(p)t} \sum_{n \in \mathbb{N}} \lambda_n^{1+p}(t) \geq e^{\Phi(p)t} \lambda_1^{1+p}(t) \tag{5.10}$$

for all $t \in \mathbb{R}_0^+$. Since $p \geq \bar{p}$, we have that $M_t(p) \to 0$ $\mathbb{P}$–a.s. as $t \to \infty$, cf. Remark 1.34. Hence, we deduce from (5.10) that

$$\Phi(p)t + \ln\left(\lambda_1(t)^{1+p}\right) \to -\infty$$

as $t \to \infty$. By the choice of $p$ this is equivalent to

$$(1+p)\left(ct + \ln(\lambda_1(t))\right) \to -\infty$$

as $t \to \infty$. Moreover, in view of the fact that $p \geq \bar{p} > \underline{p} > -1$ this implies that

$$(ct + \ln(\lambda_1(t))) \to -\infty$$

as $t \to \infty$. This implies that $\mathbb{P}(\zeta^x < \infty) = 1$ for any $x \in \mathbb{R}_0^+$, and thus it proves the the assertion. □

Notice that the statement of the previous lemma is obvious for $c \in (0, c_{\bar{p}})$ as the asymptotic speed of the largest fragment in the non–killed setting is given by $c_{\bar{p}}$, see (2.6), and thus the fragmentation process eventually crosses the killing line almost surely. However, for the critical value $c = c_{\bar{p}}$ this argument does not work as the largest fragment could approach the killing line from below without intersecting it.

Recall that $\mathcal{I}_1 = \{i \in \mathcal{I} : k(t_i) = 1\}$, that is $(t_i)_{i \in \mathcal{I}_1}$ consists of the jump times of the block containing 1 in the (unkilled) fragmentation process $\Pi$. For any $x \in (0,1)$ set

$$\tau(x) := \inf\left\{t_i \in (t_i)_{i \in \mathcal{I}_1} : |\pi_1(t_i)| \leq x\right\} \tag{5.11}$$

**Remark 5.17** Observe that

$$\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \in (0, x]) < \infty$$

for every $x \in (0,1)$ as otherwise

$$\int_{\mathcal{S}_1} (1 - s_1)\nu(d\mathbf{s}) \geq \int_{\{\mathbf{s} \in \mathcal{S}_1 : s_1 \in (0,x]\}} (1 - s_1)\nu(d\mathbf{s}) \geq (1 - x)\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \in (0, x]) = \infty,$$

which contradicts (1.7). Further, note that there exists some $x \in (0,1)$ such that

$$\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \in (0,x]) > 0, \tag{5.12}$$

as otherwise $\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 = 1) = \nu(\mathcal{S}_1)$ which contradicts (1.7). Moreover, for all $x \in (0,1)$ with $\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \in (0,x]) > 0$ [Ber96, Proposition 2 in Section 0.5] implies that under $\mathbb{P}$ the stopping time $\tau(x)$ is exponentially distributed. In particular, for any such $x \in (0,1)$ the infimum in (5.11) is actually a minimum and $\tau(x) \in (0,\infty)$ $\mathbb{P}$–almost surely. $\diamondsuit$

**Lemma 5.18** *Let $x \in \mathbb{R}_0^+$ and $y \in (1/2 \vee (1-e^{-x}), 1)$ be such that $\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \in (0,y])$. Then*

$$\mathcal{N}_{t_i}^x \in \{\emptyset, \{1\}\}$$

*$\mathbb{P}$–a.s. on $\{x + c\tau(y) < -\ln(1-y)\}$ for all $i \in \mathcal{I}_1^x$ with $t_i < \tau(y)$.*

Lemma 5.18 says that up to time $\tau(y)$ the block in $\Pi^x$ containing 1 is the only block that may be alive on the event $\{x + c\tau(y) < -\ln(1-y)\}$.

**Proof** As $y > 1/2$ we obtain

$$s_2 \leq 1 - s_1 < 1 - y < y \tag{5.13}$$

$\mathbb{P}$–a.s. for any $\mathbf{s} \in \mathcal{S}_1$ with $s_1 > y$. Moreover, since $\Delta_1(t_i) > y$ for all $t_i < \tau(y)$, we have

$$-\ln(\Delta_2(t_i)) \geq -\ln(1 - \Delta_1(t_i)) \geq -\ln(1-y) > x + c\tau(y) > x + ct_i$$

$\mathbb{P}$–a.s. on $\{x + c\tau(y) < -\ln(1-y)\}$ for all $i \in \mathcal{I}_1^x$ with $t_i < \tau(y)$. Notice that this implies that on the event $\{x+c\tau(y) < -\ln(1-y)\}$ up to time $\tau(y)$ only the largest block can possibly survive as at each jump time before $\tau(y)$ the second–largest block in the resulting dislocation is killed instantaneously. However, bearing in mind the definition of $\tau(y)$ in (5.11) it follows from (5.13) that the block containing 1 is larger than the second–largest block resulting from any dislocation before time $\tau(y)$. Consequently, on the event $\{x + c\tau(y) < -\ln(1-y)\}$ the block containing 1 is the only block that is possibly alive at some time $t \in [0, \tau(y))$. $\square$

All the following results in this chapter deal with the more interesting case that $c > c_{\bar{p}}$. That the case $c > c_{\bar{p}}$ is indeed more interesting becomes already obvious if one compares Lemma 5.16 with the following lemma.

**Lemma 5.19** *Let $c > c_{\bar{p}}$. Then*

$$\mathbb{P}(\zeta^x < \infty) \in (0,1)$$

*for all $x \in \mathbb{R}_0^+$.*

**Proof** The proof is divided into two parts. The first part shows that $\mathbb{P}(\zeta^x < \infty) < 1$ and the second part proves that $\mathbb{P}(\zeta^x < \infty) > 0$ for all $x \in \mathbb{R}_0^+$.

Part I Let us first show that $\mathbb{P}(\zeta^x < \infty) < 1$ for all $x \in \mathbb{R}_0^+$. To this end, choose some $p \in (\underline{p}, \bar{p})$ such that $c > \Phi'(p)$. As mentioned in Remark 5.8 we then have that $\psi_p'(0+) > 0$. Hence, we deduce from (5.2) that

$$\mathbb{P}^{(p)}(\tau_{1,0}^- < \infty) = 1 - \psi_p'(0+)W_p(0) = 1 - \frac{\psi_p'(0+)}{c} \in (0,1), \tag{5.14}$$

where the last equality is a consequence of Lemma 1.8 and $\psi_p'(0+) < c$ results from (5.4). Note that above we can resort to Lemma 1.8 , since $X_1$ is of bounded variation. By means of the nondecreasingness of $\mathbb{P}(\zeta^{(\cdot)} = \infty)$, equation (5.14) implies that

$$\mathbb{P}^{(p)}(\zeta^x = \infty) \geq \mathbb{P}^{(p)}(\tau_{1,0}^- = \infty) = \frac{\psi_p'(0+)}{c} \in (0,1)$$

for all $x \in \mathbb{R}_0^+$. According to Remark 1.37 this results in

$$\mathbb{P}(\zeta^x = \infty) > 0, \qquad \text{i.e.} \qquad \mathbb{P}(\zeta^x < \infty) < 1. \tag{5.15}$$

Part II For any $n \in \mathbb{N}$ and $z \in (0,1)$ define inductively

$$\tau(z,n) := \inf \{ t_i > \tau(z,n-1) : i \in \mathcal{I}_1, |\pi_1(t_i)| \leq z \},$$

where $\tau(z,0) := 0$. In order to show that $\mathbb{P}(\zeta^x < \infty) > 0$ for every $x \in \mathbb{R}_0^+$ we fix some arbitrary $y_0 \in (1/2 \vee (1-e^{-x}), 1)$ satisfying

$$\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \leq y_0) > 0.$$

Note that $y_0 > 1-e^{-x}$ implies that $-\ln(1-y_0) > x$. Furthermore, let $\epsilon \in (0, -\ln(y_0)]$. The idea to prove $\mathbb{P}(\zeta^x < \infty) > 0$ is based on bounding below by a positive constant the probability of the event that $X_1$ moves downwards by at least $\epsilon$ between the times $\tau(y_0, n-1)$ and $\tau(y_0, n)$ for all $n \leq \lfloor x/\epsilon \rfloor + 1$, so that after $\lfloor x/\epsilon \rfloor + 1$ steps $X_1$ takes a value in $(-\infty, -x)$ and thus $X_1^x(\tau(y_0, \lfloor x/\epsilon \rfloor + 1)) = 0$, i.e. on this event the block containing

1 is killed by time $\tau(y_0, \lfloor x/\epsilon \rfloor + 1)$ at the latest. To obtain a lower bound for extinction we intersect the aforementioned event with the event that for any $t \leq \tau(y_0, \lfloor x/\epsilon \rfloor + 1)$ only the block containing 1 may be alive.

For every $n \in \mathbb{N}$ set

$$E_n^1 := \{c(\tau(y_0, n) - \tau(y_0, n-1)) + \ln(y_0) < -\epsilon\} \cup \{\tau_{1,x}^- \leq \tau(y_0, n)\}$$

and

$$E_n^2 := \{X_1^x(\tau(y_0, n-1)) + c(\tau(y_0, n) - \tau(y_0, n-1)) < -\ln(1-y_0)\} \cup \{\tau_{1,x}^- \leq \tau(y_0, n)\}$$

as well as

$$E_n^3 := \{\lambda_1^x(\tau(y_0, n)) = X_1^x(\tau(y_0, n)\} \cap \{\lambda_2^x(\tau(y_0, n)) = 0\}.$$

Note that $X_1(\tau(y_0, n)) - X_1(\tau(y_0, n-1)) < -\epsilon$ or $X_1(\tau(y_0, n)) = 0$ on $E_n^1$. That is, on the event $E_n^1$ the size of $X_1^x$ is decreased by at least $\epsilon$ during the time period $[\tau(y_0, n-1), \tau(y_0, n)]$, unless $X_1^x$ is killed before or at time $\tau(y_0, n)$. Moreover, according to Lemma 5.18, on the event $\bigcap_{n=1}^k (E_n^2 \cap E_n^3)$ only the block containing 1 may be alive at any time $t \leq \tau(y_0, k)$. Recall $\tau(y_0)$ from (5.11) and observe that

$$E_1^2 = \{x + c\tau(y_0) < -\ln(1-y_0)\}.$$

Further, notice that for any $k \in \mathbb{N}$ we have $X_1^x(\tau(y_0, k)) \leq (x - k\epsilon) \vee 0 \leq x$ on $\bigcap_{n=1}^k E_n^1$, and thus

$$E_{k+1}^2 \cap \bigcap_{n=1}^k E_n^1 \supseteq \{x + c(\tau(y_0, k+1) - \tau(y_0, k)) < -\ln(1-y_0)\} \cap \bigcap_{n=1}^k E_n^1.$$

Therefore, by means of the extended fragmentation property we infer that

$$\begin{aligned}
&\mathbb{P}(\zeta^x < \infty) \\
&\geq \mathbb{P}\left( \bigcap_{n=1}^{\lfloor x/\epsilon \rfloor + 1} \left(E_n^1 \cap E_n^2 \cap E_n^3\right) \right) \\
&= \prod_{n=1}^{\lfloor x/\epsilon \rfloor + 1} \mathbb{P}\left( E_n^1 \cap E_n^2 \cap E_n^3 \,\middle|\, \bigcap_{k=1}^{n-1} \left(E_k^1 \cap E_k^2 \cap E_k^3\right) \right) \\
&\geq \mathbb{P}(E_1^1 \cap E_1^2)^{\lfloor x/\epsilon \rfloor + 1} \mathbb{P}(E_1^3)^{\lfloor x/\epsilon \rfloor + 1}
\end{aligned}$$

$$= \mathbb{P}\left(\left\{\tau(y_0) < \left[\frac{-\ln(y_0) - \epsilon}{c}\right] \wedge \left[\frac{-\ln(1-y_0) - x}{c}\right]\right\} \cup \{\tau^-_{1,x} \leq \tau(y_0)\}\right)^{\lfloor x/\epsilon \rfloor + 1}$$
$$\cdot \mathbb{P}(E^3_1)^{\lfloor x/\epsilon \rfloor + 1}$$
$$> 0,$$

where the final positivity is a consequence of $\tau(y_0)$ being exponentially distributed. □

The following auxiliary lemma will be used to prove continuity and strict monotonicity of $\varphi$. In the proof of that lemma we need to consider all the fragments at a given time, and for this purpose we need a deterministic estimate of the number of fragments alive at that time. To this end, set $N^x_t := \mathrm{card}(\mathcal{N}^x_t)$ and observe that $N^x_t < \infty$ $\mathbb{P}$–a.s. for any $t \in \mathbb{R}^+_0$. Indeed, as $\sum_{n\in\mathbb{N}} |\Pi_n(t)| \leq 1$ we infer that $|\Pi_n(t)| \geq e^{-(x+ct)}$ for at most $e^{x+ct}$–many $n \in \mathbb{N}$. That is

$$N^x_t \leq e^{x+ct} \tag{5.16}$$

for all $t \in \mathbb{R}^+_0$.

**Lemma 5.20** *Let $c > c_{\bar{p}}$. For all $0 \leq x < y < \infty$ there exists some $\alpha_{x,y} > 0$ such that*

$$\mathbb{P}(\zeta^x < \infty) - \mathbb{P}(\zeta^{x+h} < \infty) \geq \alpha_{x,y}\left(\mathbb{P}(\zeta^y < \infty) - \mathbb{P}(\zeta^{y+h} < \infty)\right)$$

*for all $h > 0$.*

**Proof** In the first part of the proof we show that for every deterministic time $t > 0$ the probability that $X_1$ reaches level $x > 0$ before time $t$ is positive. In the second part we use this fact in order to obtain a lower bound of the probability that for some $n \in \mathbb{N}$ the process $X^x_n$ hits a given level $y > x$ before some deterministic time $s > 0$. Subsequently we combine this lower bound with the previously shown estimate of the number of blocks that are alive at a given time, see (5.16), and with the positivity of the probability of extinction, which enables us to prove the assertion.

Part I According to Corollary 3.14 in [Kyp06] we have that $(\tau_{1,x})_{x\in\mathbb{R}^+_0}$ is a subordinator with either killing at an independent exponential time $\tau_e$ or with no killing in which case we set $\tau_e := \infty$. Moreover, by means of Proposition 1.7 in [Ber99] we thus infer that

$$\mathbb{P}\left(\tau^+_{1,x} < t\right) = \mathbb{P}\left(\{\tilde{\tau}^+_{1,x} < t\} \wedge \{x < \tau_e\}\right) = \mathbb{P}\left(\tilde{\tau}^+_{1,x} < t\right)\mathbb{P}\left(x < \tau_e\right) > 0 \tag{5.17}$$

holds for all $t > 0$, where $\tilde{\tau}^+_{1,x}$ is some non–killed subordinator satisfying

$$\tilde{\tau}^+_{1,x}\mathbb{1}_{\{x<\tau_e\}} = \tau^+_{1,x}\mathbb{1}_{\{x<\tau_e\}}.$$

Let us now show that

$$\forall\, t > 0 : \mathbb{P}(\tau^+_{1,x} < \tau^-_{1,0} \wedge t) > 0. \tag{5.18}$$

To this end, assume we had $\mathbb{P}(\tau^+_{1,x} < \tau^-_{1,0} \wedge t_0) = 0$ for some $t_0 > 0$. Our goal is to show that this results in a contradiction. For this purpose, set $\tau^{(2)}_0 := 0$ and for every $n \in \mathbb{N}$ define

$$\begin{aligned}\tau^{(1)}_n &:= \inf\left\{t > \tau^{(2)}_{n-1} : X_1(t) < 0\right\},\\ \tau^{(2)}_n &:= \inf\left\{t > \tau^{(1)}_n : X_1(t) = 0\right\}.\end{aligned}$$

In view of (1.3) there exists some $\delta > 0$ such that $\mathbb{P}(\tau^-_{1,0} \geq \delta) > 0$, and consequently we obtain by means of the strong Markov property that

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(\tau^{(1)}_n - \tau^{(2)}_{n-1} \geq \delta \,\middle|\, \mathscr{F}_{\tau^{(2)}_{n-1}}\right) = \sum_{n\in\mathbb{N}} \mathbb{P}\left(\tau^-_{1,0} \geq \delta\right) = \infty \tag{5.19}$$

$\mathbb{P}$–almost surely. Since $\{\tau^{(1)}_n - \tau^{(2)}_{n-1} \geq \delta\}$ is $\mathscr{F}_{\tau^{(2)}_{n-1}}$–measurable, we can apply an extended Borel–Cantelli lemma (see e.g. [Dur91, (3.2) Corollary in Chapter 4] or [Bre92, Corollary 5.29]) to deduce that

$$\begin{aligned}&\left\{\{\tau^{(1)}_n - \tau^{(2)}_{n-1} \geq \delta\} \text{ happens infinitely often}\right\}\\ &\qquad = \left\{\sum_{n\in\mathbb{N}} \mathbb{P}\left(\tau^{(1)}_n - \tau^{(2)}_{n-1} \geq \delta \,\middle|\, \mathscr{F}_{\tau^{(2)}_{n-1}}\right) = \infty\right\},\end{aligned}$$

and thus (5.19) implies that $\tau^{(1)}_n \to \infty$ $\mathbb{P}$–a.s. as $n \to \infty$. Since, by [Ber96, Theorem 12 in Section VI.3], we have that $\tau^+_{1,x} \wedge \tau^-_{1,0} < \infty$ $\mathbb{P}$–a.s., an application of the strong Markov property therefore yields that

$$\mathbb{P}\left(\tau^+_{1,x} < t_0\right) \leq \sum_{n\in\mathbb{N}} \mathbb{P}\left(\tau^{(3)}_{n,x} < \tau^{(1)}_n \wedge t_0 \,\middle|\, \mathscr{F}_{\tau^{(2)}_{n-1}}\right) = \sum_{n\in\mathbb{N}} \mathbb{P}(\tau^+_{1,x} < \tau^-_{1,0} \wedge t_0) = 0, \tag{5.20}$$

where

$$\tau^{(3)}_{n,x} := \inf\{t > \tau^{(2)}_{n-1} : X_1(t) > x\}$$

for all $n \in \mathbb{N}$. As (5.20) contradicts (5.17), we conclude that (5.18) does indeed hold true.

<u>Part II</u> Let $0 \leq x < y < \infty$ as well $s > 0$ and set

$$\tau_y^+(x) := \inf\left\{t \in \mathbb{R}_0^+ : R_1^x(t) \geq y\right\}.$$

Note that $R_1^x(\tau_y^+(x)) = y$, since $R_1^x$ does not jump upwards and thus creeps over the value $y$. Hence,

$$\tau_y^+(x) = \inf\left\{t \in \mathbb{R}_0^+ : R_1^x(t) = y\right\}. \tag{5.21}$$

Set

$$\alpha_{x,y} := \mathbb{P}\left(\tau_y^+(x) < \zeta^x \wedge s\right) \mathbb{P}(\zeta^{y+h} < \infty)^{x+cs}.$$

Observe that (5.18) and Lemma 5.19 imply that

$$\alpha_{x,y} > 0, \tag{5.22}$$

since

$$\mathbb{P}\left(\tau_y^+(x) < \zeta^x \wedge s\right) \geq \mathbb{P}\left(\tau_{1,y-x}^+ < \tau_{1,0}^- \wedge s\right).$$

Setting $\gamma := e^{x+cs} - 1$ we then have by means of the extended fragmentation property and (5.21) that

$$\begin{aligned} &\mathbb{P}(\zeta^x < \infty) - \mathbb{P}(\zeta^{x+h} < \infty) \\ &\overset{(*)}{\geq} \mathbb{P}\left(\tau_y^+(x) < \zeta^x \wedge s\right) \mathbb{P}(\zeta^y < \infty)^\gamma \Big(\mathbb{P}(\zeta^y < \infty) - \mathbb{P}(\zeta^{y+h} < \infty)\Big) \\ &= \alpha_{x,y} \Big(\mathbb{P}(\zeta^y < \infty) - \mathbb{P}(\zeta^{y+h} < \infty)\Big) \end{aligned} \tag{5.23}$$

holds true for any $h > 0$, where the exponent $\gamma$ in $(*)$ results from the estimate $N_s^x \leq e^{x+cs} = \gamma + 1$ $\mathbb{P}$–a.s., cf. (5.16). Notice that in $(*)$ we have used that the size of each block alive at time $\tau_y^+(x)$ is less than or equal to $y$ as well as the monotonicity of $\varphi$. Let us remark that the estimate in (5.23) says the following: By the extended fragmentation property the difference in the probability of survival with respect to the $x$–killing line and the probability of survival with respect to the $(x+h)$–killing line can be bounded below by the product of

- the probability that $R_1^x$ reaches the value $y \geq x$ at a finite time $\tau_y^+(x)$ before $\Pi^x$ becomes extinct,
- the probability that all the blocks, except for the one with size $y$, alive (with respect to the $x$–killing line) at time $\tau_y^+(x)$ (there are at most $\gamma$–many such blocks)

die out eventually, that is the independent copies of the killed fragmentation initiated by these blocks become extinct (with respect to the $y$–killing line),

- the difference in the probabilities of survival with respect to the $y$–killing line and with respect to the $(y+h)$–killing line respectively.

In view of (5.22) the estimate in (5.23) completes the proof. □

The following two lemmas establish some analytical properties of the function $\varphi$.

**Lemma 5.21** *Let $c > c_{\bar{p}}$. Then the function $\varphi|_{\mathbb{R}_0^+}$ is continuous.*

**Proof** We prove the assertion by contradiction. To this end, let $x \in \mathbb{R}^+$ and assume that $\varphi$ is not continuous at $x$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ with

$$x_n \to x \tag{5.24}$$

and

$$\varphi(x_n) \not\to \varphi(x) \tag{5.25}$$

as $n \to \infty$.

Notice that, due to monotonicity, $\varphi$ is continuous almost everywhere and thus we can choose some $y \in (0, x)$ such that $\varphi$ is continuous at $y$. Further, define a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_0^+$ by

$$y_n := y + x_n - x$$

and note that (5.24) implies that $y_n \to y$ as $n \to \infty$.

According to Lemma 5.20 there exists some $\alpha_{y,x} > 0$ such that

$$|\varphi(y) - \varphi(y_n)| \geq \alpha_{y,x} \, |\varphi(x) - \varphi(x_n)|$$

Therefore, we deduce from (5.25) that

$$\limsup_{n\to\infty} |\varphi(y) - \varphi(y_n)| > 0,$$

which contradicts $\varphi$ being continuous at $y$.

In order to show right–continuity of $\varphi$ at 0 recall from (1.3) that for $X_1$ the point 0 is irregular for $(-\infty, 0)$, i.e. $\mathbb{P}(\tau_0^- > 0) = 1$. Hence, we have $X_1(\tau_0^-/2) > 0$ $\mathbb{P}$–a.s., and thus

$$\varphi(0) = \varphi(X_1(\tau_0^-/2)) \leq \lim_{x\downarrow 0} \varphi(x).$$

On the other hand, the montonicity of $\varphi$ entails $\varphi(0) \geq \lim_{x\downarrow 0} \varphi(x)$, which proves that $\varphi$ is right–continuous at 0. □

**Lemma 5.22** *Let $c > c_{\bar{p}}$. Then $\varphi$ is strictly monotonically decreasing on $\mathbb{R}_0^+$.*

**Proof** For any $x \in \mathbb{R}_0^+$ set

$$\gamma_x := \ln\left(\Delta_1(\zeta^x)\left|\Pi^x_{k(\zeta^x)}(\zeta^x-)\right|\right).$$

According to Lemma 5.19 we have $\mathbb{P}(\zeta^x < \infty) > 0$, $x \in \mathbb{R}_0^+$, and hence

$$\begin{aligned}
&\mathbb{P}\left(\{\zeta^x < \infty\} \cap \bigcup_{n\in\mathbb{N}} \{x + c\zeta^x + \gamma_x \in (-n, 0)\}\right) \\
&= \mathbb{P}(\{\zeta^x < \infty\} \cap \{x + c\zeta^x + \gamma_x \in (-\infty, 0)\}) \\
&= \mathbb{P}(\zeta^x < \infty) \\
&> 0.
\end{aligned}$$

Therefore, for every $x \in \mathbb{R}_0^+$ there exists some $z > 0$ such that

$$\mathbb{P}\left(\{\zeta^x < \infty\} \cap \{x + c\zeta^x + \ln(\Delta_1(\zeta^x)) \in (-z, 0)\}\right) > 0, \tag{5.26}$$

and thus the extended fragmentation property, in conjunction with Lemma 5.19, yields that

$$\mathbb{P}(\zeta^{x+z} < \infty) < \mathbb{P}(\zeta^x < \infty). \tag{5.27}$$

Indeed, we have

$$\begin{aligned}
&\mathbb{P}(\{\zeta^x < \infty\} \cap \{\zeta^{x+z} = \infty\}) \\
&\geq \mathbb{E}\left(\mathbb{P}\left(\{\zeta^x < \infty\} \cap \{x + c\zeta^x + \gamma_x \in (-z, 0)\} \cap \{\tilde{\zeta} = \infty\} \middle| \mathscr{F}_{\zeta^x}\right)\right) \\
&\geq \mathbb{P}(\{\zeta^x < \infty\} \cap \{x + c\zeta^x + \gamma_x \in (-z, 0)\})\mathbb{P}(\zeta^0 = \infty) \qquad (5.28) \\
&> 0,
\end{aligned}$$

where conditional on $\mathscr{F}_{\zeta^x}$ the random variable $\tilde{\zeta}$ is independent of $\Pi$ and satisfies

$$\mathbb{P}\left(\tilde{\zeta} \in \cdot \middle| \mathscr{F}_{\zeta^x}\right) = \mathbb{P}\left(\zeta^y \in \cdot\right)\big|_{y = x+z+c\zeta^x+\gamma_x}$$

$\mathbb{P}$–almost surely. The first estimate in (5.28) is a consequence of the extended fragmentation property and the final positivity results from Lemma 5.19 and (5.26). Consequently, for each $x \in \mathbb{R}_0^+$ there exists some $z > 0$ such that

$$\begin{aligned}\mathbb{P}(\zeta^x < \infty) &= \mathbb{P}(\{\zeta^x < \infty\} \cap \{\zeta^{x+z} = \infty\}) + \mathbb{P}(\{\zeta^x < \infty\} \cap \{\zeta^{x+z} < \infty\}) \\ &> \mathbb{P}(\zeta^{x+z} < \infty),\end{aligned}$$

where the final estimate follows from (5.28) and $\{\zeta^{x+z} < \infty\} \subseteq \{\zeta^x < \infty\}$. Hence, we have shown that (5.27) holds true.

Observe that (5.27) implies that for any $h > 0$ and $x \in \mathbb{R}_0^+$ there exists some $y \geq x$ such that

$$\varphi(y) > \varphi(y + h). \tag{5.29}$$

Consequently, according to Lemma 5.20 there exists some $\alpha_{x,y} > 0$ such that

$$\varphi(x) - \varphi(x + h) \geq \alpha_{x,y} (\varphi(y) - \varphi(y + h)) > 0$$

for any $h > 0$, $x \in \mathbb{R}_0^+$ and $y \geq x$ satisfying (5.29), where the positivity follows from (5.29). Note that the value $y$ depends on the choice of $h$ and $x$. This shows that $\varphi$ is monotonically decreasing, and thus the assertion of the lemma is proven. □

## 5.6 A product martingale associated with killed fragmentation processes

The goal of this section is to prove Theorem 5.12. We split the proof of Theorem 5.12 into two propositions, each dealing with one of the two directions of the equivalence in Theorem 5.12.

### 5.6.1 Uniqueness of a product martingale inducing function

The first implication of Theorem 5.12 is established in the following proposition that is the subject of the present section. The converse implication will be dealt with in the subsequent section.

**Proposition 5.23** *Let $c > c_{\bar{p}}$. Furthermore, let $f : \mathbb{R} \to [0, 1]$ be a monotone function that satisfies $\lim_{x\to\infty} f(x) = 0$ and assume that the process $Z^{x,f}$ is an $\mathscr{F}$–martingale under $\mathbb{P}$. Then $f(x) = \varphi(x)$ for all $x \in \mathbb{R}_0^+$.*

Before we can tackle the proof of proposition 5.23 we need to develop several auxiliary results.

Recall the notation $N_t^x := \mathrm{card}(\mathcal{N}_t^x)$ and recall from (5.16) that $N_t^x \le e^{x+ct}$ for all $t \in \mathbb{R}_0^+$. However, notice that $e^{x+ct}$ is an upper bound of $N_t^x$, but the actual value of $N_t^x$ may possibly be below that bound. Hence, a nontrivial question is whether $N_t^x$ remains finite as $t$ tends to $\infty$. The following lemma shows that this is not the case, a fact that we make use of later on.

**Lemma 5.24** *Let $c > c_{\bar{p}}$. Then we have that*

$$\limsup_{t\to\infty} N_t^x = \infty$$

*holds true $\mathbb{P}(\cdot|\zeta^x = \infty)$–a.s. for any $x \in \mathbb{R}_0^+$.*

In order to prove Lemma 5.24 we need the following auxiliary lemma which states that for any $n \in \mathbb{N}$ there exists a stopping time such that with positive probability there are at least $n$ blocks alive at that stopping time. More precisely, we have the following result:

**Lemma 5.25** *Let $c > c_{\bar{p}}$ and $x \in \mathbb{R}_0^+$. Then for any $n \in \mathbb{N}$ there exists some $t > 0$ such that*

$$\mathbb{P}\left(N_t^x \ge n\right) > 0. \tag{5.30}$$

**Proof** In the first part of the proof we show that the probability of the event $\{N_t^x \ge 2\}$ is positive for some $t \in \mathbb{R}_0^+$ and in the second part we use this in conjunction with an induction argument to prove the assertion.

Part I Since, by Hypothesis 1.2, in the unkilled fragmentation process there are at least two blocks at the jump time, it follows from (1.7) that there exists some $y_0 \in (1/2, 1)$ such that

$$\nu(\mathbf{s} \in \mathcal{S}_1 : s_2 \ge 1 - y_0) > 0.$$

Indeed, assume $\nu(\mathbf{s} \in \mathcal{S}_1 : s_2 \ge a) = 0$ for all $a \in (0,1)$. Then $\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \ne 1) = 0$, which contradicts (1.7). Furthermore, in the light of (5.12) we assume that

$$\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \in (0, y_0]) > 0.$$

We have

$$\nu(\mathbf{s} \in \mathcal{S}_1 : s_2 \ge 1 - y_0) \le \nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \le y_0) < \infty,$$

and thus [Ber96, Proposition 2 in Section 0.5] shows that

$$\mathbb{P}\left(\Delta_2(\tau(y_0)) \ge 1 - y_0\right) = \frac{\nu(\mathbf{s} \in \mathcal{S}_1 : s_2 \ge 1 - y_0)}{\nu(\mathbf{s} \in \mathcal{S}_1 : s_1 \le y_0)} > 0, \tag{5.31}$$

where $\tau(y_0)$ is given by (5.11). Consider the fragmentation processes $\tilde{\Pi}$ obtained from the restricted dislocation measure

$$\tilde{\nu} := \nu|_{\{\mathbf{s}\in\mathcal{S}_1 : s_1 > y_0\}}.$$

According to [Ber96, Proposition 2 in Section 0.5] the stopping time $\tau(y_0)$ is exponentially distributed with parameter $q \in \mathbb{R}^+$ and independent of $\tilde{\Pi}$. Let $\tilde{X}_1^0$ be the spectrally negative Lévy process, starting from 0 and killed at the negative half–line, corresponding to $\tilde{\Pi}$ and let $\tilde{R}^{(q)}(0,\cdot)$ be the associated $q$–resolvent measure, see e.g. the proof of Theorem 8.7 in [Kyp06]. Resorting to Corollary 8.8 in [Kyp06] and to (5.31) we thus have that

$$\begin{aligned}
&\mathbb{P}\left(N^0_{\tau(y_0)} \geq 2\right)\\
&= \mathbb{P}\left(X_1^0(\tau(y_0-) \in (-\ln(y_0), -\ln(1-y_0))\right)\mathbb{P}\left(\Delta_2(\tau(y_0)) \geq 1-y_0\right)\\
&= \mathbb{P}\left(\tilde{X}_1^0(\tau(y_0) \in (-\ln(y_0), -\ln(1-y_0))\right)\mathbb{P}\left(\Delta_2(\tau(y_0)) \geq 1-y_0\right)\\
&= q\tilde{R}^{(q)}\left(0, (-\ln(y_0), -\ln(1-y_0))\right)\mathbb{P}\left(\Delta_2(\tau(y_0)) \geq 1-y_0\right) \qquad (5.32)\\
&> 0.
\end{aligned}$$

Moreover, this results in

$$\mathbb{P}\left(N_t^0 \geq 2\right) > 0 \qquad (5.33)$$

for some $t > 0$. Indeed, by means of the extended fragmentation property and Lemma 5.19 the positivity in (5.32) implies that

$$\mathbb{P}\left(N_t^0 \geq 2\right) \geq \mathbb{P}\left(\left\{N^0_{\tau(y_0)} \geq 2\right\} \wedge \{\tau(y_0) \leq t)\}\right)\mathbb{P}\left(\zeta^0 > t\right)^2 > 0$$

for some $t > 0$.

Part II We prove (5.30) by resorting to the principle of mathematical induction, see Figure 5-5. To this end, let $n \in \mathbb{N}\cup\{0\}$ and in view of (5.33) fix some $u_0 > 0$ such that

$$\mathbb{P}\left(N^0_{u_0} \geq 2\right) > 0. \qquad (5.34)$$

As the induction hypothesis assume that

$$\mathbb{P}\left(N^0_{nu_0} \geq n+1\right) > 0.$$

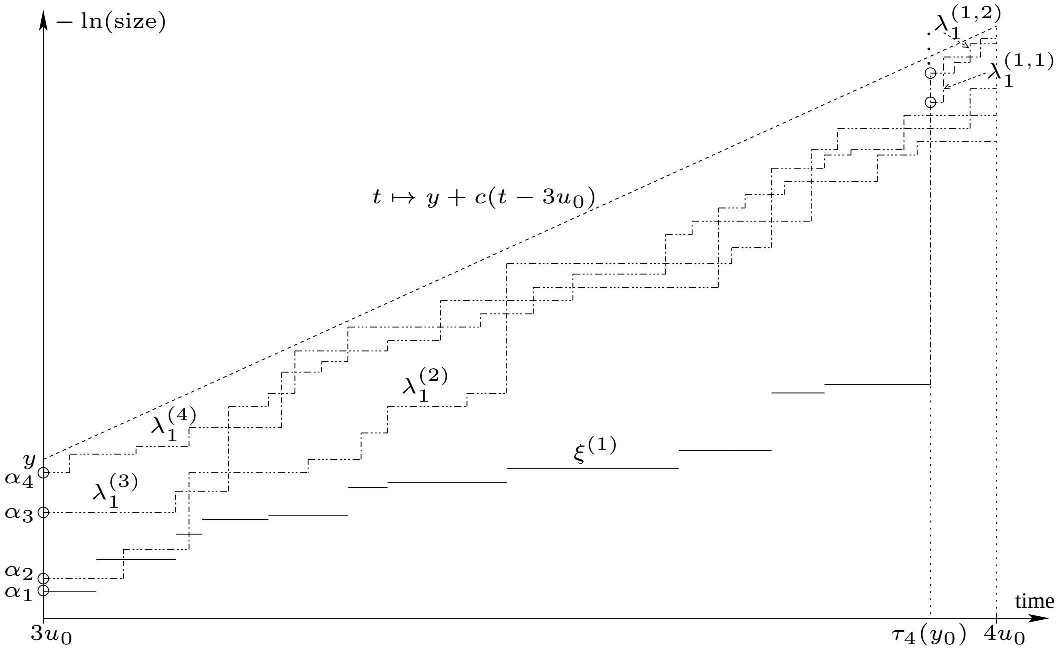



**Figure 5-5:** *Illustration of the induction argument based on (5.33) for $n = 3$ and with $y := c3u_0$ in the case of a finite dislocation measure. By induction hypothesis $N^0_{3u_0} \geq 4$. The fragmentation property implies that the blocks alive at time $3u_0$ start independent copies of $\Pi^x$ (with different initial configurations). In this illustration we depict a realisation of these independent copies. For $k \in \{1, \dots, 4\}$ set $\alpha_k := -\ln(\lambda_k^0(3u_0))$. Then $\xi^{(1)}$ is an independent copy of the subordinator $\xi$, shifted by $\alpha_1$, and $\lambda_1^{(k)}$, $k \in \{2, 3, 4\}$, is an independent copy of $(\lambda_1^{\alpha_k}(t))_{t \in [0,u_0]}$. Further, the random variable $\tau_4(y_0) - 3u_0$ is an independent copy of $\tau(y_0)$. In view of the extended fragmentation property, $\lambda_1^{(1,1)}$ and $\lambda_1^{(1,2)}$ are independent copies of $\lambda_1$, starting at $\tau_4(y_0)$ and shifted by $\xi^{(1)}(\tau_4(y_0))\Delta_1(\tau_4(y_0))$ and $\xi^{(1)}(\tau_4(y_0))\Delta_2(\tau_4(y_0))$ respectivey.*

By the fragmentation property and (5.34) as well as Lemma 5.19 this yields that

$$\begin{aligned}
\mathbb{P}\left(N^x_{(n+1)u_0} \geq n+2\right) &\geq \mathbb{P}\left(N^x_{nu_0} \geq n+1\right) \mathbb{P}\left(N^{(n)} \geq 2 \middle| \mathscr{F}_{nu_0}\right) \prod_{k=2}^{n+1} \mathbb{P}\left(\zeta_k^{(n)} > u_0 \middle| \mathscr{F}_{nu_0}\right) \\
&\geq \mathbb{P}\left(N^0_{nu_0} \geq n+1\right) \mathbb{P}\left(N^0_{u_0} \geq 2\right) \mathbb{P}\left(\zeta^0 > u_0\right)^n \\
&> 0
\end{aligned}$$

holds $\mathbb{P}$–a.s., where conditional on $\mathscr{F}_{nu_0}$ the $N^{(n)}$ and $\zeta_k^{(n)}$ are independent of any random variable involved and satisfy

$$\mathbb{P}\left(N^{(n)} \in \cdot \middle| \mathscr{F}_{nu_0}\right) = \mathbb{P}\left(N^y_{u_0} \in \cdot\right)\Big|_{y = x + cnu_0 + \ln(\lambda_1^x(nu_0))}$$

as well as

$$\mathbb{P}\left(\zeta_k^{(n)} \in \cdot \middle| \mathscr{F}_{nu_0}\right) = \mathbb{P}\left(\zeta^y \in \cdot\right)\big|_{y=x+cnu_0+\ln(\lambda_k^x(nu_0))}$$

$\mathbb{P}$–almost surely. As $\mathbb{P}\left(N_0^x \geq 1\right) = 1$, this completes the induction argument. □

Having established the previous lemma we are now in a position to tackle the proof of Lemma 5.24.

**Proof of Lemma 5.24** Fix some $k \in \mathbb{N}$ and in view of Lemma 5.25 let $t_0 > 0$ be such that

$$\mathbb{P}(N_{t_0}^0 \geq k) > 0. \tag{5.35}$$

Furthermore, for every $n \in \mathbb{N}$ define

$$E_n := \left\{\omega \in \Omega : N_{nt_0}^0(\omega) \geq k\right\}.$$

By means of the fragmentation property and (5.35) we have for any $n \in \mathbb{N}$ that

$$\mathbb{P}\left(E_n \,|\mathscr{F}_{(n-1)t_0}\right) \geq \mathbb{P}\left(N^{(n)} \geq k \,|\mathscr{F}_{(n-1)t_0}\right) \geq \mathbb{P}(N_{t_0}^0 \geq k) > 0 \tag{5.36}$$

holds $\mathbb{P}$–a.s. on $\{\zeta^0 = \infty\}$, where conditional on $\mathscr{F}_{(n-1)t_0}$ the $N^{(n)}$ are independent and satisfy

$$\mathbb{P}\left(N^{(n)} \in \cdot \middle| \mathscr{F}_{(n-1)t_0}\right) = \mathbb{P}\left(N_{t_0}^y \in \cdot\right)\big|_{y=X_{j_n}^0((n-1)\tau_0)}$$

$\mathbb{P}$–a.s. with $j_n = \min_{i \in \mathcal{N}^0_{(n-1)\tau_0}}$. As a consequence of (5.36) we obtain that

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(E_n \middle| \mathscr{F}_{(n-1)t_0}\right) = \infty \tag{5.37}$$

$\mathbb{P}$–a.s. on $\{\zeta^0 = \infty\}$.

Since $E_n$ is $\mathscr{F}_{nt_0}$–measurable, we can apply an extended Borel–Cantelli lemma (see e.g. [Dur91, (3.2) Corollary in Chapter 4] or [Bre92, Corollary 5.29]) to deduce that

$$\{E_n \text{ happens infinitely often}\} = \left\{\sum_{n\in\mathbb{N}} \mathbb{P}\left(E_n \middle| \mathscr{F}_{(n-1)t_0}\right) = \infty\right\},$$

and thus (5.37) shows that on the event $\{\zeta^0 = \infty\}$ the event $E_n$ happens for infinitely many $n \in \mathbb{N}$. Therefore,

$$\mathbb{P}\left(\limsup_{n\to\infty} E_n \middle| \zeta^0 = \infty\right) = 1,$$

where

$$\limsup_{n\to\infty} E_n := \bigcap_{n\in\mathbb{N}} \bigcup_{m=n}^{\infty} E_m.$$

Consequently, we infer that

$$\mathbb{P}\left(\limsup_{t\to\infty} N_t^0 \geq k \middle| \zeta^0 = \infty\right) = 1,$$

which proves the assertion by letting $k \to \infty$. □

The next lemma, which is not used anywhere else in this thesis, shows that $\mathbb{P}$–a.s. there is no accumulation of all the mass against the killing line, i.e. the largest fragment is bounded away from the killing line. Let $N_t^x(\epsilon)$, $\epsilon \in (0,x)$, denote the number of blocks of $\Pi^x$ whose size is a value in $(e^{-(x+ct-\epsilon)}, 1]$, see Figure 5-6.

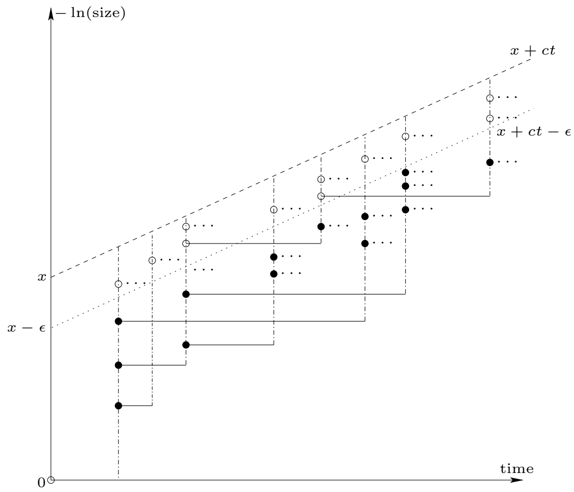



**Figure 5-6:** *Illustration of $N_t^x(\epsilon)$, that is the number of particles that at time $t$ are below the lower dotted line. In this picture these particles are indicated by a black–coloured dot.*

**Lemma 5.26** *Let $c > c_{\bar{p}}$. For any $x \in \mathbb{R}^+$ there exists some $\mathbb{R}^+$–valued random variable $\epsilon$ such that*

$$\inf_{t\in\mathbb{R}_0^+} N_t^x(\epsilon) \geq 1$$

*holds true $\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely.*

**Proof** Observe that the assertion is proven once we have shown that

$$\inf_{t\in\mathbb{R}_0^+} ((x + ct) + \ln(\lambda_1^x(t))) > 0 \tag{5.38}$$

holds $\mathbb{P}(\cdot|\zeta^x = \infty)$–a.s. for any $x \in \mathbb{R}^+$. Recall that Lemma 5.21 shows that the mapping $\varphi = \mathbb{P}(\zeta^{(\cdot)} = \infty)$ is continuous. In addition, let $x \in \mathbb{R}_0^+$ and define the event

$$A := \{\zeta^x = \infty\} \cap \left\{ \inf_{t \in \mathbb{R}_0^+} ((x + ct) + \ln(\lambda_1^x(t))) = 0 \right\}.$$

Furthermore, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_0^+$ with $x_n \uparrow x$ as $n \to \infty$. As

$$\{\zeta^{x_n} = \infty\} \cap \left\{ \inf_{t \in \mathbb{R}_0^+} ((x + ct) + \ln(\lambda_1^x(t))) = 0 \right\} = \emptyset$$

and $\{\zeta^{x_n} = \infty\} \subseteq \{\zeta^x = \infty\}$, we infer that $\mathbb{P}(\zeta^{x_n} = \infty) \leq \mathbb{P}(\{\zeta^x = \infty\} \setminus A)$. By means of the continuity of $\mathbb{P}(\zeta^{(\cdot)} = \infty)$ at $x$ we thus have

$$\mathbb{P}(\zeta^x = \infty) = \lim_{n\to\infty} \mathbb{P}(\zeta^{x_n} = \infty) \leq \mathbb{P}(\{\zeta^x = \infty\} \setminus A),$$

which implies that $\mathbb{P}(A) = 0$. Thus, $\inf_{t\in\mathbb{R}_0^+}((x + ct) + \ln(\lambda_1^x(t))) \neq 0$ $\mathbb{P}(\cdot|\zeta^x = \infty)$–a.s., but clearly we can't have $\inf_{t\in\mathbb{R}_0^+}((x + ct) + \ln(\lambda_1^x(t))) < 0$ on $\{\zeta^x = \infty\}$ as $-\ln(\lambda_1^x(t)) < x + ct$ for all $t \in \mathbb{R}_0^+$ on $\{\zeta^x = \infty\}$. We conclude that (5.38) holds true for any $x \in \mathbb{R}_0^+$, and thus for each $x \in \mathbb{R}_0^+$ there exists some $\mathbb{R}^+$–valued random variable $\epsilon$ such that

$$\inf_{t\in\mathbb{R}_0^+} N_t^x(\epsilon) \geq 1$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. □

Let us now turn to the crucial lemma that we need in order to prove Proposition 5.23.

**Lemma 5.27** *Let $c > c_{\bar{p}}$ and $x \in \mathbb{R}_0^+$. Then we have*

$$\limsup_{t\to\infty} R_1^x(t) = \infty$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$*–almost surely.*

**Proof** Let $z > x$ and set

$$\Gamma_z^x := \left\{ \omega \in \Omega : \inf\{t \in \mathbb{R}_0^+ : X_n^x(t)(\omega) \notin [0, z)\} = \infty \, \forall \, n \in \mathbb{N} \right\}.$$

Observe that [Ber96, Theorem 12 in Section VI.3] shows that the probability that the process $X_n$ stays inside the interval $(0, z)$ for an infinitely long time without leaving the interval is zero. Consequently, we have that

$$\tau^-_{n,x} < \tau^+_{1,z-x} = \infty \quad \text{on} \quad \Gamma^x_z.$$

For each $n \in \mathbb{N}$ set

$$\tau_n := \inf\{t \in \mathbb{R}^+_0 : N^x_t \geq n\}$$

and note that Lemma 5.24 implies that $\tau_n$ is a $\mathbb{P}$–a.s. finite stopping time. By means of Lemma 1.8, we thus infer from the extended fragmentation property and (5.3) that

$$\begin{aligned}
\mathbb{P}(\Gamma^x_z|\mathscr{F}_{\tau_n}) &\leq \prod_{n\in\mathcal{N}^x_{\tau_n}} \mathbb{P}(\Gamma^y_z)|_{y=X^x_n(\tau_n)} \\
&\leq \prod_{n\in\mathcal{N}^x_{\tau_n}} \mathbb{P}_{e^y}(\tau^-_{n,0} < \tau^+_{1,z})\Big|_{y=X^x_n(\tau_n)} \\
&\leq \prod_{n\in\mathcal{N}^x_{\tau_n}} \left(1 - \frac{W(X^x_n(\tau_n))}{W(z)}\right) \\
&\leq \left(1 - \frac{1}{cW(z)}\right)^{N^x_{\tau_n}} \\
&\leq \left(1 - \frac{1}{cW(z)}\right)^{n}
\end{aligned}$$

$\mathbb{P}$–a.s. on $\{\zeta^x = \infty\}$ for any $n \in \mathbb{N}$. Therefore, since $\{R^x_1(s) < z \,\forall\, s \in \mathbb{R}^+_0\} = \Gamma^x_z$ ,we have

$$\begin{aligned}
\mathbb{P}\left(\left\{\sup_{s\in\mathbb{R}^+_0} R^x_1(s) < z\right\} \cap \{\zeta^x = \infty\}\right) &= \mathbb{P}\left(\Gamma^x_z \cap \{\zeta^x = \infty\}\right) \\
&= \lim_{n\to\infty} \mathbb{E}\left(\mathbb{P}\left(\Gamma^x_z \cap \{\zeta^x = \infty\}\right| \mathscr{F}_{\tau_n})\right) \\
&= \mathbb{E}\left(\lim_{n\to\infty} \mathbb{P}\left(\Gamma^x_z \cap \{\zeta^x = \infty\}\right| \mathscr{F}_{\tau_n})\right) \\
&= 0.
\end{aligned}$$

Because $R^x_1(s) \leq x + cs$ for all $s \in \mathbb{R}^+_0$, we thus deduce that

$$\mathbb{P}\left(\left\{\sup\left\{R^x_1(s) : s \geq \frac{z-x}{c}\right\} < z\right\} \cap \{\zeta^x = \infty\}\right) = 0.$$

Consequently, resorting to the DCT and recalling that $z > x$ was chosen arbitrarily we conclude that

$$\mathbb{P}\left(\limsup_{s\to\infty} R_1^x(s)=\infty \,\middle|\, \zeta^x=\infty\right) = \lim_{z\to\infty}\mathbb{P}\left(\sup\left\{R_1^x(s) : s\geq \frac{z-x}{c}\right\}\geq z \,\middle|\, \zeta^x=\infty\right)$$
$$= 1,$$

which proves the assertion. □

We are now in a position to prove Proposition 5.23.

**Proof of Proposition 5.23** By the martingale convergence theorem we have that $Z^{x,f}$ being a nonnegative martingale implies that $Z_\infty^{x,f} := \lim_{t\to\infty} Z_t^{x,f}$ exists $\mathbb{P}$–almost surely. Since the empty product equals 1 it is immediately clear that

$$Z_\infty^{x,f} = 1 \tag{5.39}$$

holds $\mathbb{P}$–a.s. on $\{\zeta^x < \infty\}$. Moreover, according to Lemma 5.27 we have that $\limsup_{t\to\infty} R_1^x(t) = \infty$ $\mathbb{P}$–a.s. on $\{\zeta^x = \infty\}$. Since $\lim_{y\to\infty} f(y) = 0$, we thus deduce that

$$0 \leq Z_\infty^{x,f} \leq \liminf_{t\to\infty} f(R_1^x(t)) = 0 \tag{5.40}$$

$\mathbb{P}$–a.s. on $\{\zeta^x = \infty\}$. Hence, in view of (5.39) and (5.40) we infer that

$$Z_\infty^{x,f} = \mathbb{1}_{\{\zeta^x<\infty\}} \tag{5.41}$$

holds true $\mathbb{P}$–almost surely. As a consequence of $Z^{x,f}$ being a bounded, thus uniformly integrable, martingale we conclude by means of (5.41) that

$$f(x) = \mathbb{E}(Z_0^{x,f}) = \mathbb{E}(Z_\infty^{x,f}) = \mathbb{P}(\zeta^x < \infty).$$

□

### 5.6.2 Existence of a product martingale

This section is concerned with proving the second implication of Theorem 5.12, which is the content of the following proposition:

**Proposition 5.28** *Let $c > c_{\bar{p}}$. Then the process $Z^{x,\varphi}$ is an $\mathscr{F}$–martingale under $\mathbb{P}$ and* $\lim_{x\to\infty}\varphi(x) = 0$.

Recall that the first implication of Theorem 5.12 was established in Proposition 5.23 in the previous section. Combining Proposition 5.28 with Proposition 5.23 thus proves Theorem 5.12.

We shall use the following lemma:

**Lemma 5.29** *Let $x \in \mathbb{R}_0^+$ and let $f : \mathbb{R} \to \mathbb{R}_0^+$ be some function. Further, assume that*

$$\mathbb{E}\left(Z_t^{x,f}\right) = f(x). \tag{5.42}$$

*holds for all $t \in \mathbb{R}_0^+$. Then $Z^{x,f}$ is a martingale with respect to the filtration $\mathscr{F}$.*

**Proof** Let $s, t \in \mathbb{R}_0^+$. In view of the fragmentation property of $\Pi$ we have that

$$\mathbb{E}\left(Z_{t+s}^{x,f}\Big|\, \mathscr{F}_t\right) = \prod_{n \in \mathcal{N}_t^x} \mathbb{E}\left(Z^{(n)}\Big|\, \mathscr{F}_t\right) = \prod_{n \in \mathcal{N}_t^x} f(X_n^x(t)) = Z_t^{x,f}, \tag{5.43}$$

$\mathbb{P}$–a.s., where conditional on $\mathscr{F}_t$ the $Z^{(n)}$ are independent and satisfy

$$\mathbb{P}\left(Z^{(n)} \in \cdot\Big|\, \mathscr{F}_t\right) = \mathbb{P}\left(Z_s^{y,f} \in \cdot\right)\Big|_{y = X_n^x(t)}$$

$\mathbb{P}$–almost surely. Note that the second equality in (5.43) follows from (5.42). □

**Proof of Proposition 5.28** Since $\varphi$ is monotone and bounded, the limit $\lim_{x\to\infty} \varphi(x)$ exists in $[0, 1]$. Furthermore, for any $t \in \mathbb{R}_0^+$ we have $\mathcal{N}_t^x \uparrow \mathbb{N}$ $\mathbb{P}$–a.s. as $x \to \infty$, that is $\lim_{x\to\infty} \mathbb{1}_{\mathcal{N}_t^x}(n) = 1$ $\mathbb{P}$–a.s. for every $n \in \mathbb{N}$. In addition, we have that $X_n^x(t) \uparrow \infty$ $\mathbb{P}$–a.s. for any $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$ as $x \to \infty$. Resorting to the fragmentation property we deduce that

$$\varphi(x) = \mathbb{E}\left(\mathbb{P}\left(\zeta^x < \infty \right| \mathscr{F}_t)\right) = \mathbb{E}\left(\prod_{n \in \mathcal{N}_t^x} \varphi\left(X_n^x(t)\right)\right) = \mathbb{E}\left(Z_t^{x,\varphi}\right) \tag{5.44}$$

holds for all $t \in \mathbb{R}_0^+$. Hence, it follows from Lemma 5.29 that $Z^{x,\varphi}$ is a $\mathbb{P}$–martingale. Moreover, by the DCT we deduce from (5.44) that

$$\begin{aligned}\lim_{x\to\infty} \varphi(x) &= \lim_{x\to\infty} \mathbb{E}\left(\prod_{n \in \mathcal{N}_t^x} \varphi(X_n^x(t))\right) \\ &= \mathbb{E}\left(\lim_{y\to\infty} \prod_{n \in \mathcal{N}_t^y} \lim_{x\to\infty} \varphi(x)\right)\end{aligned}$$

$$= \mathbb{E}\left(\lim_{y\to\infty}\lim_{x\to\infty}\varphi(x)^{N_t^y}\right).$$

Consequently,

$$\lim_{x\to\infty}\varphi(x)\in\{0,1\}.$$

Since $\varphi$ is monotonically decreasing and $\varphi(x)\in(0,1)$ for all $x\in\mathbb{R}_0^+$, this results in $\lim_{x\to\infty}\varphi(x)=0$. □

## 5.7 The intrinsic additive martingale for killed fragmentation processes

In this section we aim at proving Theorem 5.13.

Let $c > c_{\bar{p}}$ and recall that for any $x\in\mathbb{R}_0^+$ the processes $M^x(p) := (M_t^x(p))_{t\in\mathbb{R}_0^+}$, $p\in(\underline{p},\infty)$, was defined in (5.6) and note that

$$\begin{aligned}
M_t^x(p) &= \sum_{n\in\mathcal{N}_t^x} W_p\left(X_n^x(t)\right)e^{\Phi(p)t}|B_n(t)|^{1+p}\\
&= \sum_{n\in\mathbb{N}} W_p\left(X_n^x(t)\right)e^{\Phi(p)t}|B_n(t)|^{1+p}\mathbb{1}_{\{t<\tau_{n,x}^-\}}\mathbb{1}_{\{n=\min B_n(t)\}}.
\end{aligned}$$

Moreover, recall that in Remark 5.8 we mentioned that $\psi_p'(0+)>0$ for all $p\in(\underline{p},\bar{p})$ with $\Phi'(p)<c$. The following lemma shows that under $\mathbb{P}$ the process $M^x(p)$ is a martingale for suitable $c$ and $p$.

**Lemma 5.30** *Let $c>c_{\bar{p}}$ and let $p\in(\underline{p},\bar{p})$ be such that $c>\Phi'(p)$. Further, let $x\in\mathbb{R}_0^+$. Then the process $M^x(p)$ is a $\mathbb{P}$–martingale with respect to the filtration $\mathscr{F}$.*

**Proof** Let us first show that for any $t\in\mathbb{R}_0^+$ the process $(W_p(X_1^x(s))\mathbb{1}_{\{s<\tau_{1,x}^-\}})_{s\in\mathbb{R}_0^+}$ is a $\mathbb{P}^{(p)}$–martingale with respect to $\mathscr{F}$. To this end, recall that $\mathbb{P}(\tau_{1,y}^-\in\cdot)=\mathbb{P}_{e^y}(\tau_{1,0}^-\in\cdot)$ for all $y\in\mathbb{R}_0^+$. By the Markov property of $X_1$ under $\mathbb{P}^{(p)}$ we then infer from (5.2) that

$$\begin{aligned}
\mathbb{E}^{(p)}\left(\mathbb{1}_{\{\tau_{1,x}^-=\infty\}}\middle|\mathscr{F}_s\right) &= \mathbb{P}^{(p)}\left(\tau_{1,x+X_1(s)}^-=\infty\right)\mathbb{1}_{\{s<\tau_{1,x}^-\}}\\
&= \mathbb{P}^{(p)}_{e^{x+X_1(s)}}\left(\tau_{1,0}^-=\infty\right)\mathbb{1}_{\{s<\tau_{1,x}^-\}} \qquad (5.45)\\
&= \psi'(0+)W_p(x+X_1(s))\mathbb{1}_{\{s<\tau_{1,x}^-\}}
\end{aligned}$$

holds $\mathbb{P}^{(p)}$–a.s. for any $s\in\mathbb{R}_0^+$. Note that the left–hand side of (5.45) defines a closed $\mathbb{P}^{(p)}$–martingale. Further, observe that $x+X_1(s)=X_1^x(s)$ on the event $\{s<\tau_{1,x}^-\}$.

By means of Lemma 2.6 we deduce that

$$\begin{aligned}
\mathbb{E}\left(M_s^x(p)\right) &= e^{\Phi(p)s}\mathbb{E}\left(\sum_{n\in\mathbb{N}} |B_n(s)|^{1+p} W_p(X_n^x(s))\mathbb{1}_{\{t<\tau_{n,x}^-\}}\mathbb{1}_{\{n=\min(B_n(s))\}}\right) \\
&= \mathbb{E}\left(W_p(X_1^x(s))\mathbb{1}_{\{s<\tau_{1,x}^-\}} e^{\Phi(p)s-p\xi(s)}\right) \\
&= \mathbb{E}^{(p)}\left(W_p(X_1^x(s))\mathbb{1}_{\{s<\tau_{1,x}^-\}}\right) \qquad (5.46)\\
&= \mathbb{E}^{(p)}\left(W_p(X_1^x(0))\mathbb{1}_{\{0<\tau_{1,x}^-\}}\right) \\
&= W_p(x)
\end{aligned}$$

for all $t \in \mathbb{R}_0^+$, where the penultimate equality is a consequence of the martingale property of $(W_p(X_1^x(s))\mathbb{1}_{\{s<\tau_{1,x}^-\}})_{s\in\mathbb{R}_0^+}$. Notice further that the last equality in (5.46) follows from $X_1^x(0) = x$ and $\tau_{1,x}^- > 0$ $\mathbb{P}^{(p)}$–almost surely. In view of (5.46) we infer from the fragmentation property of $\Pi$ that

$$\begin{aligned}
\mathbb{E}\left(M_{t+s}^x(p)\middle|\mathscr{F}_t\right) &= \sum_{n\in\mathbb{N}} e^{\Phi(p)t}|B_n(t)|^{1+p}\mathbb{E}\left(M^{(n)}\middle|\mathscr{F}_t\right)\mathbb{1}_{\{t<\tau_{n,x}^-\}}\mathbb{1}_{\{n=\min(B_n(t))\}} \\
&= \sum_{n\in\mathbb{N}} e^{\Phi(p)t}|B_n(t)|^{1+p} W_p(X_n^x(t))\mathbb{1}_{\{t<\tau_{n,x}^-\}}\mathbb{1}_{\{n=\min(B_n(t))\}} \\
&= M_t^x(p)
\end{aligned}$$

$\mathbb{P}$–a.s. for all $s, t \in \mathbb{R}_0^+$, where conditional on $\mathscr{F}_t$ the $M^{(n)}$ are independent and satisfy

$$\mathbb{P}\left(M^{(n)} \in \cdot\middle|\mathscr{F}_t\right) = \mathbb{P}\left(M_s^y(p) \in \cdot\right)\big|_{y=X_n^x(t)}$$

$\mathbb{P}$–almost surely. □

According to Lemma 5.30 we have that $M^x(p)$ is a nonnegative martingale and by the martingale convergence theorem we thus infer that $M_\infty^x(p) := \lim_{t\to\infty} M_t^x(p)$ exists $\mathbb{P}$–almost surely.

The second auxiliary result that we use to prove Theorem 5.13 is the following lemma.

**Lemma 5.31** *Let $c > c_{\bar{p}}$ and let $p \in (\underline{p}, \bar{p})$ be such that $c > \Phi'(p)$. Furthermore, let $x > 0$. Then $\lim_{t\to\infty}\sum_{n\in\mathcal{N}_t^x} e^{\Phi(p)t}|B_n(t)|^{1+p}$ exists $\mathbb{P}$–a.s. and $M_\infty^x(p)$ satisfies*

$$M_\infty^x(p) = \frac{1}{\psi_p'(0+)}\lim_{t\to\infty}\sum_{n\in\mathcal{N}_t^x} e^{\Phi(p)t}|B_n(t)|^{1+p}$$

$\mathbb{P}$*–almost surely.*

**Proof** In view of the monotonicity of $W_p$ and (1.5) we have

$$\sup_{y \in \mathbb{R}_0^+} W_p(y) = \lim_{y \to \infty} W_p(y) = \frac{1}{\psi_p'(0+)}, \tag{5.47}$$

and thus

$$\begin{aligned} M_t^x(p) &= \sum_{n \in \mathcal{N}_t^x} W_p\left(X_n^x(t)\right) e^{\Phi(p)t} |B_n(t)|^{1+p} \\ &\leq \sup_{y \in \mathbb{R}_0^+} W_p(y) \sum_{n \in \mathcal{N}_t^x} e^{\Phi(p)t} |B_n(t)|^{1+p} \\ &= \frac{1}{\psi_p'(0+)} \sum_{n \in \mathcal{N}_t^x} e^{\Phi(p)t} |B_n(t)|^{1+p} \end{aligned}$$

holds true for all $t \in \mathbb{R}_0^+$. Consequently,

$$M_\infty^x(p) \leq \frac{1}{\psi_p'(0+)} \liminf_{t \to \infty} \sum_{n \in \mathcal{N}_t^x} e^{\Phi(p)t} |B_n(t)|^{1+p}. \tag{5.48}$$

The remainder of the proof is concerned with the lower bound. For this purpose we fix some $\eta \in (0, c - \Phi'(p))$. At first we show that

$$\begin{aligned} &M_\infty^x(p) \\ &= \lim_{t \to \infty} \sum_{k \in \mathcal{N}_t^x} W_p(x + ct + \ln(|B_k(t)|)) e^{\Phi(p)t} |B_k(t)|^{1+p} \mathbb{1}_{\{|-\ln(|B_k(t)|) - \Phi'(p)t| \leq \eta t\}}. \end{aligned} \tag{5.49}$$

$\mathbb{P}$–almost surely. To this end, let $\varepsilon \in (0, \eta)$ and observe that for every $\varepsilon > 0$ there exists some $\delta_\varepsilon \in (0, p - \underline{p})$ such that $-\Phi'(p) \leq \delta^{-1}(\Phi(p - \delta) - \Phi(p)) + \varepsilon$ as well as $\Phi'(p) \leq \delta^{-1}(\Phi(p + \delta) - \Phi(p)) + \varepsilon$ for all $\delta \in (0, \delta_\varepsilon]$. Therefore, we have that

$$\Phi(p) - \delta\Phi'(p) - \delta\eta = \Phi(p) - \delta(\Phi'(p) + \varepsilon) - \delta(\eta - \varepsilon) \leq \Phi(p - \delta) - \delta(\eta - \varepsilon) \tag{5.50}$$

and

$$\Phi(p) + \delta\Phi'(p) - \delta\eta = \Phi(p) + \delta(\Phi'(p) - \varepsilon) - \delta(\eta - \varepsilon) \leq \Phi(p + \delta) - \delta(\eta - \varepsilon) \tag{5.51}$$

holds true for all $\delta \in (0, \delta_\varepsilon]$. Moreover, note that

$$\mathbb{1}_{\{\ln(|B_n(t)| + \Phi'(p)t + \eta t < 0\}} \leq e^{-\delta(\ln(|B_n(t)| + \Phi'(p)t + \eta t))},$$

which yields that

$$
\begin{aligned}
\sum_{n\in\mathcal{N}_t^x} e^{\Phi(p)t}|B_n(t)|^{1+p}\mathbb{1}_{\{\ln(|B_n(t)|+\Phi'(p)t+\eta t<0\}} &\leq \sum_{n\in\mathcal{N}_t^x} e^{(\Phi(p)-\delta\Phi'(p)-\delta\eta)t}|B_n(t)|^{1+p-\delta} \\
&\leq e^{-\delta(\eta-\varepsilon)t}\sum_{n\in\mathcal{N}_t^x} e^{\Phi(p-\delta)t}|B_n(t)|^{1+p-\delta} \\
&\leq e^{-\delta(\eta-\varepsilon)t}M_t^x(p-\delta) \qquad (5.52)
\end{aligned}
$$

for all $\delta \in (0, \delta_\varepsilon]$, where in the penultimate inequality we have used (5.50). Since $M^x(p-\delta)$ is a nonnegative martingale, and hence has a $\mathbb{P}$–a.s. limit, we deduce from (5.2) and (5.52) that

$$
\begin{aligned}
&\sum_{n\in\mathcal{N}_t^x} W_p(x+ct+\ln(|B_n(t)|))e^{\Phi(p)t}|B_n(t)|^{1+p}\mathbb{1}_{\{\ln(|B_n(t)|+\Phi'(p)t<-\eta t\}} \\
&\leq \psi_p'(0+)^{-1}e^{-\delta(\eta-\varepsilon)t}M_t^x(p-\delta) \qquad (5.53) \\
&\to 0
\end{aligned}
$$

$\mathbb{P}$–a.s. as $t\to\infty$. Similarly, resorting to (5.51), we conclude that

$$
\begin{aligned}
&\sum_{n\in\mathcal{N}_t^x} e^{\Phi(p)t}|B_n(t)|^{1+p}\mathbb{1}_{\{-\ln(|B_n(t)|-\Phi'(p)t+\eta t<0\}} \\
&\leq \sum_{n\in\mathcal{N}_t^x} e^{(\Phi(p)+\delta\Phi'(p)-\delta\eta)t}|B_n(t)|^{1+p+\delta} \\
&\leq e^{-\delta(\eta-\varepsilon)t}\sum_{n\in\mathcal{N}_t^x} e^{\Phi(p+\delta)t}|B_n(t)|^{1+p+\delta} \\
&\leq e^{-\delta(\eta-\varepsilon)t}M_t^x(p+\delta)
\end{aligned}
$$

holds for all $\delta \in (0, \delta_\varepsilon]$. Since $M^x(p+\delta)$ is a nonnegative martingale, and thus has a $\mathbb{P}$–a.s. limit, we obtain that

$$
\begin{aligned}
&\sum_{n\in\mathcal{N}_t^x} W_p(x+ct+\ln(|B_n(t)|))e^{\Phi(p)t}|B_n(t)|^{1+p}\mathbb{1}_{\{-\ln(|B_n(t)|)-\Phi'(p)<-\eta t\}} \\
&\leq \psi_p'(0+)^{-1}e^{-\delta(\eta-\varepsilon)t}M_t^x(p+\delta) \qquad (5.54) \\
&\to 0
\end{aligned}
$$

$\mathbb{P}$–a.s. as $t\to\infty$. In view of (5.53) and (5.54) we deduce that (5.49) holds true. Since

$\eta < c - \Phi'(p)$, we infer from (5.47) that

$$\lim_{t\to\infty} W_p\left(x + (c - \Phi'(p) - \eta)t\right) = \lim_{y\to\infty} W_p(y) = \frac{1}{\psi_p'(0+)}$$

$\mathbb{P}$–almost surely. Hence, for any $\gamma > 0$ there exists some $t_\gamma \in \mathbb{R}_0^+$ with

$$W_p\left(x + (c - \Phi'(p) - \eta)t\right) \geq \frac{1}{\psi_p'(0+)} - \gamma$$

$\mathbb{P}$–a.s. for all $t \geq t_\gamma$. Resorting to (5.49) we thus obtain that

$$M_\infty^x(p) \geq \left(\frac{1}{\psi_p'(0+)} - \gamma\right) \limsup_{t\to\infty} \sum_{n\in\mathcal{N}_t^x} e^{\Phi(p)t}|B_n(t)|^{1+p}$$

$\mathbb{P}$–a.s. for every $\gamma > 0$. Letting $\gamma \downarrow 0$, this results in

$$M_\infty^x(p) \geq \frac{1}{\psi_p'(0+)} \limsup_{t\to\infty} \sum_{n\in\mathcal{N}_t^x} e^{\Phi(p)t}|B_n(t)|^{1+p}$$

$\mathbb{P}$–a.s., which in view of (5.48) proves the assertion. □

Let us now tackle the proof of Theorem 5.13.

**Proof of Theorem 5.13** Recall first that the martingale property of $M^x(p)$ was established in Lemma 5.30.

It remains to show that the symmetric difference of $\{M_\infty^x(p) = 0\}$ and $\{\zeta^x < \infty\}$ coincide. In view of Lemma 5.31 define a function $\phi_p : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by

$$\phi_p(x) = \lim_{t\to\infty} \sum_{n\in\mathcal{N}_t^x} e^{\Phi(p)t}|B_n(t)|^{1+p}$$

for all $x \in \mathbb{R}_0^+$ and consider the function $g_p : \mathbb{R}_0^+ \to [0,1]$ given by

$$g_p(x) = \mathbb{P}(\phi_p(x) = 0)$$

for any $x \in \mathbb{R}_0^+$. Resorting to the fragmentation property we deduce that

$$g_p(x) = \mathbb{E}(\mathbb{P}(\phi_p(x) = 0|\mathscr{F}_t)) = \mathbb{E}\left(\prod_{n\in\mathcal{N}_t^x} g_p\left(X_n^x(t)\right)\right) = \mathbb{E}\left(Z_t^{x,g_p}\right)$$

holds for all $t \in \mathbb{R}_0^+$. Hence, it follows from Lemma 5.29 that $Z_t^{x,g_p}$ is a $\mathbb{P}$–martingale.

In view of Proposition 5.23 we thus deduce that

$$\mathbb{P}(\phi_p(x) = 0) = g_p(x) = \mathbb{P}(\zeta^x < \infty).$$

Since $\{\zeta^x < \infty\} \subseteq \{\phi_p(x) = 0\}$ for each $x > 0$, as the empty sum equals 0, this implies that

$$\mathbb{P}\left(\{\zeta^x < \infty\} \triangle \{\phi_p(x) = 0\}\right) = 0$$

for every $x > 0$, and thus it follows from Lemma 5.31 that

$$\mathbb{P}\left(\{\zeta^x < \infty\} \triangle \{M^x_\infty(p) = 0\}\right) = 0.$$

□

## 5.8 Asymptotic speed of the largest fragment

The final section of this chapter is devoted to the proof of Theorem 5.14. That is, in this section we deal with the asymptotic behaviour of the largest fragment in the killed fragmentation process.

**Proof of Theorem 5.14** Our approach is based on the method of proof for Corollary 1.4 in [Ber06].

For the time being, let $p \in (\underline{p}, \infty)$. In view of $W_p(x) \geq c^{-1}$ for all $x \in \mathbb{R}_0^+$, see Lemma 1.8 , we deduce that

$$\begin{aligned} c^{-1} e^{\Phi(p)t} (\lambda_1^x(t))^{1+p} &\leq c^{-1} e^{\Phi(p)t} \sum_{n \in \mathcal{N}_t^x} |B_n(t)|_n^{1+p}(t) \\ &\leq e^{\Phi(p)t} \sum_{n \in \mathcal{N}_t^x} W_p\left(X_n^x(t)\right) |B_n(t)|_n^{1+p}(t) \\ &= M_t^x(p) \end{aligned} \tag{5.55}$$

Since according to Lemma 5.30 the process $M^x(p)$ is a nonnegative $\mathbb{P}$–martingale, we have in particular that $\lim_{t\to\infty} M_t^x(p) \in \mathbb{R}_0^+$ $\mathbb{P}$–almost surely. Hence, taking the logarithm and taking the limit superior as $t \to \infty$ we deduce from (5.55) that

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\lambda_1^x(t)) \leq -\frac{\Phi(p)}{1+p},$$

and thus, since $p \in (\underline{p}, \infty)$ was chosen arbitrarily,

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\lambda_1^x(t)) \leq -\frac{\Phi(\bar{p})}{1+\bar{p}} = -\Phi'(\bar{p}), \tag{5.56}$$

$\mathbb{P}$–a.s., where the final equality follows from the definition of $\bar{p}$.

In order to show the converse inequality, recall that in the proof of Theorem 5.13 we defined $\phi_p : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by

$$\phi_p(y) = \lim_{t\to\infty} \sum_{n\in\mathcal{N}_t^y} e^{\Phi(p)t} |B_n(t)|^{1+p}$$

and showed that

$$\phi_p(y) > 0 \tag{5.57}$$

$P(\cdot|\zeta^y = \infty)$–a.s. for all $y \in \mathbb{R}_0^+$. Further, let $p \in (\underline{p}, \bar{p})$ as well as $\epsilon \in (0, p - \underline{p})$ and observe that

$$\begin{aligned}
&c^{-1} e^{\Phi(p)t} \sum_{n\in\mathcal{N}_t^x} |B_n(t)|^{1+p}(t) \\
&\leq e^{(\Phi(p)-\Phi(p-\epsilon))t} [\lambda_1^x(t)]^\epsilon e^{\Phi(p-\epsilon)t} \sum_{n\in\mathcal{N}_t^x} W_{p-\epsilon}\left(X_n^x(t)\right) |B_n(t)|^{1+p-\epsilon}(t) \\
&= e^{(\Phi(p)-\Phi(p-\epsilon))t} [\lambda_1^x(t)]^\epsilon M_t^x(p-\epsilon).
\end{aligned} \tag{5.58}$$

According to Theorem 5.13 we have $\lim_{t\to\infty} M_t^x(p-\epsilon) \in (0,\infty)$ $\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. Consequently, taking the logarithm and taking the limit superior as $t \to \infty$ we thus deduce from (5.58) in conjunction with (5.57) that

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\lambda_1^x(t)) \geq -\frac{\Phi(p) - \Phi(p-\epsilon)}{\epsilon}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. Therefore, we have

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\lambda_1^x(t)) \geq -\lim_{\varepsilon\to 0} \frac{\Phi(p) - \Phi(p-\varepsilon)}{\varepsilon} = -\Phi'(p) \tag{5.59}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. Letting $p \to \bar{p}$ and resorting to the convexity of $\Phi$, which ensures the continuity of $\Phi'$, (5.59) results in

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\lambda_1^x(t)) \geq -\Phi'(\bar{p})$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. Recalling from (5.1) that $c_{\bar{p}} = \Phi'(\bar{p})$, and bearing in mind (5.56), this proves the assertion. □

In particular, Theorem 5.14 shows that for $c > c_{\bar{p}}$ the asymptotic speed of the largest fragment $\lambda_1^x(t)$ in the killed fragmentation process is of order $t$ on the event of survival of this process. Note that this result concurs with the asymptotic speed of the right–most particle for killed branching Brownian motions, see Lemma 2 in [HHK06] . As a corollary of Theorem 5.14 we obtain the asymptotic speed of $R_1^x(t)$ on survival of $\Pi^x$.

**Corollary 5.32** *Let $c > c_{\bar{p}}$. Then we have that*

$$\lim_{t\to\infty} \frac{R_1^x(t)}{t} = c - c_{\bar{p}}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$*–a.s. for all* $x > 0$*.*

**Proof** Let $x > 0$. According to Theorem 5.14 we have that

$$\lim_{t\to\infty} \frac{\ln(\lambda_1^x(t))}{t} = -c_{\bar{p}}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. Hence, we infer that

$$\lim_{t\to\infty} \frac{R_1^x(t)}{t} = \lim_{t\to\infty} \frac{x + ct + \ln(\lambda_n^x(t))}{t} = c + \lim_{t\to\infty} \frac{\ln(\lambda_1^x(t))}{t} = c - c_{\bar{p}}$$

$\mathbb{P}(\cdot|\zeta^x = \infty)$–almost surely. □

## 5.9 Concluding remarks

In this chapter we introduced killed fragmentations. These processes form a new class of fragmentation processes that was not considered in the literature so far. Our approach follows the spirit of related considerations for other killed branching processes, in particular with regard to branching Brownian motion as in [HHK06]. In the context of branching Brownian motions several results which are comparable to those obtained here for fragmentations follow from the positive time between subsequent jumps as well as from the spatial behaviour between jumps and from well–known properties of Brownian motions. Our method is based on the close relationship between fragmentation processes and Lévy processes. Indeed, many tools that we used in this chapter are borrowed from the theory of Lévy processes as compiled in Section 1.2.

We believe that the results of this chapter are of intrinsic interest as they shed light on an interesting class of fragmentation processes. However, our main motivation for the considerations here stems from the close connection of killed fragmentations with the one–sided FKPP travelling wave equation in the context of fragmentation processes. This connection will be explained in the following chapter, where we shall use the results of the present chapter in order to obtain existence and uniqueness results for one–sided FKPP travelling waves in the setting of fragmentation processes.

# CHAPTER 6

# THE FKPP EQUATION FOR KILLED FRAGMENTATION PROCESSES

*In this chapter we prove existence and uniqueness of solutions of the one–sided FKPP travelling wave equation in the setting of fragmentation processes.*

## 6.1 Introduction

This chapter is devoted to the study of one–sided FKPP travelling waves in the setting of fragmentation processes. More precisely, we aim at studying the existence and uniqueness of solutions of the one–sided FKPP travelling wave equation for fragmentation processes. This equation, which turns out to be an integro–differential equation using the dislocation measure as integrator, has a similar interpretation as the classical FKPP travelling wave equation whose probabilistic interpretation is related to branching Brownian motion. Our main result states that there exists a constant such that for any wave speed greater than that constant there exists a unique travelling wave with thwith wave speed and for any wave speed less than or equal to that constant there is no such travelling wave. The one–sided FKPP travelling wave solutions that we obtain show similar resemblances with the one–sided solutions in the classical FKPP equation as do the two–sided travelling wave solutions for fragmentations obtained in [BHK10] with the two–sided solutions in the classical case.

Our approach is based on using killed fragmentation processes. In this respect the notions and results of Chapter 5 are crucial for our considerations here.

As in the previous chapter we consider a homogenous fragmentation process $\Pi$, satisfying Hypothesis 1.1 and Hypothesis 1.2, with $B_n$, $n \in \mathbb{N}$, and $\lambda$ being defined as on page 16 and on page 20 respectively.

*Throughout the present chapter we use the terminology* $f : \mathbb{R} \to [0,1]$ *as an abbreviated form for writing* $f : \mathbb{R} \cup \{-\infty\} \to [0,1]$ *with* $f(-\infty) = 1$, *and similarly we interpret* $u : \mathbb{R}_0^+ \times \mathbb{R} \to [0,1]$. *In addition, we adopt* $-\ln(0) := -\infty$.

In this chapter we prove existence and uniqueness of one–sided travelling waves for fragmentation processes within a certain regime of wave speeds. More specifically, the problem we are concerned with in this chapter can be roughly described as follows. Consider the following integro–differential equation

$$cf'(x) + \int_{\mathcal{S}_1} \left( \prod_{n \in \mathbb{N}} f(x + \ln(s_n)) - f(x) \right) \nu(\mathrm{d}\mathbf{s}) = 0$$

for certain $c > 0$ and all $x \in \mathbb{R}_0^+$. We are interested in solutions $f : \mathbb{R} \to [0,1]$ of this equation that satisfy

$$f|_{\mathbb{R}_0^+} \in C^1(\mathbb{R}_0^+, [0,1]) \qquad \text{and} \qquad f|_{(-\infty,0)} \equiv 1$$

as well as the boundary condition

$$\lim_{x \to \infty} f(x) = 0.$$

Roughly speaking, our main result states that there is some constant $c_0 > 0$ such that there exists a unique solution of the above boundary value problem for every $c > c_0$ and there does not exist such a solution for any $c \leq c_0$. More precisely, it turns out that this constant is given by $c_0 = c_{\bar{p}}$, where $c_{\bar{p}} > 0$ is the constant that was defined in Definition 5.6 and that played an important role in Chapter 5. In fact, that this constant appears in the present chapter as well is a consequence of the significance of killed fragmentation processes for the problem considered here.

## 6.2 Motivation – The classical FKPP equation

Let us first briefly mention related results in order to present the framework in which our main result should be seen. To this end we denote by $C^{1,2}(\mathbb{R}_0^+ \times A, [0,1])$, $A \subseteq \mathbb{R}$, the space of all functions $f : \mathbb{R}_0^+ \times A \to [0,1]$ such that $f(x, \cdot) \in C^2(A, [0,1])$ and $f(\cdot, y) \in C^1(\mathbb{R}_0^+, [0,1])$ for all $x \in \mathbb{R}_0^+$ and $y \in A$.

The classical FKPP equation in the form that is of most interest for us, cf. [McK75], is the following nonlinear, parabolic partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta(u^2 - u) \tag{6.1}$$

with $u \in C^{1,2}(\mathbb{R}_0^+ \times \mathbb{R}, [0,1])$. This equation, originally introduced by Fisher (see [Fis30] and [Fis37]) as well as by Kolmogorov, Petrovskii and Piscounov (cf. [KPP37]), has attracted much attention by analysts and probabilists alike. This kind of equation first arose in the context of a genetics model for the spread of an advantageous gene through a population and it is also satisfied by the maximal displacement of branching Brownian motion (see e.g. [McK75]). Several authors showed that this equation is closely related to dyadic branching Brownian motions, thus establishing a probabilistic link of this analytical problem. In this probabilistic interpretation the term "$\frac{1}{2}\frac{\partial^2 u}{\partial x^2}$" corresponds to the motion of the underlying Brownian motion, the "$\beta$" is the rate at which the particles split and the term "$u^2 - u$" results from the binary branching, that is two particles replace one particle at each branching time.

A solution $u$ of equation (6.1) can be interpreted in different ways. The classical work concerning this partial differential equation, such as [Fis30], [Fis37] and [KPP37], describes the wave of advance of advantageous genes. More precisely, there are two types of individuals (or genes) in a population, and $u(t,x)$ measures the frequency or concentration of the advantageous type at the space–time point $(x,t)$. In McKean's interpretation [McK75] the function $u(t,x)$ is related to a branching Brownian motion. Let $u(t,x)$ be the probability that at time $t$ the largest particle of the branching Brownian motion has a value less than $x$. Then $u$ satisfies equation (6.1). In [Fis30], [Fis37] and [KPP37] the equation describes the bulk of the population, in [McK75] it describes the most advanced particle.

The classical FKPP travelling waves are solutions of (6.1) of the form

$$u(t,x) = f(x - ct)$$

for some $f \in C^2(\mathbb{R}, [0,1])$ and some constant $c \in \mathbb{R}$. This leads to the so–called FKPP travelling wave equation with wave speed $c \in \mathbb{R}$ and $\beta > 0$

$$\begin{aligned} \frac{1}{2}f'' + cf' + \beta(f^2 - f) &= 0 \\ \lim_{x\to-\infty} f(x) &= 0 \\ \lim_{x\to\infty} f(x) &= 1, \end{aligned} \tag{6.2}$$

This travelling wave boundary value problem was studied by various authors, using both analytic as well as probabilistic techniques, and it is known that is has a unique (up to additive translation) solution $f \in C^2(\mathbb{R}, [0,1])$ if $|\rho| \geq \sqrt{2\beta}$. In the opposite case that $0 \leq |\rho| < \sqrt{2\beta}$ there is no travelling wave solution. For probabilistic approaches we refer for instance to [McK75] (cf. also [McK76]), [Bra78], [Bra83], [Uch77], [Uch78], [Nev87] as well as [CR88] and [CR90]. For our considerations the expositions in [Har99] and [Kyp04] are particularly interesting with regard to probabilistic methods dealing with the classical FKPP travelling wave equation.

More interesting with regard to our work is that the above boundary value problems was extended to continuous–time branching random walks (cf. [Kyp99]) and recently to fragmentation processes (see [BHK10]). In the setting of fragmentation processes the corresponding partial integro–differential equation is defined a follows:

$$\frac{\partial u}{\partial t}(t,x) = \int_{\mathcal{S}} \left( \prod_{n \in \mathbb{N}} u(t, x + \ln(s_n)) - u(t,x) \right) \nu(\mathrm{d}\mathbf{s}) \tag{6.3}$$

for certain $u : \mathbb{R}_0^+ \times \mathbb{R} \to [0,1]$ for which the above objects exist. We call equation (6.3) *FKPP equation for fragmentation processes*. Of particular interest are the so–called *FKPP travelling waves* to (6.3) with wave speed $c \in \mathbb{R}$, that is solutions of (6.3) which are of the form $u(t,x) = f(x - ct)$ for all $t \in \mathbb{R}_0^+$ and $x \in \mathbb{R}$. These travelling wave solutions are functions $f \in C^1(\mathbb{R}, [0,1])$ that satisfy the following FKPP travelling wave equation

$$cf'(x) + \int_{\mathcal{S}} \left( \prod_{n \in \mathbb{N}} f(x + \ln(s_n)) - f(x) \right) \nu(\mathrm{d}\mathbf{s}) = 0$$

for all $x \in \mathbb{R}$ with boundary conditions

$$\lim_{x \to -\infty} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to \infty} f(x) = 1.$$

For every $p \in (\underline{p}, \bar{p}]$ let $\mathcal{T}_2(p)$ denote the space of monotonically increasing functions $f \in C^1(\mathbb{R}, [0,1])$ satisfying the boundary conditions $\lim_{x \to -\infty} f(x) = 0$ as well as $\lim_{x \to \infty} f(x) = 1$ and such that $e^{(1+p)x}(1 - f(x))$ is monotonically increasing. In [BHK10, Theorem 1] Berestycki et. al. show that for $p \in (\underline{p}, \bar{p}]$ and

$$c_p := \frac{\Phi(p)}{1+p} \tag{6.4}$$

there exists a unique (up to additive translation) travelling wave solution in $\mathcal{T}_2(p)$ with wave speed $c_p$. Note that in view of (1.15) the definition of $c_p$ in (6.4) concurs for $p = \bar{p}$

with the definition of $c_{\bar{p}}$ in (5.1).

**Remark 6.1** Though not mentioning it directly, it results from [BHK10, Theorem 3 (ii)] in conjunction with the first part of the proof of [BHK10, Theorem 1] that monotone travelling waves do not exist for wave speeds larger than $c_{\bar{p}}$. Indeed, according to Lemma 1 in [Ber03] the mapping $p \mapsto \frac{\Phi(p)}{1+p} = c_p$ is monotonically increasing on $(\underline{p}, \bar{p}]$, and thus it follows from [BHK10] that $c_{\bar{p}}$ is the maximal travelling wave speed in this situation. ◊

In this chapter we are interested in the one–sided versions of the FKPP equation. In the classical setting the one–sided FKPP equation is the following partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta(u^2 - u)$$

on $\mathbb{R}^+ \times \mathbb{R}^+$ with $u \in C^{1,2}(\mathbb{R}_0^+ \times \mathbb{R}_0^+)$. Observe that this equation is the analogue of (6.1) for functions defined on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$. The corresponding one–sided FKPP travelling wave equation with wave speed $c \in \mathbb{R}$ is given by the differential equation

$$\frac{1}{2}f'' + cf' + \beta(f^2 - f) = 0 \tag{6.5}$$

on $\mathbb{R}^+$ for $f \in C^2(\mathbb{R}_0^+, [0,1])$ satisfying the boundary conditions

$$\lim_{x\to 0} f(x) = 1 \qquad \text{as well as} \qquad \lim_{x\to\infty} f(x) = 0. \tag{6.6}$$

By considering killed branching Brownian motion, killed upon hitting the origin, Harris et. al. proved in [HHK06] that solutions of the one–sided FKPP travelling wave boundary value problem (6.5) and (6.6) exist and are unique (up to translation) for all $c \in (-\sqrt{2\beta}, \infty)$ and there is no such travelling wave solution for $c \in (-\infty, -\sqrt{2\beta}]$. Notice that the one–sided travelling wave solutions for negative $c$ are precisely those wave speeds for which there does not exist a two–sided travelling wave. For results regarding the one–sided FKPP travelling wave equation see also [Wat65], concerning existence of a solution, as well as [Pin95] for existence and uniqueness of a solution of (6.5) and (6.6) obtained by means of analytic techniques.

Let us remark that the methods of proof for the classical FKPP travelling wave equation make use of the facts that for branching Brownian motions the time between successive jumps is exponentially distributed with a finite parameter and that branching Brownian motions have a spatial behaviour between successive jump times. For fragmentation processes the path behaviour is very different in this regard, and thus new methods need to be developed. Note further that standard stochastic analysis that is applicable for branching Brownian motions is not applicable for fragmentation processes.

## 6.3 The one–sided FKPP equation for fragmentations

Our goal is to complete the picture described in the previous section. More precisely, the problem addressed in the present chapter is to find a regime of travelling waves for which we can prove the existence of a unique one–sided FKPP travelling wave for fragmentation processes as described below.

### 6.3.1 Set–up

Consider the following initial value problem for $u : \mathbb{R}_0^+ \times \mathbb{R} \to [0, 1]$:

$$\frac{\partial u}{\partial t}(t, x) = \int_{\mathcal{S}_1} \left( \prod_{n \in \mathbb{N}} u(t, x + \ln(s_n)) - u(t, x) \right) \nu(\mathrm{d}\mathbf{s}) \tag{6.7}$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}_0^+$ as well as $u(t, \cdot)|_{(-\infty,0)} \equiv 1$ and $u(0, \cdot)|_{\mathbb{R}_0^+} = g$ for some measurable function $g : \mathbb{R}_0^+ \to [0, 1]$.

**Definition 6.2** We call equation (6.7) one–sided *FKPP equation for fragmentation processes.*

Note that (6.7) looks quite different from the classical FKPP equation. This difference results from the fact that fragmentation processes have no spatial motion except at jump times and from the more complicated jump structure of fragmentations in comparison with dyadic branching Brownian motions.

We are mainly interested in the so–called FKPP travelling wave solutions of (6.7) with wave speed $c \in \mathbb{R}_0^+$, that is in solutions of (6.7) which are of the form $u(t, x) = f(x - ct)$ for all $t, x \in \mathbb{R}_0^+$.

**Definition 6.3** A one–sided *FKPP travelling wave* is a function $f : \mathbb{R} \to [0, 1]$, with $f|_{\mathbb{R}_0^+} \in C^1(\mathbb{R}_0^+, [0, 1])$ and $f|_{(-\infty,0)} \equiv 1$, for which the mapping

$$\mathbf{s} \mapsto \prod_{n \in \mathbb{N}} f(x + \ln(s_n)) - f(x)$$

is integrable with respect to $\nu$ and that satisfies the following one–sided *FKPP travelling* *wave equation*

$$cf'(x) + \int_{\mathcal{S}_1} \left( \prod_{n \in \mathbb{N}} f(x + \ln(s_n)) - f(x) \right) \nu(\mathrm{d}\mathbf{s}) = 0 \tag{6.8}$$

for all $x \in \mathbb{R}^+$ with the boundary condition

$$\lim_{x \to \infty} f(x) = 0. \tag{6.9}$$

**Remark 6.4** Observe that the function $f \equiv 1$ solves the integro–differential equation (6.8) but does not solve the boundary condition (6.9). In fact, the motivation for this boundary condition is to exclude the trivial solution $f \equiv 1$ in order to obtain uniqueness of a solution for the equation in question. The same reasoning applies to the equations of FKPP–type considered in the previous section. The two boundary conditions there exclude the solutions $f \equiv 1$ and $f \equiv 0$ respectively. Note that here we do not need a second boundary condition in order to exclude $f \equiv 0$ as this function is not a solution of (6.8). Indeed, let $f|_{\mathbb{R}_0^+} \equiv 0$ and let $x > 0$ be such that $\nu(\mathbf{s} \in \mathcal{S}_1 : -\ln(s_1) > x) > 0$. Then

$$cf'(x) + \int_{\mathcal{S}_1} \left( \prod_{n \in \mathbb{N}} f(x + \ln(s_n)) - f(x) \right) \nu(\mathrm{d}\mathbf{s}) = \nu(\mathbf{s} \in \mathcal{S}_1 : -\ln(s_1) > x) > 0,$$

and thus $f \equiv 0$ does not solve (6.8). ◊

Below we shall need the following generalisation of the notion of a derivative for a continuous function.

**Definition 6.5** The *upper Dini derivative* $f'_+$ and the *lower Dini derivative* $f'_-$ of a continuous function $f : \mathbb{R} \to \mathbb{R}$ are defined by

$$f'_+(x) := \limsup_{h \downarrow 0} \frac{f(x+h) - f(x)}{h} \qquad \text{and} \qquad f'_-(x) := \liminf_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}$$

respectively for all $x \in \mathbb{R}$.

Note that $f'_+(x)$ and $f'_-(x)$ are well defined for any $x \in \mathbb{R}$, but may take the value $\infty$ or $-\infty$.

Let us now introduce three operators acting on a certain set of monotone functions. The definitions of these operators are inspired by the integro–differential equation (6.8).

**Definition 6.6** Let $\mathcal{D}_L$ be the set of all functions $f : \mathbb{R} \to [0,1]$, with $f|_{(-\infty,0)} \equiv 1$, for which the integral

$$\int_{\mathcal{S}_1} \left( \prod_{n\in\mathbb{N}} f(x + \ln(s_n)) - f(x) \right) \nu(\mathrm{d}\mathbf{s})$$

exists, that is its positive part or negative part is finite. Then we define an integral operator $L$ with domain $\mathcal{D}_L$ by

$$Lf(x) = \int_{\mathcal{S}_1} \left( \prod_{n\in\mathbb{N}} f(x + \ln(s_n)) - f(x) \right) \nu(\mathrm{d}\mathbf{s})$$

for each $f \in \mathcal{D}_L$ and all $x \in \mathbb{R}_0^+$. Further, let us define integro–differential operators $T^+$ and $T^-$ on $\mathcal{D}_L$ by

$$T^+ f(x) = cf'_+(x) + Lf(x)$$

as well as

$$T^- f(x) = cf'_-(x) + Lf(x)$$

for any $f \in \mathcal{D}_L$ and every $x \in \mathbb{R}_0^+$.

The following class of monotone functions plays a crucial role in the analysis of the one–sided FKPP travelling wave equation.

**Definition 6.7** For any $p > -1$ we denote by $\mathcal{D}(p)$ the set of all continuous monotonically nonincreasing functions $f : \mathbb{R} \to [0,1]$, with $f|_{(-\infty,0)} \equiv 1$, that satisfy (6.9) and for which $e^{(1+p)x} f(x)$ is monotonically nondecreasing.

For any $f : \mathbb{R} \to [0,1]$ set

$$\mathcal{C}_f := \left\{ x \in \mathbb{R}^+ : f'(x) \text{ exists} \right\}. \tag{6.10}$$

**Remark 6.8** Let us mention that for any monotone function $f : \mathbb{R}_0^+ \to [0,1]$ we have that $\mathbb{R}_0^+ \setminus \mathcal{C}_f$ is a Lebesgue null set. Recall that $X_1^x$ has only countably many discontinuities and is strictly monotone on the complement of the jump times. Therefore, $\mathbb{R}_0^+ \setminus \mathcal{C}_f$ being a Lebesgue null set implies that $X_1^x(t) \in \mathcal{C}_f$ for Lebesgue–almost all $t \in \mathbb{R}_0^+$ and every $f \in \mathcal{D}(p)$, $p > -1$. That is, the complement of

$$\mathcal{T}_f^x := \left\{ t \in \mathbb{R}_0^+ : X_1^x(t) \in \mathcal{C}_f \right\} \tag{6.11}$$

is a Lebesgue null set for any $f \in \mathcal{D}(p)$ with $p > -1$. $\Diamond$

### 6.3.2 Main results

Recall the random index set $\mathcal{N}_t^x$ defined in Definition 5.10 and furthermore recall that in Section 5.4 the process $Z^{x,f} = (Z_t^{x,f})_{t \in \mathbb{R}_0^+}$ was defined by

$$Z_t^{x,f} = \prod_{n \in \mathcal{N}_t^x} f\left(X_n^x(t)\right)$$

for all $t \in \mathbb{R}_0^+$.

Our first result reads as follows:

**Proposition 6.9** *Let $c > c_{\bar{p}}$ and let $f \in \mathcal{D}_L$ be monotone. Further, assume that $T^+ f = T^- f \equiv 0$. Then $Z_t^{x,f}$ is a martingale for all $x \in \mathbb{R}_0^+$.*

The second main result of this chapter shows in particular the more complicated converse implication of Proposition 6.9.

**Theorem 6.10** *Let $c > c_{\bar{p}}$. In addition, let $f \in \mathcal{D}(p)$ for some $p > -1$ and assume that $Z_t^{x,f}$ is a martingale. Then $f|_{\mathbb{R}_0^+} \in C^1(\mathbb{R}_0^+, [0,1])$ and $f$ solves (6.8).*

The above two results will be proven in Section 6.5.

The main goal of this chapter is to establish the existence of a unique travelling wave to (6.7) with wave speed $c$ for $c > c_{\bar{p}}$ as well as the nonexistence of such a travelling wave with wave speed $c \leq c_{\bar{p}}$. More specifically, the following theorem states that the extinction probability of the killed fragmentation process solves equation (6.8) with boundary condition (6.9) for $c > c_{\bar{p}}$. Recall $\varphi$ from Definition 5.15.

**Theorem 6.11** *If $c > c_{\bar{p}}$, then $\varphi$ is the unique monotone travelling wave to (6.7) with wave speed $c$, that is it satisfies (6.8) and (6.9). On the other hand, if $c \leq c_{\bar{p}}$, then there is no monotone travelling wave to (6.7) with wave speed $c$.*

We shall prove this theorem in Section 6.6. In the light of Theorem 6.11 it follows from Lemma 5.21 and Lemma 5.22 that for any $c > c_{\bar{p}}$ the unique one–sided FKPP travelling wave with wave speed $c$ is a continuous function that is strictly monotonically decreasing on $\mathbb{R}_0^+$. Note further that Theorem 6.11 shows that solutions of the one–sided FKPP equation exist only for wave speeds for which there does not exist a two–sided travelling wave, cf. Remark 6.1. In fact, for positive wave speeds a one–sided FKPP travelling wave solution exists if and only if there is no two–sided solution. Moreover, in view of Proposition 5.14, Theorem 6.11 shows that travelling wave solutions exist exactly for those wave speeds that are larger than the asymptotic speed of the largest fragment in the killed fragmentation on the event of survival of this killed process.

Notice that all three FKPP travelling wave boundary value problems described in Section 6.2 have initial– and final value conditions whereas here we only consider a final value problem. This difference between our situation and the classical one–sided FKPP travelling wave equation is addressed in the following remark:

**Remark 6.12** Recall the classical one–sided FKPP travelling wave boundary value problem described in Section 6.2. In that classical setting a travelling wave has to satisfy boundary value conditions at both sides of the interval $(0, \infty)$. Given that the boundary conditions are the same for the classical two–sided FKPP travelling wave boundary value problem and for the two–sided travelling wave equation for fragmentation processes, cf. Section 6.2, in our context the corresponding requirement in (6.9) should be

$$\lim_{x \to 0} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to \infty} f(x) = 0.$$

Recall that in Remark 6.4 we mentioned that in the situation considered here a condition like $\lim_{x \to 0} f(x) = 1$ in order to exclude a possible trivial solution $f \equiv 0$ is not necessary, since $f \equiv 0$ does not solve the integro–differential equation (6.8). Moreover, Lemma 5.19 shows that

$$\lim_{x \to 0} \varphi(x) = \mathbb{P}(\zeta^0 < \infty) \in (0, 1) \tag{6.12}$$

for every $c > c_{\bar{p}}$. Since our goal is to prove that for $c > c_{\bar{p}}$ the function $\varphi$ is a travelling wave to (6.7), cf. Theorem 6.11, and thus it needs to satisfy (6.9), it follows from (6.12) that in (6.9) we cannot require $\lim_{x \to 0} f(x) = 1$.

For any $c \geq c_{\bar{p}}$ and $x \in \mathbb{R}_0^+$ let $\zeta_c^x$ denote the extinction time with regard to the $x$–killing line with drift $c$. Further, let $\psi'_{c,p}(0+)$ be the Laplace exponent under $\mathbb{P}^{(p)}$ of the process $(ct + \ln(|B_1(t)|))_{t \in \mathbb{R}_0^+}$, that is

$$\psi'_{c,p}(0+) = c + \mathbb{E}^{(p)}(\ln(|B_1(1)|)) = c - \Phi'(p),$$

cf. Remark 5.8, and thus the continuity of $\Phi'$ yields that

$$\lim_{p \uparrow \bar{p}} \lim_{c \downarrow c_{\bar{p}}} \psi'_{c,p}(0+) = c_{\bar{p}} - \Phi'(\bar{p}) = 0.$$

Hence, the estimate in (6.12) now gives us the upper bound equal to 1, that is

$$\lim_{c\downarrow c_{\bar{p}}}\lim_{x\to 0}\mathbb{P}(\zeta_c^x < \infty) \leq 1,$$

and therefore it may be possible that

$$\lim_{c\downarrow c_{\bar{p}}}\lim_{x\to 0}\mathbb{P}(\zeta_c^x < \infty) = 1. \tag{6.13}$$

Let us now show that (6.13) does indeed hold true. For this purpose let $x \in \mathbb{R}_0^+$ and assume that $\zeta_{c_{\bar{p}}}^x < \infty$. Then we have that $-\ln(\lambda_1^x(\zeta_{c_{\bar{p}}}^x)) > x + c_{\bar{p}}\zeta_{c_{\bar{p}}}^x$. However, this implies that $\zeta_{c_{\bar{p}}}^x$ is also the extinction time for all

$$c \in \left(c_{\bar{p}}, \frac{-\ln(\lambda_1^x(\zeta_{c_{\bar{p}}}^x)) - x}{\zeta_{c_{\bar{p}}}^x}\right),$$

and since Lemma 5.16 implies that $\mathbb{P}(\zeta_{c_{\bar{p}}}^x < \infty) = 1$, we thus infer that

$$\lim_{c\downarrow c_{\bar{p}}}\mathbb{P}(\zeta_c^x < \infty) = \mathbb{P}\left(\bigcup_{c>c_{\bar{p}}}\{\zeta_c^x < \infty\}\right) = 1,$$

which proves (6.13). ◊

## 6.4 The finite activity case

An approach to solve the classical one–sided FKPP equation with boundary condition $u(0,x) = g(x)$ for some suitable function $g : \mathbb{R}_0^+ \to [0,1]$ is to show that the function $u : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to [0,1]$ given by

$$u(t,x) = \mathbb{E}\left(\prod_{n\in\mathbb{N}} g(x + X_n(t))\right)$$

for all $t, x \in \mathbb{R}_0^+$ is a solution of the considered boundary value problem, where the $X_n(t)$, $n \in \mathbb{N}$, are the positions of the particles at time $t$ in the branching Brownian motion.

In this section we show that for fragmentations with a finite dislocation measure $\nu$ the same approach works for the initial value problem (6.7) with boundary condition $u(0,x) = g(x)$. More precisely, we aim at proving that for any measurable function

$g : \mathbb{R}_0^+ \to [0,1]$ the function $u : \mathbb{R}_0^+ \times \mathbb{R} \to [0,1]$ defined by

$$u(t,x) = \mathbb{E}\left(\prod_{n\in\mathcal{N}_t^x} g(x+\ln(|B_n(t)|))\right) \qquad \text{and} \qquad u(t,\cdot)|_{(-\infty,0)} \equiv 1 \tag{6.14}$$

for all $t,x \in \mathbb{R}_0^+$ solves equation (6.7) with boundary condition $u(0,\cdot)|_{\mathbb{R}_0^+} = g$.

In addition to proving that the function $u$ defined by (6.14) solves equation (6.7), the proposition that we shall consider in the present section also provides a relatively short proof that the extinction probability $\varphi$, cf. Definition 5.15, of the killed fragmentation process solves equation (6.8) in the special case that the dislocation measure $\nu$ is finite. Since, according to Theorem 5.12, for $c > c_{\bar{p}}$ the function $\varphi$ also satisfies the boundary condition (6.9), it thus shows that $\varphi$ is an FKPP travelling wave solution of (6.7) with wave speed $c > c_{\bar{p}}$ in the finite activity case. The major part of this chapter, cf. Theorem 6.11, is concerned with the proof that this latter statement holds true in the general case of an infinite dislocation measure.

**Proposition 6.13** *Assume that* $\nu(\mathcal{S}_1) < \infty$ *and let* $c > 0$. *Then every function* $u : \mathbb{R}_0^+ \times \mathbb{R} \to [0,1]$ *defined by (6.14), for some measurable* $g : \mathbb{R}_0^+ \to [0,1]$, *is a solution of (6.7) with the boundary condition*

$$u(0,\cdot)|_{\mathbb{R}_0^+} = g. \tag{6.15}$$

*Moreover, the function* $\varphi$ *solves (6.8).*

**Proof** Let $g : \mathbb{R}_0^+ \to [0,1]$ be some measurable function and consider the function $u : \mathbb{R}_0^+ \times \mathbb{R} \to [0,1]$ defined by (6.14). We first aim at showing that $u$ is a solution of (6.7) with the boundary condition (6.15). To this end, note first that the boundary condition (6.15) is trivially satisfied by the definition $u$. In order to show that $u$ solves (6.7), define a function $u_s : \mathbb{R}_0^+ \times \mathbb{R} \to [0,1]$, $s \in \mathbb{R}_0^+$, by

$$u_s(t,x) = \mathbb{E}\left(\prod_{n\in\mathcal{N}_t^x} g(x - cs + \ln(|B_n(t)|))\right) \qquad \text{and} \qquad u_s(t,\cdot)|_{(-\infty,0)} \equiv 1$$

for every $t,x \in \mathbb{R}_0^+$. Let $\mathcal{K}_t^x$ denote the set of all indices such that the block $\Pi_k^x(t)$ is alive at time $t \in \mathbb{R}_0^+$ for any $k \in \mathcal{K}_t^x$. In particular, note that

$$k \in \mathcal{K}_t^x \iff \Pi_k^x(t) \neq \emptyset \iff \min\left(\Pi_k^x(t)\right) \in \mathcal{N}_t^x$$

holds for all $k \in \mathbb{N}$ and $t, x \in \mathbb{R}_0^+$. Then the fragmentation property yields that

$$
\begin{aligned}
&u(s+t,x)\\
&= \mathbb{E}\left(\prod_{k\in\mathcal{K}^x_{s+t}} g(x+\ln(|\Pi^x_k(s+t)|))\right)\\
&= \mathbb{E}\left(\mathbb{E}\left(\prod_{i\in\mathcal{K}^x_s}\prod_{j\in\mathcal{K}^{(i)}} g\left(x+\ln\left(|\Pi^x_i(s)|\cdot\left|\Pi^{(i)}_j\right|\right)\right)\middle|\mathscr{F}_s\right)\right) \qquad (6.16)\\
&= \mathbb{E}\left(\prod_{i\in\mathcal{K}^x_s}\mathbb{E}\left(\prod_{j\in\mathcal{K}^{y_i}_t} g\left(x+\ln\left(|\Pi^x_i(s)|\right)+\ln\left(\left|\Pi^{y_i}_j(t)\right|\right)\right)\right)\Bigg|_{y_i=x+cs+\ln(|\Pi^x_i(s)|)}\right)\\
&= \mathbb{E}\left(\prod_{k\in\mathcal{K}^x_s} u_s\left(t, x+cs+\ln(|\Pi^x_k(s)|)\right)\right)
\end{aligned}
$$

$\mathbb{P}$–a.s. for all $s, t, x \in \mathbb{R}_0^+$, where under $\mathscr{F}_s$ the $\mathcal{K}^{(i)}$ [resp. $\Pi_j^{(i)}$] are independent and each having the same distribution as $\mathcal{K}_t^{y_i}$ [resp. $\Pi_j^{y_i}(t)$] with $y_i = x + cs + \ln(|\Pi_i^x(s)|)$. Recall from Definition 5.4 that $(t_i)_{i\in\mathcal{I}^x}$ are the jump times of $\Pi^x$, and in view of the finiteness of the dislocation measure we assume without loss of generality that $\mathcal{I}^x = \mathbb{N}$ and that $\mathcal{I}^x$ is ordered such that $0 < t_i < t_j$ for all $i, j \in \mathcal{I}^x$ with $i < j$. Since $t_1^x$ is exponentially distributed with parameter $\nu(\mathcal{S}_1)$ and $\Delta(t_1^x)$ has distribution $\nu(\cdot)/\nu(\mathcal{S}_1)$, we infer from the compensation formula for Poisson random measures that

$$
\begin{aligned}
\mathbb{P}(t_1^x \le h, \Delta(t_1^x) \in \mathrm{d}\mathbf{s}) &= \mathbb{E}\left(\sum_{[0,\infty)} \mathbb{1}_{\{t\le h\}}\mathbb{1}_{\{\Delta(t)\in \mathrm{d}\mathbf{s}\}}\right)\\
&= \int_{[0,h]} \mathrm{d}t \int_{\mathcal{S}_1} \mathbb{1}_{\{\mathbf{u}\in \mathrm{d}\mathbf{s}\}}\nu(\mathrm{d}\mathbf{u}) \qquad (6.17)\\
&= h\nu(\mathrm{d}\mathbf{s}).
\end{aligned}
$$

Resorting to the extended fragmentation property we deduce from a similar argument that

$$
\begin{aligned}
\mathbb{P}(t_2^x \le h) &= \mathbb{E}\left(\mathbb{P}\left(t_2^x \le h \middle| \mathscr{F}_{t_1^x}\right)\right)\\
&\le \int_{(0,h]} \mathbb{P}(t_1^x \in \mathrm{d}u)\mathbb{P}(t_1^x \le h-u)\;\mathrm{d}u
\end{aligned}
$$

$$\leq \int_{(0,h]} \mathbb{P}(t_1^x \in \mathrm{d}u)\mathbb{P}(t_1^x \leq h)\ \mathrm{d}u \tag{6.18}$$

$$= \left( \int_{(0,h]} \nu(\mathcal{S}_1) e^{-t\nu(\mathcal{S}_1)} \right)^2$$

$$= \left(1 - e^{-h\nu(\mathcal{S}_1)}\right)^2.$$

Then (6.17) and (6.18) result in

$$\lim_{h\to 0} \frac{\mathbb{P}(t_1^x \leq h, t_2^x > h, \Delta(t_1^x) \in \mathrm{d}\mathbf{s})}{h}$$
$$= \lim_{h\to 0} \frac{\mathbb{P}(t_1^x \leq h, \Delta(t_1^x) \in \mathrm{d}\mathbf{s})}{h} - \lim_{h\to 0} \frac{\mathbb{P}(t_2^x \leq h, \Delta(t_1^x) \in \mathrm{d}\mathbf{s})}{h} \tag{6.19}$$
$$= \nu(\mathrm{d}\mathbf{s}).$$

Observe that Fatou's lemma for limit superiors yields that

$$\limsup_{h\downarrow 0} \frac{(u(t,x) - u_h(t,x+ch))\mathbb{P}(t_1^x > h)}{h}$$
$$\leq \limsup_{h\downarrow 0} \mathbb{E}\left( \frac{\prod_{n\in\mathcal{N}_t^x} g(x+\ln(|B_n(t)|)) - \prod_{n\in\mathcal{N}_t^{x+ch}} g(x+\ln(|B_n(t)|))}{h} \right) \tag{6.20}$$
$$\leq \mathbb{E}\left( \limsup_{h\downarrow 0} \frac{\prod_{n\in\mathcal{N}_t^x} g(x+\ln(|B_n(t)|)) - \prod_{n\in\mathcal{N}_t^{x+ch}} g(x+\ln(|B_n(t)|))}{h} \right)$$

Since $\nu$ is finite, for any $h > 0$ there are only finitely many $n \in \mathcal{N}_t^{x+ch} \setminus \mathcal{N}_t^x$. Hence, consider

$$\delta_0 := \max_{n\in\mathcal{N}_t^{x+ch_0}\setminus\mathcal{N}_t^x} \left(x + c\tau_{n,x}^- + \ln(|B_n(\tau_{n,x}^-)|)\right) \in (-ch_0, 0)$$

for some $h_0 > 0$. Then we have that $\mathcal{N}_t^{x+ch} = \mathcal{N}_t^x$ for all $h < {}^{-\delta_0}/_{c} < h_0$, and thus

$$\limsup_{h\downarrow 0} \frac{\prod_{n\in\mathcal{N}_t^x} g(x+\ln(|B_n(t)|)) - \prod_{n\in\mathcal{N}_t^{x+ch}} g(x+\ln(|B_n(t)|))}{h} = 0.$$

Consequently, it follows from (6.20) that

$$\limsup_{h\downarrow 0} \frac{(u_h(t,x) - u(t,x+ch))\mathbb{P}(t_1^x > h)}{h} = 0. \tag{6.21}$$

Since by means of (6.16) we have

$$
\begin{aligned}
u(t+h,x) &= \mathbb{E}\left(\prod_{k\in\mathcal{K}_h^x} u_h(t,x+ch+\ln(|\Pi_k^x(h)|))\right) \\
&= u_h(t,x+ch)\mathbb{P}(t_1^x > h) + \mathbb{E}\left(\mathbb{1}_{\{t_2^x\le h\}}\prod_{k\in\mathcal{K}_h^x} u_h(t,x+ch+\ln(|\Pi_k^x(h)|))\right) \\
&\quad + \int_{\mathcal{S}_1}\prod_{n\in\mathbb{N}:s_n>0} u_h(t,x+ch+\ln(s_n))\mathbb{P}(t_1^x\le h, t_2^x>h, \Delta(t_1^x)\in \mathrm{d}\mathbf{s}),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) \\
&= \lim_{h\downarrow 0}\frac{u(t+h,x)-u(t,x)}{h} \\
&= \int_{\mathcal{S}_1}\left(\lim_{h\downarrow 0}\prod_{n\in\mathbb{N}:s_n>0} u_h(t,x+ch+\ln(s_n)) - u(t,x)\right)\lim_{h\downarrow 0}\frac{\mathbb{P}(t_1^x\le h, t_2^x>h,\Delta(t_1^x)\in\mathrm{d}\mathbf{s})}{h} \\
&\quad + \lim_{h\downarrow 0}\frac{(u_h(t,x+ch)-u(t,x))\mathbb{P}(t_1^x>h)}{h} \qquad (6.22)\\
&\quad + \lim_{h\downarrow 0}\frac{\mathbb{E}\left(\mathbb{1}_{\{t_2^x\le h\}}\left(\prod_{k\in\mathcal{K}_h^x} u_h(t,x+ch+\ln(|\Pi_k^x(h)|)) - u(t,x)\right)\right)}{h} \\
&= \int_{\mathcal{S}_1}\left(\prod_{n\in\mathbb{N}:s_n>0}\lim_{h\downarrow 0} u_h(t,x+ch+\ln(s_n)) - u(t,x)\right)\nu(\mathrm{d}\mathbf{s}) \\
&= \int_{\mathcal{S}_1}\left(\prod_{n\in\mathbb{N}:s_n>0} u(t,x+\ln(s_n)) - u(t,x)\right)\nu(\mathrm{d}\mathbf{s}),
\end{aligned}
$$

where the second equality is obtained by applying the DCT and the third equality is a consequence of (6.17), (6.18) and (6.19). Note that in the third equality we can interchange the limit with the product, since the product has only finitely many factors that are not equal to 1. Indeed, since $u|_{(-\infty,0)}\equiv 1$ we only need to consider those factors with $-\ln(s_n) < x+ch$. Since $\sum_{n\in\mathbb{N}} s_n \le 1$, there are only at most $e^{x+ch}$–many $n\in\mathbb{N}$ satisfying $s_n > e^{-(x+ch)}$. The final equality in (6.22) results from (6.21).

In view of (6.22) we conclude that $u$ solves (6.7).

It remains to show that $\varphi$ is a solution of (6.8). For this purpose, observe that the fragmentation property, in conjunction with the tower property for conditional expectations, yields that

$$\varphi(x-ct) = \mathbb{E}\left(\prod_{n\in\mathcal{N}_t^x} \mathbb{P}\left(\zeta^{x-ct+ct+y} < \infty\right)\Big|_{y=\ln(|B_n(t)|)}\right)$$
$$= \mathbb{E}\left(\prod_{n\in\mathcal{N}_t^x} \varphi(x+\ln(|B_n(t)|))\right),$$

and thus $u : \mathbb{R}_0^+ \times \mathbb{R} \to [0,1]$ given by $u(t,x) := \varphi(x-ct)$ satisfies (6.14). Hence, according to our above considerations $u$ is a solution of (6.7) and consequently we conclude that $\varphi$ solves (6.8). □

## 6.5 Concurrence of FKPP travelling wave solutions and product martingales

In this section we prove Proposition 6.9 and Theorem 6.10. For this purpose, let us start with the following auxiliary result:

**Lemma 6.14** *Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subseteq [0,1]$ be sequences in $[0,1]$. Then*

$$\left|\prod_{n\in\mathbb{N}} a_n - \prod_{n\in\mathbb{N}} b_n\right| \le \sum_{n\in\mathbb{N}} |a_n - b_n|.$$

**Proof** We first show by induction that

$$\left|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i\right| \le \sum_{i=1}^n |a_i - b_i| \tag{6.23}$$

for all $n \in \mathbb{N}$. To this end, assume that $|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i| \le \sum_{i=1}^n |a_i - b_i|$ for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\left|\prod_{i=1}^{n+1} a_i - \prod_{i=1}^{n+1} b_i\right| &\le \sum_{i=1}^n |a_i - b_i| = \left|a_{n+1}\prod_{i=1}^n a_i - b_{n+1}\prod_{i=1}^n b_i\right| \\ &\le |a_{n+1} - b_{n+1}| \prod_{i=1}^n a_i + b_{n+1}\left|\prod_{n\in\mathbb{N}} a_n - \prod_{n\in\mathbb{N}} b_n\right| \\ &\le |a_{n+1} - b_{n+1}| + \sum_{i=1}^n |a_i - b_i| = \sum_{i=1}^{n+1} |a_i - b_i|.\end{aligned}$$

Since (6.23) trivially holds for $n = 1$, we conclude that (6.23) holds true. Now the assertion of the lemma follows by taking the limit as $n \to \infty$. $\square$

Let us point out that according to Theorem 7.21 in [Rud87] the fundamental theorem of calculus holds for Lebesgue integrals, that is

$$H(t) - H(0) = \int_{[0,t]} h(s) \, \mathrm{d}s \tag{6.24}$$

for all $t \in \mathbb{R}_0^+$ and any $H$ that is differentiable on [0,1] and whose derivative $h$ is bounded and Lebesgue–integrable.

**Proof of Proposition 6.9**

Observe that the jump times of $Z^{x,f}$ coincide with $(t_i)_{i\in\mathcal{I}^x}$, the jump times of $\Pi^x$. Consequently, the monotonicity of $f$ implies that $(Z_t^{x,f})_{t\in\mathbb{R}_0^+\setminus(t_i)_{i\in\mathcal{I}^x}}$ is monotone, and since $\mathcal{I}^x$ is countable, we thus conclude that $Z^{x,f}$ is differentiable almost everywhere with respect to the Lebesgue measure. That is, the complement of

$$\mathcal{T}_Z^{x,f} := \left\{ t \in \mathbb{R}_0^+ : Z^{x,f} \text{ is differentiable at } t \right\} \tag{6.25}$$

is a Lebesgue null set. Let us introduce the notation

$$\dot{Z}_t^{x,f} := \left. \frac{\mathrm{d}}{\mathrm{d}s} Z_s^{x,f} \right|_{s=t} \tag{6.26}$$

for all $t \in \mathcal{T}_Z^{x,f}$. In view of (6.24) we then obtain that

$$\mathbb{E}\left(Z_t^{x,f}\right) - \mathbb{E}\left(Z_0^{x,f}\right) = \mathbb{E}\left( \int_{(0,t)} \dot{Z}_s^{x,f} \, \mathrm{d}s + \sum_{i\in\mathcal{I}^x : t_i \le t} \Delta Z_{t_i}^{x,f} \right), \tag{6.27}$$

where $\Delta Z_t^{x,f} := Z_t^{x,f} - Z_{t-}^{x,f}$. Define a function $\psi : \mathbb{R}_0^+ \times \mathcal{S}_1 \times \mathbb{N} \to \mathbb{R}$ by

$$\psi(u, \mathbf{s}, k) := \prod_{i\in\mathcal{N}_{u-}^x\setminus\{k\}} f(X_i^x(u-)) \prod_{j\in\mathbb{N}} f(X_k^x(u-) + \ln(s_j)) - \prod_{n\in\mathcal{N}_{u-}^x} f(X_n^x(u-)).$$

With $N$ being the Poisson random measure associated with the underlying Poisson point process, we deduce from (6.27), in conjunction with the DCT, the compensation formula for Poisson point processes and Fubini's theorem as well as Tonelli's theorem, that

$$\mathbb{E}\left(Z_t^{x,f}\right) - \mathbb{E}\left(Z_0^{x,f}\right) \tag{6.28}$$

$$= \mathbb{E}\left(\int_{(0,t)} \dot{Z}^{x,f}_{u-}\ \mathrm{d}u\right) + \mathbb{E}\left(\int_{(0,t)}\int_{\mathcal{S}_1}\int_{\mathbb{N}} \psi(u,\mathbf{s},k)N(\mathrm{d}u,\mathrm{d}\mathbf{s},\mathrm{d}k)\right)$$
$$= \int_{(0,t)} \mathbb{E}\left(\dot{Z}^{x,f}_{u-}\right)\ \mathrm{d}u + \int_{(0,t)} \mathbb{E}\left(\int_{\mathbb{N}}\int_{\mathcal{S}_1} \psi(u,\mathbf{s},k)\nu(\mathrm{d}\mathbf{s})\sharp(\mathrm{d}k)\right)\ \mathrm{d}u.$$

Observe that

$$\begin{aligned}
\dot{Z}^{x,f}_{u-} &\leq \sum_{k\in\mathcal{N}^x_{u-}} cf'_+(X^x_k(u-)) \prod_{i\in\mathcal{N}^x_{u-}\setminus\{k\}} f(X^x_i(u-)) \\
&= \sum_{k\in\mathcal{N}^x_{u-}} cf'_+(X^x_k(u-))\frac{1}{f(X^x_k(u-))} \prod_{i\in\mathcal{N}^x_{u-}} f(X^x_i(u-)) \qquad (6.29)\\
&= \prod_{i\in\mathcal{N}^x_{u-}} f(X^x_i(u-)) \sum_{k\in\mathcal{N}^x_{u-}} \frac{cf'_+(X^x_k(u-))}{f(X^x_k(u-))},
\end{aligned}$$

and similarly

$$\dot{Z}^{x,f}_{u-} \geq \prod_{i\in\mathcal{N}^x_{u-}} f(X^x_i(u-)) \sum_{k\in\mathcal{N}^x_{u-}} \frac{cf'_-(X^x_k(u-))}{f(X^x_k(u-))}, \qquad (6.30)$$

$\mathbb{P}$–a.s. for any $u \in \mathcal{T}^{x,f}_Z$. Furthermore,

$$\begin{aligned}
&\sum_{k\in\mathcal{N}^x_{u-}} \int_{\mathcal{S}_1} \psi(u,\mathbf{s},k)\nu(\mathrm{d}\mathbf{s}) \\
=\ & \sum_{k\in\mathcal{N}^x_{u-}} \prod_{i\in\mathcal{N}^x_{u-}\setminus\{k\}} f(X^x_i(u-)) \int_{\mathcal{S}_1} \left[\prod_{j\in\mathbb{N}} f(X^x_k(u-)+\ln(s_j)) - f(X^x_k(u-))\right]\nu(\mathrm{d}\mathbf{s}) \\
=\ & \prod_{i\in\mathcal{N}^x_{u-}} f(X^x_i(u-)) \sum_{k\in\mathcal{N}^x_{u-}} \frac{1}{f(X^x_k(u-))} Lf(X^x_k(u-)) \qquad (6.31)
\end{aligned}$$

$\mathbb{P}$–a.s. for every $u \in \mathbb{R}^+$, which in conjunction with (6.28) and (6.29) results in

$$\begin{aligned}
&\mathbb{E}\left(Z^{x,f}_t\right) - \mathbb{E}\left(Z^{x,f}_0\right) \\
&\leq \int_{(0,t)} \mathbb{E}\left(\prod_{i\in\mathcal{N}^x_u} f(X^x_i(u)) \sum_{k\in\mathcal{N}^x_u} \frac{1}{f(X^x_k(u))} T^+ f(X^x_k(u))\right) \mathrm{d}u \\
&= 0,
\end{aligned}$$

where the final equality results from $T^+f \equiv 0$. Analogously, making us of $T^- \equiv 0$, we deduce from (6.28), (6.30) and (6.31) that

$$\mathbb{E}\left(Z_t^{x,f}\right) - \mathbb{E}\left(Z_0^{x,f}\right) \geq \int_{(0,t)} \mathbb{E}\left(\prod_{i\in\mathcal{N}_u^x} f(X_i^x(u)) \sum_{k\in\mathcal{N}_u^x} \frac{1}{f(X_k^x(u))} T^- f(X_k^x(u))\right) \mathrm{d}u = 0.$$

Hence, we infer that

$$\mathbb{E}\left(Z_t^{x,f}\right) = \mathbb{E}\left(Z_0^{x,f}\right) = f(x).$$

The assertion now follows from Lemma 5.29. □

A first approach to try proving Theorem 6.10 might be to follow the lines of the proof of Theorem 1 in [BHK10]. However, that proof relies on $f$ being continuously differentiable and in order to use that idea we would at least need that the set of discontinuities of the derivative of the differentiable function in question is a Lebesgue null set. However, in general the set of such discontinuities may have positive Lebesgue measure. Indeed, Example 3.5 in [WH93] shows that the derivative may not be Riemann integrable. Nonetheless, many ideas of our method to prove Theorem 6.10 are taken from [BHK10].

Our proof of Theorem 6.10 is based on two auxiliary results that we are now going to develop. Afterwards, having these auxiliary results on hand, we shall tackle the proof of Theorem 6.10.

**Lemma 6.15** *Let $f \in \mathcal{D}(p)$ for some $p > -1$ and let $a, b \in \mathbb{R}_0^+$. Then*

$$\sup_{x\in[a,b]} \left| \prod_{n\in\mathbb{N}} f(x + \ln(s_n)) - f(x) \right| \in \mathscr{L}^1(\nu).$$

**Proof** We infer from Lemma 6.14 that

$$\int_{\mathcal{S}_1} \sup_{x\in[a,b]} \left| \prod_{n\in\mathbb{N}} f(x + \ln(s_n)) - f(x) \right| \nu(\mathrm{d}\mathbf{s}) \tag{6.32}$$
$$\leq \int_{\mathcal{S}_1} \sup_{x\in[a,b]} |f(x+\ln(s_1)) - f(x)|\, \nu(\mathrm{d}\mathbf{s}) + \int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}\setminus\{1\}} \sup_{x\in[a,b]} |f(x+\ln(s_n)) - 1|\, \nu(\mathrm{d}\mathbf{s})$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(x) + 2(1-x)] = \frac{1}{x} - 2$$

and $\ln(1) + 2(1-1) = 0$, we have that

$$-\ln(x) \le 2(1-x)$$

for all $x \in [2^{-1}, 1]$. Therefore, for every $\epsilon \in (0, 2^{-1}]$ we have

$$-\ln(s_1) \le 2(1-s_1) \tag{6.33}$$

for all $(s_n)_{n\in\mathbb{N}} \in \mathcal{S}_1$ with $1 - s_1 \le \epsilon$.

Moreover, since $x \mapsto e^{(1+p)x} f(x)$ is nondecreasing, we infer by taking its derivative that

$$0 \le \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(1+p)x} f(x)\right) = (1+p)e^{(1+p)x} f(x) + e^{(1+p)x} f'(x),$$

and thus

$$-(1+p) \le -(1+p)f(x) \le f'(x) \le 0 \tag{6.34}$$

for every $x \in \mathcal{C}_f$. Let $x \in \mathbb{R}_0^+$ as well as $\epsilon \in (0, 2^{-1}]$ and recall the definition of $\mathcal{C}_f$ in (6.10). By the Mean Value Theorem we have for any $x \in \mathbb{R}_0^+$ and $\mathbf{s} \in \mathcal{S}_1$ that

$$|f(x + \ln(s_1)) - f(x)| \le -\ln(s_1) \sup_{y \in \mathcal{C}_f} f'(y) \le -\ln(s_1)(1+p),$$

where the final estimate results from (6.34). Furthermore, let $\epsilon > 0$ and notice that $\nu(\{\mathbf{s} \in \mathcal{S}_1 : 1 - s_1 \ge \epsilon\}) < \infty$, cf. Remark 5.17. Hence, resorting to (6.33) we infer that

$$\begin{aligned}
&\int_{\mathcal{S}_1} \sup_{x\in[a,b]} |f(x+\ln(s_1)) - f(x)|\,\nu(\mathrm{d}\mathbf{s}) \\
&\le \int_{\{\mathbf{s}\in\mathcal{S}_1 : 1-s_1\ge\epsilon\}} \sup_{x\in[a,b]} |f(x+\ln(s_1)) - f(x)|\,\nu(\mathrm{d}\mathbf{s}) \\
&\qquad + \int_{\{\mathbf{s}\in\mathcal{S}_1 : 1-s_1<\epsilon\}} \sup_{x\in[a,b]} |f(x+\ln(s_1)) - f(x)|\,\nu(\mathrm{d}\mathbf{s}) \\
&\le \nu(\{\mathbf{s}\in\mathcal{S}_1 : 1-s_1\ge\epsilon\}) - (1+p)\int_{\{\mathbf{s}\in\mathcal{S}_1 : 1-s_1<\epsilon\}} \ln(s_1)\nu(\mathrm{d}\mathbf{s}) \\
&\le \nu(\{\mathbf{s}\in\mathcal{S}_1 : 1-s_1\ge\epsilon\}) + 2(1+p)\int_{\mathcal{S}_1} (1-s_1)\nu(\mathrm{d}\mathbf{s}) \\
&< \infty,
\end{aligned}$$

which shows that the first term on the right–hand side of (6.32) is finite. In order to deal with the second term on the right–hand side of (6.32), note that $f|_{(-\infty,0)} \equiv 1$ and

$f|_{[0,\infty)} \in [0,1]$ yields that

$$\begin{aligned}\int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}\setminus\{1\}} \sup_{x\in[a,b]} |1-f(x+\ln(s_n))|\nu(\mathrm{d}\mathbf{s}) &\le \int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}\setminus\{1\}} |1-f(b+\ln(s_n))|\nu(\mathrm{d}\mathbf{s}) \\ &\le \int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}\setminus\{1\}} e^{(b+\ln(s_n))}\nu(\mathrm{d}\mathbf{s}) \\ &= e^b \int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}\setminus\{1\}} s_n\nu(\mathrm{d}\mathbf{s}) \\ &< \infty\end{aligned}$$

for all $x > 0$. Observe that the finiteness follows from (1.7), since

$$\begin{aligned}\int_{\mathcal{S}_1} \sum_{n\in\mathbb{N}\setminus\{1\}} s_n\nu(\mathrm{d}\mathbf{s}) &= \int_{\mathcal{S}_1} \left((1-s_1) + \left(\sum_{n\in\mathbb{N}} s_n - 1\right)\right)\nu(\mathrm{d}\mathbf{s}) \\ &\le \int_{\mathcal{S}_1} (1-s_1)\nu(\mathrm{d}\mathbf{s}) \\ &< \infty.\end{aligned}$$

Consequently, we also have the finiteness of the second term on the right–hand side in (6.32). $\square$

Note that

$$|Lf(x)| \le \int_{\mathcal{S}_1} \left|\prod_{n\in\mathbb{N}} f(x+\ln(s_n)) - f(x)\right| \nu(\mathrm{d}\mathbf{s})$$

for any $f \in \mathcal{D}(p)$, and thus Lemma 6.15 implies that $Lf$ is bounded for every $f \in \mathcal{D}(p)$.

**Lemma 6.16** *Let $f \in \mathcal{D}(p)$ for some $p > -1$. Then the function $Lf$ is continuous.*

**Proof** Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}_0^+$ with $x_k \to x$ as $k \to \infty$. Observe that

$$\begin{aligned}&\int_{\mathcal{S}_1} \sup_{k\ge k_0} \left|\prod_{n\in\mathbb{N}} f(x-\ln(s_n)) - f(x) - \prod_{n\in\mathbb{N}} f(x_k-\ln(s_n)) + f(x_k)\right| \nu(\mathrm{d}\mathbf{s}) \\ &\le \int_{\mathcal{S}_1} \left|\prod_{n\in\mathbb{N}} f(x-\ln(s_n)) - f(x)\right| \nu(\mathrm{d}\mathbf{s}) \qquad (6.35) \\ &\quad + \int_{\mathcal{S}_1} \sup_{\delta\in(0,\epsilon)} \left|\prod_{n\in\mathbb{N}} f(x+\delta-\ln(s_n)) - f(x+\delta)\right| \nu(\mathrm{d}\mathbf{s}),\end{aligned}$$

where $k_0 \in \mathbb{N}$ is chosen such that $|x - x_k| \le \epsilon$ for all $k \ge k_0$. According to Lemma 6.15 both of the integrals on the right–hand side of (6.35) are finite. Hence, the DCT is applicable to deduce that

$$\begin{aligned}
&\lim_{k\to\infty} |Lf(x) - Lf(x_k)| \\
&\le \int_{\mathcal{S}_1} \lim_{k\to\infty} \left| \prod_{n\in\mathbb{N}} f(x - \ln(s_n)) - f(x) - \prod_{n\in\mathbb{N}} f(x_k - \ln(s_n)) + f(x_k) \right| \nu(\mathrm{d}\mathbf{s}) \\
&\le \int_{\mathcal{S}_1} \lim_{k\to\infty} \left| \prod_{n\in\mathbb{N}} f(x - \ln(s_n)) - \prod_{n\in\mathbb{N}} f(x_k - \ln(s_n)) \right| \nu(\mathrm{d}\mathbf{s}) \\
&\qquad + \int_{\mathcal{S}_1} \lim_{k\to\infty} |f(x) - f(x_k)| \, \nu(\mathrm{d}\mathbf{s}) \\
&\overset{(*)}{\le} \int_{\mathcal{S}_1} \left| \prod_{n\in\mathbb{N}} f(x - \ln(s_n)) - \prod_{n\in\mathbb{N}} \lim_{k\to\infty} f(x_k - \ln(s_n)) \right| \nu(\mathrm{d}\mathbf{s}) \\
&\qquad + \int_{\mathcal{S}_1} \lim_{k\to\infty} |f(x) - f(x_k)| \, \nu(\mathrm{d}\mathbf{s}) \\
&= 0,
\end{aligned}$$

where the final equality from the continuity of $f$. Notice that as in (6.22) we can interchange the limit and the product in $(*)$, since only finitely many factors of the product differ from 1. This proves the continuity of $Lf$. □

Below we shall make frequent use of the deterministic estimate $N_t^x \le e^{x+ct}$ on the number of particles alive at time $t \in \mathbb{R}_0^+$, cf. (5.16). Moreover, for any $x \in \mathbb{R}_0^+$ recall the stopping time $\tau(x)$ that was defined in (5.11) by

$$\tau(x) = \inf \{ t_i \in (t_i)_{i\in\mathcal{I}_1} : |\pi_1(t_i)| \le x \} .$$

**Proof of Theorem 6.10** The proof is divided into three parts. In Part I we provide an equivalent characterisation of $f'$ on $\mathcal{C}_f$ and in the second part we give an estimate of $Lf$. Finally, in the third part we combine the first two parts to show that $-cf' = Lf$ on $\mathcal{C}_f$. Having shown this equality on $\mathcal{C}_f$ we can then deduce that the assertion of Theorem 6.10 holds true.

Part I Fix some $x \in \mathcal{C}_f$ and let $\tau : \Omega \to (0, 1]$ be an $\mathscr{F}$–stopping time.

Observe that $z \mapsto e^{(1+p)z} f(z)$ being nondecreasing implies that

$$e^{(1+p)(z+h)} f(z + h) \ge e^{(1+p)z} f(z),$$

which is equivalent to

$$f(z+h) \geq e^{-(1+p)h} f(z),$$

for all $h, z \in \mathbb{R}_0^+$. Consequently, we have

$$f(z) - f(z+h) \leq f(z)\left(1 - e^{-(1+p)h}\right) \leq 1 - e^{-(1+p)h}. \tag{6.36}$$

for all $z \in \mathbb{R}_0^+$.

By means of the extended fragmentation property we have

$$\begin{aligned}
&\frac{f(x) - f(x+h)}{h} \\
&= \frac{1}{h}\mathbb{P}\left(\{\zeta^x < \infty\} \cap \left\{\zeta^{x+h} = \infty\right\}\right) \\
&= \frac{1}{h}\mathbb{E}\left(\mathbb{P}\left(\{\zeta^x < \infty\} \cap \left\{\zeta^{x+h} = \infty\right\} \middle| \mathscr{F}_\tau\right)\right) \qquad (6.37)\\
&= \frac{1}{h}\mathbb{E}\left(\mathbb{P}\left(\left[\bigcap_{n \in \mathcal{N}_\tau^x} \left\{\zeta^{(n,0)} < \infty\right\}\right] \cap \left[\bigcup_{n \in \mathcal{N}_\tau^{x+h}} \left\{\zeta^{(n,h)} = \infty\right\}\right] \middle| \mathscr{F}_\tau\right)\right) \\
&= \frac{1}{h}\mathbb{E}\left(\mathbb{P}\left(\bigcup_{n \in \mathcal{N}_\tau^{x+h}} \left[\left\{\zeta^{(n,h)} = \infty\right\} \cap \bigcap_{k \in \mathcal{N}_\tau^x} \left\{\zeta^{(k,0)} < \infty\right\}\right] \middle| \mathscr{F}_\tau\right)\right)
\end{aligned}$$

for any $h > 0$, where conditional on $\mathscr{F}_\tau$ the $\zeta^{(n,r)}$, $r \in \{0, h\}$, are independent and satisfy

$$\mathbb{P}\left(\zeta^{(n,r)} \in \cdot \middle| \mathscr{F}_\tau\right) = \mathbb{P}\left(\zeta^{y+r} \in \cdot\right)\Big|_{y = X_n^x(\tau)}$$

$\mathbb{P}$–almost surely. We remark that the above–mentioned independence only means that $\zeta^{(n,r)}$ is independent of $\zeta^{(k,r)}$ for any $k, n \in \mathbb{N}$ with $k \neq n$. However, for any $n \in \mathbb{N}$ the random variables $\zeta^{(n,0)}$ and $\zeta^{(n,h)}$ are not independent. Making use of the independence, conditional on $\mathscr{F}_\tau$, of the sequence $(\zeta^{(k,0)})_{k \in \mathbb{N}}$ we obtain from (6.36) that

$$\begin{aligned}
&\mathbb{P}\left(\left\{\zeta^{(n,h)} = \infty\right\} \cap \bigcap_{k \in \mathcal{N}_\tau^x} \left\{\zeta^{(k,0)} < \infty\right\} \middle| \mathscr{F}_\tau\right) \\
&= \mathbb{P}\left(\left\{\zeta^{(n,h)} = \infty\right\} \cap \left\{\zeta^{(n,0)} < \infty\right\} \middle| \mathscr{F}_\tau\right) \prod_{k \in \mathcal{N}_\tau^x \setminus \{n\}} \mathbb{P}\left(\zeta^{(k,0)} < \infty \middle| \mathscr{F}_\tau\right) \\
&\leq \mathbb{P}\left(\left\{\zeta^{y+h} = \infty\right\} \cap \{\zeta^y < \infty\}\right)\Big|_{y = X_n^x(\tau)} \\
&= f(X_n^x(\tau)) - f(X_n^x(\tau) + h) \qquad (6.38)
\end{aligned}$$

$$\leq 1 - e^{-(1+p)h}$$

holds $\mathbb{P}$–a.s. for all $h > 0$ and $n \in \mathbb{N}$. Moreover, by $\sigma$–subadditivity we have

$$\begin{aligned}
&\mathbb{P}\left( \bigcup_{n \in \mathcal{N}_\tau^{x+h}} \left\{ \zeta^{(n,h)} = \infty \right\} \cap \bigcap_{k \in \mathcal{N}_\tau^x} \left\{ \zeta^{(k,0)} < \infty \right\} \middle| \mathscr{F}_\tau \right) \\
&= \mathbb{P}\left( \bigcup_{n \in \mathbb{N}} \left\{ n \in \mathcal{N}_\tau^{x+h} \right\} \cap \left\{ \zeta^{(n,h)} = \infty \right\} \cap \bigcap_{k \in \mathcal{N}_\tau^x} \left\{ \zeta^{(k,0)} < \infty \right\} \middle| \mathscr{F}_\tau \right) \\
&\leq \sum_{n \in \mathbb{N}} \mathbb{1}_{\{n \in \mathcal{N}_\tau^{x+h}\}} \mathbb{P}\left( \left\{ \zeta^{(n,h)} = \infty \right\} \cap \bigcap_{k \in \mathcal{N}_\tau^x} \left\{ \zeta^{(k,0)} < \infty \right\} \middle| \mathscr{F}_\tau \right) \qquad (6.39) \\
&= \sum_{n \in \mathcal{N}_\tau^{x+h}} \mathbb{P}\left( \left\{ \zeta^{(n,h)} = \infty \right\} \cap \bigcap_{k \in \mathcal{N}_\tau^x} \left\{ \zeta^{(k,0)} < \infty \right\} \middle| \mathscr{F}_\tau \right)
\end{aligned}$$

$\mathbb{P}$–a.s. for every $h > 0$. For any $h > 0$ define a random variable $\phi_{\geq}(h, \tau)$ by

$$\phi_{\geq}(h, \tau) := \frac{1}{h} \mathbb{1}_{\{\zeta^x < \infty\} \cap \left\{\zeta^{x+h} = \infty\right\}} \mathbb{1}_{\left\{\tau \geq \tau_{1,x}^- \wedge \tau(1-\varepsilon)\right\}}.$$

Consequently, in conjunction with (5.16) and the reasoning in (6.37), the estimates provided by (6.38) and (6.39) result in

$$\begin{aligned}
&\mathbb{E}\left(\phi_{\geq}(h, \tau)\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(\phi_{\geq}(h, \tau) \middle| \mathscr{F}_\tau\right)\right) \\
&= \mathbb{E}\left( \frac{1}{h} \mathbb{1}_{\left\{\tau \geq \tau_{1,x}^- \wedge \tau(1-\varepsilon)\right\}} \mathbb{P}\left( \bigcup_{n \in \mathcal{N}_\tau^{x+h}} \left\{ \zeta^{(n,h)} = \infty \right\} \cap \bigcap_{k \in \mathcal{N}_\tau^x} \left\{ \zeta^{(k,0)} < \infty \right\} \middle| \mathscr{F}_\tau \right) \right) \\
&\leq \mathbb{E}\left( \mathbb{1}_{\left\{\tau \geq \tau_{1,x}^- \wedge \tau(1-\varepsilon)\right\}} \sum_{n \in \mathcal{N}_\tau^{x+h}} \frac{1}{h} \mathbb{P}\left( \left\{ \zeta^{(n,h)} = \infty \right\} \cap \bigcap_{k \in \mathcal{N}_\tau^x} \left\{ \zeta^{(k,0)} < \infty \right\} \middle| \mathscr{F}_\tau \right) \right) \\
&\leq \mathbb{E}\left( \mathbb{1}_{\left\{\tau \geq \tau_{1,x}^- \wedge \tau(1-\varepsilon)\right\}} N_\tau^{x+h} \frac{1 - e^{-(1+p)h}}{h} \right) \\
&\leq e^{x+h+c\tau} \frac{1 - e^{-(1+p)h}}{h} \mathbb{P}\left( \tau \geq \tau_{1,x}^- \wedge \tau(1-\varepsilon) \right)
\end{aligned}$$

for all $h > 0$. Therefore, by L'Hôpital's rule we infer that

$$\begin{aligned}
\limsup_{h\downarrow 0} \mathbb{E}\left(\phi_{\geq}(h,\tau)\right) &\leq \limsup_{h\downarrow 0} \left( e^{x+h+c\tau} \frac{1-e^{-(1+p)h}}{h} \right) \mathbb{P}\left(\tau \geq \tau_{1,x}^{-} \wedge \tau(1-\varepsilon)\right) \\
&= (1+p) \lim_{h\downarrow 0} e^{-(1+p)h} e^{x+c\tau} \mathbb{P}\left(\tau \geq \tau_{1,x}^{-} \wedge \tau(1-\varepsilon)\right) \\
&\leq (1+p) e^{x+c} \mathbb{P}\left(\tau \geq \tau_{1,x}^{-} \wedge \tau(1-\varepsilon)\right). \qquad (6.40)
\end{aligned}$$

For the time being, we fix some $h \in (0,1]$ and for the remainder of the proof let $\varepsilon \in (0, e^{-(x+1+c)})$. Since $\tau \leq 1$ we have according to Lemma 5.18 that

$$N_{\tau}^{x+h} \leq 1 \qquad (6.41)$$

on $\{\tau < \tau(1-\varepsilon)\}$. Hence, (6.37) results in

$$\begin{aligned}
&\frac{f(x)-f(x+h)}{h} \\
&= \frac{1}{h}\mathbb{E}\left(\mathbb{P}\left(\left\{\zeta^{(1,0)} < \infty\right\} \cap \left\{\zeta^{(1,h)} = \infty\right\} \Big| \mathscr{F}_{\tau}\right) \mathbb{1}_{\{\tau < \tau_{1,x}^{-} \wedge \tau(1-\varepsilon)\}}\right) \qquad (6.42) \\
&\quad + \frac{1}{h}\mathbb{P}\left(\{\zeta^{x} < \infty\} \cap \left\{\zeta^{x+h} = \infty\right\} \cap \left\{\tau \geq \tau_{1,x}^{-} \wedge \tau(1-\varepsilon)\right\}\right)
\end{aligned}$$

$\mathbb{P}$–a.s., where, as above, conditional on $\mathscr{F}_{\tau}$ the $\zeta^{(1,r)}$, $r \in \{0,h\}$, are independent and satisfy

$$\mathbb{P}\left(\zeta^{(1,r)} \in \cdot \,\Big|\, \mathscr{F}_{\tau}\right) = \mathbb{P}\left(\zeta^{y+r} \in \cdot\right)$$

with $y = X_1^x(\tau)$. The reason for the appearance of $\tau_{1,x}^{-}$ in the indicator functions in (6.42) will become clear below. Of course, (6.42) would hold true also without the bound $\tau_{1,x}^{-}$, but below we are going to use this decomposition in the form given here.

Furthermore, observe that

$$\begin{aligned}
\mathbb{P}\left(\left\{\zeta^{(1,0)} < \infty\right\} \cap \left\{\zeta^{(1,h)} = \infty\right\} \Big| \mathscr{F}_{\tau}\right) &= \mathbb{P}\left(\{\zeta^{y} < \infty\} \cap \left\{\zeta^{y+h} = \infty\right\}\right)\Big|_{y=X_1^x(\tau)} \\
&= f(X_1^x(\tau)) - f(X_1^x(\tau)+h) \qquad (6.43)
\end{aligned}$$

$\mathbb{P}$–almost surely. In the light of the definition of $\phi_{\geq}(h,\tau)$ and (6.42) as well as (6.43) let us define for any $h > 0$ a random variable $\phi_{<}(h,\tau)$ by

$$\phi_{<}(h,\tau) := \frac{f(X_1^x(\tau)) - f(X_1^x(\tau)+h)}{h} \mathbb{1}_{\{\tau < \tau_{1,x}^{-} \wedge \tau(1-\varepsilon)\}}.$$

Resorting to Fatou's lemma we infer from (6.42) and (6.43) that

$$\begin{aligned}
-f'(x) &= \lim_{h\downarrow 0} \frac{f(x)-f(x+h)}{h} \\
&\geq \mathbb{E}\left( \liminf_{h\downarrow 0} \frac{f(X_1^x(\tau)) - f(X_1^x(\tau)+h)}{h} \mathbb{1}_{\{\tau<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}} \right) \\
&\quad + \liminf_{h\downarrow 0} \mathbb{E}\left(\phi_{\geq}(h,\tau)\right) \qquad (6.44) \\
&\geq \mathbb{E}\left( \liminf_{h\downarrow 0} \phi_{<}(h,\tau) \right)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
-\liminf_{h\downarrow 0} \phi_{<}(h,\tau) &= -\liminf_{h\downarrow 0} \frac{f(X_1^x(\tau)) - f(X_1^x(\tau)+h)}{h} \mathbb{1}_{\{\tau<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}} \\
&= \limsup_{h\downarrow 0} \frac{f(X_1^x(\tau)+h) - f(X_1^x(\tau))}{h} \mathbb{1}_{\{\tau<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}} \qquad (6.45) \\
&= f'_+(X_1^x(\tau)) \mathbb{1}_{\{\tau<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}
\end{aligned}$$

and analogously

$$\limsup_{h\downarrow 0} \phi_{<}(h,\tau) = -f'_-(X_1^x(\tau)) \mathbb{1}_{\{\tau<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}. \qquad (6.46)$$

Using (6.45) we deduce from (6.44) that

$$f'(x) \leq \mathbb{E}\left( -\liminf_{h\downarrow 0} \phi_{<}(h,\tau) \right) = \mathbb{E}\left( f'_+(X_1^x(\tau)) \right). \qquad (6.47)$$

Notice that (6.36) yields that

$$\sup_{h\in\mathbb{R}^+} \frac{f(z)-f(z+h)}{h} \leq \sup_{h\in\mathbb{R}^+} \frac{1-e^{-(1+p)h}}{h} < \infty \qquad (6.48)$$

as well as

$$\begin{aligned}
\left|f'_+(z)\right| &= \limsup_{h\downarrow 0} \frac{f(z)-f(z+h)}{h} \\
&\leq \lim_{h\downarrow 0} \frac{1-e^{-(1+p)h}}{h} \\
&= \lim_{h\downarrow 0} \left( (1+p)e^{-(1+p)h} \right) \qquad (6.49) \\
&= 1+p
\end{aligned}$$

for any $z \in \mathbb{R}_0^+$. Therefore, we can apply Fatou's lemma for limit superiors in order to obtain by means of (6.40) and (6.42) as well as (6.43) that

$$\begin{aligned} -f'(x) &= \lim_{h\downarrow 0} \frac{f(x)-f(x+h)}{h} \\ &\leq \mathbb{E}\left(\limsup_{h\downarrow 0} \frac{f(X_1^x(\tau)) - f(X_1^x(\tau)+h)}{h} \mathbb{1}_{\{\tau<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}\right) \\ &\qquad + \limsup_{h\downarrow 0}\left(\frac{1}{h}\mathbb{P}\left(\{\zeta^x<\infty\}\cap\left\{\zeta^{x+h}=\infty\right\}\cap\left\{\tau\geq\tau_{1,x}^-\wedge\tau(1-\varepsilon)\right\}\right)\right) \\ &= \mathbb{E}\left(\limsup_{h\downarrow 0}\phi_<(h,\tau)\right) + \limsup_{h\downarrow 0}\mathbb{E}\left(\phi_\geq(h,\tau)\right) \end{aligned} \tag{6.50}$$

Recall the definitions of $\mathcal{T}_f^x$ and $\mathcal{T}_Z^{x,f}$ in (6.11) and (6.25) respectively. Let $(a_n)_{n\in\mathbb{N}}$ be a random $(0,1]$–valued sequence with $a_n \downarrow 0$ as $n\to\infty$ and such that $x+a_n \in \mathcal{C}_f$ as well as

$$\tau_{1,a_n}^+ \in \mathcal{T}_f^x \cap \mathcal{T}_Z^{x,f}$$

for any $n\in\mathbb{N}$. Note that such a sequence exists, since the complement of $\mathcal{T}_f^x \cap \mathcal{T}_Z^{x,f}$ is a Lebesgue null set, see (6.25) and Remark 6.8, and $\mathcal{T}_f^x \cap \mathcal{T}_Z^{x,f}$ is therefore dense in $\mathbb{R}_0^+$. The requirement $\tau_{1,a_n}^+ \in \mathcal{T}_Z^{x,f}$ on the event $\{\tau_{1,a_n}^+ \leq 1\}$ will be needed in Part II of this proof. Furthermore, define a positive $\mathscr{F}$–stopping time $\tau_N : \Omega \to \mathbb{R}^+$ by

$$\tau_N := \tau_{1,a_n}^+ \mathbb{1}_{\{\tau_{1,a_n}^+\leq 1\}} + \tau(1-\varepsilon)\mathbb{1}_{\{\tau_{1,a_n}^+>1\}}\Big|_{n=N} \tag{6.51}$$

for all $N:\Omega\to\mathbb{N}$. Because $X_1^x$ cannot jump upwards, it is necessarily creeping over the level $x+a_n$, if it ever hits the interval $[x+a_n,\infty)$, and thus we have

$$X_1^x(\tau_n) = x + a_n \in \mathcal{C}_f \tag{6.52}$$

$\mathbb{P}$–a.s. on $\{\tau_n < \tau_{1,x}^-\}$.

In view of (6.40), (6.46), (6.47), (6.49), (6.50) and (6.52) we conclude that

$$\begin{aligned} &\left|\mathbb{E}\left(f'_+(X_1^x(\tau_n))\right) - f'(x)\right| \\ &\leq \left|\mathbb{E}\left(f'_+(X_1^x(\tau_n))\mathbb{1}_{\{\tau_n<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}} + \limsup_{h\downarrow 0}\phi_<(h,\tau_n)\right)\right| \\ &\qquad + \left|\mathbb{E}\left(f'_+(X_1^x(\tau_n))\mathbb{1}_{\{\tau_n\geq\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}\right) + \limsup_{h\downarrow 0}\mathbb{E}\left(\phi_\geq(h,\tau_n)\right)\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \left|\mathbb{E}\left(f'_+(X_1^x(\tau_n))\mathbb{1}_{\{\tau_n<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}} - f'_-(X_1^x(\tau_n))\mathbb{1}_{\{\tau_n<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}\right)\right| \\
&\qquad \mathbb{E}\left(\left|f'_+(X_1^x(\tau_n))\right|\mathbb{1}_{\{\tau_n\geq\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}\right) + \limsup_{h\downarrow 0}\mathbb{E}\left(\phi_{\geq}(h,\tau_n)\right) \\
&\leq \mathbb{E}\left(\left|f'_+(X_1^x(\tau_n)) - f'_-(X_1^x(\tau_n))\right|\mathbb{1}_{\{\tau_n<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}\right) \qquad (6.53) \\
&\qquad + 2(1+p)e^{x+c}\mathbb{P}\left(\tau_n \geq \tau_{1,x}^-\wedge\tau(1-\varepsilon)\right) \\
&= 2(1+p)e^{x+c}\mathbb{P}\left(\tau_n\geq\tau_x^-\wedge\tau(1-\varepsilon)\right)
\end{aligned}
$$

holds for all $n\in\mathbb{N}$. Note that for the final equality above we have used (6.52) as well as

$$
f'_+(X_1^x(\tau_n)) = f'(X_1^x(\tau_n)) = f'_-(X_1^x(\tau_n)) \tag{6.54}
$$

on $\{\tau_n<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}$.

Theorem 6.5 in [Kyp06] shows that for $X_1$ the point 0 is regular for $(0,\infty)$, that is $\inf\{t\in\mathbb{R}_0^+ : X_1(t)\in(0,\infty)\} = 0$ $\mathbb{P}$–almost surely. Hence, it is immediately clear that $\tau_{1,a_n}^+\downarrow 0$, and thus $\tau_n\downarrow 0$, $\mathbb{P}$–a.s. as $n\to\infty$. Let us further point out that by means of the right–continuity of $X_1$ we have that $\tau_{1,x}^->0$ $\mathbb{P}$–almost surely. Moreover, Remark 5.17 shows that $\tau(1-\varepsilon)$ is exponentially distributed, and therefore $\tau(1-\varepsilon)>0$ $\mathbb{P}$–almost surely. Consequently, we deduce from (6.53) and the DCT that

$$
\begin{aligned}
\lim_{n\to\infty}\left|-f'(x)+\mathbb{E}\left(f'_+(X_1^x(\tau_n))\right)\right| &\leq 2(1+p)e^{x+c}\lim_{n\to\infty}\mathbb{P}\left(\tau_n\geq\tau_x^-\wedge\tau(1-\varepsilon)\right) \\
&= 0 \qquad (6.55)
\end{aligned}
$$

for every $x\in\mathcal{C}_f$.

<u>Part II</u> Let $g_1$ be some function that is almost everywhere, with respect to the Lebesgue measure, differentiable and whose derivative is bounded. In addition, let $G_2$ be the antiderivative of some bounded function $g_2$. Then

$$
g_1' - g_2 \equiv 0 \tag{6.56}
$$

if $\int_{[0,t]}(g_1'-g_2)(s)\ \mathrm{d}s = 0$ for all $t\in\mathbb{R}_0^+$. Indeed, let $H=g_1-G_2$ and set $h:=H'$, where $H'$ denotes the derivative of $H$. Since $g_1'$ and $g_2$ are bounded, the assumption

$$
\int_{[0,t]} h(s)\ \mathrm{d}s = \int_{[0,t]}(g_1'-g_2)(s)\ \mathrm{d}s = 0
$$

for all $t \in \mathbb{R}_0^+$ does then yield by means of (6.24) that $H \equiv H(0)$, i.e. $H$ is constant. Consequently,

$$g_1' - g_2 = h = H' \equiv 0,$$

which proves (6.56).

Let $x > 0$ and consider a function $\kappa$ defined on the space of bounded positive $\mathscr{F}$–stopping times by

$$\kappa(\tau) = \mathbb{E}\left(Z_\tau^{x,f} \mathbb{1}_{\{\tau \in \mathcal{T}_Z^{x,f}\}}\right)$$

for every bounded positive $\mathscr{F}$–stopping time $\tau$. Further, denote the derivative of $\kappa$ by $\kappa'$ and recall from (6.25) and (6.26) the notations $\mathcal{T}_Z^{x,f}$ as well as

$$\dot{Z}_t^{x,f} = \frac{\mathrm{d}}{\mathrm{d}s} Z_s^{x,f}\bigg|_{s=t}$$

for any $t \in \mathcal{T}_Z^{x,f}$.

Observe that by the DCT and (5.16) as well as (6.36) we have

$$\begin{aligned}
\sup_{t\in[0,1]} \left|\dot{Z}_t^{x,f} \mathbb{1}_{\{t\in\mathcal{T}_Z^{x,f}\}}\right| &\leq \sup_{t\in[0,1]} \sum_{n\in\mathcal{N}_t^x} \left|cf'_-(X_n^x(t))\right| \prod_{k\in\mathcal{N}_t^x\setminus\{n\}} f(X_k^x(t)) \\
&\leq c(1+p) \sup_{t\in[0,1]} e^{x+ct} \qquad (6.57)\\
&\leq c(1+p)e^{x+c},
\end{aligned}$$

where we have used that $f \leq 1$. In view of (6.28) and (6.31) it thus follows from the martingale property of $Z^{x,f}$, Fubini's theorem and the DCT that

$$\begin{aligned}
0 &= \mathbb{E}(Z_t^{x,f}) - \mathbb{E}(Z_0^{x,f}) \\
&= \int_{[0,t]} \mathbb{E}\left(\dot{Z}_u^{x,f} \mathbb{1}_{\{u\in\mathcal{T}_Z^{x,f}\}} + \sum_{n\in\mathcal{N}_u^x} \prod_{k\in\mathcal{N}_u^x\setminus\{n\}} f(X_k^x(u)) Lf(X_n^x((u)-))\right) \mathrm{d}u \\
&= \int_{[0,t]} \kappa'(u)\, \mathrm{d}u \qquad (6.58)\\
&\quad - \int_{[0,t]} \mathbb{E}\left(-\sum_{n\in\mathcal{N}_u^x} \prod_{k\in\mathcal{N}_u^x\setminus\{n\}} f(X_k^x((\tau\wedge t-u))) Lf(X_n^x((u)-))\right) \mathrm{d}u
\end{aligned}$$

holds for all $t \in [0,1]$.

We aim at applying (6.56) to (6.58). For this purpose we need to show that the two integrands on the right–hand side in (6.58) are bounded. To this end, observe first that by the DCT and (6.57) we have

$$\sup_{t\in[0,1]} \left|\kappa'(t)\right| \leq \mathbb{E}\left( \sup_{t\in[0,1]} \left| \dot{Z}_t^{x,f} \mathbb{1}_{\{t\in\mathcal{T}_Z^{x,f}\}} \right| \right) < \infty. \tag{6.59}$$

Moreover, Lemma 6.15 in conjunction with (5.16) yields that

$$\begin{aligned}
&\sup_{t\in[0,1]} \left| \mathbb{E}\left( \sum_{n\in\mathcal{N}_t^x} \prod_{k\in\mathcal{N}_t^x\setminus\{n\}} f(X_k^x((t)))Lf(X_n^x((t)-)) \right) \right| \\
&\leq \sup_{t\in[0,1]} \mathbb{E}\left( N_t^x \sup_{y\in[0,x+c(t)]} Lf(y)^{N_t^x-1} \right) \\
&\leq e^{x+c} \sup_{y\in[0,x+c]} Lf(y)^{e^{x+c}} \vee 1 \\
&< \infty.
\end{aligned}$$

Hence, we deduce from (6.56) and (6.58) that

$$\mathbb{E}\left( \dot{Z}_t^{x,f} \mathbb{1}_{\{t\in\mathcal{T}_Z^{x,f}\}} \right) = \kappa' t = -\mathbb{E}\left( \sum_{n\in\mathcal{N}_t^x} \prod_{k\in\mathcal{N}_t^x\setminus\{n\}} f(X_k^x(t))Lf(X_n^x(t-)) \right) \tag{6.60}$$

for all $t \in [0,1]$. As the complement of $\mathcal{T}_Z^{x,f}$ is $\mathbb{P}$–a.s. a Lebesgue null set, we infer in view of [Kyp06, Corollary 8.8] that there exists some deterministic Lebesgue null set $\mathcal{T}_0 \subseteq [0,1]$ such that $\mathbb{P}(t \in \mathcal{T}_Z^{x,f}) = 1$ for every $t \in [0,1] \setminus \mathcal{T}_0$. Therefore, since $(Z_t^{x,f})_{t\in[0,1]}$ is a uniformly integrable martingale, (6.57) yields that also $(\dot{Z}_t^{x,f} \mathbb{1}_{\{t\in\mathcal{T}_Z^{x,f}\}})_{t\in[0,1]\setminus\mathcal{T}_0}$ is a martingale. Hence, (6.60) shows that on $[0,1] \setminus \mathcal{T}_0$ the map $t \mapsto \sum_{n\in\mathcal{N}_t^x} \prod_{k\in\mathcal{N}_t^x\setminus\{n\}} f(X_k^x(t))Lf(X_n^x(t-))$ has constant expectation. Moreover, since $(\sum_{n\in\mathcal{N}_t^x} \prod_{k\in\mathcal{N}_t^x\setminus\{n\}} f(X_k^x(t))Lf(X_n^x(t-)))_{t\in[0,1]\setminus\mathcal{T}_0}$ has independent increments, this process thus is a martingale. Consequently, it follows from (6.60) in conjunction with the optional sampling theorem that

$$\mathbb{E}\left( \dot{Z}_\tau^{x,f} \right) = -\mathbb{E}\left( \sum_{n\in\mathcal{N}_\tau^x} \prod_{k\in\mathcal{N}_\tau^x\setminus\{n\}} f(X_k^x(\tau))Lf(X_n^x(\tau-)) \right) \tag{6.61}$$

for any $\mathscr{F}$–stopping time $\tau : \Omega \to (0,1] \cap \mathcal{T}_Z^{x,f} \setminus \mathcal{T}_0$.

Consider a sequence $(\tau_n)_{n\in\mathbb{N}}$ of positive $\mathscr{F}$–stopping times defined as in (6.51) with the additional assumption on the sequence $(a_n)_{n\in\mathbb{N}}$ that

$$\tau_{1,a_n}^{+}\mathbb{1}_{\{\tau_{1,a_n}^{+}\leq 1\}} \in \mathcal{T}_f^x \cap \mathcal{T}_Z^{x,f} \setminus \mathcal{T}_0,$$

which is possible because $\mathcal{T}_0$ is a Lebesgue null set. In addition, let $\epsilon > 0$ as well as $\varepsilon \in (0, e^{-(x+c)})$. Since, according to Lemma 6.16, $Lf$ is a continuous function and because $X_n^x(t) \to x$ as $t \to 0$, it follows from (5.16) and $\tau_n \downarrow 0$ as $n \to \infty$ that there exists some random variable $N : \Omega \to \mathbb{N}$ such that

$$Lf(x) \geq \mathbb{E}\left(\sum_{i\in\mathcal{N}_{\tau_{n\wedge N}}^x}\prod_{j\in\mathcal{N}_{\tau_{n\wedge N}}^x\setminus\{i\}} f(X_j^x(\tau_{n\wedge N}))Lf(X_i^x(\tau_{n\wedge N}-))\right) - \epsilon \tag{6.62}$$

and

$$Lf(x) \leq \mathbb{E}\left(\sum_{i\in\mathcal{N}_{\tau_{n\wedge N}}^x}\prod_{j\in\mathcal{N}_{\tau_{n\wedge N}}^x\setminus\{i\}} f(X_j^x(\tau_{n\wedge N}))Lf(X_i^x(\tau_{n\wedge N}-))\right) + \epsilon \tag{6.63}$$

for all $n \in \mathbb{N}$. Let $N : \Omega \to \mathbb{N}$ be a random variable satisfying (6.62) as well as (6.63). Then (6.61) and (6.62) imply that

$$\begin{aligned}
Lf(x) &\geq \mathbb{E}\left(\sum_{i\in\mathcal{N}_{\tau_{n\wedge N}}^x}\prod_{j\in\mathcal{N}_{\tau_{n\wedge N}}^x\setminus\{i\}} f(X_j^x(\tau_{n\wedge N}))Lf(X_i^x(\tau_{n\wedge N}-))\right) - \epsilon \\
&= -\mathbb{E}\left(\dot{Z}_{\tau_{n\wedge N}}^{x,f}\right) - \epsilon \qquad (6.64)\\
&\geq -\mathbb{E}\left(\sum_{i\in\mathcal{N}_{\tau_{n\wedge N}}^x} cf'_+(X_i^x(\tau_{n\wedge N}))\prod_{j\in\mathcal{N}_{\tau_{n\wedge N}}^x\setminus\{i\}} f(X_j^x(\tau_{n\wedge N}))\right) - \epsilon \\
&\geq -\mathbb{E}\left(cf'(X_1^x(\tau_{n\wedge N}))\mathbb{1}_{\{\tau_{n\wedge N}<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}\right) - \epsilon,
\end{aligned}$$

and similarly, resorting to (6.63) instead of (6.62), we also obtain

$$\begin{aligned}
Lf(x) &\leq \mathbb{E}\left(\sum_{i\in\mathcal{N}_{\tau_{n\wedge N}}^x}\prod_{j\in\mathcal{N}_{\tau_{n\wedge N}}^x\setminus\{i\}} f(X_j^x(\tau_{n\wedge N}))Lf(X_i^x(\tau_{n\wedge N}-))\right) + \epsilon \\
&\leq -\mathbb{E}\left(cf'_-(X_1^x(\tau_{n\wedge N}))\mathbb{1}_{\{\tau_{n\wedge N}<\tau_{1,x}^-\wedge\tau(1-\varepsilon)\}}\right) \qquad (6.65)\\
&\qquad + c(1+p)e^{x+c}\mathbb{P}\left(\tau_{n\wedge N} \geq \tau_{1,x}^- \wedge \tau(1-\varepsilon)\right) + \epsilon,
\end{aligned}$$

where the last equality results from (6.41) and (6.59). It now follows from (6.64) as well as (6.65) that

$$
\begin{aligned}
&\left|Lf(x) + \mathbb{E}\left(cf'_+(X_1^x(\tau_{n\wedge N}))\right) + \epsilon\right| \\
&\leq \left|Lf(x) + \mathbb{E}\left(cf'_+(X_1^x(\tau_{n\wedge N}))\mathbb{1}_{\{\tau_{n\wedge N}<\tau_{1,x}^-\wedge\rho_\epsilon\wedge\tau(1-\varepsilon)\}}\right) + \epsilon\right| \\
&\qquad - \mathbb{E}\left(cf'_+(X_1^x(\tau_{n\wedge N}))\mathbb{1}_{\{\tau_{n\wedge N}\geq\tau_{1,x}^-\wedge\rho_\epsilon\wedge\tau(1-\varepsilon)\}}\right) \\
&\leq \mathbb{E}\left(\left|cf'_+(X_1^x(\tau_{n\wedge N})) - cf'_-(X_1^x(\tau_{n\wedge N}))\right|\mathbb{1}_{\{\tau_{n\wedge N}<\tau_{1,x}^-\wedge\rho_\epsilon\wedge\tau(1-\varepsilon)\}}\right) \\
&\qquad + 2c(1+p)e^{x+c}\mathbb{P}\left(\tau_{n\wedge N}\geq\tau_{1,x}^-\wedge\tau(1-\varepsilon)\right) + 2\epsilon \\
&= 2c(1+p)e^{x+c}\mathbb{P}\left(\tau_{n\wedge N}\geq\tau_{1,x}^-\wedge\tau(1-\varepsilon)\right) + 2\epsilon,
\end{aligned}
$$

where we have used (6.54) as well as $\sup_{y\in\mathbb{R}_0^+}|f'_+(y)| \leq 1+p \leq (1+p)e^{x+c}$, see (6.48). Similarly to the reasoning preceding (6.55) we thus conclude that

$$\lim_{n\to\infty}\left|-\mathbb{E}\left(cf'_+(X_1^x(\tau_n))\right) - \epsilon - Lf(x)\right| \leq 2\epsilon. \tag{6.66}$$

<u>Part III</u> For the time being, let $x \in \mathcal{C}_f$ and let $(\tau_n)_{n\in\mathbb{N}}$ be given by (6.51). In view of (6.55) and (6.66) an application of the triangle inequality results in

$$
\begin{aligned}
&\left|-cf'(x) - Lf(x)\right| \\
&\leq c\lim_{n\to\infty}\left|-f'(x) + \mathbb{E}\left(f'_+(X_1^x(\tau_n))\right)\right| + \epsilon + \lim_{n\to\infty}\left|-\mathbb{E}\left(cf'_+(X_1^x(\tau_n))\right) - \epsilon - Lf(x)\right| \\
&\leq 3\epsilon
\end{aligned}
$$

for any $\epsilon > 0$. Letting $\epsilon \to 0$ this shows that

$$-cf' = Lf \text{ on } \mathcal{C}_f. \tag{6.67}$$

Since $Lf$ is continuous, cf. Lemma 6.16, this implies that

$$\lim_{t\downarrow 0} f'(x+t) = \lim_{t\uparrow 0} f'(x+t)$$

for all $x \in \mathbb{R}^+$, where the limits are taken over $t \in \mathbb{R}$ that satisfy $x+t \in \mathcal{C}_f$, and thus the right–derivative and left–derivative coincide for every $x \in \mathbb{R}^+$. Hence, $f'$ exits on $\mathbb{R}^+$ and is continuous, that is $f|_{\mathbb{R}_0^+} \in C^1(\mathbb{R}_0^+, [0,1])$. Consequently, $f'_+ = f'_- = f'$, and thus we deduce from (6.67) that $f$ solves (6.8). □

## 6.6 Existence and uniqueness of one–sided FKPP travelling waves

This section is devoted to the proof of Theorem 6.11. We break this proof down into two sub–results dealing with the cases $c \le c_{\bar{p}}$ and $c > c_{\bar{p}}$ separately.

**Lemma 6.17** *Let $c \le c_{\bar{p}}$. Then there does not exist a monotone travelling wave to (6.7) with wave speed $c$.*

**Proof** Let $x \in \mathbb{R}_0^+$ and let $f \in \mathcal{D}_L$ be a monotone function satisfying (6.8). Then Proposition 6.9 yields that $(Z_t^{x,f})_{t\in\mathbb{R}_0^+}$ is a uniformly integrable martingale and hence the martingale limit $Z_\infty^{x,f} := \lim_{t\to\infty} Z_t^{x,f}$ satisfies

$$\mathbb{E}\left(Z_\infty^{x,f}\right) = \mathbb{E}\left(Z_0^{x,f}\right) = f(x). \tag{6.68}$$

Since $c \ge c_{\bar{p}}$, we have according to Lemma 5.16 that $\mathbb{P}(\zeta^x < \infty) = 1$. As the empty product equals 1, we thus infer that

$$Z_\infty^{x,f} = \lim_{t\to\infty} \prod_{n\in\mathcal{N}_t^x} f(X_n^x(t)) = 1$$

$\mathbb{P}$–almost surely. In view of (6.68) this implies that $f \equiv 1$. Consequently, there does not exist a monotone function $f \in \mathcal{D}_L$ that satisfies (6.8) and (6.9). □

**Proposition 6.18** *Let $c > c_{\bar{p}}$. Then $\varphi$ is the unique monotone travelling wave to (6.7) with wave speed $c$.*

**Proof** The outline of the proof is as follows. We first show in Part I of the proof that any travelling wave $f$ to (6.7) with wave speed $c$ must satisfy $f = \varphi$, that is if there exists any one–sided FKPP travelling wave then it is necessarily unique. In the second part of the proof we then show that $\varphi$ does indeed satisfy (6.8) and (6.9).

Part I Assume that $f \in \mathcal{D}_L$ is a monotone function that solves (6.8) and (6.9). Then Proposition 6.9 shows that $(Z_t^{x,f})_{t\in\mathbb{R}_0^+}$ is a nonnegative martingale, and thus we infer from Proposition 5.23 that $f = \varphi$, which proves the uniqueness part.

Part II Our goal is to apply Theorem 6.10. To this end we need to establish that

$$\varphi \in \mathcal{D}(p) \tag{6.69}$$

for some $p > -1$. For this purpose, observe first that $\varphi$ is monotone and that Theorem 5.12 yields that $\varphi$ satisfies (6.9). Further, according to Theorem 5.11 we have that $\varphi|_{\mathbb{R}_0^+}$ is continuous. Hence, in order to prove (6.69) it only remains to show that the map $x \mapsto e^{(1+p)x}\varphi(x)$ is nondecreasing for some $p > -1$. To this end, fix some $p > \underline{p}$ with $\Phi'(p) < c$. For the time being, let $x \in \mathbb{R}_0^+$. Since $M_\infty^x(p)$ is nonnegative, we deduce by resorting to the DCT that

$$\mathbb{P}(M_\infty^x(p) = 0) = \mathbb{E}\left(\lim_{y\to\infty} e^{-zyM_\infty^x(p)}\right) = \lim_{y\to\infty} \mathbb{E}\left(e^{-zyM_\infty^x(p)}\right) = \lim_{y\to\infty} \mathcal{L}_y(z),$$

holds for all $z > 0$, where $\mathcal{L}_y(\cdot) := \mathbb{E}(e^{-(\cdot)yM_\infty^x(p)})$ is the Laplace transform of $yM_\infty^x(p)$. With $z := e^{-(1+p)x}$ we thus infer that

$$e^{(1+p)x}\mathbb{P}(M_\infty^x(p) = 0) = \lim_{y\to\infty} \frac{\mathcal{L}_y(z)}{z}. \tag{6.70}$$

According to [Fel71, (2.6) in XIII.2] we have that $z \mapsto {}^{\mathcal{L}_y(z)}/_{z}$, $y > 0$, is the Laplace transform of the cumulative distribution function of $yM_\infty^x(p)$. Hence, $z \mapsto {}^{\mathcal{L}_y(z)}/_{z}$ is monotonically decreasing for every $y > 0$, and thus also the map $z \mapsto \lim_{y\to\infty} {}^{\mathcal{L}_y(z)}/_{z}$ is monotonically decreasing. Consequently, in view of (5.7) and (6.70) this shows that the mapping

$$x \mapsto e^{(1+p)x}\varphi(x) = e^{(1+p)x}\mathbb{P}(\zeta^x < \infty) = e^{(1+p)x}\mathbb{P}(M_\infty^x(p) = 0)$$

is monotonically increasing on $\mathbb{R}_0^+$, and thus (6.69) holds true for any $p > \underline{p}$ with $\Phi'(p) < c$. Furthermore, the fragmentation property yields that

$$\varphi(x) = \mathbb{E}(\mathbb{P}(\zeta^x < \infty|\mathscr{F}_t)) = \mathbb{E}\left(\prod_{n\in\mathcal{N}_t^x} \varphi(X_n^x(t))\right) = \mathbb{E}(Z_t^{x,\varphi}),$$

which by means of Lemma 5.29 shows that $Z^{x,\varphi}$ is a martingale. In view of (6.69) we deduce from Theorem 6.10 that $\varphi$ solves the integro–differential equation (6.8). This proves the existence part, since $\varphi$ satisfies (6.9). □

Theorem 6.11 can now easily be obtained by combining the previous two results.

**Proof of Theorem 6.11** Lemma 6.17 proves the nonexistence of FKPP travelling waves for speeds $c \leq c_{\bar{p}}$ and Proposition 6.18 proves the existence and uniqueness of such travelling waves for wave speeds above the critical value $c_{\bar{p}}$. □

## 6.7 Concluding remarks

In the literature the FKPP equation is usually considered in the classical setting, that is of the form (6.2) for the two–sided case and (6.5) for the one–sided situation. Although looking different from the partial differential equation in (6.2), the FKPP equation for fragmentation processes as in (6.3) is not very surprising bearing in mind the interpretation of that partial differential equation as well as older work on continuous–time branching random walks as in [Kyp99]. For fragmentation processes the FKPP equation was introduced in [BHK10] and there the existence and uniqueness of two–sided solutions were studied. Similarly to the classical case, the one–sided case needs to be treated differently from the two–sided one and also the resulting travelling waves are of a different form. As in the two–sided situation it turns out that the form of the one–sided travelling wave solutions for fragmentation processes are of a similar form to those for the one–sided classical FKPP equation. Making use of killed fragmentation processes to study the one–sided FKPP travelling wave equation for fragmentation processes is motivated by the use of killed dyadic branching Brownian motion to deal with one–sided solutions in the classical situation, cf. [HHK06]. The method of using killed fragmentation processes is not straightforward as they have different properties compared to branching Brownian motions. In particular the lack of a positive time between jumps needs to be taken care of. The results on killed fragmentation processes needed to obtain results regarding the one–sided FKPP travelling wave equation are provided by Chapter 5.

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