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ASYMPTOTIC PROPERTIES OF GAUSSIAN PROCESSES

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We study separable mean zero Gaussian processes $X(t)$ with correlation $\rho(t, s)$ for which $1 - \rho(t, s)$ is asymptotic to a regularly varying (at zero) function of $|t - s|$ with exponent $0 < \alpha \leqq 2$. For such processes, we obtain the asymptotic distribution of the maximum of $X(t)$. This result is used to obtain a result for $X(t)$ as $t \to \infty$ similar to the so-called law of the iterated logarithm.

0. Introduction. Let $\{X(t), -\infty < t < \infty\}$ be a real separable Gaussian process defined on a probability space $(\Omega, \mathscr{A}, P)$. We assume $EX(t) \equiv 0$, $v^2(t) = EX^2(t) > 0$, and the covariance function $r(t, s) = E(X(t)X(s))$ is continuous with respect to t and s. And we set $\rho(t, s) = r(t, s)/(v(t)v(s))$. In this paper, we are concerned with Gaussian processes whose correlation functions satisfy

$$(0.1) \quad \rho(t, s) = 1 - |t - s|^\alpha H(|t - s|) + o(|t - s|^\alpha H(|t - s|)) \quad \text{as} \quad |t - s| \to 0 .$$

where $0 < \alpha \leqq 2$ and H varies slowly at zero. The existence of such correlation functions has been established by Pitman [9].

In Section 2, we extend a result of Pickands [7], which gives the asymptotic distribution of the maximum $Z(t) = \max_{0 \leqq s \leqq t} X(s)$, to condition (0.1) with $0 < \alpha \leqq 2$. Pickands treated stationary Gaussian processes whose covariance functions $\rho(|t - s|) \equiv \rho(t, s)$ satisfy condition (0.1) with $0 < \alpha \leqq 2$ for $H(|t - s) \equiv a$ constant. Such studies have been done for Hölder continuity of sample functions by Marcus [6], Kôno [5], and Sirao and Watanabe [11]. Our efforts using Pickands' methods to give the asymptotic distribution of $Z(t)$ for the case $\alpha = 0$ were not successful. A technical error in [7] (Lemma 2.8 is not true) is corrected herein.

In Section 3, we use the result of Section 2 to obtain the extension of the 0-1 behavior treated in Watanabe [13] and Qualls and Watanabe [10] to our

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present case. The method in [10] turns out to be powerful, while the method in [13] does not seem to be successful in our case. Section 4 treats the non-stationary case; in Section 2 and Section 3, we assume stationarity. In Section 5, we give the limiting distribution of the number of "ε-upcrossings" and that of the extreme value $Z(t)$ as $t \to \infty$.

1. Preliminaries. We list some definitions and properties of regularly varying functions that will be required in the following sections. Some general references on regular variation are Karamata [4], Adamovic [1], and Feller [3].

DEFINITION 1.1. A positive function $H(x)$ defined for $x > 0$ varies slowly at zero, if for all $t > 0$,

$$\lim_{x\to 0} \frac{H(tx)}{H(x)} = 1 . \tag{1.1}$$

DEFINITION 1.2. A positive function $Q(x)$ defined for $x > 0$ varies regularly at zero with exponent $\alpha \geqq 0$, if for all $t > 0$,

$$\lim_{x\to 0} \frac{Q(tx)}{Q(x)} = t^\alpha . \tag{1.2}$$

A function $Q(x)$ satisfies (1.2) if and only if $Q(x) = x^\alpha H(x)$, where $H(x)$ varies slowly.

Let $Q(x)$ vary regularly with exponent $\alpha \geqq 0$ and $H(x)$ vary slowly at zero. Then, the following properties hold.

(1.3) The limits (1.1) and (1.2) converge uniformly in t on any compact subsect of the half line $(0, \infty)$.

(1.4) For any $\varepsilon > 0$, we have that

$$\lim_{x\to 0} x^{-\varepsilon} H(x) = \infty \quad \text{and} \quad \lim_{x\to 0} x^{\varepsilon} H(x) = 0 .$$

(1.5) The function $H(x)$ varies slowly at zero if and only if

$$H(x) = a(x) \exp\{\textstyle\int_x^1 \varepsilon(t)/t\,dt\} ,$$

where $\varepsilon(x) \to 0$ and $a(x) \to A$ as $x \to 0$ $(0 < A < \infty)$.

DEFINITION 1.3. The slowly varying function $H(x)$ is said to be "normalized" if $a(x) \equiv A$ in property (1.5) above.

(1.6) If $H(x)$ is a "normalized" slowly varying function at zero, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$t^\varepsilon \leqq \frac{H(tx)}{H(x)} \leqq t^{-\varepsilon}$$

for all positive $t < 1$ and all $x > 0$ such that $tx < \delta$.

(1.7) If $H(x)$ is a "normalized" slowly varing function at zero, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$t^{-\varepsilon} \leqq \frac{H(tx)}{H(x)} \leqq t^{\varepsilon}$$

for all $x > 0$ and all $t > 1$ such that $tx < \delta$.

(1.8) If $H(x)$ is a "normalized" slowly varying function at zero then for any $\alpha > 0$ the function $x^{\alpha}H(x)$ is monotone increasing near zero.

2. The asymptotic distribution of the maximum. For the study of the asymptotic distribution of $Z(t) = \sup_{0 \leqq s \leqq t} X(s)$ in this section, we assume that the process $X(t)$ is stationary in addition to its covariance function $\rho(s) \equiv \rho(t, t+s)$ satisfying condition (0.1) with $0 < \alpha \leqq 2$. We are assuming each $X(t)$ has mean 0 and variance 1. The non-stationary case is discussed in Section 4. Without loss of generality, we also assume the slowly varying function $H(s)$ in condition (0.1) is "normalized". See Section 1 for definitions. Let $\sigma^2(s) \equiv E\{X(t+s) - X(t)\}^2 = 2(1-\rho(s))$, define $\tilde{\sigma}^2(s) = 2|s|^{\alpha}H(s)$, $A_1(t) = \inf_{0<s\leqq t} \sigma(s)/\tilde{\sigma}(s)$ and $A_2(t) = \sup_{0<s\leqq t} \sigma(s)/\tilde{\sigma}(s)$.

The theorem of this section is an extension of Pickands' result [7]. Since Pickands' methods apply in our case, we will only sketch the proof emphasizing the points of difference.

THEOREM 2.1. *If condition* (0.1) *with* $0 < \alpha \leqq 2$ *holds*, $\sigma^2(s) > 0$ *for* $s \neq 0$, *and* $\tilde{\sigma}(\cdot)$ *is defined as above, then*

$$\text{(2.1)} \quad \lim_{x\to\infty} \frac{P[Z(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} = H_{\alpha} \equiv \lim_{T\to\infty} T^{-1} \textstyle\int_0^{\infty} e^s P[\sup_{0<t<T} Y(t) > s]ds\,,$$

and $0 < H_{\alpha} < \infty$, *where* $\{Y(t), 0 \leqq t < \infty\}$ *is a non-stationary Gaussian process with* $Y(0) = 0$ a.s., $E\{Y(t)\} = -|t|^{\alpha}/2$, $\mathrm{Cov}\{Y(t_1), Y(t_2)\} = (|t_1|^{\alpha} + |t_2|^{\alpha} - |t_1 - t_2|^{\alpha})|2$, *and* $\psi(x) = (2\pi)^{-\frac{1}{2}}x^{-1}\exp(-x^2/2)$.

REMARK 2.1. By property (1.8) in Section 1, the $\tilde{\sigma}(\cdot)$ defined above (or any "normalized" regularly varying function of positive exponent) is monotone on some small interval $(0, \delta)$. This useful fact does not seem to be well known in this context. Of course, any function $\hat{\sigma}(\cdot)$ with $\hat{\sigma}(s) \sim \sigma(s)$ as $s \to 0$ (or $\sigma(\cdot)$ itself) that is monotone near the origin can be used in Theorem 2.1. In fact, we can show $\hat{\sigma}^{-1}(1/x) \sim \tilde{\sigma}^{-1}(1/x)$ as $x \to \infty$.

REMARK 2.2. The condition that $\sigma^2(s) > 0$ for $s \neq 0$ excludes the periodic case. However, if $\rho(s)$ is periodic with period s_0, then Theorem 2.1 holds with t in the denominator of (2.1) replaced by $\tau = \min(t, s_0)$. See the remarks in [10].

The proof of Theorem 2.1 is accomplished by a series of lemmas; and in particular Lemma 2.3 is a useful discrete version of Theorem 2.1. The connection between the constants of Theorem 2.1 and Lemma 2.3 is that $H_\alpha = \lim_{a\to 0} H_\alpha(a)/a$. The proof of the first lemma below illustrates the central idea behind the discrete version of Theorem 2.1. For this discrete version, we need a partition of the time interval $(0, t)$ with mesh size $\Delta(x)$ approaching 0 at the proper rate as $x \to \infty$. Let $\Delta(x) = \tilde{\sigma}^{-1}(1/x)$ for all $x \geqq 1/\tilde{\sigma}(\tilde{\delta})$.

LEMMA 2.1. *If the conditions of Theorem* 2.1 *hold, then for* $a > 0$,

$$\liminf_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \geqq a^{-1}(1 - 2\sum_{k=1}^{\infty} (1 - \Phi(\tfrac{1}{2}(ka)^{\alpha/2})))\,,$$

where $Z_x(t) = \max_{0\leqq k\leqq m} X(ka \cdot \Delta(x))$, $m = [t/(a\Delta(x))]$, $[\cdot]$ *denotes the greatest integer function, and* $\Phi(\cdot)$ *is the standard Gaussian distribution function.*

PROOF. This lemma corresponds to Pickands' Lemma 2.4 [7]. Defining the events $B_k = [X(ka \cdot \Delta(x)) > x]$ and using stationarity, we have

$$\begin{aligned} P[Z_x(t) > x] &\geqq \textstyle\sum_{k=0}^{m} PB_k - \sum\sum_{0\leqq j<k\leqq m} P(B_j \cap B_k) \\ &\geqq (m+1)(PB_0 - \textstyle\sum_{k=1}^{m} P(B_0 \cap B_k))\,. \end{aligned}$$

Now from [7], Lemma 2.3, we record that

$$P(B_0 \cap B_k) \leqq 2\psi(x)\{1 - \Phi(x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}})\}\,, \tag{2.2}$$

where $\rho = \rho(ka \cdot \Delta(x))$, and obtain

$$\begin{aligned} &\liminf_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \\ &\quad\geqq a^{-1}(1 - \limsup_{x\to\infty} 2\textstyle\sum_{k=1}^{m} \{1 - \Phi(x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}})\})\,. \end{aligned} \tag{2.3}$$

To study $2\sum_{k=1}^{m}\{1 - \Phi(x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}})\}$ partition the sum into three parts according to (i) $ka \leqq 1$, (ii) $ka > 1$, $ka \cdot \Delta(x) <$ some δ, and (iii) $ka > 1$, $ka \cdot \Delta(x) \geqq \delta$. First, $\lim_{x\to\infty} \sum^{(i)} 2(1 - \Phi) = \sum^{(i)} \lim_{x\to\infty} 2(1-\Phi)$.

We may ignore the third sum $\sum^{(iii)}$. For $ka \cdot \Delta(x) \geqq \delta$, there exists a positive κ such that $1 - \rho \geqq \kappa$, and

$$\begin{aligned} \textstyle\sum^{(iii)}\left\{1 - \Phi\left(x\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}}\right)\right\} &\leqq \textstyle\sum^{(iii)}\{1 - \Phi(x(\kappa/2)^{\frac{1}{2}})\} \\ &\leqq m\psi(x(\kappa/2)^{\frac{1}{2}}) \leqq \frac{t}{a\Delta(x)} \exp(-\kappa x^2/4) \to 0 \end{aligned}$$

as $x \to \infty$.

For positive δ_1 sufficiently small, $A_1(\delta_1) > 0$. For $\sum^{(ii)}$ when $ka \cdot \Delta(x) < \delta_1$ estimate

$$\begin{aligned} x\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}} &\geqq \frac{x}{2}\sigma(ka \cdot \Delta(x)) = \frac{1}{2}\frac{\sigma(ka\cdot\Delta(x))}{\tilde{\sigma}(\Delta(x))} \\ &\geqq \frac{A_1(\delta_1)}{2}\frac{\tilde{\sigma}(ka\cdot\Delta(x))}{\tilde{\sigma}(\Delta(x))} = \frac{A_1(\delta_1)}{2}(ka)^{\alpha/2}\left[\frac{H(ka\cdot\Delta(x))}{H(\Delta(x))}\right]^{\frac{1}{2}}. \end{aligned}$$

By property (1.7) in Section 1 and for $ka > 1$, there is a positive δ_α such that $H(ka \cdot \Delta(x))/H(\Delta(x)) \geqq (ka)^{-\alpha/2}$ provided $ka \cdot \Delta(x) < \delta_\alpha$. Take $\delta = \min(\delta_\alpha, \delta_1, t)$. Consequently

$$\inf_{T \leqq x < \infty} x\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1}{2}} \geqq \frac{A_1(\delta)}{2}(ka)^{\alpha/4}$$

for $ka > 1$, $ka \cdot \Delta(x) < \delta$ and T large.

Finally, defining $a_k(x) = 2\{1 - \Phi(x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}}\}$ for $ka \cdot \Delta(x) < \delta$ and $a_k(x) \equiv 2\{1 - \Phi(2^{-1}(ka)^{\alpha/2})\}$ for $ka \cdot \Delta(x) \geqq \delta$, we have

$$\sum_{k>a^{-1}}^{\infty} \sup_{T \leqq x < \infty} a_k(x) \leqq \sum_{k>a^{-1}} 2\left\{1 - \Phi\left(\frac{A_1(\delta)}{2}(ka)^{\alpha/4}\right)\right\} < \infty .$$

It follows that

$$\begin{aligned}(2.4) \qquad \limsup_{x\to\infty} \sum_{k=1}^{\infty} a_k(x) &\leqq \sum_{k=1}^{\infty} \limsup_{x\to\infty} a_k(x) \\ &= \sum_{k=1}^{\infty} 2(1 - \Phi(2^{-1}(ka)^{\alpha/2})) < \infty ,\end{aligned}$$

since

$$x(1-\rho)^{\frac{1}{2}}(1+\rho)^{-\frac{1}{2}} \sim \frac{x}{2}\sigma(ka \cdot \Delta(x)) \sim \frac{1}{2}\frac{\tilde{\sigma}(ka \cdot \Delta(x))}{\tilde{\sigma}(\Delta(x))} \to \frac{1}{2}(ka)^{\alpha/2}$$

as $x \to \infty$. Applying (2.4) in (2.3) completes the proof. $\square$

The partition corresponding to $Z_x(t)$ above was made to depend on a, and the lower estimate of the distribution of $Z_x(t)$ (as well as the upper estimate) depends on a. Since we wish to take $a \downarrow 0$, it is seen that Lemma 2.1 is not sharp enough to obtain the desired result. Hence Lemma 2.2 will be needed.

LEMMA 2.2. *If the conditions of Theorem* 2.1 *hold, then for* $a > 0$

$$\begin{aligned}\lim_{x\to\infty} &\frac{P[Z_x(na \cdot \Delta(x)) > x]}{\psi(x)} \\ &= H_\alpha(n, a) \equiv 1 + \int_0^\infty e^s P[\max_{1\leqq k\leqq n} Y(ka) > s]ds < \infty .\end{aligned}$$

PROOF. Simplifying Pickands' proof ([7] Lemma 2.2), we have

$$\begin{aligned}&P[Z_x(na \cdot \Delta(x)) > x] \\ &\quad = P[X(0) > x] + P[X(0) \leqq x, \max_{1\leqq k\leqq n} X(ka \cdot \Delta(x)) > x] .\end{aligned}$$

The second term equals

$$\int_{-\infty}^{x} P[\max_{1\leqq k\leqq n} X(ka \cdot \Delta(x)) > x/X(0) = u]\phi(u)du ,$$

where $\phi(u)$ is the standard Gaussian function. Substituting $u = x - s/x$, and defining $Y_1(t) = x(X(t \cdot \Delta(x)) - x) + s$, we obtain

$$\begin{aligned}&\psi(x) \int_0^\infty e^s P[\max_{1\leqq k\leqq n} X(ka \cdot \Delta(x)) > x/X(0) = x - s/x] \exp(-s^2/(2x^2))ds \\ &\quad = \psi(x) \int_0^\infty e^s P[\max_{1\leqq k\leqq n} Y_1(ka) > s/X(0) = x - s/x] \exp(-s^2/(2x^2))ds .\end{aligned}$$

Note that

$$
\begin{aligned}
E(Y_1(t)/X(0) = x - s/x) &= x(\rho(t \cdot \Delta(x))(x - s/x) - x) + s \\
&= -x^2(1 - \rho(t \cdot \Delta(x)) + s(1 - \rho(t \cdot \Delta(x))) \\
&= -x^2\tilde{\sigma}^2(\Delta(x)) \cdot |t|^\alpha/2 + o(1) \\
&= -|t|^\alpha/2 + o(1) \quad \text{as} \quad x \to \infty\,;
\end{aligned}
$$

and that

$$
\begin{aligned}
&\operatorname{Cov}(Y_1(t_1), Y_1(t_2)/X(0) = x - s/x) \\
&\quad = x^2[\rho((t_2 - t_1) \cdot \Delta(x)) - \rho(t_1 \cdot \Delta(x))\rho(t_2 \cdot \Delta(x))] \\
&\quad = x^2/2[-\tilde{\sigma}^2(\Delta(x))|t_2 - t_1|^\alpha + \tilde{\sigma}^2(\Delta(x))|t_1|^\alpha + \tilde{\sigma}^2(\Delta(x))|t_2|^\alpha - \tilde{\sigma}^4(\Delta(x))|t_1t_2|^\alpha/2] \\
&\qquad + o(1) \\
&\quad = \tfrac{1}{2}[-|t_2 - t_1|^\alpha + |t_1|^\alpha + |t_2|^\alpha] + o(1) \quad \text{as} \quad x \to \infty\,.
\end{aligned}
$$

Consequently $P[\max_{1 \leqq k \leqq n} Y_1(ka) > s/X(0) = x - s/x] \to P[\max_{1 \leqq k \leqq n} Y(ka) > s]$ as $x \to \infty$, and an application of Boole's inequality and the Lebesgue dominated convergence theorem completes the proof. $\square$

LEMMA 2.3. *If the conditions of Theorem* 2.1 *hold, then for* $a > 0$

$$
\lim_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} = \frac{H_\alpha(a)}{a}\,, \tag{2.5}
$$

where $0 < H_\alpha(a) \equiv \lim_{n\to\infty} H_\alpha(n, a)/n < \infty$, $Z_x(t) = \max_{0 \leqq k \leqq m} X(ka \cdot \Delta(x))$, *and* $m = [t/(a\Delta(x))]$.

PROOF. This lemma corresponds to [7], Lemma 2.5. For each nonnegative integer k, let $B_k = [X(ka \cdot \Delta(x)) > x]$, and for an arbitrary positive integer n, let $A_k = \bigcup_{j=(k-1)n}^{kn-1} B_j$. Then

$$
P[\textstyle\bigcup_{k=1}^{m'} A_k] \leqq P[Z_x(t) > x] \leqq P[\bigcup_{k=1}^{m'+1} A_k]\,, \tag{2.6}
$$

where $m' = [(m + 1)/n]$. By stationarity, $P(A_k) = P(A_1)$ for all $k \geqq 1$. Consequently

$$
P[Z_x(t) > x] \leqq \textstyle\sum_{k=1}^{m'+1} P(A_k) = (m' + 1)P(A_1)\,.
$$

Now using Lemma 2.2, we obtain

$$
\limsup_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \leqq \limsup_{x\to\infty} \frac{(m' + 1)PA_1}{(t/\Delta(x))\psi(x)} = H_\alpha(n - 1, a)/na\,. \tag{2.7}
$$

On the other hand, (2.6) and stationarity imply

$$
\begin{aligned}
P[Z_x(t) > x] &\geqq \textstyle\sum_{k=1}^{m'} P(A_k) - \sum \sum_{1 \leqq k < j \leqq m'} P(A_k \cap A_j) \\
&\geqq m'P(A_1) - m' \textstyle\sum_{j=2}^{m'} P(A_1 \cap A_j) \\
&\geqq m'\{PA_1 - \textstyle\sum_{k=0}^{n-1} \sum_{l=n}^{m} P(B_k \cap B_l)\}\,.
\end{aligned} \tag{2.8}
$$

As in the proof of Lemma 2.1, inequality (2.2) applied to $P(B_k \cap B_l) = P(B_0 \cap B_{l-k})$ and inequality (2.8) yield

$$(2.9) \qquad \liminf_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)}$$
$$\geqq \{H_\alpha(n-1, a) - \limsup_{x\to\infty} \textstyle\sum_{k=0}^{n-1} \sum_{l=n}^{m} P(B_k \cap B_l)/\psi(x)\}/na$$
$$\geqq \{H_\alpha(n-1, a) - \textstyle\sum_{k=0}^{n-1} \sum_{l=n}^{\infty} d_{l-k}\}/na$$

where $d_j = 2\{1 - \Phi(\frac{1}{2}(ja)^{\alpha/2})\}$. By (2.4) the $\sum_{j=0}^{\infty} d_j < \infty$, and therefore $\lim_{n\to\infty} \sum_{k=0}^{n-1} \sum_{l=n}^{\infty} d_{l-k}/na = 0$, by Kronecker's lemma.

Combining (2.7) and (2.9), we have

$$\limsup_{n\to\infty} \frac{H_\alpha(n-1, a)}{na} \leqq \liminf_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)}$$
$$\leqq \limsup_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)}$$
$$\leqq \liminf_{n\to\infty} \frac{H_\alpha(n-1, a)}{na},$$

and the conclusion of Lemma 2.3.

Now (2.7) implies $H_\alpha(a) < \infty$. By Lemma 2.1, $H_\alpha(a) > 0$ for a sufficiently large, say for all $a >$ some a_0. For any $a > 0$, there exists an integer m such that $ma > a_0$. Now $H_\alpha(n, am) \leqq H_\alpha(nm, a)$ implies $H_\alpha(am) \leqq mH_\alpha(a)$ and $H_\alpha(a) > 0$. □

LEMMA 2.4. *Under the same conditions as Theorem* 2.1, *it follows for* $a > 0$ *and* $2^{-\alpha/4} < b < 1$ *that*

$$\limsup_{x\to\infty} \frac{P[X(0) \leqq x - \gamma/x, Z(a\Delta(x - \gamma/x)) > x]}{\psi(x)} \leqq M(a, \gamma),$$

where

$$M(a, \gamma) = (a/2)^{\alpha/2} \textstyle\sum_{k=0}^{\infty} 2^{k(1-\alpha/2)} R(\gamma(1-b)(2/a)^{\alpha/2}(2^{\alpha/2}b)^k - 2^{-1}(a/2)^{\alpha/2}2^{-\alpha k/2}),$$

and

$$R(x) = \textstyle\int_x^\infty (1 - \Phi(s))ds.$$

Furthermore, for $\gamma = a^\beta$ *with* $0 \leqq \beta < \alpha/2$ *it follows that*

$$\lim_{a\to 0} \frac{M(a, a^\beta)}{a} = 0.$$

PROOF. Note that

$$[X(0) \leqq x - \gamma/x, Z(\alpha\Delta(x - \gamma/x)) > x] \subseteqq \bigcup_{k=0}^{\infty} D_k \quad \text{and} \quad D_k \subseteqq \bigcap_{j=0}^{2^k-1} E_{j,k},$$

where

$$D_k = [\max_{0\leqq j\leqq 2^k-1} X(ja\Delta(x - \gamma/x)/2^k) \leqq x - \gamma b^k/x,$$
$$\max_{0\leqq j\leqq 2^{k+1}-1} X(ja\Delta(x - \gamma/x)/2^{k+1}) > x - \gamma b^{k+1}/x]$$

and

$$E_{j,k} = [X(ja\Delta(x-\gamma/x)/2^k) \leqq x - \gamma b^k/x\,,$$
$$X((2j+1)a\Delta(x-\gamma/x)/2^{k+1}) > x - \gamma b^{k+1}/x]\,.$$

By using [7] Lemma 2.6, we obtain $P(E_{j,k}) \leqq \psi(x)x\rho^{-1}(1-\rho^2)^{\frac{1}{2}}R(y)$, where

$$\rho = \rho(a\Delta(x-\gamma/x)/2^{k+1})\,,$$

and

$$y = y(x) = \gamma(1-b)b^k\rho x^{-1}(1-\rho^2)^{-\frac{1}{2}} - x(1+\rho)^{-1}(1-\rho^2)^{\frac{1}{2}}\,.$$

Consequently

$$(2.10)\qquad \limsup_{x\to\infty} \frac{P[X(0) \leqq x-\gamma/x,\ Z(a\Delta(x-\gamma/x)) > x]}{\psi(x)}$$
$$\leqq \limsup_{x\to\infty} \sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1} P(E_{j,k})/\psi(x)$$
$$\leqq \limsup_{x\to\infty} \sum_{k=0}^{\infty} 2^k x\rho^{-1}(1-\rho^2)^{\frac{1}{2}}R(y)\,.$$

In order to apply the technique used in Lemma 2.1, we need to show

$$\sum_{k=0}^{\infty} 2^k \sup_{T\leqq x<\infty} x\rho^{-1}(1-\rho^2)^{\frac{1}{2}}R(y) < \infty \quad \text{for some} \quad T > 0\,.$$

That this sum is finite follows from the following estimates for all $x \geqq T$ sufficiently large:

$$\begin{aligned}
&x\rho^{-1}(1-\rho^2)^{\frac{1}{2}}\\
&\quad\leqq x\rho^{-1}\sigma(a\Delta(x-\gamma/x)/2^{k+1}) \leqq xS^{-1}A_2\tilde{\sigma}(a\Delta(x-\gamma/x)/2^{k+1})\\
&\quad\leqq S^{-1}A_2x(x-\gamma/x)^{-1}\tilde{\sigma}(a\Delta(x-\gamma/x)/2^{k+1})/\tilde{\sigma}(\Delta(x-\gamma/x))\\
&\quad\leqq S^{-1}A_2(1-\gamma/x^2)^{-1}(a2^{-k-1})^{\alpha/2}(H(a\Delta(x-\gamma/x)/2^{k+1})/H(\Delta(x-\gamma/x)))^{\frac{1}{2}}\\
&\quad\leqq S^{-1}A_2 2(a2^{-k-1})^{\alpha/2}(a2^{-k-1})^{-\alpha/4} \qquad \text{by (1.6) in Section 1}\\
&\quad= 2S^{-1}A_2(a2^{-k-1})^{\alpha/4}\,,
\end{aligned}$$

and similarly

$$\begin{aligned}
y(x) &\geqq \gamma(1-b)b^kSx^{-1}/\sigma - x(1+S)^{-1}\sigma\\
&\geqq (1-\gamma/x^2)\{\gamma(1-b)b^kSA_2^{-1}\tilde{\sigma}(\Delta(x-\gamma/x))/\tilde{\sigma}(a\Delta(x-\gamma/x)2^{-k-1})\\
&\qquad - (1-\gamma/x^2)^{-2}(1+S)^{-1}A_2\tilde{\sigma}(\alpha\Delta(x-\gamma/x)2^{-k-1})/\tilde{\sigma}(\Delta(x-\gamma/x))\}\\
&\geqq 2^{-1}\{\gamma(1-b)b^kSA_2^{-1}(2^k2/a)^{\alpha/4} - 4(1+S)^{-1}A_2(2^k2/a)^{-\alpha/4}\}\,.
\end{aligned}$$

Here $S = \inf_{0\leqq s\leqq a\Delta(x-\gamma/x)} \rho(s) \geqq 1 - \frac{1}{2}$ and $A_2 = A_2(a\Delta(x-\gamma/x)) \leqq 1 + \frac{1}{2}$ for all $x >$ some large T.

Therefore, (2.10) yields

$$\limsup_{x\to\infty} \frac{P[X(0) \leqq x-\gamma/x,\ Z(a\Delta(x-\gamma/x)) > x]}{\psi(x)}$$
$$\leqq \sum_{k=0}^{\infty} \limsup_{x\to\infty} 2^k x\rho^{-1}(1-\rho^2)^{\frac{1}{2}}R(y) = M(a,\gamma)$$

since $x(1-\rho^2)^{\frac{1}{2}} \to (a/2)^{\alpha/2}2^{-k\alpha/2}$ as $x\to\infty$.

In order to see that $M(a, a^\beta) \to 0$ as $a \to 0$, use the estimates $R(x) \leqq \psi(x)/x \leqq \exp(-x^2/2)$ for $x^2 \geqq (2\pi)^{-1}$; note that one may disregard any finite number of the leading terms of $M(a, \gamma)$; and then factor $\exp(-ca^{-(\alpha-2\beta)})$ out of the infinite sum $M(a, a^\beta)$. Here $c > 0$ is a properly chosen constant.

PROOF OF THEOREM 2.1. Lemma 2.8 in [7] is not true, though Pickands' basic concepts for the proof of his Lemma 2.9 stand under the required closer examination. (In a private communication Pickands outlined a verification of his Lemma 2.9 without the use of Lemma 2.8.) We give an independently developed proof of our Theorem 2.1 that avoids use of his Lemma 2.8. Define $H_\alpha^+ = \limsup_{x\to\infty} P[Z(t) > x]/(t\psi(x)/\Delta(x))$ and $H_\alpha^- = \liminf_{x\to\infty} P[Z(t) > x]/(t\psi(x)/\Delta(x))$. Now since $\Delta(x - \gamma/x)/\Delta(x) \to 1$ and $\psi(x - \gamma/x)/\psi(x) \to \exp(\gamma)$ as $x \to \infty$ for $\gamma > 0$, we see from Lemma 2.3 that $\lim_{x\to\infty} P[Z_{x-\gamma/x}(t) > x - \gamma/x]/(t\psi(x)/\Delta(x)) = \exp(\gamma) \cdot H_\alpha(a)/a$, where $Z_{x-\gamma/x}(t) = \max_{0\leq k\leq m} X(ka \cdot \Delta \times (x - \gamma/x))$ and the integer $m = [t/(a\Delta(x - \gamma/x))]$. For $\gamma > 0$, we obtain

$$
\begin{aligned}
H_\alpha^- - e^\gamma \frac{H_\alpha(a)}{a} &\leqq H_\alpha^+ - e^\gamma \frac{H_\alpha(a)}{a} \\
&= \limsup_{x\to\infty} \frac{P[Z(t) > x] - P[Z_{x-\gamma/x}(t) > x - \gamma/x]}{t\psi(x)/\Delta(x)} \\
&\leqq \limsup_{x\to\infty} \frac{P[Z(t) > x,\ Z_{x-\gamma/x}(t) \leqq x - \gamma/x]}{t\psi(x)/\Delta(x)} \\
&\leqq \limsup_{x\to\infty} \frac{P[X(0) \leqq x - \gamma/x,\ Z(a\Delta(x - \gamma/x)) > x]}{a\psi(x)} \\
&\leqq \frac{M(a, \gamma)}{a} \quad \text{by Lemma 2.4.}
\end{aligned}
\tag{2.11}
$$

Using $P[Z_x(t) > x] \leqq P[Z(t) > x]$ and (2.11), we have

$$
0 \leqq H_\alpha^- - \frac{H_\alpha(a)}{a} \leqq H_\alpha^+ - \frac{H_\alpha(a)}{a} \leqq \frac{M(a, \gamma)}{a} + (e^\gamma - 1)\frac{H_\alpha(a)}{a}. \tag{2.12}
$$

On examining (2.12) we see that H_α^- and H_α^+ are finite, and then that lim sup and lim inf of $H_\alpha(a)/a$ as $a \to 0$ must be finite. Now choosing $\gamma = a^\beta$ with $0 < \beta < \alpha/2$ in (2.12) so that $M(a, \gamma)/a \to 0$ as $a \to 0$, we obtain that $\lim_{a\to0} H_\alpha(a)/a$ exists and that $H_\alpha \equiv H_\alpha^+ = H_\alpha^- = \lim_{a\to0} H_\alpha(a)/a$. Of course H_α is finite and $H_\alpha \geqq H_\alpha(a)/a > 0$.

Define $H_\alpha^+(T) = \limsup_{x\to\infty} P[Z(T\Delta(x)) > x]/(T\psi(x))$. We now develop an expression for H_α via the estimate

$$
\begin{aligned}
|H_\alpha - H_\alpha^+(na)| \leqq |H_\alpha - H_\alpha(a)/a| &+ |H_\alpha(a)/a - H_\alpha(n, a)/(na)| \\
&+ |H_\alpha(n, a)/(na) - H_\alpha^+(na)|\,.
\end{aligned}
\tag{2.13}
$$

Considering the last term in (2.13) and noting that

$$[Z(na\Delta(x)) > x] \subseteqq [Z(na\Delta(x - \gamma/x)) > x]$$

for all large x, we obtain from Lemmas 2.3 and 2.4 that

$$(2.14)\quad H_\alpha^+(na) - e^\gamma \frac{H_\alpha(n, a)}{na}$$
$$\leqq \liminf_{x\to\infty} \frac{P[Z(na\Delta(x-\gamma/x)) > x] - P[Z_{x-\gamma/x}(na\Delta(x-\gamma/x)) > x-\gamma/x]}{na\psi(x)}$$
$$\leqq \frac{M(a, \gamma)}{a} \quad \text{for all} \quad \gamma > 0\,.$$

Using $P[Z_x(na\Delta(x)) > x] \leqq P[Z(na\Delta(x)) > x]$ and (2.14), we have

$$(2.15)\qquad 0 \leqq H_\alpha^+(na) - \frac{H_\alpha(n, a)}{na} \leqq \frac{M(a, \gamma)}{a} + (e^\gamma - 1)\frac{H_\alpha(n, a)}{na}$$

for all $a > 0$, and all positive integers n, and all $\gamma > 0$.

Again choosing $\gamma = a^\beta$, $0 < \beta < \alpha/2$, we see that there exists $\delta > 0$ such that for all $a < \delta$ $M(a, a^\beta)/a < \varepsilon/6$, and $\exp(a^\beta) - 1 \leqq \varepsilon/(12H_\alpha)$, and $|H_\alpha - H_\alpha(a)/a| < \varepsilon/3$. Since $H_\alpha(n, a)/(na) \leqq 2H_\alpha(a)/a \leqq 2H_\alpha$ for all $n \geqq$ some n_a, we obtain from (2.15) that $|H_\alpha^+(na) - H_\alpha(n, a)/(na)| \leqq \varepsilon/3$ for all $a < \delta$ and all $n \geqq$ some n_a depending on a. Consequently for a fixed $a_0 < \delta$ inequality (2.13) yields

$$|H_\alpha - H_\alpha^+(na_0)| \leqq \varepsilon/3 + \left|\frac{H_\alpha(a_0)}{a_0} - \frac{H_\alpha(n, a_0)}{na_0}\right| + \varepsilon/3 \quad \text{for all} \quad n \geqq n_{a_0}$$
$$\leqq \varepsilon \quad \text{for all} \quad n \geqq \text{some } n_1 \geqq n_{a_0}\,.$$

Since for $n = [T/a_0]$, $na_0 \leqq T \leqq (n+1)a_0$ and $na_0T^{-1}H_\alpha^+(na_0) \leqq H_\alpha^+(T) \leqq (na_0 + a_0)T^{-1}H_\alpha^+(na_0 + a_0)$, we have $|H_\alpha - H_\alpha^+(T)| < \varepsilon$ for all $T \geqq$ some T_0 or $H_\alpha = \lim_{T\to\infty} H_\alpha^+(T)$.

Finally for arbitrary fixed $T > 0$, we consider $n = [T/a]$ with $a \downarrow 0$ for the particular sequence $a_j = T2^{-j}$. Now by Lemma 2.3 and monotone convergence

$$\lim_{a_j\to 0} \frac{H_\alpha(n_j, a_j)}{n_j a_j} = \lim_{a_j\to 0} \frac{1}{T}\left(1 + \int_0^\infty e^s P[\max_{1\leqq k\leqq n} Y(ka) > s]ds\right)$$
$$= \frac{1}{T}\left(1 + \int_0^\infty e^s P[\sup_{0<t<T} Y(t) > s]ds\right).$$

Taking the same limit as $a_j \to 0$ in inequality (2.15) with $\gamma = a_j^\beta$, $0 < \beta < \alpha/2$, we obtain that

$$H_\alpha^+(T) = \frac{1}{T}\left(1 + \int_0^\infty e^s P[\sup_{0<t<T} Y(t) > s]ds\right). \qquad \square$$

REMARK 2.3. Pickands' Theorem 2.1 [7] concerning the expected number of ε-upcrossings is hereby generalized also.

3. An asymptotic 0-1 behavior. In this section, we use the results of Section

2 to obtain an extension of the results in Qualls and Watanabe [10]. We again postpone discussion of the non-stationary case to Section 4. Using the notation of Section 2, we have

THEOREM 3.1. *If $\rho(t)$ satisfies* (0.1) *with* $0 < \alpha \leqq 2$, $\tilde{\sigma}(\cdot)$ *is defined as in Section* 2, *and*

$$(3.1) \qquad \rho(t) = 0(t^{-\gamma}) \quad as \quad t \to \infty\,, \quad for\ some \quad \gamma > 0\,;$$

then, for any positive non-decreasing function $\phi(t)$ on some interval $[a, \infty)$,

$$PE_\phi \equiv P\{\exists\, t_0(\omega) > a : X(t) \leqq \phi(t) \text{ for all } t \geqq t_0\} = 1 \quad or \quad 0$$

as the integral

$$I(\phi) \equiv \int_a^\infty (\phi(t)\tilde{\sigma}^{-1}(1/\phi(t)))^{-1} \exp(-\phi^2(t)/2)dt$$

converges or diverges.

REMARK 3.1. Monotone $\tilde{\sigma}(\cdot)$ other than the one defined in Section 2 can be used; see Remark 2.1. Note that condition (3.1) implies $\rho(t)$ is not periodic; consequently Theorem 2.1 is applicable.

PROOF. For every $\varepsilon > 0$, assumption (0.1) implies that $s^{(\alpha+\varepsilon)/2} \leqq \tilde{\sigma}(s) \leqq s^{(\alpha-\varepsilon)/2}$ and that $s^{2/(\alpha-\varepsilon)} \leqq \tilde{\sigma}^{-1}(s) \leqq s^{2/(\alpha+\varepsilon)}$ for all positive $s \leqq$ some δ. In particular, the integrand of $I(\phi)$ is eventually a decreasing function of ϕ.

(1) *The case when $I(\phi) < \infty$.*

Let $t_n = n\Delta$, where $\Delta > 0$ and $n = 0, 1, 2, \cdots$. By Theorem 2.1 and for fixed $\Delta > 0$, we have

$$\begin{aligned}
&\sum_{n=n_0}^\infty P\{\sup_{t_n \leqq t \leqq t_{n+1}} X(t) \geqq \phi(t_n)\} \\
&\quad \leqq C_1 \sum_{n=n_0}^\infty (t_{n+1} - t_n)(\phi(t_n)\tilde{\sigma}^{-1}(1/\phi(t_n))^{-1} \exp(-\phi^2(t_n)/2) \\
&\quad = C_1 \sum_{n=n_0}^\infty (t_n - t_{n-1})(\phi(t_n)\tilde{\sigma}^{-1}(1/\phi(t_n))^{-1} \exp(-\phi^2(t_n)/2) \\
&\quad \leqq C_1 \int_{n_0\Delta}^\infty (\phi(t)\tilde{\sigma}^{-1}(1/\phi(t)))^{-1} \exp(-\phi^2(t)/2)dt < \infty\,,
\end{aligned}$$

for n_0 sufficiently large. Here $C_1 > 0$ is a certain constant. So, the Borel-Cantelli lemma yields

$$P\{\exists\, n_\phi(\omega) : \sup_{t_n \leqq t \leqq t_{n+1}} X(t) \leqq \phi(t_n) \text{ for all } n \geqq n_\phi\} = 1\,;$$

and consequently $PE_\phi = 1$. □

(2) *The case when $I(\phi) = \infty$.*

For this part of the proof, we need the following lemma.

LEMMA 3.1. *If Theorem* 3.1 *when $I(\phi) = \infty$ holds under the additional assumption that*

$$2 \log t \leqq \phi^2(t) \leqq 3 \log t\,, \qquad for\ all\ large \quad t\,,$$

then it holds without this additional assumption.

PROOF. From the bounds on $\tilde{\sigma}^{-1}(\cdot)$ given above, there are positive constants C_2 and C_3 such that

$$(3.2) \qquad C_2 \int_a^\infty \phi(t)^{2/(\alpha+\varepsilon)-1} \exp(-\phi^2(t)/2)dt \\ \leqq I(\phi) \leqq C_3 \int_a^\infty \phi(t)^{2/(\alpha-\varepsilon)-1} \exp(-\phi^2(t)/2)dt .$$

When $0 < \alpha < 2$, choose $\varepsilon > 0$ such that $\alpha + \varepsilon < 2$ and $\alpha - \varepsilon > 0$. When $\alpha = 2$, it is well known that the $H(s)$ in $\sigma^2(s)$ cannot tend to zero; consequently, we may choose $\varepsilon = 0$ in the left-hand side of (3.2) when $\alpha = 2$. We obtain for $0 < \alpha \leqq 2$ that

$$(3.3) \qquad I(\phi) \geqq C_2' \int_a^\infty \exp(-\phi^2(t)/2)dt \equiv C_2' J(\phi) .$$

Let $\phi(t)$ be an arbitrary positive non-decreasing function such that $I(\phi) = \infty$. Let $\hat{\phi}(t) = \min(\max(\phi(t), (2\log t)^{\frac{1}{2}}), (3 \log t)^{\frac{1}{2}})$. To show $I(\hat{\phi}) = \infty$, we may assume $\phi(t)$ crosses $u(t) = (2\log t)^{\frac{1}{2}}$ infinitely often as $t \to \infty$. Otherwise, either $\phi \leqq u$ and $I(\hat{\phi}) = I(u) = \infty$, or $\phi > u$ and $I(\hat{\phi}) \geqq I(\phi) = \infty$, for some large a.

The proof of Lemma 1.4 in [10] now shows that $J(\hat{\phi}) = \infty$; and by (3.3) that $I(\hat{\phi}) = \infty$.

That $P[X(t) > \hat{\phi}(t) \text{ i.o.}] = 1$ implies $P[X(t) > \phi(t) \text{ i.o.}] = 1$ follows from "Theorem 3.1 when $I(v) < \infty$" with $v(t) = (3 \log t)^{\frac{1}{2}}$; details are given in Lemma 4.1 in [13]. □

The proof of the second part of Theorem 3.1 now proceeds in the same way as in Qualls and Watanabe [10]. We will use the same notation as in [10].

Define a sequence of intervals by $I_n = [n\Delta, n\Delta + \beta]$ for $\Delta > 0$ and $0 < \beta < \Delta$. Let $G_k = \{t_{k,\nu} = k\Delta + \nu/n_k : \nu = 0, 1, \cdots, [\beta n_k]\}$ be points in I_k where $n_k = [(\tilde{\sigma}^{-1}(1/\phi(k\Delta + \beta)))^{-1}]$. Let $E_k = [\max_{s \in G_k} X(s) \leqq \phi(k\Delta + \beta)]$. Now using Lemma 2.3 in the same way as in [10], we see that $I(\phi) = \infty$ implies $\sum P(E_k^c) = \infty$.

So, we only need to prove the asymptotic independence of the E_k's, that is,

$$(3.4) \qquad \lim_{m\to\infty} \lim_{n\to\infty} |P(\bigcap_m^n E_k) - \prod_m^n PE_k| = 0 .$$

Now by the use of Lemma 3.1 and the bounds on $\tilde{\sigma}^{-1}(\cdot)$, we have $\phi^2(k\Delta + \beta) \geqq 2\log(k\Delta + \beta)$ and

$$n_k \leqq \phi(k\Delta + \beta)^{2/(\alpha-\varepsilon)} \leqq (3\log(k\Delta + \beta))^{1/(\alpha-\varepsilon)} ,$$

where $\varepsilon > 0$ and $\alpha - \varepsilon > 0$. Now the proof of (3.4) given in [10] applies without change. □

REMARK 3.2. By the use of inequalities (3.2) and Theorem 3.1, we can easily show that for every $\varepsilon > 0$,

$$P\left[\frac{1}{2} + \frac{1}{\alpha+\varepsilon} \leqq \limsup_{x\to\infty} \frac{(2\log t)^{\frac{1}{2}}(Z(t) - (2\log t)^{\frac{1}{2}})}{\log\log t} \leqq \frac{1}{2} + \frac{1}{\alpha - \varepsilon}\right] = 1 ;$$

and consequently

$$P\left[\limsup_{x\to\infty} \frac{(2\log t)^{\frac{1}{2}}(Z(t)-(2\log t)^{\frac{1}{2}})}{\log\log t} = \frac{1}{2}+\frac{1}{\alpha}\right] = 1\,.$$

It is interesting that this is true whatever H may be (as long as it satisfies the assumptions of Theorem 3.1).

REMARK 3.3. Generally speaking, it seems to be difficult to compute $I(\phi)$ in the criterion of Theorem 3.1 in concrete examples. Of course, inequalities (3.2) may be used except in the critical cases.

4. The non-stationary case. It is not surprising that Slepian's result ([12], Theorem 1) can be used to generalize Section 3. See Section 2 of [10]. It is more interesting that Slepian's result can be used to generalize Section 2.

Let $X(t)$ be a separable Gaussian process with zero mean function and correlation function $\rho(t, s)$. We adopt the notation of Section 2 with modifications to the non-stationary case. At first, we only assume that

$$1 - C_1 h^\alpha H(h) \leqq \rho(\tau, \tau+h) \leqq 1 - C_2 h^\alpha H(h) \tag{4.1}$$

for $0 < h < \delta_{12}$ and $0 \leqq \tau \leqq t$, where $0 < \alpha \leqq 2$ and H is slowly varying at zero. Without loss of generality, we take H to be "normalized". There exist separable zero mean stationary processes $Y_1(t)$ and $Y_2(t)$ with covariance functions satisfying $q_1(h) \sim 1 - C_3 h^\alpha H(h)$ and $q_2(h) \sim 1 - C_4 h^\alpha H(h)$ as $h \to 0$, respectively. For $C_4 < C_2 < C_1 < C_3$, we have

$$q_1(h) \leqq \rho(\tau, \tau+h) \leqq q_2((h) \quad \text{for} \quad 0 < h < \delta_{34} \tag{4.2}$$

and $0 \leqq \tau \leqq t$. We shall use the subscripts 1 and 2 throughout to correspond to the stationary processes $Y_1(\cdot)$ and $Y_2(\cdot)$, respectively.

In order to exclude any type of periodic case, we assume

$$\kappa = \sup\{\rho(\tau, \tau+h) : \delta_{34} \leqq h, 0 \leqq \tau + h \leqq t\} < 1\,.$$

THEOREM 4.1. *If $X(\cdot)$ satisfies* (4.1) *and $\kappa < 1$, then for $a > 0$*

$$\begin{aligned} &C_2^{1/\alpha}\frac{H_\alpha(a)}{a} - 2(C_1^{1/\alpha} - C_2^{1/\alpha})\frac{H_\alpha(a)}{a} \\ &\quad \leqq \liminf_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\tilde\sigma^{-1}(1/x)} \leqq \limsup_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\tilde\sigma^{-1}(1/x)} \\ &\quad \leqq C_1^{1/\alpha}\frac{H_\alpha(a)}{a}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &C_2^{1/\alpha}H_\alpha - 2(C_1^{1/\alpha} - C_2^{1/\alpha})H_\alpha \\ &\quad \leqq \liminf_{x\to\infty} \frac{P[Z(t) > x]}{t\psi(x)/\tilde\sigma^{-1}(1/x)} \leqq \limsup_{x\to\infty} \frac{P[Z(t) > x]}{t\psi(x)/\tilde\sigma^{-1}(1/x)} \\ &\quad \leqq C_1^{1/\alpha}H_\alpha\,, \end{aligned} \tag{4.4}$$

where $\tilde\sigma^2(h) \equiv 2|h|^\alpha H(h)$.

Moreover if $X(\cdot)$ satisfies (0.1) *with* $0 < \alpha \leqq 2$ *and* $\kappa < 1$, *then for* $a > 0$

$$\lim_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} = \frac{H_\alpha(a)}{a}\,, \tag{4.5}$$

and

$$\lim_{x\to\infty} \frac{P[Z(t) > x]}{t\psi(x)/\tilde{\sigma}^{-1}(1/x)} = H_\alpha\,. \tag{4.6}$$

PROOF. The proof consists of showing that the results only depend on the "local condition" for $\rho(\tau, \tau + h)$ instead of on the total time interval $(0, t)$. Let the integer M be large enough that $\delta = t/M$ is less than $\delta_{34}/2$. Define

$$A_j = [\max_{(j-1)\delta \leqq ka\cdot\Delta(x) < j\delta} X(ka \cdot \Delta(x)) > x]\,.$$

That $PA_j \leqq P_1A_j \equiv P_1A_1$ is Slepian's result in the non-stationary case together with the fact that P_1 is a stationary measure. Since $\Delta(x) \sim C_3^{1/\alpha}\Delta_1(x)$ as $x \to \infty$, Theorem 2.1 (or rather Lemma 2.3) yields

$$\limsup_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \leqq \limsup_{x\to\infty} \frac{P_1[Z_x(\delta) > x]}{\delta\psi(x)/\Delta(x)} = C_3^{1/\alpha}\frac{H_\alpha(a)}{a}\,. \tag{4.7}$$

Similarly

$$\limsup_{x\to\infty} \frac{P[Z(t) > x]}{t\psi(x)/\Delta(x)} \leqq \limsup_{x\to\infty} \frac{P_1[Z(\delta) > x]}{\delta\psi(x)/\Delta(x)} = C_3^{1/\alpha}H_\alpha\,. \tag{4.8}$$

For lower bounds, we consider

$$P[Z_x(t) > x] \geqq \sum_{j=1}^{M} PA_j - \sum\sum_{1\leqq i<j\leqq M} P(A_i \cap A_j)\,. \tag{4.9}$$

For $j - i \geqq 2$ in the double sum, we use a well-known device (see, e.g., Lemma 1.5 in [10]) to obtain

$$\begin{aligned}|P(A_i \cap A_j) - PA_iPA_j| &\leqq C \sum_{k=1}^{m} \sum_{l=1}^{m} |\rho| \frac{\exp(-x^2/(1+\rho))}{(1-\rho^2)^{\frac{1}{2}}} \\ &\leqq K^1m^2 \exp(-x^2/(1+\kappa))\,,\end{aligned}$$

where $m = [t/(a\Delta(x))]$.

Dividing by $t\psi(x)/\Delta(x)$, we see that the error term and PA_iPA_j approach zero as $x \to \infty$; and therefore we may ignore this part of the double sum. For $j - i = 1$,

$$\sum_{j=1}^{M-1} P(A_j \cap A_{j+1}) < M(2P_1(A_1) - P_2(A_1 \cup A_2))\,. \tag{4.10}$$

Using (4.10) in (4.9), we have

$$\begin{aligned}\liminf_{x\to\infty} \frac{P[Z(t) > x]}{t\psi(x)/\Delta(x)} &\geqq \liminf_{x\to\infty} \frac{P[Z_x(t) > x]}{t\psi(x)/\Delta(x)} \\ &\geqq \liminf_{x\to\infty} \frac{P_2[Z_x(\delta) > x]}{\delta\psi(x)/\Delta(x)} \\ &\quad - 2\limsup_{x\to\infty}\left\{\frac{P_1[Z_x(\delta) > x]}{\delta\psi(x)/\Delta(x)} - \frac{P_2[Z_x(2\delta) > x]}{2\delta\psi(x)/\Delta(x)}\right\} \\ &= C_4^{1/\alpha}\frac{H_\alpha(a)}{a} - 2(C_3^{1/\alpha} - C_4^{1/\alpha})\frac{H_\alpha(a)}{a}\,.\end{aligned} \tag{4.11}$$

Choosing $C_3 = C_1$, $C_4 = C_2$, and letting $a \to 0$ in (4.7), (4.8) and (4.11), we obtain (4.3) and (4.4). Choosing $C_1 = C_2 = 1$ in (4.3) and (4.4), we obtain (4.5) and (4.6). □

Of course, Theorem 3.1 of Section 3 can be generalized easily to a result analogous to Theorems 2.1 and 2.3 of [10]. We write the following theorem without proof.

THEOREM 4.2. *If $X(\cdot)$ satisfies (4.1) with $0 < \alpha \leqq 2$ for $0 < h < \delta$ and all $\tau > T$, and*

$$(4.12) \quad \rho(\tau, \tau + s) = O(s^{-\gamma}) \quad \text{uniformly in} \quad \tau \quad \text{as} \quad s \to \infty \quad \text{for some} \quad \gamma > 0 ,$$

then, for any positive non-decreasing function $\phi(t)$ on some $[a, \infty)$,

$$P[X(t) > v(t)\phi(t) \text{ i.o. in } t] = 0 \quad or \quad 1$$

as the integral $I(\phi) < \infty$ or $= \infty$.

5. Comments. By use of the results in Section 2, we can easily obtain the extension of Theorem 3.2 in Pickands [8] to our case. We will state only the result, because his proof is applicable by just changing $x^{2/\alpha}$ to $(\tilde{\sigma}^{-1}(1/x))^{-1}$.

THEOREM 5.1. *Let $\{X(t), -\infty < t < \infty\}$ satisfy the conditions in Theorem 2.1. Also we assume that*

$$\lim_{t\to\infty} \rho(t) \log t = 0 \quad or \quad \textstyle\int_{-\infty}^{\infty} \rho^2(t)dt < \infty .$$

Let $N(\varepsilon, y, t)$ be the number of "ε-upcrossings" of the level y in the interval $(0, t)$. An "ε-upcrossing" of the level y is said to have occured at t_0 if $X(t_0) = x$ and $X(t) < x$, for all t such that $t_0 - \varepsilon \leqq t < t_0$. Then

$$\lim_{y\to\infty} P(N(\varepsilon, y, \lambda/\mu) = k) = e^{-\lambda}\lambda^k/k! , \qquad k = 0, 1, 2, \cdots,$$

where $\mu \equiv E(N(\varepsilon, y, t))/t$ has the same value for all t. Furthermore,

$$\mu \sim (2\pi)^{-\frac{1}{2}}(\tilde{\sigma}^{-1}(1/y))^{-1}y^{-1} \exp(-y^2/2) .$$

as $y \to \infty$, where H_α is given by (2.1).

By using the above theorem, we can prove the following.

THEOREM 5.2 *Under the same assumptions as in Theorem 5.1, we have for all x, $-\infty < x < \infty$*

$$\lim_{t\to\infty} P((A(t))^{-1}(Z(t) - B(t)) \leqq x) = \exp(-e^{-x}) ,$$

where $A(t) = (2 \log t)^{-\frac{1}{2}}$ and $B(t) = (2 \log t)^{\frac{1}{2}} - (2 \log t)^{-\frac{1}{2}} \log (2H_\alpha^{-1}(\pi \log t)^{\frac{1}{2}} \tilde{\sigma}^{-1}(1/(2 \log t)^{\frac{1}{2}}))$.

PROOF. As in the proof of Theorem 2.1 in Pickands [8], it is sufficient to prove that

$$\lim_{t\to\infty} P(N(\varepsilon, A(t)x + B(t), t) = 0) = \exp(-e^{-x})$$

for all x. In order to prove this, by the preceding theorem, it is sufficient to show

$$\lim_{t\to\infty} t\mu = e^{-x},$$

where $\mu \sim H_\alpha(2\pi)^{-\frac{1}{2}}(\tilde{\sigma}^{-1}(1/y))^{-1}y^{-1}\exp(-y^2/2)$ and $y = A(t)x + B(t)$. But

$$\begin{aligned} y^2/2 &= x^2/(4\log t) + \log t + (4\log t)^{-1}\log(2H_\alpha^{-1}(\pi\log t)^{\frac{1}{2}}\tilde{\sigma}^{-1}(1/(2\log t)^{\frac{1}{2}})) \\ &\quad + x - x(2\log t)^{-1}\log(2H_\alpha^{-1}(\pi\log t)^{\frac{1}{2}}\tilde{\sigma}^{-1}(1/(2\log t)^{\frac{1}{2}})) \\ &\quad - \log(2H_\alpha^{-1}(\pi\log t)^{\frac{1}{2}}\tilde{\sigma}^{-1}(1/(2\log t)^{\frac{1}{2}})) \\ &= \log t + x - \log(2H_\alpha^{-1}(\pi\log t)^{\frac{1}{2}}\tilde{\sigma}^{-1}(1/(2\log t)^{\frac{1}{2}})) + o(1) \end{aligned}$$

as $t \to \infty$.

So

$$\exp(-y^2/2) \sim t^{-1}e^{-x}2H_\alpha^{-1}(\pi\log t)^{\frac{1}{2}}\tilde{\sigma}^{-1}((2\log t)^{-\frac{1}{2}}).$$

Since $y \sim (2\log t)^{\frac{1}{2}}$ as $t \to \infty$, obviously we have

$$(\tilde{\sigma}^{-1}(1/y))^{-1}y^{-1} \sim (\tilde{\sigma}^{-1}((2\log t)^{-\frac{1}{2}}))^{-1}(2\log t)^{-\frac{1}{2}},$$

and consequently,

$$\begin{aligned} t\mu &\sim t(2\pi)^{-\frac{1}{2}}H_\alpha(\tilde{\sigma}^{-1}((2\log t)^{-\frac{1}{2}}))^{-1}(2\log t)^{-\frac{1}{2}}t^{-1}e^{-x}2H_\alpha^{-1}(\pi\log t)^{\frac{1}{2}}\tilde{\sigma}^{-1}((2\log t)^{-\frac{1}{2}}) \\ &= \exp(-x) \quad \text{as} \quad t \to \infty. \end{aligned}$$

The theorem is proved.

REMARK. Berman [2] proves Theorem 5.2 under different conditions which do not seem to be weaker than ours.

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