DISSERTATION

Asymptotic properties of solutions to wave equations with time-dependent dissipation

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Suche das Einfache und mißtraue ihm.
(Alfred North Whitehead)

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Freiberg, April 2005 Jens Wirth

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1 Introduction

1.1 Background

In mathematical physics, hyperbolic partial differential equations are used to describe evolutionary processes with the property that information propagate with a finite speed. One of the simplest and therefore standard models is that of the *free wave equation*

$$u_{tt} - c^2 \Delta u = 0, \tag{1.1.1}$$

where c denotes the speed of propagation and $\Delta = \sum_{i=1}^n \partial_i^2$ the usual Laplacian in Euclidean space. This equation arises together with certain initial and boundary conditions if one models the oscillatory behaviour of vibrating strings, membranes or the propagation of sound. Here $u = u(t,x)$ denotes a displacement or a pressure and thus a time-dependent scalar field. In electrodynamics the unknowns are the electric and the magnetic field, which satisfy in vacuum a related equation. The investigation of this problem is related to names like J.R. d'Alembert, who solved the problem in one space dimension, or J. Fourier who developed the method of Fourier series, while studying such kinds of problems. In three space dimensions S.-D. Poisson understood the relation to geometry and spherical means and gave a first explicit representation of solutions, later generalized by G.R. Kirchhoff.

In general, one can not expect that (1.1.1) models real-world problems. Oscillations of vibrating strings and membranes are described by quasi-linear equations due to a relation between length/area deformation and energy, the above given equation is the linearization at its trivial solution $u \equiv 0$. A second idealisation is that in the above equation we excluded the influence of matter, the strings oscillate in vacuum, the acoustics is considered inside an ideal gas and electrodynamics far away from matter and charges.

The propagation of electro-magnetic waves inside matter but without free charges, as it takes place inside conductors, is described by the so-called *telegraph equation*

$$\vec{E}_{tt} - c^2 \Delta \vec{E} + \frac{\sigma}{\epsilon} \vec{E}_t = 0, \tag{1.1.2}$$

where $c^2 \mu \epsilon = 1$ and σ denotes the conductibility. Similar, for oscillations of membranes a dissipating environment leads to the occurance of this kind of first order term.

For the study of wave and damped wave equations the introduction of a so-called *hyperbolic energy* is an important step. Inspired by physics one defines

$$E(u;t) = \frac{1}{2} \int \left(c^2 |\nabla u|^2 + |u_t|^2 \right) \mathrm{d}x \tag{1.1.3}$$

for a solution $u = u(t,x)$ of the wave equation (1.1.1) or the *damped wave equation*

$$u_{tt} - c^2 \Delta u + b u_t = 0 \tag{1.1.4}$$

with positive coefficient $b > 0$. Then, if one integrates over $\mathbb{R}^n$ or assumes Dirichlet boundary conditions $u(t,x)|_{\partial\Omega} = 0$ on a domain Ω with sufficiently regular boundary, integration by parts yields immediately

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u;t) = \int \big(c^2\nabla u\cdot\nabla u_t + u_t u_{tt}\big)\mathrm{d}x = \int u_t \Box_c u \mathrm{d}x = -b\int |u_t|^2 \mathrm{d}x \le 0 \tag{1.1.5}$$

for real-valued solutions. In the case of complex solutions the calculations are similar. As usual we denote by $\Box_c = \partial_t^2 - c^2\Delta$ the *d'Alembertian*. Later on we will set $c=1$ and omit the c in the notation. Thus, for free waves the energy is preserved,

$$E(u;t) = E(u;0), \tag{1.1.6}$$

while for damped waves it is monotonically decreasing. This simple calculation gives no information whether it tends to zero or remains positive for all times. The question for the precise decay rate and the asymptotic description of the solutions may be an old question, but it is still a young and active research area.

1.2 Objectives

The aim of this thesis is to give a contribution to this research field by studying special classes of time-dependent dissipation terms and their influence on the asymptotic properties of the solutions. To be more concrete, we investigate the Cauchy problem

$$\Box u + b(t)u_t = 0, \qquad u(0,\cdot) = u_1, \quad \mathrm{D}_t u(0,\cdot) = u_2, \tag{1.2.1}$$

where $\Box = \Box_1 = \partial_t^2 - \Delta$ and $\mathrm{D}_t = -i\partial_t$ in the case of Schwartz or Sobolev data. Main tasks are

- to understand structural properties of the solution in terms of structural properties of its representation,
- the derivation of energy and more general L^p–L^q decay estimates for dual indices p and q,
- asymptotic descriptions of the solutions and, related to that, the sharpness of the obtained decay estimates,
- ideas to classify dissipation terms related to their influence on the representations.

Energy decay estimates for variable coefficient dissipation terms are available from the literature under several assumptions. We refer only to some of the most cited references, the concrete relation to our results will be given later throughout the main part of the thesis. Basic results for the Cauchy problem or initial boundary value problems on exterior domains are the weighted energy inequalities based on the paper of A. Matsumura, [Mat77], later reconsidered by H. Uesaka in [Ues80] or F. Hirosawa and H. Nakazawa, [HN03]. Furthermore, K. Mochizuki gave in [Moc77] and later together with H. Nakazawa in [MN96] the answer to the question under which conditions the hyperbolic energy tends to zero.

So the main objective of this thesis is the derivation of the more general L^p–L^q estimates for solutions. These estimates rely on more structural properties of representations of solutions than estimates in the L^2-scale and can not be deduced by the same methods as the above mentioned results. Our approach is based on the one hand on explicit representations in a special case and on the other hand on asymptotic representations combined with an extensive phase plane analysis under more general assumptions, mostly adapted from the treatment of degenerate hyperbolic problems. For completeness we mention the book of K. Yagdjian, [Yag97], and the consideration of wave equations with increasing speed of propagation by M. Reissig and K. Yagdjian, [RY00], for the combination with dissipation and mass terms [Rei01] and [HR03]. The method we used is based on the Fourier transform and Fourier multiplier representations (also called WKB representations) of solutions, therefore we consider only purely time-dependent dissipation terms. The consideration of time and spatial dependencies in the coefficient in the language of pseudo-differential and Fourier integral operators yields essential difficulties in connection with the time asymptotics and is therefore not considered here.

The schedule is as follows. First, we give a short overview on known and now merely classical results for L^p–L^q estimates in the case of free and damped wave equations. Furthermore, we will sketch some of the main results of the thesis related to these classical results. This will complete this introductory chapter. Later on in Chapter 2 we study one of the most important examples for time-dependent dissipation, the scale-invariant or the Euler-Poisson-Darboux type equation. It turns out that this example provides us with a lot of ideas and gives some feeling for the more general results proven later. Chapters 3 and 4 contain the main ideas and provide the solution representations for the two occurring cases of dissipative wave equations and their applications to derive L^p–L^q decay estimates together with their sharpness. Later on, in Chapter 5 we are concerned with estimates for solutions and estimates for higher order energies. Furthermore, the so-called diffusive or parabolic structure of damped wave equations will be considered there.

1.3 Asymptotic properties for special model equations

1.3.1 The Cauchy problem for free waves

As mentioned before, for free waves the hyperbolic energy $E(u;t)$ is a preserved quantity. In contrast to this the solution spreads out with a constant speed of propagation. This means, if the data are given within a ball of radius R, after time t the solution lives in a ball of radius $R + ct$. In odd-dimensional space the Huygens' principle is valid and free waves have also a backward wave front, and therefore, the solution vanishes inside a ball of radius $ct - R$ for $ct \geq R$.

The conservation of energy heuristically gives for this enlarging region a decay of the solutions in $L^\infty(\mathbb{R}^n)$. The decay rate may be guessed from the spreading of this angular domain.[1] If one assumes that the solution is bounded and distributes in a uniform way one may guess

$$||(\partial_t, \nabla)u(t,\cdot)||_\infty^2 \ \operatorname{meas}\big(B(R+ct) \setminus B(ct)\big) \sim \mathrm{const} \tag{1.3.1}$$

and therefore

$$||(\partial_t, \nabla)u(t,\cdot)||_\infty \sim (1+t)^{-\frac{n-1}{2}}, \qquad n \text{ odd.} \tag{1.3.2}$$

Of course, this reasoning is incomplete and can only be used to get the $\gtrsim$ part of the statement.

A rigorous proof of the $\lesssim$ estimate can be deduced from explicit representations of solutions like the Kirchhoff formula and was given by W. von Wahl, [vW71]. Using representations by Fourier multipliers these estimates arise in papers of W. Littman, [Lit73], R.S. Strichartz, [Str70], P. Brenner, [Bre75], and H. Pecher, [Pec76], to name just a few of the most cited references. The usual form of these L^p–L^q *estimates* is obtained by interpolating the L^∞-estimate for the derivatives with the simple

[1] This intuitive motivation for the L^∞-decay of free waves is taken from a talk of O. Liess given at the Workshop "Pseudo-differential Methods for Evolution Equations" at the Bimestre Intensivo "Microlocal Analysis and Related Subjects" at the Universitá di Torino/Politecnico di Torino, May-June 2003.

conservation property for the hyperbolic energy and reads as

$$||(\partial_t,\nabla)u(t,\cdot)||_{L^q} \le C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\{||u_1||_{W_p^{N_p+1}} + ||u_2||_{W_p^{N_p}}\} \tag{1.3.3}$$

for $p\in[1,2]$, q the corresponding dual index, i.e. $pq=p+q$, and $N_p = \left\lceil n\left(\frac{1}{p}-\frac{1}{q}\right)\right\rceil + 1$. This formulation of the estimate is taken from the book of R. Racke, [Rac92, Chapter 2].

It may be extended to more general p and q forming a not necessarily dual pair, furthermore the regularity may be improved for $p\in(1,2]$ using Besov spaces.

1.3.2 Damped wave equations

Now we turn to the consideration of solutions to (1.1.4), where the hyperbolic energy is monotonically decreasing. The first one who proved sharp decay estimates for solutions to the Cauchy problem was A. Matsumura, [Mat76]. He showed, that

$$||\mathrm{D}_t^k\mathrm{D}_x^\alpha u(t,\cdot)||_{L^2} \le C(1+t)^{-k-\frac{|\alpha|}{2}}\{||u_1||_{H^{k+|\alpha|}} + ||u_2||_{H^{k+|\alpha|-1}}\} \tag{1.3.4}$$

and an L^2–L^∞ estimate of the form

$$||\mathrm{D}_t^k\mathrm{D}_x^\alpha u(t,\cdot)||_{L^\infty} \le C(1+t)^{-\frac{n}{4}-k-\frac{|\alpha|}{2}}\{||u_1||_{H^{\lceil n/2\rceil+k+|\alpha|}} + ||u_2||_{H^{\lceil n/2\rceil+k+|\alpha|-1}}\}. \tag{1.3.5}$$

Both estimates can be improved by assuming a further L^p-property of the data, $p\in[1,2]$. For the complete structure of these improved estimates we refer to the original paper of Matsumura, [Mat76] or the discussions later on in this thesis.

These decay estimates show a difference in the decay order for time and spatial derivatives like for parabolic equations. We remark, that estimates (1.3.4) and (1.3.5) coincide in the decay order with the corresponding estimates for the heat equation given e.g. in the paper of G. Ponce, [Pon85].

In particular the estimates imply a decay rate for the hyperbolic energy of the form

$$E(u;t) = \mathcal{O}(t^{-1}), \qquad t\to\infty. \tag{1.3.6}$$

The estimates of Matsumura hint to an underlying parabolic structure. The works of Yang H. and A.J. Milani, [YM00], K. Nishihara, [Nis97], [Nis03], [MN03], and T. Narazaki, [Nar04], make this relation more precise. The observation is referred to as the *diffusion phenomenon* and goes back to a result of Hsiao L. and Liu T.-P., [HL92], for the compressible flow through porous media.

If we consider the two Cauchy problems

$$\begin{cases}\Box u + u_t = 0,\\ u(0,\cdot)=u_1, \quad \mathrm{D}_t u(0,\cdot)=u_2,\end{cases} \qquad \text{and} \qquad \begin{cases} w_t = \Delta w,\\ w(0,\cdot) = w_0 = u_1 + iu_2,\end{cases}$$

the solutions behave asymptotically equivalent in the sense that, [YM00, Theorem 2.1],

$$||u(t,\cdot)-w(t,\cdot)||_{L^\infty} = \mathcal{O}(t^{-\frac{n}{2}-1}), \qquad t\to\infty, \tag{1.3.7}$$

or, in three space dimensions and with v related to the solution of the free wave equation to data $u_1,u_2\in L^p(\mathbb{R}^n)$, $p\in[1,\infty]$, [Nis03, Theorem 1.1],

$$||u(t,\cdot)-w(t,\cdot)-e^{-\frac{t}{2}}v(t,\cdot)||_{L^q} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-1}||(u_1,u_2)||_{L^p} \tag{1.3.8}$$

for $p \leq q$. This means, the norm of the difference $(u - w)$ decays faster than the norm of u and w itself (where the decay rates obtained by A. Matsumura are sharp). Furthermore, if we exclude the case $p = q = 2$, the solutions of the free and also of the damped wave equations are less regular than L^q. Thus at least some weak singularities cancel on the left hand side and are thus described by the free wave equation and decay like $e^{-\frac{t}{2}}$.

Recently, T. Narazaki, [Nar04], generalized this result of K. Nishihara to arbitrary space dimensions $n \geq 2$. He showed that the restriction of the solution to small frequencies can be described by the heat equation, while large frequencies behave up to an exponential decay factor like a modification of free waves.

One consequence of these estimates is that for the semi-linear damped wave equation

$$\Box u + u_t = |u|^p, \tag{1.3.9}$$

the critical exponent for global existence of small data solutions is given by the Fujita exponents $p_c(n) = 1 + 2/n$, [Fuj66], like for parabolic equations. This result is due to G. Todorova and B. Yordanov, [TY01], and independently to [Nis03] for $n = 3$.

1.3.3 Damped wave equations on domains

For completeness we will give two remarks on damped wave equations on domains. Let for this $\Omega \subseteq \mathbb{R}^n$ be a domain with smooth boundary. Then for the initial boundary value problem

$$\begin{cases} \Box u + u_t = 0, & x \in \Omega, t \geq 0, \\ u(0,\cdot) = u_1, \quad \mathrm{D}_t u(0,\cdot) = u_2, & x \in \Omega, \\ u(t,x) = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \tag{1.3.10}$$

where for simplicity the compatibility conditions $u_1\big|_{\partial\Omega}, u_2\big|_{\partial\Omega} = 0$ are assumed[2], the situation is quite different and results depend on geometric properties of the domain.

If the domain Ω is bounded, the hyperbolic energy decays exponentially, i.e. there exists a constant c such that

$$E(u;t) = \mathcal{O}(e^{-ct}), \qquad t \to \infty. \tag{1.3.11}$$

This result is merely classical and proven for general dissipative systems with variable coefficients by M.E. Taylor and J. Rauch, [RT74]. For the case of constant dissipation it can be obtained by the energy method combined with Friedrichs inequality.

The situation appears to be quite different, if the domain is exterior. In this case the energy decay rate is the same as for the Cauchy problem. Furthermore, following R. Ikehata, [Ike02], the diffusion phenomenon is also valid in this case. He proved for the solution u of (1.3.10) and w of the corresponding parabolic problem

$$\begin{cases} w_t = \Delta w, & x \in \Omega, t \geq 0, \\ w(0,\cdot) = w_0 = u_1 + i u_2, & x \in \Omega, \\ w(t,x) = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \tag{1.3.12}$$

the L^2-estimate

$$||u(t,\cdot) - w(t,\cdot)||_{L^2} \leq C(1+t)^{-\frac{1}{2}} [\log(e+t)]^{-1} \left\{ ||u_1||_{H^2} + ||u_2||_{H^1} \right\}, \tag{1.3.13}$$

[2] as long as the data are regular enough to give these conditions a meaning

while the solutions u and w are in general only bounded (and not decaying). In view of the estimates by K. Nishihara, [Nis03], and in general T. Narazaki, [Nar04], R. Ikehata conjectured that the sharp rate for the exterior problem will also be $(1+t)^{-1}$.

1.4 Selected results

We will conclude this introductory chapter with several selected results of the thesis. For simplicity, and in order to make the situation not to complicate, we assume here that the coefficient function $b = b(t)$ is a positive, smooth and monotone function of t, which satisfies

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}t^k} b(t)\right| \le C_k b(t) \left(\frac{1}{1+t}\right)^k \tag{1.4.1}$$

for all $k \in \mathbb{N}_0$.

The first result is mentioned for completeness. It states that integrable coefficients are asymptotically negligible.

Result 1. *Assume* $\int_0^\infty b(t)\mathrm{d}t < \infty$. *Then the solutions of* (1.2.1) *are asymptotically free.* [Theorem 3.1]

In fact, this result is a special case of the following one. We denote by

$$\lambda(t) = \exp\left\{\frac{1}{2}\int_0^t b(\tau)\mathrm{d}\tau\right\} \tag{1.4.2}$$

an auxiliary function.

Result 2. *Assume* $\limsup_{t\to\infty} tb(t) < 1$. *Then the solution* $u = u(t,x)$ *of* (1.2.1) *satisfies the* L^p*–*L^q *estimate*

$$||(\partial_t,\nabla)u(t,\cdot)||_{L^q} \le C\frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{||u_1||_{W_p^{N_p+1}} + ||u_2||_{W_p^{N_p}}\right\} \tag{1.4.3}$$

for $p \in (1,2]$, q *the corresponding dual index and* $N_p > \left(\frac{1}{p}-\frac{1}{q}\right)$. [Theorem 3.24]
Furthermore, $\lambda(t)u(t,x)$ *is asymptotically free.* [Theorem 3.26]

We will refer to dissipation terms, which lead to this kind of estimates, as *non-effective (weak) dissipation.* Non-effectivity means that the asymptotic properties are still described by the free wave equation, at least after modifying it by the energy decay rate $\lambda^{-1}(t)$.

Related to this case is the coefficient function considered in Chapter 2, $b(t) = \frac{\mu}{1+t}$. Then the resulting L^q–L^q decay estimate depends on the size of the coefficient μ and the value of p. For large values of μ there is some relation to the following case, referred to as *effective dissipation.*

Result 3. *Assume* $tb(t) \to \infty$ *as* $t \to \infty$. *Then the solution* $u = u(t,x)$ *of* (1.2.1) *satisfies the* L^p*–*L^q *estimate*

$$||(\partial_t,\nabla)u(t,\cdot)||_{L^q} \le C\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\left\{||u_1||_{W_p^{N_p+1}} + ||u_2||_{W_p^{N_p}}\right\} \tag{1.4.4}$$

for $p \in (1,2]$, q *the corresponding dual index and* $N_p > n\left(\frac{1}{p}-\frac{1}{q}\right)$. [Theorem 4.25]

In this case the question of sharpness of the above estimate is closely related to the diffusion phenomenon. Note that for the corresponding parabolic surrogate,

$$w_t = \frac{1}{b(t)} \Delta w, \tag{1.4.5}$$

the above given L^p–L^q estimate is sharp in the decay order. In order to state the sharpness of Result 3, we therefore show that the difference $v(t,x) = u(t,x) - w(t,x)$ decays faster than the above given rate. For $p = 2$ this is done in Theorem 5.22 and Corollary 5.23.

In the case that $1/b(t)$ becomes integrable, Result 3 gives no decay at all. This case will be referred to as the case of *over-damping* and is characterised by the following remarkable property.

Result 4. *Assume* $\int_0^\infty \frac{\mathrm{d}\tau}{b(\tau)} < \infty$*. Then the solution* $u = u(t,x)$ *of* (1.2.1) *with data from* $L^2(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$ *converges as* $t \to \infty$ *to the asymptotic state*

$$u(\infty, x) = \lim_{t\to\infty} u(t,x)$$

in $L^2(\mathbb{R}^n)$*. Furthermore, this limit is non-zero for non-zero initial data.* [Theorem 4.27]

2 Scale-invariant weak dissipation

In this chapter we are concerned with the Cauchy problem

$$\Box u + \frac{\mu}{1+t} u_t = 0, \qquad u(0,\cdot) = u_1, \quad \mathrm{D}_t u(0,\cdot) = u_2 \tag{2.0.1}$$

with a special choice of a time-dependent dissipation term. The coefficient μ is non-negative real number. The main result is collected in Theorem 2.1 and allows us to conclude energy and L^p–L^q decay estimates, Theorems 2.7 and 2.8, (published in [Wir04]) and also L^p–L^q estimates for higher order energies, Theorem 2.9.

This Cauchy problem is of particular interest for several reasons. At first this equation has more symmetries than other problems with variable coefficient dissipation. If we apply an hyperbolic scaling, i.e. if we substitute the variables according to

$$\tilde{u}(t,x) = u(\sigma(t+1) - 1, \sigma x) \tag{2.0.2}$$

with $\sigma > 0$, the function $\tilde{u}$ satisfies the same problem with related data. We say, equation (2.0.1) is scale-invariant. As will be seen later, this implies that we can compute explicit representations of solutions in terms of known *special functions*.

Problem (2.0.1) is closely related to the Euler-Poisson-Darboux (EPD) equation

$$\Delta u = u_{tt} + \frac{\mu}{t} u_t, \tag{2.0.3}$$

studied by L. Euler, [Eul70, Sectio secunda, Caput IV], and G. Darboux, [Dar89, Libre IV, Chapitre III], and later L. Asgeirsson, [Asg36], in connection with the theory of spherical means. For an detailed exposition of classical results for the Euler-Poisson-Darboux equation see the paper of A. Weinstein, [Wei54], and the literature cited therein. More closely related to our approach is the treatment of the Euler-Poisson-Darboux equation in the book of R.W. Carroll and R.E. Showalter, [CS76], where convolution representations of solutions to the (singular) Cauchy problem for the EPD equation were given.

The special Cauchy problem considered in this chapter turns out to be the basic example for a wave equation with time-dependent dissipation. On page 30 we give an interpretation of one of the main results and relate it to the forthcoming considerations of this thesis.

2.1 Multiplier Representation

Reduction to Bessel's equation. At first we construct the fundamental solution of the corresponding ordinary differential equation in the Fourier image. Let $\hat{u}(t,\xi) = \mathcal{F}_{x\to\xi}[u]$ be the partial Fourier transform,

$$\hat{u}(t,\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t,x)\mathrm{d}x. \tag{2.1.1}$$

Then $\hat{u}$ satisfies the ordinary differential equation

$$\hat{u}_{tt} + |\xi|^2 \hat{u} + \tfrac{\mu}{1+t} \hat{u}_t = 0. \tag{2.1.2}$$

Following K. Taniguchi and Y. Tozaki, [TT80], we use the relation of this differential equation to *Bessel's equation* in order to construct a system of linearly independent solutions. We substitute $\tau = (1+t)|\xi|$ and get

$$\hat{u}_{\tau\tau} + \tfrac{\mu}{\tau} \hat{u}_\tau + \hat{u} = 0. \tag{2.1.3}$$

If we make the ansatz $\hat{u} = \tau^\rho w(\tau)$ this leads to

$$\begin{aligned} 0 =&\rho(\rho-1)\tau^{\rho-2} w + 2\rho\tau^{\rho-1} w' + \tau^\rho w'' \\ &+ \frac{\mu}{\tau}(\rho \tau^{\rho-1} w + \tau^\rho w') + \tau^\rho w \\ &+ \tau^{\rho-2}\big(\tau^2 w'' + (\mu+2\rho)\tau w' + (\tau^2 + \rho(\rho-1+\mu))w\big), \end{aligned}$$

i.e. by the choice of $\mu + 2\rho = 1$,

$$\rho = -\frac{\mu-1}{2}, \tag{2.1.4}$$

and hence $\rho - 1 + \mu = -\rho$, we get Bessel's differential equation

$$\tau^2 w'' + \tau w' + (\tau^2 - \rho^2) w = 0 \tag{2.1.5}$$

of order $\pm\rho$. Note that our assumption on μ implies $\rho \in (-\infty, \frac{1}{2}]$. A system of linearly independent solutions of (2.1.5) is given by the pair of Hankel functions $\mathcal{H}^\pm_\rho(\tau)$. For details on these functions we refer to the treatment in the book of G.N. Watson, [Wat22], or the short overview on basic properties contained in Appendix B.1.

Hence

$$w_+(\tau) = \tau^\rho \mathcal{H}^+_\rho(\tau), \qquad w_-(\tau) = \tau^\rho \mathcal{H}^-_\rho(\tau), \tag{2.1.6}$$

with ρ determined by (2.1.4) gives a pair of linearly independent solutions of (2.1.3).

Representation of the Fourier multiplier. We are interested in particular solutions $\Phi_1(t, t_0, \xi)$ and $\Phi_2(t, t_0, \xi)$ of (2.1.2) subject to initial conditions

$$\Phi_1(t_0, t_0, \xi) = 1, \qquad \mathrm{D}_t \Phi_1(t_0, t_0, \xi) = 0, \tag{2.1.7a}$$

$$\Phi_2(t_0, t_0, \xi) = 0, \qquad \mathrm{D}_t \Phi_2(t_0, t_0, \xi) = 1, \tag{2.1.7b}$$

where the parameter $t_0 > -1$ describes the initial time level in order to obtain a representation $\hat{u}(t,\xi) = \Phi_1(t, 0, \xi)\hat{u}_1(\xi) + \Phi_2(t, 0, \xi)\hat{u}_2(\xi)$. We collect these Φ_i in the fundamental matrix

$$\Phi(t, t_0, \xi) = \begin{pmatrix} \Phi_1(t, t_0, \xi) & \Phi_2(t, t_0, \xi) \\ \mathrm{D}_t\Phi_1(t, t_0, \xi) & \mathrm{D}_t\Phi_2(t, t_0, \xi) \end{pmatrix}. \tag{2.1.8}$$

For $w_\pm(t, \xi) = w_\pm((1+t)|\xi|)$ we have the following initial values

$$w_+(t_0, \xi) = (1+t_0)^\rho |\xi|^\rho \mathcal{H}^+_\rho\big((1+t_0)|\xi|\big), \tag{2.1.9a}$$

$$\partial_t w_+(t_0, \xi) = (1+t_0)^\rho |\xi|^{\rho+1} \mathcal{H}^+_{\rho-1}\big((1+t_0)|\xi|\big), \tag{2.1.9b}$$

$$w_-(t_0, \xi) = (1+t_0)^\rho |\xi|^\rho \mathcal{H}^-_\rho\big((1+t_0)|\xi|\big), \tag{2.1.9c}$$

$$\partial_t w_-(t_0, \xi) = (1+t_0)^\rho |\xi|^{\rho+1} \mathcal{H}^-_{\rho-1}\big((1+t_0)|\xi|\big), \tag{2.1.9d}$$

which follow straightforward from the recurrence relations for Bessel functions. For instance we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)^\rho|\xi|^\rho\mathcal{H}^+_\rho((1+t)|\xi|)\right]\big|_{t=t_0}\\
&\quad=\rho(1+t)^{\rho-1}|\xi|^\rho\mathcal{H}^+_\rho((1+t)|\xi|)\big|_{t=t_0}+(1+t)^\rho|\xi|^\rho(\mathcal{H}^+_\rho)'((1+t)|\xi|)|\xi|\big|_{t=t_0}\\
&\quad=\Big((1+t_0)|\xi|\big(\mathcal{H}^+_\rho\big)'((1+t_0)|\xi|)+\rho\mathcal{H}^+_\rho((1+t_0)|\xi|)\Big)|\xi|^\rho(1+t_0)^{\rho-1}\\
&\quad=(1+t_0)^\rho|\xi|^{\rho+1}\mathcal{H}^+_{\rho-1}\big((1+t_0)|\xi|\big).
\end{aligned}$$

From these initial values we determine the constants $C_{i\pm}(t_0,\xi)$ in

$$\Phi_i(t,t_0,\xi)=C_{i+}(t_0,\xi)w_+(t,\xi)+C_{i-}(t_0,\xi)w_-(t,\xi),\qquad i=1,2, \tag{2.1.10}$$

such that (2.1.7) holds. This means, they have to satisfy

$$\begin{pmatrix} w_+(t_0,\xi) & w_-(t_0,\xi)\\ \mathrm{D}_tw_+(t_0,\xi) & \mathrm{D}_tw_-(t_0,\xi)\end{pmatrix}\begin{pmatrix} C_{1+}(t_0,\xi) & C_{2+}(t_0,\xi)\\ C_{1-}(t_0,\xi) & C_{2-}(t_0,\xi)\end{pmatrix}=I. \tag{2.1.11}$$

Hence, we have

$$\begin{pmatrix} C_{1+}(t_0,\xi) & C_{2+}(t_0,\xi)\\ C_{1-}(t_0,\xi) & C_{2-}(t_0,\xi)\end{pmatrix}=\frac{i}{\det W(t_0)}\begin{pmatrix} \mathrm{D}_tw_-(t_0,\xi) & -w_-(t_0,\xi)\\ -\mathrm{D}_tw_+(t_0,\xi) & w_+(t_0,\xi)\end{pmatrix}, \tag{2.1.12}$$

where

$$\begin{aligned}
\det W(t_0)&=\det\begin{pmatrix} w_+(t_0,\xi) & w_-(t_0,\xi)\\ \partial_tw_+(t_0,\xi) & \partial_tw_-(t_0,\xi)\end{pmatrix}\\
&=(1+t_0)^{2\rho}|\xi|^{2\rho+1}\det\begin{pmatrix}\mathcal{H}^+_\rho\big((1+t_0)|\xi|\big) & (\mathcal{H}^+_\rho)'\big((1+t_0)|\xi|\big)\\ \mathcal{H}^-_\rho\big((1+t_0)|\xi|\big) & (\mathcal{H}^-_\rho)'\big((1+t_0)|\xi|\big)\end{pmatrix}^T\\
&=-\frac{4i}{\pi}|\xi|^{2\rho}(1+t_0)^{2\rho-1},
\end{aligned}\tag{2.1.13}$$

using formula (B.1.8). Thus, we obtain for the fundamental solution

$$\begin{aligned}
\Phi_1(t,t_0,\xi)=&\frac{i\pi}{4}|\xi|^{-2\rho}(1+t_0)^{1-2\rho}\\
&\Big\{(1+t_0)^\rho|\xi|^{\rho+1}\mathcal{H}^-_{\rho-1}\big((1+t_0)|\xi|\big)\,(1+t)^\rho|\xi|^\rho\mathcal{H}^+_\rho\big((1+t)|\xi|\big)\\
&\quad-(1+t_0)^\rho|\xi|^{\rho+1}\mathcal{H}^+_{\rho-1}\big((1+t_0)|\xi|\big)\,(1+t)^\rho|\xi|^\rho\mathcal{H}^-_\rho\big((1+t)|\xi|\big)\Big\}\\
=&\frac{i\pi}{4}|\xi|\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\Big\{\mathcal{H}^-_{\rho-1}\big((1+t_0)|\xi|\big)\mathcal{H}^+_\rho\big((1+t)|\xi|\big)\\
&\quad-\mathcal{H}^+_{\rho-1}\big((1+t_0)|\xi|\big)\mathcal{H}^-_\rho\big((1+t)|\xi|\big)\Big\}
\end{aligned}\tag{2.1.14a}$$

and similarly

$$\begin{aligned}
\Phi_2(t,t_0,\xi)=&\frac{\pi}{4}\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\Big\{\mathcal{H}^-_\rho\big((1+t_0)|\xi|\big)\mathcal{H}^+_\rho\big((1+t)|\xi|\big)\\
&\quad-\mathcal{H}^+_\rho\big((1+t_0)|\xi|\big)\mathcal{H}^-_\rho\big((1+t)|\xi|\big)\Big\}.
\end{aligned}\tag{2.1.14b}$$

We collect the results in the following theorem.

Theorem 2.1. *Assume that $u = u(t,x)$ solves the Cauchy problem* (2.0.1) *for data $u_1, u_2 \in \mathcal{S}'(\mathbb{R}^n)$. Then the Fourier transform $\hat{u}(t,\xi)$ can be represented as*

$$\hat{u}(t,\xi) = \sum_{j=1,2} \Phi_j(t,0,\xi)\hat{u}_j(\xi),$$

where the multipliers Φ_j are given by[1]

$$\Phi_1(t,t_0,\xi) = \frac{i\pi}{4}|\xi|\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{H}^-_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{H}^-_\rho\big((1+t)|\xi|\big)\\ \mathcal{H}^+_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{H}^+_\rho\big((1+t)|\xi|\big)\end{vmatrix}$$

and

$$\Phi_2(t,t_0,\xi) = \frac{\pi}{4}\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{H}^-_\rho\big((1+t_0)|\xi|\big) & \mathcal{H}^-_\rho\big((1+t)|\xi|\big)\\ \mathcal{H}^+_\rho\big((1+t_0)|\xi|\big) & \mathcal{H}^+_\rho\big((1+t)|\xi|\big)\end{vmatrix}.$$

Time derivatives. Next we need the time derivatives of these functions. Derivation with respect to t leads to

$$\mathrm{D}_t\Phi_1(t,t_0,\xi) = \frac{\pi}{4}|\xi|^2\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{H}^-_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{H}^-_{\rho-1}\big((1+t)|\xi|\big)\\ \mathcal{H}^+_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{H}^+_{\rho-1}\big((1+t)|\xi|\big)\end{vmatrix}$$

using $\rho\mathcal{H}^+_\rho(z) + z(\mathcal{H}^+_\rho)'(z) = z\mathcal{H}^+_{\rho-1}(z)$ and similarly for the second multiplier.

Corollary 2.2. *The time derivatives of the multipliers Φ_j are given by*

$$\mathrm{D}_t\Phi_1(t,t_0,\xi) = \frac{\pi}{4}|\xi|^2\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{H}^-_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{H}^-_{\rho-1}\big((1+t)|\xi|\big)\\ \mathcal{H}^+_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{H}^+_{\rho-1}\big((1+t)|\xi|\big)\end{vmatrix}$$

and

$$\mathrm{D}_t\Phi_2(t,t_0,\xi) = \frac{i\pi}{4}|\xi|\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{H}^-_\rho\big((1+t_0)|\xi|\big) & \mathcal{H}^-_{\rho-1}\big((1+t)|\xi|\big)\\ \mathcal{H}^+_\rho\big((1+t_0)|\xi|\big) & \mathcal{H}^+_{\rho-1}\big((1+t)|\xi|\big)\end{vmatrix}.$$

It is possible to obtain a similar expression for higher order time derivatives by induction.

Corollary 2.3. *The higher order time derivatives of the multipliers Φ_j are given by*

$$\mathrm{D}^k_t\Phi_1(t,t_0,\xi) = \frac{i\pi}{4}C_k|\xi|^{k+1}\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{H}^-_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{H}^-_{\rho-k}\big((1+t)|\xi|\big)\\ \mathcal{H}^+_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{H}^+_{\rho-k}\big((1+t)|\xi|\big)\end{vmatrix}$$

and

$$\mathrm{D}^k_t\Phi_2(t,t_0,\xi) = -\frac{\pi}{4}C_k|\xi|^k\frac{(1+t)^\rho}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{H}^-_\rho\big((1+t_0)|\xi|\big) & \mathcal{H}^-_{\rho-k}\big((1+t)|\xi|\big)\\ \mathcal{H}^+_\rho\big((1+t_0)|\xi|\big) & \mathcal{H}^+_{\rho-k}\big((1+t)|\xi|\big)\end{vmatrix}$$

with $C_k = \rho^{k-1}\frac{\Gamma(\rho)}{\Gamma(\rho-k+1)}(-i)^k$ for $\rho \notin -\mathbb{N}_0$ (and the corresponding analytic continuation for the negative integers).

[1] $|\cdot|$ stands for determinants ...

Representation by real-valued functions. If we use the definition of $\mathcal{H}^{\pm}_{\rho}$ by the real-valued Bessel and Weber given in (B.1.6), we obtain an alternative characterisation of Φ by these functions

$$\Phi_1(t,t_0,\xi) = -\frac{\pi}{2}|\xi|\frac{(1+t)^{\rho}}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{J}_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{J}_{\rho}\big((1+t)|\xi|\big)\\ \mathcal{Y}_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{Y}_{\rho}\big((1+t)|\xi|\big)\end{vmatrix}, \tag{2.1.15a}$$

$$\Phi_2(t,t_0,\xi) = i\frac{\pi}{2}\frac{(1+t)^{\rho}}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{J}_{\rho}\big((1+t_0)|\xi|\big) & \mathcal{J}_{\rho}\big((1+t)|\xi|\big)\\ \mathcal{Y}_{\rho}\big((1+t_0)|\xi|\big) & \mathcal{Y}_{\rho}\big((1+t)|\xi|\big)\end{vmatrix}, \tag{2.1.15b}$$

$$\mathrm{D}_t\Phi_1(t,t_0,\xi) = -i\frac{\pi}{2}|\xi|^2\frac{(1+t)^{\rho}}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{J}_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{J}_{\rho-1}\big((1+t)|\xi|\big)\\ \mathcal{Y}_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{Y}_{\rho-1}\big((1+t)|\xi|\big)\end{vmatrix}, \tag{2.1.15c}$$

$$\mathrm{D}_t\Phi_2(t,t_0,\xi) = -\frac{\pi}{2}|\xi|\frac{(1+t)^{\rho}}{(1+t_0)^{\rho-1}}\begin{vmatrix}\mathcal{J}_{\rho}\big((1+t_0)|\xi|\big) & \mathcal{J}_{\rho-1}\big((1+t)|\xi|\big)\\ \mathcal{Y}_{\rho}\big((1+t_0)|\xi|\big) & \mathcal{Y}_{\rho-1}\big((1+t)|\xi|\big)\end{vmatrix}. \tag{2.1.15d}$$

In the case of non-integral ρ the representation can be simplified to

$$\begin{aligned}\Phi_1(t,t_0,\xi) &= \frac{\pi}{2}\csc(\rho\pi)|\xi|\frac{(1+t)^{\rho}}{(1+t_0)^{\rho-1}}\\ &\qquad\begin{vmatrix}\mathcal{J}_{-\rho+1}\big((1+t_0)|\xi|\big) & \mathcal{J}_{-\rho}\big((1+t)|\xi|\big)\\ -\mathcal{J}_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{J}_{\rho}\big((1+t)|\xi|\big)\end{vmatrix}, \qquad (2.1.16a)\\ \Phi_2(t,t_0,\xi) &= i\frac{\pi}{2}\csc(\rho\pi)\frac{(1+t)^{\rho}}{(1+t_0)^{\rho-1}}\\ &\qquad\begin{vmatrix}\mathcal{J}_{-\rho}\big((1+t_0)|\xi|\big) & \mathcal{J}_{-\rho}\big((1+t)|\xi|\big)\\ \mathcal{J}_{\rho}\big((1+t_0)|\xi|\big) & \mathcal{J}_{\rho}\big((1+t)|\xi|\big)\end{vmatrix}, \qquad (2.1.16b)\\ \mathrm{D}_t\Phi_1(t,t_0,\xi) &= i\frac{\pi}{2}\csc(\rho\pi)|\xi|^2\frac{(1+t)^{\rho}}{(1+t_0)^{\rho-1}}\\ &\qquad\begin{vmatrix}\mathcal{J}_{-\rho+1}\big((1+t_0)|\xi|\big) & \mathcal{J}_{-\rho+1}\big((1+t)|\xi|\big)\\ \mathcal{J}_{\rho-1}\big((1+t_0)|\xi|\big) & \mathcal{J}_{\rho-1}\big((1+t)|\xi|\big)\end{vmatrix}, \qquad (2.1.16c)\\ \mathrm{D}_t\Phi_2(t,t_0,\xi) &= -\frac{\pi}{2}\csc(\rho\pi)|\xi|\frac{(1+t)^{\rho}}{(1+t_0)^{\rho-1}}\\ &\qquad\begin{vmatrix}\mathcal{J}_{-\rho}\big((1+t_0)|\xi|\big) & \mathcal{J}_{-\rho+1}\big((1+t)|\xi|\big)\\ -\mathcal{J}_{\rho}\big((1+t_0)|\xi|\big) & \mathcal{J}_{\rho-1}\big((1+t)|\xi|\big)\end{vmatrix}. \qquad (2.1.16d)\end{aligned}$$

In the first and in the last formula we used $\csc(\rho\pi-\pi) = -\csc(\rho\pi)$.

2.2 Estimates

We use the isomorphism (order reduction)

$$\langle \mathrm{D}\rangle^s : L_{p,s}(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

where $\langle \mathrm{D}\rangle$ is the Fourier multiplier with symbol $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, to characterise the *Sobolev spaces of fractional order*[2] over $L^p(\mathbb{R}^n)$, $p\in(1,\infty)$. Note, that $\langle \mathrm{D}\rangle^s$ defines for all $s\in\mathbb{R}$ isomorphisms of the Schwartz space $\mathcal{S}'(\mathbb{R}^n)$.

[2] Bessel potential spaces were introduced by N. Aronszajn in the basic articles [AS61] and [AMS63]. For functional analytic properties and relations to other scales of spaces we refer to the book of T. Runst and W. Sickel, [RS96].

The representation of the fundamental matrix $\Phi(t,0,\xi)$ (as well as the knowledge about strictly hyperbolic problems) imply a natural regularity difference for the data of one Sobolev order. Therefore we define the following two operators corresponding to the Cauchy problem (2.0.1). On the one hand we are interested in the solution itself. Let therefore

$$\mathbb{S}(t,D) : (u_1, \langle \mathrm{D} \rangle^{-1} u_2) \mapsto u(t,\cdot) \tag{2.2.1}$$

be the *solution operator*. We used the order reductions in such a way that

$$\mathbb{S}(t,D) : L^2(\mathbb{R}^n, \mathbb{R}^2) \to L^2(\mathbb{R}^n).$$

On the other hand, we are interested in energy estimates. We define the *energy operator*

$$\mathbb{E}(t,\mathrm{D}) : (\langle D \rangle u_1, u_2) \mapsto (|\mathrm{D}|u(t,\cdot), \partial_t u(t,\cdot)), \tag{2.2.2}$$

with $\mathbb{E}(t,\mathrm{D}) : L^2(\mathbb{R}^n,\mathbb{R}^2) \to L^2(\mathbb{R}^n,\mathbb{R}^2)$. For both operators we will give norm estimates from L^p scale to L^q with dual indices p and q, i.e. $p+q=pq$ and $p \le q$, and compare the obtained estimates with the known L^p–L^q estimates for wave and damped wave equations.

Properties of Bessel functions. To obtain norm estimates for the operator families $\mathbb{S}(t,D)$ and $\mathbb{E}(t,\mathrm{D})$, we have to review some of the main properties of Bessel functions for small and large arguments. For details we refer to [Wat22, §3.13 ,§3.52, §10.6 and §7.2].

Proposition 2.4.

1. *The function*
$$\Lambda_\nu(\tau) = \tau^{-\nu} \mathcal{J}_\nu(\tau)$$
is entire in ν and τ. Furthermore, $\Lambda_\nu(0) = \frac{2^\nu}{\Gamma(1+\nu)} \neq 0$ for $\nu \notin \{-1,-2,-3,\dots\}$.

2. *Weber's function $\mathcal{Y}_n(\tau)$ satisfies for integral n*
$$\mathcal{Y}_n(\tau) = \frac{2}{\pi} \mathcal{J}_n(\tau) \log \tau + A_n(\tau),$$
where $\tau^n A_n(\tau)$ is entire and non-zero for $\tau = 0$.

3. *The Hankel functions $\mathcal{H}^\pm_\nu(\tau)$ with $\tau \ge K$ can be written as*
$$\mathcal{H}^\pm_\nu(\tau) = e^{\pm i\tau} a^\pm_\nu(\tau),$$
where $a^\pm_\nu \in S^{-\frac{1}{2}}(K,\infty)$ is a classical symbol of order $-1/2$ on each interval (K,∞), $K>0$.

4. *For small arguments, $0 < \tau \le c < 1$, we have*
$$\left|\mathcal{H}^\pm_\nu(\tau)\right| \lesssim \begin{cases} \tau^{-|\nu|}, & \nu \neq 0, \\ -\log \tau, & \nu = 0. \end{cases}$$

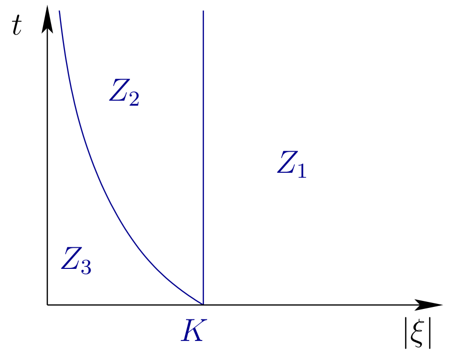

Figure 2.1: Sketch of the used decomposition of the phase space.

2.2.1 Consideration for a model operator

The model operator. Due to the special structure of the fundamental matrix $\Phi(t, t_0, \xi)$, and hence of the multipliers corresponding to $\mathbb{S}(t, D)$ and $\mathbb{E}(t, \mathrm{D})$, we consider the time dependent model multiplier

$$\Psi_{k,s,\rho,\delta}(t,\xi) = |\xi|^k \langle\xi\rangle^{s+1-k} \begin{vmatrix} \mathcal{H}^-_\rho(|\xi|) & \mathcal{H}^-_{\rho+\delta}\big((1+t)|\xi|\big) \\ \mathcal{H}^+_\rho(|\xi|) & \mathcal{H}^+_{\rho+\delta}\big((1+t)|\xi|\big) \end{vmatrix} \tag{2.2.3}$$

parameterised by $k, s, \rho, \delta \in \mathbb{R}$. Again we can write $\Psi_{k,s,\rho,\delta}(t, |\xi|)$ in terms of the real-valued Bessel functions of first and second kind. Similar to (2.1.15) and (2.1.16) we have

$$\Psi_{k,s,\rho,\delta}(t,\xi) = 2i|\xi|^k \langle\xi\rangle^{s+1-k} \begin{vmatrix} \mathcal{J}_\rho\big(|\xi|\big) & \mathcal{J}_{\rho+\delta}\big((1+t)|\xi|\big) \\ \mathcal{Y}_\rho\big(|\xi|\big) & \mathcal{Y}_{\rho+\delta}\big((1+t)|\xi|\big) \end{vmatrix} \tag{2.2.4a}$$

$$= 2i \csc(\rho\pi)\, |\xi|^k \langle\xi\rangle^{s+1-k} \begin{vmatrix} \mathcal{J}_{-\rho}\big(|\xi|\big) & \mathcal{J}_{-\rho-\delta}\big((1+t)|\xi|\big) \\ (-1)^\delta \mathcal{J}_\rho\big(|\xi|\big) & \mathcal{J}_{\rho+\delta}\big((1+t)|\xi|\big) \end{vmatrix}, \tag{2.2.4b}$$

the last line holds for $\rho \notin \mathbb{Z}$ and $\rho + \delta \notin \mathbb{Z}$.

In order to understand the model multiplier we subdivide the phase space $\mathbb{R}_+ \times \mathbb{R}^n$ into three zones,

$$Z_1 = \{K \le |\xi|\}, \qquad Z_2 = \{|\xi| \le K \le (1+t)|\xi|\}, \qquad Z_3 = \{(1+t)|\xi| \le K\}, \tag{2.2.5}$$

as sketched in Figure 2.2.1. This decomposition reflects algebraic properties of the representation of the multiplier (and therefore it is different from the decompositions used later on).

L^2–L^2 estimates for the model multiplier. By Plancherel's theorem L^2–L^2 estimates of the operator $\Psi_{k,s,\rho,\delta}(t, \mathrm{D})$ correspond to L^∞ estimates of the corresponding multiplier $\Psi_{k,s,\rho,\delta}(t,\xi)$. Furthermore, Proposition B.1.1. implies that the operator norm coincides with the L^∞-norm of its multiplier.

Lemma 2.5. *It holds $\Psi_{k,s,\rho,\delta}(t,\cdot) \in L^\infty(\mathbb{R}^n)$ for all $t \ge 0$ if and only if $s \le 0$ and $k \ge |\delta|$. Furthermore the estimate*

$$\big|\big|\Psi_{k,s,\rho,\delta}(t,\cdot)\big|\big|_\infty \lesssim \begin{cases} (1+t)^{-\frac12}, & |\rho| - k \le -\frac12, \\ (1+t)^{|\rho|-k}, & \rho \ne 0, |\rho| - k \ge -\frac12, \\ (1+t)^{-k}\log(e+t), & \rho = 0, k \le \frac12, \end{cases}$$

is valid.

Proof. We subdivide the proof into three parts corresponding to the three zones Z_1, Z_2 and Z_3.

$\boxed{Z_1}$ We use Proposition 2.4.3 together with the definition of the zone $K \le |\xi| \le (1+t)|\xi|$. Thus the multiplier is bounded in Z_1 iff $s \le 0$. It satisfies

$$\big|\Psi_{k,s,\rho,\delta}(t,\xi)\big| \lesssim (1+t)^{-\frac{1}{2}}$$

under this assumption.

$\boxed{Z_2}$ For $\rho \ne 0$ we can use Proposition 2.4.4 to conclude

$$\begin{aligned}\big|\Psi_{k,s,\rho,\delta}(t,\xi)\big| &\lesssim |\xi|^{k-|\rho|}(1+t)^{-\frac{1}{2}}|\xi|^{-\frac{1}{2}}\\ &\lesssim \begin{cases}(1+t)^{-\frac{1}{2}}, & |\rho|-k \le -\frac{1}{2},\\ (1+t)^{|\rho|-k}, & |\rho|-k \ge -\frac{1}{2}.\end{cases}\end{aligned}$$

For $\rho = 0$ we have to modify this estimate by the $\log$ term

$$\begin{aligned}\big|\Psi_{k,s,0,\delta}(t,\xi)\big| &\lesssim |\xi|^k \log\frac{2K}{|\xi|}\,(1+t)^{-\frac{1}{2}}|\xi|^{-\frac{1}{2}}\\ &\lesssim \begin{cases}(1+t)^{-\frac{1}{2}}, & k > \frac{1}{2},\\ (1+t)^{-k}\log(e+t), & k \le \frac{1}{2}.\end{cases}\end{aligned}$$

$\boxed{Z_3}$ In this zone we use the representation of $\Psi_{k,s,\rho,\delta}(t,\xi)$ in terms of real-valued functions given by (2.2.4). For non-integral ρ and $\rho+\delta$ we can use the representation by Bessel functions of first kind to conclude[3]

$$\begin{aligned}&\big|\Psi_{k,s,\rho,\delta}(t,\xi)\big|\\ &\lesssim \Big||\xi|^{\rho}\mathcal{J}_{-\rho}(|\xi|)\,\big((1+t)|\xi|\big)^{-\rho-\delta}\mathcal{J}_{\rho+\delta}\big((1+t)|\xi|\big)\,(1+t)^{\rho+\delta}|\xi|^{k+\delta}\Big|\\ &\quad+\Big||\xi|^{-\rho}\mathcal{J}_{\rho}(|\xi|)\,\big((1+t)|\xi|\big)^{\rho+\delta}\mathcal{J}_{-\rho-\delta}\big((1+t)|\xi|\big)\,(1+t)^{-\rho-\delta}|\xi|^{k-\delta}\Big|\\ &=(1+t)^{|\rho|-k}\bigg(\Big|\Lambda_{-\rho}(|\xi|)\Lambda_{\rho+\delta}\big((1+t)|\xi|\big)(1+t)^{\rho-|\rho|}\big((1+t)|\xi|\big)^{k+\delta}\Big|\\ &\quad+\Big|\Lambda_{\rho}(|\xi|)\Lambda_{-\rho-\delta}\big((1+t)|\xi|\big)(1+t)^{-\rho-|\rho|}\big((1+t)|\xi|\big)^{k-\delta}\Big|\bigg)\\ &\lesssim(1+t)^{|\rho|-k}.\end{aligned}$$

The condition $k \ge |\delta|$ is necessary and sufficient for the boundedness in ξ.

For integral values of ρ or $\rho+\delta$ we use Weber's functions and Proposition 2.4.2. We sketch the estimate if both ρ and $\rho+\delta$ are integral. Then, we have

$$\begin{aligned}\Psi_{k,s,\rho,\delta} = &-\frac{4i}{\pi}|\xi|^k\,\langle\xi\rangle^{s+1-k}\log(1+t)\mathcal{J}_{\rho}(|\xi|)\mathcal{J}_{\rho+\delta}\big((1+t)|\xi|\big)\\ &+2i|\xi|^k\,\langle\xi\rangle^{s+1-k}\begin{vmatrix}\mathcal{J}_{\rho}\big(|\xi|\big) & \mathcal{J}_{\rho+\delta}\big((1+t)|\xi|\big)\\ A_{\rho}\big(|\xi|\big) & A_{\rho+\delta}\big((1+t)|\xi|\big)\end{vmatrix}\end{aligned}$$

[3] By estimating the difference structure of the multiplier by triangle inequality we do not lose information. For non-integral ρ the leading terms of the series expansions do not cancel.

and, hence,

$$\begin{aligned}\left|\Psi_{k,s,\rho,\delta}(t,\xi)\right| &\lesssim \log(e+t)\ (1+t)^{-|\rho|-k} + (1+t)^{|\rho|-k}\\ &\lesssim \begin{cases}(1+t)^{|\rho|-k}, & \rho\neq 0,\\ (1+t)^{-k}\log(e+t), & \rho=0.\end{cases}\end{aligned}$$

If only one of both indices is an integer we have to mix the representations.[4] □

L^p–L^q estimates for the model multiplier. Using the stationary phase method combined with a decomposition of the phase space into zones, we obtain also L^p–L^q estimates. We consider the model operator

$$\Psi_{k,s,\rho,\delta}(t,\mathrm{D}) : u(x) \mapsto \mathcal{F}^{-1}\big[\Psi_{k,s,\rho,\delta}(t,\xi)\hat{u}(\xi)\big](x) \tag{2.2.6}$$

from $L_{p,r}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, (p,q) a dual pair with $p\le q$. We choose r to be the smallest value, such that the operator has this mapping property. For all fixed $t\ge 0$ the multiplier $\Psi_{k,s,\rho,\delta}(t,\xi)$ can be decomposed into a sum of functions, which consist for large ξ of a phase $e^{\pm it|\xi|}$ and a further symbol of order zero.

Theorem 2.6. *Assume $p\in(1,2]$, q such that $pq=p+q$. Let further $k\ge|\delta|$. Then the model operator* (2.2.6) *satisfies the norm estimate*

$$\begin{aligned}&||\Psi_{k,s,\rho,\delta}(t,\mathrm{D})||_{p,r\to q}\\ &\lesssim \begin{cases}(1+t)^{-\frac{n-1}{2}\left(\frac1p-\frac1q\right)-\frac12}, & d>\frac12,\\ (1+t)^{-n\left(\frac1p-\frac1q\right)+|\rho|-k}, & \rho\neq0, d\le\frac12,\\ (1+t)^{-n\left(\frac1p-\frac1q\right)+\theta\epsilon-k}(\log(e+t))^{1-\theta}, & \rho=0, d<\frac12+\epsilon, \epsilon>0,\end{cases}\end{aligned}$$

for $d=\frac{n+1}{2}\left(\frac1p-\frac1q\right)+k-|\rho|$ and $r=n\left(\frac1p-\frac1q\right)+s$. The interpolating constant θ in the last case is given by $\theta=\frac{n+1}{2\epsilon+1-2k}\left(\frac1p-\frac1q\right)$.

Proof. Again we decompose $\mathbb{R}_t\times\mathbb{R}^n_\xi$ into three zones. For this, we use a smooth cut-off function $\psi\in C^\infty(\mathbb{R}_+)$ with $\psi'\le0$, $\psi(r)=1$ for $r<1/2$ and $\psi(r)=0$ for $r>2$. Using this function, we define

$$\begin{aligned}\phi_1(t,\xi)&=1-\psi(|\xi|/K),\\ \phi_2(t,\xi)&=\psi(|\xi|/K)\big(1-\psi\big((1+t)|\xi|/K\big)\big),\\ \phi_3(t,\xi)&=\psi(|\xi|/K)\psi\big((1+t)|\xi|/K\big),\end{aligned}$$

such that $\phi_1(t,\xi)+\phi_2(t,\xi)+\phi_3(t,\xi)=1$. Thus we can decompose the multiplier $\Psi_{k,s,\rho,\delta}$ into the sum $\sum_{i=1,2,3}\phi_i(t,\xi)\Psi_{k,s,\rho,\delta}(t,\xi)$ and estimate each of the summands. We prove the estimate for $r=0$, i.e. we restrict the proof to the corresponding value

$$s=-n\left(\frac1p-\frac1q\right).$$

[4] For later reference we use only integral values of δ.

The remaining cases reduce to this one in an obvious way.

$\boxed{Z_1}$ Using Proposition 2.4 we decompose the representation of

$$\phi_1(t,\xi)\Psi_{k,s,\rho,\delta}(t,\xi)$$

as sum of two multipliers of the form

$$e^{\pm it|\xi|}a(|\xi|)b\big((1+t)|\xi|\big)$$

with symbols $a \in S^{s+\frac{1}{2}}(K/2,\infty)$ and $b \in S^{-\frac{1}{2}}(K/2,\infty)$ and $\operatorname{supp} a, b \in (K/2,\infty)$. We follow P. Brenner, [Bre75], to estimate the corresponding Fourier integral operator. The key tool is a dyadic decomposition together with Littman's lemma, see Appendix B.2, Lemma B.3. Let $\chi \in C_0^\infty(\mathbb{R}_+)$ be non-negative with support contained in $[1/2, 2]$ and

$$\sum_{j=-\infty}^{\infty} \chi(2^j r) = 1, \qquad \text{for } r \neq 0. \tag{2.2.7}$$

Such a function exists, a proof may be found in the paper of L. Hörmander, [Hör60, Lemma 2.3] on Fourier multiplier. Let further $\chi_j(\xi) = \chi(2^{-j}\xi/K)$.

We obtain an L^p–L^q estimate for this operator by interpolating L^1–L^∞ and L^2–L^2 estimates with Riesz-Thorin interpolation theorem. For this, we define

$$I_j = \Big|\Big|\mathcal{F}^{-1}\Big[\chi_j(\xi)e^{\pm it|\xi|}a(|\xi|)b\big((1+t)|\xi|\big)\Big]\Big|\Big|_\infty \tag{2.2.8a}$$

and

$$\tilde{I}_j = \Big|\Big|\chi_j(\xi)e^{\pm it|\xi|}a(|\xi|)b\big((1+t)|\xi|\big)\Big|\Big|_\infty \tag{2.2.8b}$$

and estimate these norms. They correspond to operator norms of the dyadic components of the operator. For all $j < 0$ we have $I_j = \tilde{I}_j = 0$. For I_j we perform the substitution $\xi = 2^j K\eta$ and obtain

$$\begin{aligned}
I_j &\leq C2^{jn}\Big|\Big|\mathcal{F}^{-1}\Big[e^{\pm it2^jK\eta}a(2^jK\eta)b((1+t)2^jK\eta)\chi(|\eta|)\Big]\Big|\Big|_\infty \\
&\leq C2^{jn}(1+2^jKt)^{-\frac{n-1}{2}}\sum_{|\alpha|\leq M}||\mathrm{D}^\alpha a(2^jK\eta)b((1+t)2^jK\eta)\chi(|\eta|)||_\infty \\
&\leq C2^{jn}(1+2^jKt)^{-\frac{n-1}{2}} \\
&\qquad \sum_{|\alpha+\beta|\leq M}\sup_{1/2\leq|\eta|\leq 2}(2^jK|\eta|)^{s+\frac{1}{2}-|\alpha|}2^{j|\alpha|}((1+t)2^jK|\eta|)^{-\frac{1}{2}-|\beta|}((1+t)2^j)^{|\beta|} \\
&\leq C2^{j(n+s)}(1+2^jKt)^{-\frac{n-1}{2}}(1+t)^{-\frac{1}{2}},
\end{aligned}$$

where in the first step we used Lemma B.3. From $C_K(1+t) \leq (1+2^jKt) \leq C_K' 2^j(1+t)$ we get finally

$$I_j \leq C2^{j(n+s)}(1+t)^{-\frac{n}{2}}. \tag{2.2.9}$$

For $\tilde{I}_j$ we obtain

$$\begin{aligned}
\tilde{I}_j &\leq C\sup_{\eta\in\operatorname{supp}\chi}\phi_1(2^jK\eta)|2^jK\eta|^{s+\frac{1}{2}}|(1+t)2^jK\eta|^{-\frac{1}{2}} \\
&\leq C2^{js}(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{2.2.10}$$

The estimates (2.2.9) and (2.2.10) correspond to L^1–L^∞ and L^2–L^2 estimates for the dyadic component of the model operator (2.2.6). Interpolation leads to

$$\begin{aligned}\big|\big|\mathcal{F}^{-1}\left[\phi_1(t,\xi)\chi_j(\xi)\Psi_{k,s,\rho,\delta}(t,\xi)\hat{u}(\xi)\right]\big|\big|_q& \\ \leq C 2^{j\left(n\left(\frac{1}{p}-\frac{1}{q}\right)+s\right)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}||u||_p& \qquad (2.2.11)\end{aligned}$$

for all $p \in (1,2]$, $p+q=pq$. Finally, we use Lemma B.4 to conclude for $n\left(\frac{1}{p}-\frac{1}{q}\right)+s \leq 0$ the estimate

$$\big|\big|\mathcal{F}^{-1}\left[\phi_1(t,\xi)\Psi_{k,s,\rho,\delta}(t,\xi)\hat{u}(\xi)\right]\big|\big|_q \leq C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}||u||_p. \qquad (2.2.12)$$

$\boxed{Z_2}$ In this zone we subdivide each summand of the multiplier (2.2.3) in two ξ-dependent factors

$$|\xi|^{|\rho|+\varepsilon}\mathcal{H}^{\mp}_{\rho}(|\xi|)\phi_{21}(\xi), \qquad (2.2.13a)$$

$$\big((1+t)|\xi|\big)^{k-|\rho|-\varepsilon}\mathcal{H}^{\pm}_{\rho+\delta}\big((1+t)|\xi|\big)\phi_{22}(t,\xi), \qquad (2.2.13b)$$

and the remaining factor $(1+t)^{|\rho|-k+\varepsilon}$, where

$$\phi_{21}(\xi)=\psi(|\xi|/K) \qquad \text{and} \qquad \phi_{22}(t,\xi)=1-\psi((1+t)|\xi|/K), \qquad (2.2.14)$$

such that $\phi_2(t,\xi)=\phi_{21}(\xi)\phi_{22}(t,\xi)$. The constant ε will be chosen later. The first multiplier is time independent and satisfies

$$\begin{aligned}|\xi|^{|\rho|+\varepsilon}\mathcal{H}^{\mp}_{\rho}(|\xi|)\phi_{21}(\xi) \approx |\xi|^{|\rho|+\varepsilon}&\mathcal{H}^{\mp}_{|\rho|}(|\xi|)\phi_{21}(\xi)\\ =&(1\pm i\cot|\rho|\pi)|\xi|^{2|\rho|+\epsilon}|\xi|^{-|\rho|}\mathcal{J}_{|\rho|}(|\xi|)\phi_{21}(\xi)\\ &\mp i\csc|\rho|\pi\ |\xi|^{\epsilon}|\xi|^{|\rho|}\mathcal{J}_{-|\rho|}(|\xi|)\phi_{21}(\xi), \qquad \rho\notin\mathbb{Z},\\ =&|\xi|^{2|\rho|+\varepsilon}|\xi|^{-|\rho|}\mathcal{J}_{|\rho|}(|\xi|)\phi_{21}(\xi)\\ &\pm i|\xi|^{2|\rho|+\varepsilon}\ \log|\xi|\ \ |\xi|^{-|\rho|}\mathcal{J}_{|\rho|}(|\xi|)\phi_{21}(\xi)\\ &\pm i|\xi|^{\varepsilon}|\xi|^{|\rho|}A_{|\rho|}(|\xi|)\phi_{21}(\xi), \qquad \rho\in\mathbb{Z},\end{aligned}$$

where $\tau^{\mp|\rho|}\mathcal{J}_{\pm|\rho|}(\tau)$ and $\tau^{|\rho|}A_{|\rho|}(\tau)$ are entire. By $\approx$ we denote equality up to a multiplicative constant here.

From the Marcinkiewicz multiplier theorem, see the book of E.M. Stein, [Ste70, Chapter IV.3 Theorem 3], it follows that

$$|\xi|^{\varepsilon}\phi_{21}(|\xi|)\in M_p^p \qquad \forall\varepsilon\geq 0, \qquad (2.2.15a)$$

$$|\xi|^{\varepsilon}\log|\xi|\ \phi_{21}(|\xi|)\in M_p^p \qquad \forall\varepsilon > 0 \qquad (2.2.15b)$$

for all $p\in(1,\infty)$. Thus, we conclude with the algebra property of multiplier spaces, see [Hör60, Corollary 1.4], that the first multiplier belongs to M_p^p for $p\in(1,\infty)$ if $\varepsilon\geq 0$ and $\rho\neq 0$ (or for $\rho=0$ if $\varepsilon>0$).

For the second multiplier we prove an L^p–L^q estimate. For this we use again a dyadic decomposition. Let χ be like in the discussion of Z_1 and

$$\chi_j(t,\xi)=\chi\big(2^{-j}(1+t)|\xi|/K\big), \qquad (2.2.16a)$$

$$\chi_0(t,\xi)=1-\sum_{j>0}\chi_j(t,\xi). \qquad (2.2.16b)$$

We estimate

$$I_j = \left|\left|\mathcal{F}^{-1}\left[\chi_j(t,\xi)\big((1+t)|\xi|\big)^{k-|\rho|-\varepsilon}\mathcal{H}^{\pm}_{\rho+\delta}\big((1+t)|\xi|\big)\phi_{22}(t,\xi)\right]\right|\right|_{\infty} \tag{2.2.17}$$

and

$$\tilde{I}_j = \left|\left|\chi_j(t,\xi)\big((1+t)|\xi|\big)^{k-|\rho|-\varepsilon}\mathcal{H}^{\pm}_{\rho+\delta}\big((1+t)|\xi|\big)\phi_{22}(t,\xi)\right|\right|_{\infty}. \tag{2.2.18}$$

For $j>0$ we have $\phi_{22}(t,\xi)=1$ on $\operatorname{supp}\chi_j$. Hence, using the substitution $(1+t)\xi = 2^jK\eta$, we get the estimate

$$\begin{aligned}
I_j &= \left|\left|\mathcal{F}^{-1}\left[e^{i(1+t)|\xi|}a\big((1+t)\xi\big)\psi_j(t,\xi)\right]\right|\right|_{\infty}\\
&\le C2^{jn}(1+t)^{-n}\left|\left|e^{i2^jK\eta}a(2^jK\eta)\psi(|\eta|)\right|\right|_{\infty}\\
&\le C2^{jn}(1+t)^{-n}(1+2^jK)^{-\frac{n-1}{2}}(2^jK)^{-\frac{1}{2}+k-|\rho|-\varepsilon}\\
&\le C2^{j\left(\frac{n+1}{2}+k-|\rho|-\varepsilon-\frac{1}{2}\right)}(1+t)^{-n},
\end{aligned}$$

where $a\in S^{-\frac{1}{2}+k-|\rho|-\varepsilon}$. For I_0 we obtain a similar estimate in the same way. For $\tilde{I}_j$ we have

$$\begin{aligned}
\tilde{I}_j &\le C\sup_{\eta\in\operatorname{supp}\psi}(2^jK\eta)^{k-|\rho|-\varepsilon-\frac{1}{2}}\\
&\le C2^{j(k-|\rho|-\varepsilon-\frac{1}{2})}.
\end{aligned} \tag{2.2.19}$$

Interpolation leads to

$$\begin{aligned}
&\left|\left|\mathcal{F}^{-1}\left[\psi_j(t,\xi)\big((1+t)|\xi|\big)^{k-|\rho|-\varepsilon}\mathcal{H}^{\pm}_{\rho+\delta}\big((1+t)|\xi|\big)\phi_{22}(t,\xi)\hat{u}(\xi)\right]\right|\right|_{q}\\
&\qquad\le C2^{j\left(\frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+k-|\rho|-\varepsilon-\frac{1}{2}\right)}(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}||u||_p,
\end{aligned} \tag{2.2.20}$$

which gives for

$$\varepsilon \ge \frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}+k-|\rho|$$

the L^p–L^q estimate

$$\begin{aligned}
&\left|\left|\mathcal{F}^{-1}\left[\big((1+t)|\xi|\big)^{k-|\rho|-\varepsilon}\mathcal{H}^{\pm}_{\rho+\delta}\big((1+t)|\xi|\big)\phi_{22}(t,\xi)\hat{u}(\xi)\right]\right|\right|_{q}\\
&\qquad\le C(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}||u||_p.
\end{aligned} \tag{2.2.21}$$

The ‘regularity’ ε is determined from both multipliers, hence the optimal choice is

$$\varepsilon = \max\left\{0, \tfrac{n+1}{2}\left(\tfrac{1}{p}-\tfrac{1}{q}\right)-\tfrac{1}{2}+k-|\rho|\right\}. \tag{2.2.22}$$

under the assumption $\rho\neq 0$. For $\rho=0$ the choice $\varepsilon=0$ is not possible. Therefore, we have to exclude $k\le\frac{1}{2}$. We postpone this exceptional case.

Multiplication of the multipliers corresponds to a concatenation of the corresponding operators. Hence, we have

$$\begin{aligned}
&\left|\left|\mathcal{F}^{-1}\left[\phi_2(t,\xi)\Psi_{k,s,\rho,\delta}(t,\xi)\hat{u}(\xi)\right]\right|\right|_q\\
&\qquad\le C(1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2},\,-n\left(\frac{1}{p}-\frac{1}{q}\right)+|\rho|-k\}}||u||_p
\end{aligned} \tag{2.2.23}$$

for $\rho \neq 0$ or $\rho = 0$ and $k > 1/2$.

$\boxed{Z_2 \text{ for } \rho = 0 \text{ and } k \leq 1/2}$ In this exceptional case we get an estimate for all dual p and q by interpolation. From $\frac{n}{2} + k > 0$ we can follow the previously used reasoning to conclude an estimate for dual p, q with

$$\frac{n+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2} + k > 0.$$

If we interpolate the corresponding result with the previously proven L^2 estimate in this zone, we obtain

$$\left|\left|\mathcal{F}^{-1}\left[\phi_2(t,\xi)\Psi_{k,s,0,\delta}(t,\xi)\hat{u}(\xi)\right]\right|\right|_q \leq C(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)+\theta\epsilon-k}(\log(e+t))^{1-\theta}||u||_p \tag{2.2.24}$$

for $\frac{n+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2} + k = \epsilon$. The interpolating constant θ is given by

$$\theta = \frac{n+1}{2\epsilon + 1 - 2k}\left(\frac{1}{p} - \frac{1}{q}\right).$$

$\boxed{Z_3}$ We use the estimate

$$\left|\phi_3(t,\xi)\Psi_{k,s,\rho,\delta}(t,\xi)\right| \lesssim \begin{cases} (1+t)^{|\rho|-k} & ,\rho \neq 0, \\ (1+t)^{-k}\log(e+t) & ,\rho = 0 \end{cases}$$

together with the definition of the zone Z_3 to conclude the estimate

$$\begin{aligned}
&||\mathcal{F}[\phi_3(t,\xi)\Psi_{k,s,\rho,\delta}(t,\xi)\hat{u}(\xi)]||_q \\
&\quad \leq \left|\left|\phi_3(t,\cdot)\Psi_{k,s,\rho,\delta}(t,\cdot)\hat{u}(\xi)\right|\right|_p \\
&\quad \leq ||\phi_3(t,\cdot)||_{1/\left(\frac{1}{p}-\frac{1}{q}\right)}||\hat{u}||_q||\Psi_{k,s,\rho,\delta}(t,\cdot)||_\infty \\
&\quad \lesssim ||u||_p(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \begin{cases} (1+t)^{|\rho|-k} & ,\rho \neq 0, \\ (1+t)^{-k}\log(e+t) & ,\rho = 0. \end{cases}
\end{aligned}$$

Under our assumptions on p and q this estimate is weaker than the estimates in the zones Z_1 and Z_2. □

2.2.2 Estimates for the solution

If we compare (2.1.14) with (2.2.3), we obtain the representation

$$\Phi_1(t,0,\xi) = \frac{i\pi}{4}(1+t)^\rho\Psi_{1,0,\rho-1,1}(t,\xi), \tag{2.2.25a}$$

$$\Phi_2(t,0,\xi) = \frac{\pi}{4}(1+t)^\rho\Psi_{0,-1,\rho,0}(t,\xi), \tag{2.2.25b}$$

$$\mathrm{D}_t\Phi_1(t,0,\xi) = \frac{\pi}{4}(1+t)^\rho\Psi_{2,1,\rho-1,0}(t,\xi), \tag{2.2.25c}$$

$$\mathrm{D}_t\Phi_2(t,0,\xi) = \frac{i\pi}{4}(1+t)^\rho\Psi_{1,0,\rho,-1}(t,\xi) \tag{2.2.25d}$$

of the entries of the fundamental matrix in terms of our model multiplier. Thus we can apply the estimates of Lemma 2.5 to get a priori estimates for the solution $u = u(t,x)$ of (2.0.1). This gives

$$||u(t,\cdot)||_2 \lesssim ||u_1||_2 + ||u_2||_{H^{-1}} \begin{cases} (1+t)^{2\rho}, & \rho \in (0,\frac{1}{2}), \\ \log(e+t), & \rho = 0, \\ 1, & \rho < 0. \end{cases} \tag{2.2.26}$$

Together with Theorem 2.6 we conclude the following statement. For convenience we give the L^2–L^2 estimate separately.

Theorem 2.7. *1. The solution operator $\mathbb{S}(t,\mathrm{D})$ satisfies the L^2–L^2 estimate*

$$||\mathbb{S}(t,\mathrm{D})||_{2\to 2} \lesssim \begin{cases} (1+t)^{1-\mu}, & \mu\in(0,1), \\ \log(e+t), & \mu=1, \\ 1, & \mu>1. \end{cases}$$

2. The solution operator $\mathbb{S}(t,\mathrm{D})$ satisfies the L^p–L^q estimate

$$||\mathbb{S}(t,\mathrm{D})||_{p,r\to q}$$
$$\lesssim \begin{cases} (1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mu}{2},\,-n\left(\frac{1}{p}-\frac{1}{q}\right)+1-\mu\}}, & \mu\in(0,1), \\ (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}, & \mu=1,\delta>\frac{1}{2} \\ (1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)+\theta\epsilon}(\log(e+t))^{1-\theta}, & \mu=1,\delta<\frac{1}{2}+\epsilon,\epsilon>0 \\ (1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mu}{2},\,-n\left(\frac{1}{p}-\frac{1}{q}\right)\}}, & \mu>1, \end{cases}$$

for $p\in(1,2]$, q with $pq=p+q$, $\delta=\frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$ and $r=n\left(\frac{1}{p}-\frac{1}{q}\right)$.

The interpolating constant θ in the third case is given by $\theta=\frac{2\delta}{2\epsilon+1}$.

The L^2–L^2 estimate stated in this theorem is better than corresponding results obtained by weighted energy inequalities. The naive way to obtain estimates for solutions by integrating estimates for the time derivative would imply only the rate

$$||\mathbb{S}(t,\mathrm{D})||_{2\to 2} \lesssim (1+t)^{1-\frac{\mu}{2}}, \tag{2.2.27}$$

for $\mu<2$, cf. formula (2.2.33).

The dependence of the decay rate from the parameter μ and the index p is sketched in Figure 2.2 in order to illustrate the different cases from Theorem 2.7.

Equations with increasing speed of propagation. The obtained estimates imply an exceptional behaviour of the case $\mu=1$. This case is related to the consideration of A. Galstian given in [Gal03] for wave equations with exponentially increasing speed of propagation. For the sake of completeness we will give this relation.

If one considers the Cauchy problem

$$v_{tt}-\lambda^2(t)\Delta v=0, \qquad v(0,\cdot)=v_1, \quad \mathrm{D}_t v(0,\cdot)=v_2 \tag{2.2.28}$$

with positive coefficient $\lambda=\lambda(t)$, one can apply a change of variables, which reduces it to a problem with constant speed of propagation and a dissipative term. We introduce the new time variable

$$t'=\Lambda(t)=\int_{t_0}^{t}\lambda(s)\mathrm{d}s, \qquad \Lambda'(t)=\lambda(t)>0, \tag{2.2.29}$$

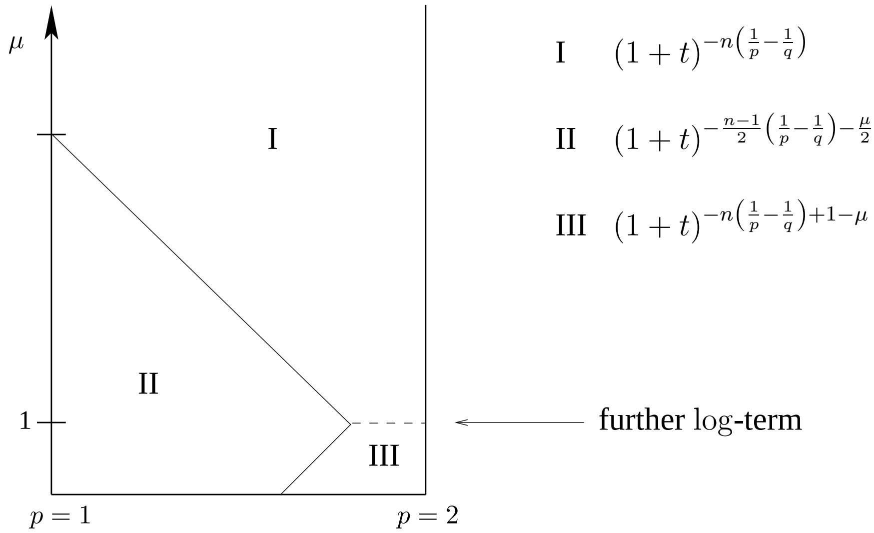

Figure 2.2: Relation between the decay-rates and the parameter μ and index p for the estimate of $\mathbb{S}(t, \mathrm{D})$. At the common boundary of I and III the $\log$-term occurs.

such that $\partial_t = \lambda(t)\partial_{t'}$ and $\partial_t^2 = \lambda^2(t)\partial_{t'}^2 + \lambda'(t)\partial_{t'}$. Thus the problem rewrites in the new variables

$$\lambda^2(t)\left[\Box' v + \frac{\lambda'(t)}{\lambda^2(t)}\partial_{t'} v\right] = 0, \tag{2.2.30}$$

equivalent to an equation with dissipative term

$$b(t') = \frac{\lambda'(\Lambda^{-1}(t'))}{\lambda^2(\Lambda^{-1}(t'))}. \tag{2.2.31}$$

Following M. Reissig and K. Yagdjian, [RY00], it is natural to assume for the increasing behaviour of $\lambda(t)$

$$\frac{\lambda'}{\lambda} \sim \frac{\lambda}{1+\Lambda}, \qquad \text{in our notation} \qquad b(t) \sim \frac{1}{1+t}. \tag{2.2.32}$$

So the case of scale invariant weak dissipation is naturally related to wave equations with increasing in time speed of propagation.

This gives (for the right choice of t_0) the correspondence

$$\begin{aligned} &\lambda(t) = (1+t)^\ell, \quad \ell \geq 0, && \Longleftrightarrow && b(t) = \frac{\mu}{1+t}, \quad \mu = \frac{\ell}{\ell+1} \in (0,1), \\ &\lambda(t) = e^t, && \Longleftrightarrow && b(t) = \frac{1}{1+t}, \quad \mu = 1, \end{aligned}$$

the first line corresponds to the approach of M. Reissig, [Rei97], the second one to the paper of A. Galstian, [Gal03].

2.2.3 Estimates for the energy

For the first derivatives we obtain

$$||u_t(t,\cdot)||_2 \leq C_1(1+t)^{\rho-\frac{1}{2}}||u_1||_{H^1} + C_2(1+t)^{\max\{\rho-\frac{1}{2},-1\}}||u_2||_2, \tag{2.2.33}$$

$$||\nabla u(t,\cdot)||_2 \leq C_1(1+t)^{\rho-\frac{1}{2}}||u_1||_{H^1} + C_2(1+t)^{\max\{\rho-\frac{1}{2},-1\}}||u_2||_2, \tag{2.2.34}$$

which reestablish already known results on the energy decay for this model problem, see the papers of A. Matsumura, [Mat77], H. Uesaka, [Ues80] and the recent considerations of F. Hirosawa and H. Nakazawa [HN01, Example 2.1].

We collect the energy estimates in the following theorem.

Theorem 2.8. *1. The energy operator $\mathbb{E}(t,\mathrm{D})$ satisfies the L^2–L^2 estimate*

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2} \lesssim \begin{cases} (1+t)^{-\frac{\mu}{2}}, & \mu \in (0,2], \\ (1+t)^{-1}, & \mu > 2. \end{cases}$$

2. The energy operator $\mathbb{E}(t,\mathrm{D})$ satisfies the L^p–L^q estimate

$$||\mathbb{E}(t,\mathrm{D})||_{p,r\to q} \lesssim (1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mu}{2},\, -n\left(\frac{1}{p}-\frac{1}{q}\right)-1\}}$$

for $p \in (1,2]$, q with $pq = p+q$ and $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Remark 2.1. It should be remarked that it is essential to use the H^1-norm on the right-hand side of the energy estimate (or the normalisation by $\langle \mathrm{D}\rangle^{-1}$ in the definition of $\mathbb{E}(t,\mathrm{D})$). Otherwise we get for the usual energy from Lemma 2.5 only the trivial (and in view of this Lemma also sharp!) norm estimate

$$E(u;t) \lesssim E(u;0).$$

This implies that information about the size of the datum u_1 is *necessary* for precise a priori estimates of the energy.

Remark 2.2. If we fix a pair of initial data, then one can even obtain

$$\lim_{t\to\infty} (1+t)^2 E(u;t) = 0$$

for $\mu > 2$. This result follows from the considerations in the paper of F. Hirosawa and H. Nakazawa, [HN03, Theorem 1.2].

Remark 2.3. The estimates of Theorem 2.8 coincide for $\mu = 0$ (i.e. $\rho = 1/2$) with the well-known L^p–L^q estimates for the wave equation, recalled in Section 1.3.1.

For $\mu = 2$ (i.e. $\rho = -1/2$) we can reduce the Cauchy problem (2.0.1) to the Cauchy problem for the wave equation setting

$$v(t,x) = (1+t)\,u(t,x), \tag{2.2.35}$$

such that $\Box v = 0$. Thus, the solutions behave for $\mu = 2$ like free waves multiplied by the energy decay rate $(1+t)^{-1}$. Together with the a priori estimate $||v(t,\cdot)||_2 \lesssim (1+t)$, the decay rate for the energy in this case follows immediately from the conservation of energy for free waves. The above transformation goes back to S.-D. Poisson and was a basic step in his reasoning to deduce explicit representations for solutions in three dimensional space.

Interpretation. What conclusions can we draw from the statement of Theorem 2.8? If we start with the L^2–L^2 estimate we see that two different cases occur. On the one hand, for small values of μ the dissipation term has a direct influence on the decay rate for the hyperbolic energy, while for large values of μ the decay rate is independent of the size of the coefficient. The second statement makes this difference more precise. For small values of μ the L^p–L^q decay rate corresponds to the hyperbolic decay rate with exponent $\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$ known from the free wave equation, which is altered by some additional decay coming from the influence of the dissipative term.

For large values of μ the decay exponent has a completely different structure. No relation to the free wave equation becomes apparent, and, till now, we have no explanation for this factor, except the calculations done so far. One of the aims of the next two chapters is to understand this paradigm shift from small to large size of the dissipation term.

The example considered in this chapter will turn out to be the basic example for time-dependent dissipation terms. In Chapter 3 dissipation terms will be considered, which lead to estimates, related to the above given ones, for small values of μ; this case will be referred to as *non-effective weak dissipation*. Also the close relation to the free wave equation will be made more precise there.

In Chapter 4 the opposite case is treated. We will classify dissipation terms, which lead to similar estimates like in this chapter for large values of μ. In these cases the dissipation term will be called *effective*, because it alters the asymptotic properties of the solutions in a significant way.

In the next section we will obtain a related statement to Theorem 2.8 for estimates of higher order; it can be seen that the same change in the behaviour occurs.

2.2.4 Energy estimates of higher order

The proven estimates for the model multiplier (2.2.3) enable us to conclude also energy estimates of higher order. By this, we mean estimates for $\mathrm{D}_t^\ell|\mathrm{D}|^{k-\ell}u$ depending on Sobolev norms of the data.

We consider the operator family

$$\mathbb{E}_\ell^k(t,\mathrm{D}) : (\langle \mathrm{D}\rangle^k u_1, \langle \mathrm{D}\rangle^{k-1} u_2)^T \mapsto \mathrm{D}_t^\ell|\mathrm{D}|^{k-\ell}u, \tag{2.2.36}$$

with $\ell \le k$, normalised in the such a way, that $\mathbb{E}_\ell^k(t,\mathrm{D}) : L^2 \to L^2$. Using Corollary 2.3 we can represent the corresponding Fourier multiplier as a matrix with entries given by multiples of

$$(1+t)^\rho \Psi_{k+1,0,\rho-1,1-\ell}(t,\xi), \qquad (1+t)^\rho \Psi_{k,0,\rho,-\ell}(t,\xi). \tag{2.2.37}$$

Theorem 2.6 gives now immediately the corresponding norm estimates. We exclude the case $\ell = 0$ corresponding to the exceptional estimate for $\mathbb{S}(t,\mathrm{D})$.

Theorem 2.9. *Assume $k,\ell \in \mathbb{N}$, $1 \le \ell \le k$.*

1. *The operator $\mathbb{E}_\ell^k(t)$ satisfies the L^2–L^2 estimate*

$$||\mathbb{E}_\ell^k(t,\mathrm{D})||_{2\to 2} \lesssim \begin{cases} (1+t)^{-\frac{\mu}{2}}, & \mu \in [0,2k) \\ (1+t)^{-k}, & \mu \ge 2k \end{cases}$$

2. *The operator $\mathbb{E}_\ell^k(t,\mathrm{D})$ satisfies the L^p–L^q estimate*

$$||\mathbb{E}_\ell^k(t,\mathrm{D})||_{p,r\to q} \lesssim (1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mu}{2},\, -n\left(\frac{1}{p}-\frac{1}{q}\right)-k\}}$$

for $p \in (1,2]$, q with $pq = p+q$ and $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

For $k = 1$ and $\ell \in \{0,1\}$ the estimates correspond to the energy estimate. We see that for large values of μ taking higher order derivatives improves the decay rate a finite number of steps. Like for the free wave equation, for small μ such an improvement does not take place.

In the case of constant dissipation this improvement for higher order derivatives was observed by A. Matsumura, [Mat76]. There all derivatives influence the decay rate.

2.3 Conclusions

We want to draw some conclusions from the derivation of estimates in the case of scale-invariant weak dissipation. The first observation is that the difference in the decay orders for small and for large values of μ originates in different areas of the phase space.

One may conclude that for small values of the parameter μ large frequencies give the important contribution to the asymptotic behaviour, while for sufficiently large μ the interior zones $\{|\xi| \le K\}$ become of greater importance. This is sketched in Figure 2.3.

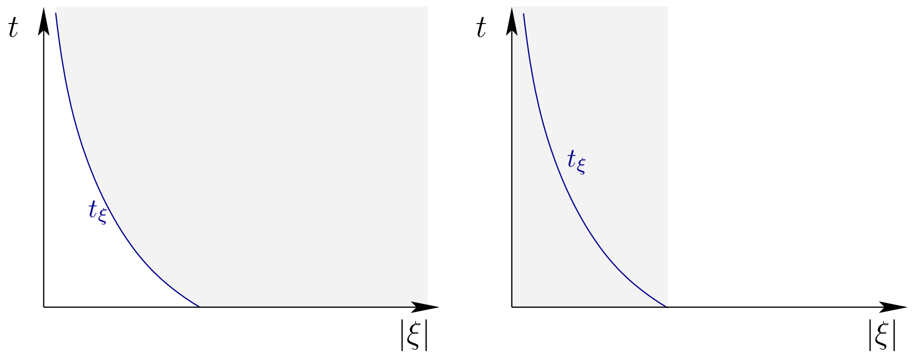

Figure 2.3: Part of the phase space responsible for the energy decay, on the left for small μ, on the right for large μ.

The critical value μ^* for this change depends on the estimate under consideration. In Table 2.1 this dependence is given for the case of L^2–L^2 estimates. In general, we have to distinguish between sub- and supercritical cases for the energy estimates. In the sub-critical cases we claim that the decay rates are determined by the behaviour for large frequencies while for supercritical cases small frequencies play the essential role.

Estimate	μ^*	Reference	decay rate	
			for $\mu < \mu^*$	for $\mu > \mu^*$
solution	$\mu^* = 1$	Theorem 2.7.1	$t^{1-\mu}$	1
energy	$\mu^* = 2$	Theorem 2.8.1	$t^{-\mu/2}$	t^{-1}
higher order energy	$\mu^* = 2k$	Theorem 2.9.1	$t^{-\mu/2}$	t^{-k}

Table 2.1: Critical values μ^* in dependence on the estimate, case of L^2–L^2 estimates.

For the more general L^p–L^q estimates, the critical values of μ are sketched in Figure 2.4. For higher order energy estimate the picture is essentially the same like for $\mathbb{E}(t, \mathrm{D})$, except that the critical line moves upwards.

We want to fix the main strategies for the following chapters. We will consider more general variable coefficient dissipation terms, but we want to *remain in one of the cases*, i.e. we do not want to touch the above given critical values. The two starting questions are:

Task 1. Which estimates are valid for the solutions of the Cauchy problem (1.2.1), if the coefficient is given by $b(t) = \frac{\mu(t)}{1+t}$ with

$$\mu(t) \to 0, \qquad t \to \infty.$$

In this case, we expect that large frequencies determine the asymptotic behaviour of solutions and we will refer to it as *non-effective (weak) dissipation*.

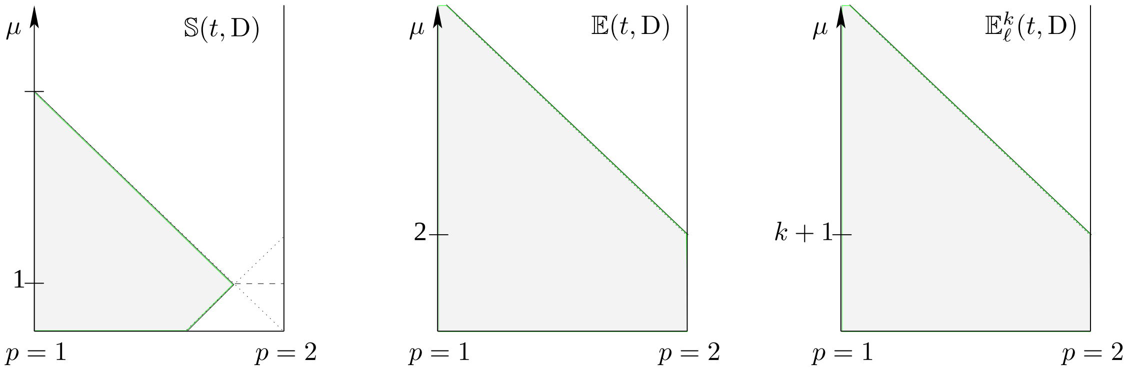

Figure 2.4: Critical values μ^* in dependence on the estimate, case of L^p–L^q estimates for dual p and q. In the shaded region the estimate is determined by the hyperbolic zone. The dashed line in the left picture corresponds to the occurance of logarithmic terms.

Non-effective dissipation terms will be considered in Chapter 3. The precise assumptions on the coefficient function are also given there. Basic examples under consideration will be

Example 2.4.

$$b(t) = \frac{\mu}{(1+t)^\kappa} \qquad \text{with} \ \ \kappa > 1,$$

Example 2.5.

$$b(t) = \frac{\mu}{(1+t)\log(e+t)\cdots\log^{[m]}(e^{[m]}+t)}$$

with iterated logarithms $\log^{[0]}\tau = \tau$ and $\log^{[m+1]}\tau = \log\log^{[m]}\tau$ and corresponding iterated exponentials.

Task 2. What kind of asymptotic properties possess solutions to the Cauchy problem (1.2.1), if the coefficient is given by $b(t) = \frac{\mu(t)}{1+t}$ with

$$\mu(t) \to \infty, \qquad\qquad t \to \infty.$$

In this case we expect that the main influence arises from smaller frequencies, thus the dissipation term influences the asymptotic properties much stronger than in the previous task and we will refer to this case as the case of *effective dissipation* .

Effective dissipation is the content of Chapter 4. Basic example is

Example 2.6.

$$b(t) = \frac{\mu}{(1+t)^\kappa} \qquad \text{with} \ \ \kappa < 1.$$

2.4 Application to the Euler-Poisson-Darboux equation

We want to give some comments to the related singular problem, the *Euler-Poisson-Darboux equation*

$$\Box v + \frac{\mu}{t} v_t = 0 \tag{2.4.1}$$

with parameter $\mu \in \mathbb{R}$. We restrict the consideration to results related to Fourier representations of solutions, classical counterparts are given in the paper of A. Weinstein, [Wei54].

We assume $v = v(t,x) \in C(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^n))$. Similar to the reasoning in Section 2.1 we obtain for the partial Fourier transform $\hat{v}(t,\xi)$ with $t > 0$ the representation[5]

$$(t|\xi|)^\rho \hat{v}(t,\xi) = C_+(\xi)\mathcal{H}_\rho^+(t|\xi|) + C_-(\xi)\mathcal{H}_\rho^-(t|\xi|), \qquad \rho = \frac{\mu-1}{2}, \tag{2.4.2}$$

with suitable $C_\pm(\xi) \in \mathcal{S}'(\mathbb{R}^n)$. For non-integral ρ we can replace this representation by a corresponding one using Bessel functions of first kind,

$$(t|\xi|)^\rho \hat{v}(t,\xi) = C_1(\xi)\mathcal{J}_\rho(t|\xi|) + C_2(\xi)\mathcal{J}_{-\rho}(t|\xi|), \tag{2.4.3}$$

while for integral ρ

$$(t|\xi|)^\rho \hat{v}(t,\xi) = \tilde{C}_1(\xi)\mathcal{J}_\rho(t|\xi|) + \tilde{C}_2(\xi)\mathcal{Y}_\rho(t|\xi|). \tag{2.4.4}$$

We want to sketch how to draw conclusions for the asymptotic behaviour as $t \to +0$ in dependence on μ in the spaces $X^{s,k} = \bigcap_{j=0}^k C^j(\overline{\mathbb{R}_+}, H^{s-j}(\mathbb{R}^n))$:

- For $\mu \le 1$ we have $\rho \le 0$ and for non-integral ρ, i.e. μ no odd (negative) number, the power $(t\xi)^\rho$ cancels the singularity of the Bessel functions and so the function $\hat{v}$ is continuous up to $t = 0$. Furthermore, $(t|\xi|)^{-\rho}\mathcal{J}_{-\rho}(t|\xi|)$ is $-\mu$ times continuously differentiable, while the $(1-\mu)$'th derivative remains only bounded.

- For the exceptional integers $\mu = 1, -1, -3, -5, \dots$ the functions $\mathcal{J}_\rho$ and $\mathcal{J}_{-\rho}$ are linearly dependent and we have to take a further logarithmic term into account. In this case the $(1-\mu)$'th derivative tends to infinity like $\log t$ as $t \to +0$. Especially for $\mu = 1$ we have the same logarithmic behaviour of the solution (near $t = 0$ and for $t \to \infty$ as observed in the L^2–L^2 estimate of Theorem 2.7).

- If $\mu > 1$ the power $(t|\xi|)^{-\rho}$ does not cancel the singularities any more. So in this case the solutions are continuous up to $t = 0$ only under the assumption $C_2(\xi) \equiv 0$, and then the solutions are smooth up to $t = 0$.

Thus we obtain the following dependence between the regularity of solutions and the value of the parameter μ.

Theorem 2.10. *Assume $v = v(t,x) \in C((0,\infty), H^s(\mathbb{R}^n))$ is a solution of the Euler-Poisson-Darboux equation to the parameter $\mu \in \mathbb{R}$. Then the following statements are valid.*

1. *If $\mu \in (-\infty, 1) \setminus (2\mathbb{Z}+1)$, then $v \in X^{s,k}$ for $k = \lfloor 1-\mu \rfloor$.*
2. *If $\mu = 1 - 2\kappa$ with $\kappa \in \mathbb{N}_0$, then $v \in X^{s,2\kappa-1}$ and*
$$||\partial_t^{2\kappa} v(t,\cdot)||_{H^{s-2\kappa}} \lesssim -\log t$$
for $t \ll 1$.
3. *If $\mu > 1$, then $v(t,x)$ extends by continuity up to $t = 0$ if and only if it has the form $\hat{v}(t,\xi) = C(\xi)(t|\xi|)^{-\rho}\mathcal{J}_\rho(t|\xi|)$; and then it is smooth in t, i.e. $v \in X^{s,\infty}$.*

The logarithmic singularity occurring for the exceptional odd integers $\mu = 1 - 2\kappa$, $\kappa \in \mathbb{N}_0$, cancels if we assume $u(t,\cdot)$ to be polyharmonic of order κ, i.e.

$$\Delta^\kappa u(t,\cdot) = 0$$

for all t. In this case the solutions are even smooth. This follows from [CS76, Remark 1.4.8].

[5] We used $-\rho$ in Section 2.1, but this should not cause any confusion.

3 Non-effective weak dissipation

We will employ the translation invariance of the Cauchy problem. This implies, that a partial Fourier transform with respect to the spatial variables may be used to reduce the partial differential equation in $u(t,x)$,

$$\square u + b(t)u_t = 0, \tag{3.0.1}$$

to an ordinary differential equation for $\hat{u}(t,\xi)$ parameterised by the frequency parameter $|\xi|$,

$$\hat{u}_{tt} + b(t)\hat{u}_t + |\xi|^2\hat{u} = 0.$$

Its solution can be represented in the form

$$\hat{u}(t,\xi) = \Phi_1(t,\xi)\hat{u}_1 + \Phi_2(t,\xi)\hat{u}_2$$

in terms of the Cauchy data u_1 and u_2 with suitable functions (Fourier multipliers) Φ_1 and Φ_2. Our aim is to derive structural properties of the functions Φ_1 and Φ_2 in order to decide asymptotic properties of the solutions.

In general estimates for Φ are complicated to obtain directly from the equation; so the natural starting point is to rewrite the second order equation as system for the micro-energy $(|\xi|\hat{u}, \mathrm{D}_t\hat{u})^T$ or a modified one and to use a diagonalization technique to simplify the structure and to estimate its fundamental solution.

The main results of this chapter are the solution representation of Theorem 3.15 together with its consequences for the L^p–L^q decay, Theorem 3.24. Furthermore the sharpness of these results follows from a modified scattering theory given by Theorem 3.26.

The chapter starts with scattering results for integrable coefficients $b(t) \in L^1(\mathbb{R}_+)$ in order to show, how the constructive approach may be used to represent the Møller wave operator. The main result is Theorem 3.1.

3.1 Scattering theorems

We start by characterising these coefficient functions $b = b(t)$, which lead to free solutions, this means,

> the solution $u = u(t,x)$ of
>
> $$\square u + b(t)u_t = 0, \qquad u(0,\cdot) = u_1, \qquad \mathrm{D}_t u(0,\cdot) = u_2 \tag{3.1.1}$$
>
> behaves in certain function spaces (in x) for $t \to \infty$ like the solution of the corresponding free problem
>
> $$\square \tilde{u} = 0, \qquad \tilde{u}(0,\cdot) = \tilde{u}_1, \qquad \mathrm{D}_t\tilde{u}(0,\cdot) = \tilde{u}_2 \tag{3.1.2}$$
>
> to (in some sense related) data $(\tilde{u}_1, \tilde{u}_2)$.

The operator relating (u_1, u_2) to $(\tilde{u}_1, \tilde{u}_2)$ will be denoted as *Møller wave operator* following the conventions from the scattering theory for wave and Schrödinger equations, see e.g the book of R.B. Melrose, [Mel95, Chapter 3.3] or the basic works of P.D. Lax and R.S. Phillips, [LP73], on dissipative systems.

Scattering results for damped wave equations are special non-decay to zero results for the energy and go back to considerations of K. Mochizuki, [Moc77], [MN96], for x-dependent dissipation terms. Recently, H. Nakazawa, [Nak], gave the sharp result for isotropic dissipation terms.[1] Their results are based on the scattering theory of T. Kato, [Kat66]. Independently, there exists an abstract approach to scattering theories for contraction semigroups by H. Neidhardt, [Nei85] and [Nei89].

3.1.1 Results in L^2-scale

The main result is contained in the following theorem, its proof follows the general philosophy of our approach to construct the main term(s) of the solution representation explicitly. We denote by $E(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ the *energy space* and use the order reducing isomorphism $(u_1, u_2) \in E(\mathbb{R}^n)$ if and only if $(|\mathrm{D}|u_1, u_2) \in L^2(\mathbb{R}^n)$.

Theorem 3.1. *Assume the coefficient $b = b(t)$ satisfies $b \in L^1(\mathbb{R}_+)$.*

Then there exists an isomorphism $W_+ : E \to E$ of the energy space, such that for the solution $u = u(t,x)$ of (3.1.1) *to data $(u_1, u_2) \in E$ and the solution $\tilde{u} = \tilde{u}(t,x)$ of* (3.1.2) *to data $(\tilde{u}_1, \tilde{u}_2)^T = W_+(u_1, u_2)^T$ the asymptotic equivalence*

$$||(u, \mathrm{D}_t u) - (\tilde{u}, \mathrm{D}_t \tilde{u})||_E \to 0 \qquad \textit{as } t \to \infty \tag{3.1.3}$$

holds.

Proof. We subdivide the proof into several steps and construct the operator W_+ explicitly in terms of the solution representation. We restrict ourselves to the case $n \geq 2$, for $n = 1$ the same arguments are valid if we replace $|\xi|$ by ξ or $-\xi$.

For this, let $U = (|\xi|\hat{u}, \mathrm{D}_t \hat{u})^T$. Then U satisfies

$$\mathrm{D}_t U = \begin{pmatrix} & |\xi| \\ |\xi| & \end{pmatrix} U + \begin{pmatrix} & \\ & ib(t) \end{pmatrix} U.$$

We consider the first matrix as principal part and the second one as remainder. The remainder is due to our assumption integrable.

Step 1. We diagonalize the main part. Therefore we use the diagonalizer

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad M^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{3.1.4}$$

and consider $U^{(0)} = M^{-1}U$. We get

$$\mathrm{D}_t U^{(0)} = M^{-1} \begin{pmatrix} & |\xi| \\ |\xi| & \end{pmatrix} M U^{(0)} + M^{-1} \begin{pmatrix} & \\ & ib(t) \end{pmatrix} M U^{(0)} = \mathcal{D}(\xi) U^{(0)} + R(t) U^{(0)}, \tag{3.1.5}$$

where

$$\mathcal{D}(\xi) = \begin{pmatrix} |\xi| & \\ & -|\xi| \end{pmatrix} \tag{3.1.6}$$

[1] See the discussion on page 40 of this section.

is diagonal and the remainder satisfies $R(t) \in L_1(\mathbb{R}_+, \mathbb{C}^{2\times 2})$.
Step 2. We start with the fundamental solution to the diagonal main part $\mathrm{D}_t - \mathcal{D}(\xi)$. It is given by

$$\mathcal{E}_0(t-s,\xi) = \exp\{i(t-s)\mathcal{D}(\xi)\} = \begin{pmatrix} e^{i(t-s)|\xi|} & \\ & e^{-i(t-s)|\xi|} \end{pmatrix}. \tag{3.1.7}$$

The matrix $M\mathcal{E}_0(t,s,\xi)M^{-1}$ is the multiplier corresponding to the unitary operator

$$S_0(t-s,\mathrm{D}) : (|\mathrm{D}|\tilde{u}(s), \mathrm{D}_t\tilde{u}(s))^T \mapsto (|\mathrm{D}|\tilde{u}(t), \mathrm{D}_t\tilde{u}(t))^T \tag{3.1.8}$$

for free waves $\Box\tilde{u} = 0$.
Step 3. Now we construct the fundamental solution to $\mathrm{D}_t - \mathcal{D}(\xi) - R(t)$. Let therefore,

$$\mathcal{R}(t,s,\xi) = \mathcal{E}_0(s-t,\xi)R(t)\mathcal{E}_0(t-s,\xi).$$

Using Theorem B.5, it follows that

$$\mathcal{Q}(t,s,\xi) = I + \sum_{k=1}^{\infty} i^k \int_s^t \mathcal{R}(t_1,s,\xi) \int_s^{t_1} \mathcal{R}(t_2,s,\xi) \dots \int_s^{t_{k-1}} \mathcal{R}(t_k,s,\xi)\mathrm{d}t_k \dots \mathrm{d}t_1$$

solves the Cauchy problem

$$\mathrm{D}_t\mathcal{Q}(t,s,\xi) - \mathcal{R}(t,s,\xi)\mathcal{Q}(t,s,\xi) = 0, \qquad \mathcal{Q}(s,s,\xi) = I.$$

With $\mathcal{Q}(t,s,\xi)$ we can express the fundamental solution to the system (3.1.5). Let therefore $\mathcal{E}(t,s,\xi) = \mathcal{E}_0(t,s,\xi)\mathcal{Q}(t,s,\xi)$. Then we obtain

$$\begin{aligned} \mathrm{D}_t(\mathcal{E}_0\mathcal{Q}) = (\mathrm{D}_t\mathcal{E}_0)\mathcal{Q} + \mathcal{E}_0(\mathrm{D}_t\mathcal{Q}) &= \mathcal{D}(\xi)\mathcal{E}_0\mathcal{Q} + \mathcal{E}_0\mathcal{R}(t,s,\xi)\mathcal{Q} \\ &= \mathcal{D}(\xi)\mathcal{E}_0\mathcal{Q} + R(t)\mathcal{E}_0\mathcal{Q} \end{aligned}$$

and $\mathcal{E}_0(s,s,\xi)\mathcal{Q}(s,s,\xi) = I$. Thus, $\mathcal{E}(t,s,\xi)$ is the desired fundamental solution. Hence, the matrix-valued function $M\mathcal{E}(t,s,\xi)M^{-1}$ is the multiplier of the operator

$$S(t,s,\mathrm{D}) : (|\mathrm{D}|u(s), \mathrm{D}_t u(s))^T \mapsto (|\mathrm{D}|u(t), \mathrm{D}_t u(t))^T$$

for solutions u to $\Box u + b(t)u_t = 0$.
Step 4. We estimate this fundamental solution. We do this step by step. At first we have

$$||\mathcal{E}_0(t,s,\xi)|| = 1.$$

We can estimate uniformly in ξ and therefore in the multiplier space $M_2^2(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The next estimate is

$$||\mathcal{R}(t,s,\cdot)||_\infty \le ||R(t)|| \in L^1(\mathbb{R}_+),$$

which will be used to estimate $\mathcal{Q}(t,s,\xi)$. We apply (B.3.4). Combined with the series representation of $\mathcal{Q}$ we get

$$\begin{aligned} ||\mathcal{Q}(t,s,\cdot) - I||_\infty &\le \sum_{k=1}^{\infty} \frac{1}{k!} \left(\int_s^t ||R(\tau)||\mathrm{d}\tau \right)^k \\ &= \exp\left\{ \int_s^t ||R(\tau)||\mathrm{d}\tau \right\} - 1 \lesssim 1 \end{aligned}$$

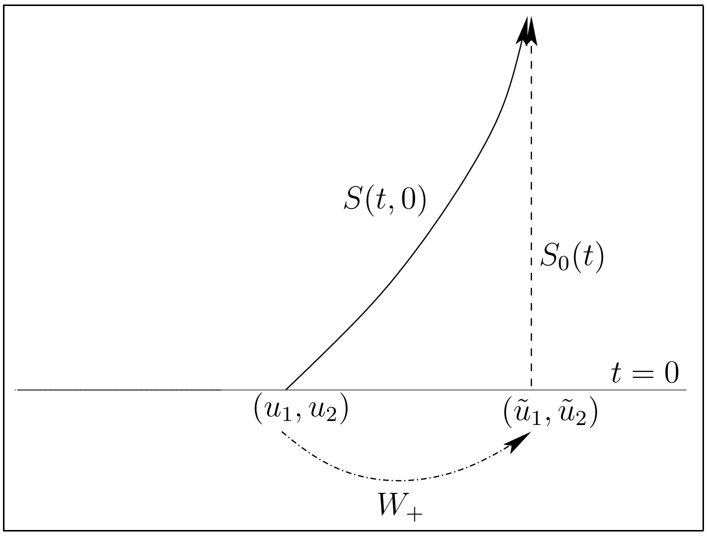

Figure 3.1: Sketch of operators related the definition of the Møller wave operator $W_+(\mathrm{D})$.

and therefore

$$||\mathcal{E}(t,s,\cdot)||_\infty \lesssim 1.$$

Step 5. We are interested in the Møller wave operator $W_+(\mathrm{D})$. Therefore, we consider data (u_1,u_2) from the energy space and apply the solution operator $S(t,0,\mathrm{D})$. Then we go back to the initial line using the solution operator of the homogeneous problem $S_0(-t,\mathrm{D})$. This gives data to the homogeneous wave equation which produce a solution coinciding with u at the time level t. Now we let $t\to\infty$ and define

$$\lim_{t\to\infty} S_0(-t,\mathrm{D})S(t,0,\mathrm{D}) = W_+(\mathrm{D}), \tag{3.1.9}$$

compare also Figure 3.1. If this limit exists in the strong sense, it is called the Møller wave operator, [Mel95, Chapter 3.3].

It holds on the operator level

$$S_0(-t,\mathrm{D})S(t,0,\mathrm{D}) = M\mathcal{E}_0(0,t,\mathrm{D})\mathcal{E}(t,0,\mathrm{D})M^{-1} = M\mathcal{Q}(t,0,\mathrm{D})M^{-1}$$

and thus, it is equivalent to decide whether the limit

$$\lim_{t\to\infty}\mathcal{Q}(t,0,\xi)$$

exists in an appropriate sense. We prove the existence in $L^\infty(\mathbb{R}^n_\xi) = M^2_2$, that means we prove norm-convergence on the operator-level. Therefore, we consider the difference

$$\begin{aligned}
&\mathcal{Q}(t,0,\xi)-\mathcal{Q}(s,0,\xi)\\
&\quad=\sum_{k=1}^\infty i^k\Big[\int_0^t \mathcal{R}(t_1,0,\xi)\int_0^{t_1}\mathcal{R}(t_2,0,\xi)\dots\int_0^{t_{k-1}}\mathcal{R}(t_k,0,\xi)\mathrm{d}t_k\dots\mathrm{d}t_1\\
&\qquad\qquad-\int_0^s \mathcal{R}(t_1,0,\xi)\int_0^{t_1}\mathcal{R}(t_2,0,\xi)\dots\int_0^{t_{k-1}}\mathcal{R}(t_k,0,\xi)\mathrm{d}t_k\dots\mathrm{d}t_1\Big]\\
&\quad=\sum_{k=1}^\infty i^k\int_s^t \mathcal{R}(t_1,0,\xi)\int_0^{t_1}\mathcal{R}(t_2,0,\xi)\dots\int_0^{t_{k-1}}\mathcal{R}(t_k,0,\xi)\mathrm{d}t_k\dots\mathrm{d}t_1.
\end{aligned}$$

If we apply $||\cdot||_\infty$ on both sides and use (B.3.4) to estimate the integrals we get

$$\begin{aligned} ||\mathcal{Q}(t,0,\cdot)-\mathcal{Q}(s,0,\cdot)||_\infty &\le \sum_{k=1}^\infty \int_s^t ||R(t_1)|| \frac{1}{(k-1)!}\left(\int_0^{t_1} ||R(\tau)||\mathrm{d}\tau\right)^{k-1} \mathrm{d}t_1 \\ &\le \int_s^t ||R(t_1)|| \sum_{k=0}^\infty \frac{1}{k!}\left(\int_0^{t_1} ||R(\tau)||\mathrm{d}\tau\right)^{k} \mathrm{d}t_1 \\ &= \int_s^t ||R(t_1)|| \exp\left\{\int_0^{t_1} ||R(\tau)||\mathrm{d}\tau\right\} \mathrm{d}t_1 \to 0 \end{aligned}$$

as $t,s\to\infty$ from the integrability of $R(t)$. Thus, it is a Cauchy sequence and therefore the limit exists in the Banach space $L^\infty(\mathbb{R}^n)$. We define

$$W_+(\xi) = \lim_{t\to\infty} M\mathcal{Q}(t,0,\xi)M^{-1} \in L^\infty(\mathbb{R}^n).$$

Step 6. The operator W_+ has the desired property: On the Fourier level we have

$$\begin{aligned} (|\xi|\hat{u}, \mathrm{D}_t\hat{u})^T - (|\xi|\hat{\tilde{u}}, \mathrm{D}_t\hat{\tilde{u}})^T &= M\mathcal{E}_0\mathcal{Q}M^{-1}(|\xi|\hat{u}_1,\hat{u}_2)^T - M\mathcal{E}_0 M^{-1}(|\xi|\hat{\tilde{u}}_1,\hat{\tilde{u}}_2)^T \\ &= M\mathcal{E}_0M^{-1}\left[M\mathcal{Q}M^{-1}-W_+\right](|\xi|\hat{u}_1,\hat{u}_2)^T \end{aligned}$$

and the term in brackets tends to 0 as $t\to\infty$. Thus, (3.1.3) follows.
Step 7. The transpose of the inverse of $\mathcal{Q}(t,s,\xi)$ satisfies the related equation

$$\mathrm{D}_t\mathcal{Q}^{-T}(t,s,\xi) + \mathcal{R}^T(t,s,\xi)\mathcal{Q}^{-T}(t,s,\xi) = 0, \qquad \mathcal{Q}^{-T}(s,s,\xi) = I.$$

Thus we can estimate $\mathcal{Q}^{-T}$ in a similar style as $\mathcal{Q}$, especially we can prove that

$$\lim_{t\to\infty} \mathcal{Q}^{-1}(t,s,\xi)$$

exists. Furthermore $\lim_{t\to\infty}\mathcal{Q}^{-1}(t,s,\xi) = \lim_{t\to\infty}[\mathcal{Q}(t,s,\xi)]^{-1}$. Thus, the matrix $W_+(\xi)$ is invertible in $L^\infty(\mathbb{R}^n)$ or equivalently on the operator level $W_+ = W_+(\mathrm{D})$ is invertible in $L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$. □

Corollary 3.2. *Under the assumptions of Theorem 3.1 it holds*

$$||(u,\mathrm{D}_tu)-(\tilde{u},\mathrm{D}_t\tilde{u})||_E \lesssim ||(u_1,u_2)||_E \int_t^\infty b(\tau)\mathrm{d}\tau \tag{3.1.10}$$

and the occurring constant depends only on $||b||_1$.

Proof. The statement follows directly from

$$\mathcal{Q}(\infty,0,\xi)-\mathcal{Q}(t,0,\xi) = \sum_{k=1}^\infty i^k \int_t^\infty \mathcal{R}(t_1,0,\xi)\int_0^{t_1}\mathcal{R}(t_2,0,\xi)\dots\int_0^{t_{k-1}}\mathcal{R}(t_k,0,\xi)\mathrm{d}t_k\dots\mathrm{d}t_1$$

and

$$||\mathcal{Q}(\infty,0,\cdot)-\mathcal{Q}(t,0,\cdot)||_\infty \le \int_t^\infty ||R(t_1)||\exp\left\{\int_0^{t_1}||R(\tau)||\mathrm{d}\tau\right\}\mathrm{d}t_1 \lesssim \int_t^\infty b(\tau)\mathrm{d}\tau,$$

where $\mathcal{Q}(\infty,s,\xi) = \lim_{t\to\infty}\mathcal{Q}(t,s,\xi)$. □

Example 3.1. If we consider the special case $b(t) = (1+t)^{-\kappa}$ with $\kappa > 1$, the assumptions of Theorem 3.1 are satisfied. The convergence rate from Corollary 3.2 for these examples is $\mathcal{O}(t^{1-\kappa})$.

Example 3.2. We can come sharper to the borderline case using the coefficient function

$$b(t) = \frac{1}{(e^{[m]}+t)\log(e^{[m]}+t)\cdots\log^{[m-1]}(e^{[m]}+t)\big(\log^{[m]}(e^{[m]}+t)\big)^{\gamma}}, \qquad \gamma > 1$$

with $e^{[0]} = 1$, $e^{[k+1]} = e^{e^{[k]}}$, $\log^{[0]}(\tau) = \tau$ and $\log^{[k+1]}(\tau) = \log(\log^{[k]}(\tau))$ for sufficiently large τ. In this case it holds

$$\int_t^\infty b(\tau)\mathrm{d}\tau = \frac{1}{1-\gamma}\big(\log^{[m]}(e^{[m]}+t)\big)^{1-\gamma},$$

thus the convergence rate can be of arbitrarily small logarithmic order. This example is essentially the same as the comparison function of K. Mochizuki and H. Nakazawa, [MN96].

Comparison of results. If we compare our result, Theorem 3.1, with the results K. Mochizuki and H. Nakazawa contained in [Moc77], [MN96], [Nak], we see two essential differences. On the one hand, the results in the cited papers are for dissipation depending (essentially) on the x-variable, while our result is for t-depending coefficients. Besides this difference, the conditions imposed on the coefficient function are closely related:

- we use $b(t) \in L^1(\mathbb{R}_+)$,
- in [Nak] the condition is $|b(x)| \le a(|x|)$ with $a \in L^1(\mathbb{R}_+)$ and sufficiently small L^1-norm is used.

On the other hand, the results differ in the strength of the convergence to the wave operator. In the cited papers the limit exists as strong limit, while our assumption enables us to prove convergence in the operator norm. The reason for this difference is not only related to the approach, if the influence comes from the x-variable one can not expect the result to be uniform in the data in general. This follows from the finite speed of propagation, if the dissipation is concentrated in one region of the space and we consider data supported in a different part the time when the dissipation influences the solution depends on the spatial distance of these regions.

Most of the results presented in this section can be generalized to coefficients $b = b(t,x)$ with $b \in L^1L^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$. The calculations are closely related and contained in the preprint [Wir02]. On the other hand, the results of K. Mochizuki and H. Nakazawa are valid, if we assume the estimates uniform in t.

3.1.2 Results in L^q-scale, $q \ge 2$

We proved that the energy density $(|\mathrm{D}|u, \mathrm{D}_t u)$ behaves asymptotically in $L^2(\mathbb{R}^n)$ like the energy density of a solution to the free wave equation. It is natural to ask for an extension of this result to other L^q-spaces. At least for $q \ge 2$ this is possible, as the following theorem implies. The argument is heavily based on the x-independence of the coefficient and the translation invariance of the solution operator.

Theorem 3.3. *Let $E^s = (|\mathrm{D}|^{-1}H^s) \times H^s$. Then the previously defined Møller wave operator acts $W_+ : E^s \to E^s$ for all $s \in \mathbb{R}$ and for the solution $u = u(t,x)$ of* (3.1.1) *to data $(u_1, u_2) \in E^s$ and the corresponding free solution $\tilde{u}(t,x)$ of* (3.1.2) *to data $W_+(u_1,u_2)$ it holds*

$$||(u, \mathrm{D}_t u) - (\tilde{u}, \mathrm{D}_t\tilde{u})||_{E^s} \lesssim ||(u_1,u_2)||_{E^s}\int_t^\infty b(\tau)\mathrm{d}\tau.$$

Proof. We just have to replace Step 6 of the proof of Theorem 3.1. We have to include the multiplier $\langle\xi\rangle^s$ defining the Sobolev norm. This gives on the Fourier level

$$\begin{aligned}\langle\xi\rangle^s\big((|\xi|\hat{u},\mathrm{D}_t\hat{u})-(|\xi|\hat{\tilde{u}},\mathrm{D}_t\hat{\tilde{u}})\big)&=\langle\xi\rangle^s M\mathcal{E}_0M^{-1}\left[M\mathcal{Q}M^{-1}-W_+\right](|\xi|\hat{u}_1,\hat{u}_2)\\&=M\mathcal{E}_0M^{-1}\left[M\mathcal{Q}M^{-1}-W_+\right]\langle\xi\rangle^s(|\xi|\hat{u}_1,\hat{u}_2)\end{aligned}$$

by the commutation property of Fourier multipliers. Now the assumptions imply the boundedness of $||\langle\xi\rangle^s(|\xi|\hat{u}_1,\hat{u}_2)||_2$ and like in the previous proof $||M\mathcal{Q}M^{-1}-W_+||_\infty\to 0$ satisfies the above given estimate. ☐

Corollary 3.4. *Under the assumptions of Theorem 3.3 with* $s\geq n\left(\frac{1}{2}-\frac{1}{q}\right)$ *for* $q\in[2,\infty)$ *and* $s>\frac{n}{2}$ *for* $q=\infty$ *it holds*

$$||(|\mathrm{D}|u,\mathrm{D}_tu)-(|\mathrm{D}|\tilde{u},\mathrm{D}_t\tilde{u})||_q\lesssim||(u_1,u_2)||_{E^s}\int_t^\infty b(\tau)\mathrm{d}\tau.$$

The assumed regularity for the data is natural in view of the L^2–L^q estimates for the wave equation given by S. Klainerman, [Kla85]. Similarly, for L^∞–L^∞ estimates a Sobolev regularity of at least $\lceil\frac{n}{2}\rceil$ is required. So the use of Sobolev embedding does not destroy the quality of the estimate. What we cannot conclude by this method is whether the wave operators are bounded on L^q itself.

3.2 Objectives and strategies

There remains a gap between the case of integrable coefficients and the scale invariant case discussed in Chapter 2. As examples one may take the following coefficient functions originating from the paper of K. Mochizuki and H. Nakazawa, [MN96], see also Examples 3.2 and 2.5.

Example 3.3. Let $\mu>0$ and $m\geq 1$. Then we consider

$$b(t)=\frac{\mu}{(1+t)\log(e+t)\cdots\log^{[m]}(e^{[m]}+t)}$$

and ask for (sharp) energy and L^p–L^q decay estimates for the solutions of (3.0.1).

To answer this question we follow partly the consideration in Section 3.1 and apply a diagonalization procedure to derive expressions of the leading terms of the representation of solutions. In opposite to these considerations we cannot stop after diagonalizing the principal part, some of the lower order remainder terms influence the asymptotic properties.

The diagonalization procedure is essentially based on the approach used in a joint paper of M. Reissig and K. Yagdjian, [RY00], for wave equations with variable speed of propagation or by K. Yagdjian in [Yag97], [Ягд89], for the case of weakly hyperbolic problems. Basic idea is the construction of a WKB representation of the solutions to the Fourier transformed equation. As known from the theory of ordinary differential equations, see e.g. the book of M. Fedoryuk, [Fed93], we need assumptions for derivatives of the coefficient function to construct these representations.

Assumptions. We make the following assumptions on the coefficient function $b=b(t)$:

(A1) positivity $b(t)\geq 0$,

(A2) monotonicity $b'(t)<0$,

(A3) $b^2(t) \lesssim -b'(t)$,

which allow us to conclude energy estimates. Assumption (A3) implies a minimal decay rate of the coefficient. Integrating both sides of the inequality yields

$$t \lesssim \int_0^t \frac{-b'(t)}{b^2(t)} \mathrm{d}t = \frac{1}{b(t)} - \frac{1}{b(0)}, \tag{3.2.1}$$

and thus $tb(t)$ remains bounded.

Basic ideas, zones. Similar to the proof of Theorem 3.1 we can consider the vector-valued function $\tilde{U} = (|\xi|\hat{u}, \mathrm{D}_t\hat{u})^T$, such that

$$\mathrm{D}_t\tilde{U} = \begin{pmatrix} & |\xi| \\ |\xi| & ib(t) \end{pmatrix} \tilde{U}. \tag{3.2.2}$$

Contrary to the consideration in Section 3.1, we cannot say, that $|\xi|$ is the dominating entry in the coefficient matrix. We have to relate the size of $b(t)$ to the size of $|\xi|$. This leads to a decomposition of the phase space. In Figure 3.2 this idea is sketched.

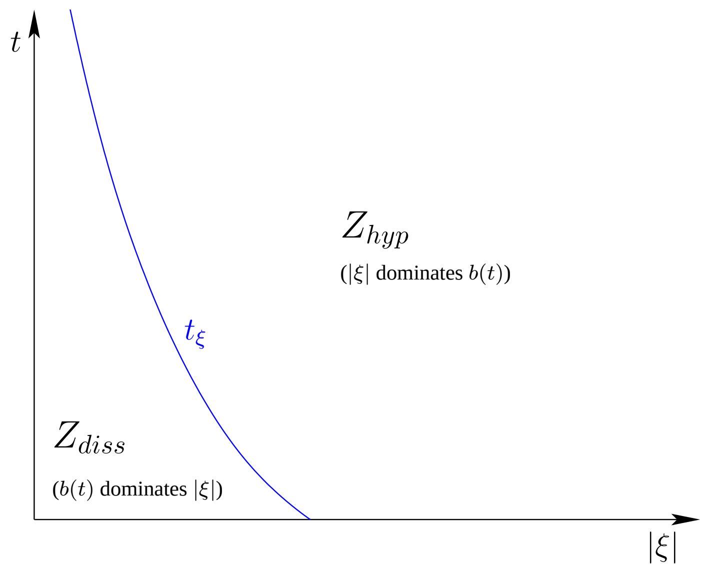

Figure 3.2: Idea behind the definition of zones for the hyperbolic case.

It is possible to replace Assumptions (A2) and (A3) by Assumption

(A4)$_\ell$ for all numbers $k \leq \ell$ it holds

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}t^k} b(t)\right| \leq C_k \left(\frac{1}{1+t}\right)^{1+k}.$$

This allows us to use coefficients which are not monotonous. For the derivation of L^p–L^q estimates Assumption (A4)$_\ell$ is necessary for sufficiently many time derivatives, while for the consideration in L^2-scale Assumptions (A2) and (A3) seem to be more appropriate. The notation (A4)$_\infty$ will be shortened to (A4).

For later reference, we distinguish between a

- *low regularity theory*:
  we assume $b \in C^1$ with (A1) to (A3) and a
- *high regularity theory*:
  with $b \in C^\ell$ for sufficiently large ℓ and (A1) together with (A4)$_\ell$.

Definition 3.1. *We call the dissipation term $b(t)u_t$ in equation* (3.0.1) *non-effective, if $b(t)$ satisfies Assumptions (A1) – (A3) or (A1) and (A4)$_\ell$ together with the asymptotic bound* $\limsup_{t\to\infty} tb(t) < 1$.

The last condition is related to the exceptional behaviour of the case $b(t) = (1+t)^{-1}$ observed in Chapter 2, Theorem 2.7, and allows us to exclude the critical cases arisen in these considerations.

Low regularity theory. We subdivide the phase space into zones corresponding to dominating entries of the coefficient matrix. For this we use the monotonicity of the function $b = b(t)$ and define implicitly t_ξ by

$$Nb(t_\xi) = |\xi| \tag{3.2.3}$$

for small $|\xi|$ and with a suitably chosen constant N. Furthermore,

$$Z_{hyp}(N) := \{\,(t,\xi)\,|\,t \geq t_\xi\,\}, \qquad Z_{diss}(N) := \{\,(t,\xi)\,|\,0 \leq t \leq t_\xi\,\}. \tag{3.2.4}$$

In the *hyperbolic zone* $Z_{hyp}(N)$ the entries of first order, $|\xi|$, dominate the dissipation, $ib(t)$, and we will use a diagonalization technique to construct an equivalent system with $\pm|\xi|$ as the main diagonal part, some lower order terms arising from the dissipation and an integrable and not necessarily diagonal remainder. The essential point of the low regularity theory is the integrability of $-b'(t)$ and, therefore, also of $b^2(t)$.

In the *dissipative zone* $Z_{diss}(N)$ the main contribution comes from $ib(t)$ and we will use a reformulation as an integral equation to conclude estimates there.

The precise choice of the zone constant N depends on the number of diagonalization steps. In the low regularity theory the restriction $N > \frac{1}{4}$ follows from the precise structure of the coefficient matrices in formula (3.3.12).

High regularity theory. If we forget about the monotonicity of $b(t)$ we have to change the decomposition of the phase space. Instead of (3.2.3) we will use

$$(1+t_\xi)|\xi| = N \tag{3.2.5}$$

with suitable constant N for the definition of the zones. This is related to the estimate of Assumption (A4) and the introduction of symbol classes in $Z_{hyp}(N)$ (similar symbol classes were used e.g. in [RY00] or [Yag97]). Again the precise choice of the zone constant is given later. The existence of suitable constants is guaranteed by Lemma 3.12 and it increases with the number of applied diagonalization steps.

Definition 3.2. *The time-dependent Fourier multiplier $a(t,\xi)$ belongs to the hyperbolic symbol class $S_N^{\ell_1,\ell_2}\{m_1,m_2\}$ with restricted smoothness ℓ_1,ℓ_2, if it satisfies the symbol estimates*

$$\left|\mathrm{D}_t^k \mathrm{D}_\xi^\alpha a(t,\xi)\right| \leq C_{k,\alpha}|\xi|^{m_1-|\alpha|}\left(\frac{1}{1+t}\right)^{m_2+k} \tag{3.2.6}$$

for all $(t,\xi) \in Z_{hyp}(N)$ and all natural numbers $k \leq \ell_1$ and multi-indices $|\alpha| \leq \ell_2$.

We fix the notation $S_N\{m_1,m_2\}$ for the class $S_N^{\infty,\infty}\{m_1,m_2\}$. Obviously, it holds

$$S_N\{m_1,m_2\} \hookrightarrow S_N^{\ell_1,\ell_2}\{m_1,m_2\} \hookrightarrow S_N^{\ell_1',\ell_2'}\{m_1,m_2\} \qquad \forall\, \ell_1' \le \ell_1,\ \ell_2' \le \ell_2.$$

Using (3.2.5), the embedding rule

$$S_N^{\ell_1,\ell_2}\{m_1-k,m_2+\ell\} \hookrightarrow S_N^{\ell_1,\ell_2}\{m_1,m_2\} \qquad \forall\, \ell \ge k \ge 0 \tag{3.2.7}$$

follows, which will be essentially used in the diagonalization scheme.

Definition 3.2 extends immediately to matrix-valued Fourier multipliers. The rules of the symbolic calculus are obvious and collected in the following proposition.

Proposition 3.5. *1. $S_N^{\ell_1,\ell_2}\{m_1,m_2\}$ is a vector space,*

2. $S_N^{\ell_1,\ell_2}\{m_1,m_2\} \cdot S_N^{\ell_1,\ell_2}\{m_1',m_2'\} \hookrightarrow S_N^{\ell_1,\ell_2}\{m_1+m_1',m_2+m_2'\}$,

3. $\mathrm{D}_t^k \mathrm{D}_\xi^\alpha S_N^{\ell_1,\ell_2}\{m_1,m_2\} \hookrightarrow S_N^{\ell_1-k,\ell_2-|\alpha|}\{m_1-|\alpha|,m_2+k\}$,

4. $S_N^{0,0}\{-1,2\} \hookrightarrow L_\xi^\infty L_t^1(Z_{hyp})$.

The symbol estimates with restricted smoothness are sufficient to deduce mapping properties in L^p spaces[2]. We give one auxiliary result following directly from Marcinkiewicz multiplier theorem, [Ste70, Chapter IV.3, Theorem 3].

Proposition 3.6. *Each $a \in S_N^{0,\lceil \frac{n}{2} \rceil}\{0,m\}$ with $\operatorname{supp} a \subseteq Z_{hyp}(N)$ gives rise to an operator $a(t,\mathrm{D}) : L^p \to L^p$ for all $p \in (1,\infty)$ with norm estimate*

$$||a(t,\mathrm{D})||_{p\to p} \lesssim \left(\frac{1}{1+t}\right)^m .$$

Formulation in system form. Like in the considerations of Chapter 2 the two components of the energy behave differently in the dissipative zone. This can be seen as a reason to consider not the vector $\tilde{U}$ defined as above, but the *micro-energy*

$$U = (h(t,\xi)\hat{u}, \mathrm{D}_t\hat{u})^T \tag{3.2.8}$$

with

$$h(t,\xi) = Nb(t)\phi_{diss,N}(t,\xi) + |\xi|\phi_{hyp,N}(t,\xi) \tag{3.2.9}$$

in the low regularity approach and $b(t)$ replaced by $\frac{1}{1+t}$ in the high regularity one. Here and thereafter, we denote by $\phi_{diss,N}(t,\xi)$ the characteristic function of the dissipative zone and by $\phi_{hyp,N}(t,\xi)$ the characteristic function of the hyperbolic zone, or a smooth surrogate $\phi_{diss,N}(t,\xi) = \chi((1+t)|\xi|/N)$ with $\chi \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\chi = B_2$ and $\chi \equiv 1$ on $B_{\frac{1}{2}}$ together with $\phi_{diss,N} + \phi_{hyp,N} = 1$. Remark that $\phi_{hyp,N} \in S_{hyp,N}\{0,0\}$ by this definition and $\operatorname{supp}(\partial_t,\nabla)\phi_{hyp,N} \subseteq Z_{hyp}(N/2) \cap Z_{diss}(2N)$.

Our aim is to prove estimates and structural properties for the fundamental solution $\mathcal{E}(t,s,\xi)$ to the corresponding system $\mathrm{D}_t U = A(t,\xi)U$.

[2]We are speaking about Fourier multiplier only, so no essential difficulties can arise by this lack of smoothness.

3.3 Representation of solutions

3.3.1 The dissipative zone

In the dissipative zone we use the positivity of the coefficient function $b(t)$. The essential idea is to write the problem as an *Volterra integral equation*. The approach works in the low regularity theory and in the high regularity theory as well. Remark that in the first case $tb(t)$ remains bounded and therefore the dissipative zone can only be 'smaller'.

In the dissipative zone the micro-energy (3.2.8) reduces to

$$U = \left(\frac{N}{1+t}\hat{u}, \mathrm{D}_t\hat{u} \right)^T$$

and thus we have to solve the system

$$\mathrm{D}_t\mathcal{E}(t,s,\xi) = A(t,\xi)\mathcal{E}(t,s,\xi) = \begin{pmatrix} \frac{i}{1+t} & \frac{N}{1+t} \\ \frac{(1+t)|\xi|^2}{N} & ib(t) \end{pmatrix} \mathcal{E}(t,s,\xi), \qquad \mathcal{E}(s,s,\xi) = I \tag{3.3.1}$$

in order to get $U(t,\xi) = \mathcal{E}(t,s,\xi)U(s,\xi)$.

We will use the auxiliary function

$$\lambda(t) = \exp\left\{ \frac{1}{2}\int_0^t b(\tau)\mathrm{d}\tau \right\}, \tag{3.3.2}$$

related to the entry $ib(t)$ of the coefficient matrix. It plays an essential role in the description of the energy decay as will be seen later.

In order to understand the influence of the different entries, we need a relation between $b(t)$ and t. We distinguish the following two cases. Recall that the first one is part of the definition of the notion of non-effective dissipation.

(C1) It holds $\limsup_{t\to\infty} tb(t) < 1$.

(C2) It holds $\liminf_{t\to\infty} tb(t) > 1$.

The remaining gap corresponds to the exceptional case $b(t) = \frac{1}{1+t}$ from Chapter 2. In this case we have to modify the estimates for the fundamental solution and logarithmic terms have to occur. In the calculation we use the following two consequences of Assumptions (C1), (C2).

Proposition 3.7. *1. Assumptions (A1), (C1) imply for the auxiliary function $\lambda(t)$ defined by* (3.3.2)

$$\int_0^t \frac{\mathrm{d}\tau}{\lambda^2(\tau)} \sim \frac{t}{\lambda^2(t)}$$

and $\frac{t}{\lambda^2(t)}$ is monotonous increasing for large t and tends to infinity.

2. Assumption (C2) implies $\lambda^{-2}(t) \in L^1(\mathbb{R}_+)$ with

$$\int_t^\infty \frac{\mathrm{d}\tau}{\lambda^2(\tau)} \lesssim \frac{1+t}{\lambda^2(t)}.$$

Furthermore $\frac{t}{\lambda^2(t)}$ is monotonous decreasing for large t.

Proof. Conditions like (C1) and (C2) imply polynomial bounds for $\lambda(t)$. It holds

$$\frac{1+t}{\lambda^2(t)} = \exp\left\{\int_0^t \left(b(\tau) - \frac{1}{1+\tau}\right)\mathrm{d}\tau\right\},$$

and the integrand is strictly negative for large τ under (C1) or positive under (C2) and behaves like $\frac{1}{1+\tau} \notin L^1(\mathbb{R}_+)$. Thus, in the first case the expression tends to zero, while in the second one to infinity. Hence, under Condition (C1) we have $\lambda^2(t) \lesssim 1+t$, while (C2) implies $1+t \lesssim \lambda^2(t)$.

Part 1. Integration by parts yields

$$\int_0^t \frac{\mathrm{d}\tau}{\lambda^2(\tau)} = \frac{t}{\lambda^2(t)} + \int_0^t \frac{\tau b(\tau)}{\lambda^2(\tau)}\mathrm{d}\tau.$$

On the one hand, the right-hand side is larger than $t\lambda^{-2}(t)$ by Assumption (A1). On the other hand we conclude from $tb(t) \le c < 1$ for $t > t_0$ that

$$\int_0^t \frac{\tau b(\tau)}{\lambda^2(\tau)}\mathrm{d}\tau \le \int_0^{t_0} \frac{\tau b(\tau)}{\lambda^2(\tau)}\mathrm{d}\tau + c\int_{t_0}^t \frac{\mathrm{d}\tau}{\lambda^2(\tau)}, \le C + c\int_0^t \frac{\mathrm{d}\tau}{\lambda^2(\tau)}$$

and the statement follows from

$$\int_0^t \frac{\mathrm{d}\tau}{\lambda^2(\tau)} \le \frac{1}{1-c}\left(C + \frac{t}{\lambda^2(t)}\right) \lesssim \frac{t}{\lambda^2(t)}.$$

For small t the statement can be concluded from $\lambda^2(t) \sim 1$.

Monotonicity is a consequence of

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{t}{\lambda^2(t)} = \frac{1-tb(t)}{\lambda^2(t)}$$

and $tb(t) < 1$ for $t \gg 1$.

Part 2. From $\liminf_{t\to\infty} tb(t) > 1+\epsilon$ we conclude

$$\lambda^2(t) = \exp\left\{\int_0^t b(\tau)\mathrm{d}\tau\right\} \gtrsim (1+t)^{1+\epsilon},$$

which implies integrability of $\lambda^{-2}(t)$. Furthermore, it follows for $t \gg 1$ with $tb(t) > 1+\epsilon$

$$\epsilon\int_t^\infty \frac{\mathrm{d}\tau}{\lambda^2(\tau)} \le \int_t^\infty \frac{\tau b(\tau)-1}{\lambda^2(\tau)}\mathrm{d}\tau = \frac{t}{\lambda^2(t)}$$

and the statement is proven. ☐

Lemma 3.8. *Assume (A1) and (C1). Then*

$$||\mathcal{E}(t,s,\xi)|| \lesssim \frac{\lambda^2(s)}{\lambda^2(t)}, \qquad t_\xi \ge t \ge s. \tag{3.3.3}$$

Proof. If we denote by $v(t,s,\xi)$ and $w(t,s,\xi)$ the entries of a column of $\mathcal{E}(t,s,\xi)$, then equation (3.3.1) implies

$$v(t,s,\xi) = \frac{1+s}{1+t}\eta_1 - i\frac{N}{1+t}\int_s^t w(\tau,s,\xi)\mathrm{d}\tau, \tag{3.3.4a}$$

$$w(t,s,\xi) = \frac{\lambda^2(s)}{\lambda^2(t)}\eta_2 - i\frac{|\xi|^2}{N\lambda^2(t)}\int_s^t (1+\tau)\lambda^2(\tau)v(\tau,s,\xi)\mathrm{d}\tau, \tag{3.3.4b}$$

where $\eta = (\eta_1,\eta_2) = (1,0)^T$ for the first and $\eta = (0,1)^T$ for the second column. If we multiply it by the weight factor $\frac{\lambda^2(t)}{\lambda^2(s)}$, we obtain

$$\frac{\lambda^2(t)}{\lambda^2(s)}v(t,s,\xi) = \frac{\lambda^2(t)}{\lambda^2(s)}\frac{1+s}{1+t}\eta_1 - i\frac{N\lambda^2(t)}{1+t}\int_s^t \frac{1}{\lambda^2(\tau)}\left(\frac{\lambda^2(\tau)}{\lambda^2(s)}w(\tau,s,\xi)\right)\mathrm{d}\tau,$$
$$\frac{\lambda^2(t)}{\lambda^2(s)}w(t,s,\xi) = \eta_2 - i\frac{|\xi|^2}{N}\int_s^t (1+\tau)\left(\frac{\lambda^2(\tau)}{\lambda^2(s)}v(\tau,s,\xi)\right)\mathrm{d}\tau.$$

The aim is now, to prove well-posedness of this system of Volterra integral equations in $L^\infty\{t \geq s,\ (t,\xi),(s,\xi) \in Z_{diss}\}$ and, therefore, a uniform bound on its solution. This follows by Theorem B.9 applied to the equation

$$\frac{\lambda^2(t)}{\lambda^2(s)}w(t,s,\xi) = \eta_2 - i\frac{|\xi|^2}{N}\int_s^t (1+s)\frac{\lambda^2(\tau)}{\lambda^2(s)}\eta_1\mathrm{d}\tau + |\xi|^2\int_s^t \lambda^2(\tau)\int_s^\tau \frac{1}{\lambda^2(\theta)}\left(\frac{\lambda^2(\theta)}{\lambda^2(s)}w(\theta,s,\xi)\right)\mathrm{d}\theta\mathrm{d}\tau,$$

obtained by plugging the first equation into the second one. Proposition 3.7.1 implies the condition on the right-hand side and on the integral kernel

$$|\xi|^2\int_s^t (1+s)\frac{\lambda^2(s)}{\lambda^2(\tau)}\mathrm{d}\tau \lesssim |\xi|^2(1+t)^2 \lesssim 1, \qquad \int_s^t |\xi|^2\int_s^\tau \frac{\lambda^2(\tau)}{\lambda^2(\theta)}\mathrm{d}\tau\mathrm{d}\theta \lesssim 1$$

on $Z_{diss}(N)$ and uniform in $s \leq t$. Furthermore, the first integral equation implies the desired bound on $v(t,s,\xi)$. □

Lemma 3.9. *Assume (A1) and (C2). Then*

$$||\mathcal{E}(t,s,\xi)|| \lesssim \frac{1+s}{1+t}, \qquad t_\xi \geq t \geq s. \tag{3.3.5}$$

Proof. We estimate the columns separately. Again they satisfy the integral equations (3.3.4).

We start by estimating the first one. If we plug the second integral equation into the first one, we obtain

$$\frac{1+t}{1+s}v(t,s,\xi) = 1 - |\xi|^2\int_s^t\int_s^\tau \underbrace{\frac{\lambda^2(\theta)}{\lambda^2(\tau)}}_{\leq 1}\frac{1+\theta}{1+s}v(\theta,s,\xi)\mathrm{d}\theta\mathrm{d}\tau$$

and Theorem B.9 together with the definition of the zone yields well-posedness of this equation in $L^\infty(Z_{diss}\cap\{t \geq s\})$. Now, the second integral equation may be used to deduce the same bound for $w(t,s,\xi)$

$$\frac{1+t}{1+s}w(t,s,\xi) \lesssim |\xi|^2\int_s^t \lambda^2(\tau)\mathrm{d}\tau\frac{1+t}{\lambda^2(t)} \lesssim 1.$$

For the second column we obtain similarly

$$\frac{1+t}{1+s}v(t,s,\xi) = -iN\frac{\lambda^2(s)}{1+s}\int_s^t \frac{\mathrm{d}\tau}{\lambda^2(\tau)} - |\xi|^2\int_s^t\int_s^\tau \frac{\lambda^2(\theta)}{\lambda^2(\tau)}\frac{1+\theta}{1+s}v(\theta,s,\xi)\mathrm{d}\theta\mathrm{d}\tau$$

and again Theorem B.9 is applicable, because the first summand is uniformly bounded by Proposition 3.7.2. For $w(t,s,\xi)$ we use the second integral equation to conclude the desired bound, it holds

$$\frac{|\xi|^2}{\lambda^2(t)}\int_s^t \lambda^2(\tau)(1+t)\mathrm{d}\tau \lesssim |\xi|^2(1+t)^2 \lesssim 1.$$

$\square$

Further results for the high regularity case. In order to perform a perfect diagonalization in the hyperbolic zone, it is essential to find symbol estimates for $\mathcal{E}(t_\xi,0,\xi)$ for $|\xi|\le N$.

Lemma 3.10. *Assume that (A1), (A4)$_\ell$ and (C1) hold. Then for $|\xi|\le N$ the symbol-like estimate*

$$||\mathrm{D}_\xi^\alpha\mathcal{E}(t_\xi,0,\xi)|| \le C_\alpha\frac{1}{\lambda^2(t_\xi)}|\xi|^{-|\alpha|}$$

is valid for all $|\alpha|\le\ell+1$.

Proof. It holds $\mathrm{D}_t\mathcal{E} = A\mathcal{E}$ with

$$A(t,\xi) = \begin{pmatrix} & \frac{N}{1+t} \\ \frac{1+t}{N}|\xi|^2 & ib(t)\end{pmatrix}, \qquad ||A(t,\xi)|| \lesssim \frac{1}{1+t}.$$

Thus for $|\alpha|=1$ we get

$$\mathrm{D}_t\mathrm{D}_\xi^\alpha\mathcal{E} = \mathrm{D}_\xi^\alpha(A\mathcal{E}) = (\mathrm{D}_\xi^\alpha A)\mathcal{E} + A(\mathrm{D}_\xi^\alpha\mathcal{E})$$

or using Duhamel's formula together with the initial condition $\mathrm{D}_\xi^\alpha\mathcal{E}(0,0,\xi)=0$

$$\mathrm{D}_\xi^\alpha\mathcal{E}(t,0,\xi) = \int_0^t \mathcal{E}(t,\tau,\xi)(\mathrm{D}_\xi^\alpha A(\tau,\xi))\mathcal{E}(\tau,0,\xi)\mathrm{d}\tau.$$

Now the known estimates for $\mathcal{E}(t,s,\xi)$, equation (3.3.3), imply together with $||\mathrm{D}_\xi^\alpha A(t,\xi)|| \lesssim 1$ the desired statement $||\mathrm{D}_\xi^\alpha\mathcal{E}(t,0,\xi)|| \lesssim t \lesssim |\xi|^{-1}$ for $(t,\xi)\in Z_{diss}(N)$.

For $|\alpha|=\ell>1$ we use Leibniz rule to get similar representations containing all derivatives of order less than $|\alpha|$ under the integral and use induction over ℓ. From the estimates

$$(\mathrm{D}_\xi^{\alpha_1}A)(\mathrm{D}_\xi^{\alpha_2}\mathcal{E}) \lesssim \frac{1}{\lambda^2(t)}|\xi|^{1-|\alpha_1|-|\alpha_2|}$$

for $|\alpha_1|+|\alpha_2|\le\ell$, formula (3.3.3) and from the first statement we conclude

$$||\mathrm{D}_\xi^\alpha\mathcal{E}(t,0,\xi)|| \lesssim \int_0^t \frac{1}{\lambda^2(\tau)}|\xi|^{1-\ell}\mathrm{d}\tau \lesssim \frac{1}{\lambda^2(t)}|\xi|^{-|\alpha|}$$

by the aid of Proposition 3.7.1. Application of the equation itself and using (A4)$_\ell$ to estimate

$$||\mathrm{D}_t^k A(t,\xi)|| \lesssim \left(\frac{1}{1+t}\right)^{k+1}, \qquad k\le\ell,$$

implies

$$||\mathrm{D}_t^k \mathrm{D}_\xi^\alpha \mathcal{E}(t,0,\xi)|| \lesssim \frac{1}{\lambda^2(t)} \left(\frac{1}{1+t}\right)^k |\xi|^{-|\alpha|}.$$

Finally, with the estimate for the zone boundary t_ξ,

$$|\mathrm{D}_\xi^\alpha t_\xi| \lesssim |\xi|^{-1-|\alpha|}, \qquad |\xi| \le N, \tag{3.3.6}$$

the statement follows. □

This result can be reformulated in the following form. The multiplier $\lambda^2(t_\xi)\mathcal{E}(t_\xi,0,\xi)$ is an element of the *homogeneous symbol class*

$$\dot{S}^0_{(\ell-1)} = \{ \ m \in C^\infty(\mathbb{R}^n \setminus \{0\}) \mid \forall |\alpha| \le \ell - 1 \ : \ |\mathrm{D}_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \ \} \tag{3.3.7}$$

of restricted smoothness $\ell - 1$. Thus as consequence of the Marcinkiewicz multiplier theorem the Fourier multiplier with symbol $\lambda^2(t_\xi)\mathcal{E}(t_\xi,0,\xi)$ maps L^p into L^p for all $p \in (1,\infty)$, if $\ell \ge \lceil \frac{n}{2} \rceil - 1$.

3.3.2 The hyperbolic zone: low regularity theory

We assume (A1) – (A3) and restrict our considerations to the hyperbolic zone

$$Z_{hyp}(N) = \{ \ (t,\xi) \mid |\xi| \ge N b(t) \ \}$$

with suitably chosen zone constant N. In this zone the micro-energy (3.2.8) coincides with the usual hyperbolic energy. Thus, we consider

$$U = (|\xi|\hat{u}, \mathrm{D}_t \hat{u})^T$$

with

$$\mathrm{D}_t U = A(t,\xi) U = \begin{pmatrix} & |\xi| \\ |\xi| & ib(t) \end{pmatrix} U.$$

We apply two transformations to this system. In a first step we diagonalize the homogeneous principal part. After that, we perform one further diagonalization step to make the remainder integrable over the hyperbolic zone.

Step 1. We denote by M the matrix

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{3.3.8}$$

consisting of eigenvectors of the homogeneous principal part of $A(t,\xi)$ with inverse

$$M^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{3.3.9}$$

Then for $U^{(0)} = M^{-1} U$ we get the system (cf. page 36, where we did exactly the same)

$$\mathrm{D}_t U^{(0)} = \big(\mathcal{D}(\xi) + R(t)\big) U^{(0)} \tag{3.3.10}$$

with

$$\mathcal{D}(\xi) = \begin{pmatrix} |\xi| & \\ & -|\xi| \end{pmatrix}, \qquad R(t) = \frac{ib(t)}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.3.11}$$

If $b(t)$ is integrable, we are done. If $b(t)$ is not integrable we perform one further diagonalization step in the hyperbolic zone.

Step 2. Following K. Yagdjian, [Yag97], and M. Reissig / K. Yagdjian, [RY00], we denote $F_0 = \operatorname{diag} R$ and

$$N^{(1)} = \begin{pmatrix} & \frac{R_{12}}{\tau_1-\tau_2} \\ \frac{R_{21}}{\tau_2-\tau_1} & \end{pmatrix} = \begin{pmatrix} & \frac{ib(t)}{4|\xi|} \\ -\frac{ib(t)}{4|\xi|} & \end{pmatrix},$$

$$B^{(1)} = \mathrm{D}_t N^{(1)} - (R-F_0)N^{(1)} = \begin{pmatrix} & \frac{b'(t)}{4|\xi|} \\ -\frac{b'(t)}{4|\xi|} & \end{pmatrix} - \begin{pmatrix} -\frac{b^2(t)}{8|\xi|} & \\ & \frac{b^2(t)}{8|\xi|} \end{pmatrix},$$

$$N_1 = I + N^{(1)}, \qquad \det N_1 = 1 - \frac{b^2(t)}{16|\xi|^2}.$$

Especially, $\det N_1 \neq 0$ in $Z_{hyp}(N)$ for zone constant $N > \frac{1}{4}$. Furthermore, the norm satisfies $||N_1|| = 1 + \frac{b(t)}{4|\xi|} \lesssim 1$ in $Z_{hyp}(N)$. Thus N_1 is invertible with uniformly bounded inverse matrix on $Z_{hyp}(N)$. Thus, if we define

$$R_1(t,\xi) = -N_1^{-1} B^{(1)}(t,\xi),$$

a simple calculation shows the (operator[3]) identity

$$\big(\mathrm{D}_t - \mathcal{D}(\xi) - R(t)\big) N_1(t,\xi) = N_1(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(\xi) - F_0(t) - R_1(t,\xi)\big). \tag{3.3.12}$$

Indeed, we have by the construction given above, that

$$[\mathcal{D}(\xi), N^{(1)}(t,\xi)] = F_0(t) - R(t),$$

and hence,

$$(\mathrm{D}_t - \mathcal{D} - R)N_1 = \mathrm{D}_t N^{(1)} + N_1 \mathrm{D}_t - \mathcal{D}N^{(1)} - \mathcal{D} - RN^{(1)} - R = -N_1 R_1 + N_1 \mathrm{D}_t - N_1 \mathcal{D} - N_1 F_0.$$

Step 3. In a third step, we estimate the fundamental solution of the transformed system

$$\big(\mathrm{D}_t - \mathcal{D}(\xi) - F_0(t) - R_1(t,\xi)\big)\mathcal{E}_1(t,s,\xi) = 0, \qquad \mathcal{E}_1(s,s,\xi) = I \in \mathbb{C}^{2\times 2}. \tag{3.3.13}$$

From (A2) and (A3) we conclude that $b^2(t)$ is dominated by $-b'(t)$, and therefore,

$$\int_{t_\xi}^{\infty} ||R_1(\tau,\xi)|| \mathrm{d}\tau \lesssim \int_{t_\xi}^{\infty} \frac{-b'(\tau)}{|\xi|} \mathrm{d}\tau = \frac{b(t_\xi)}{|\xi|} = \frac{1}{N}.$$

Furthermore, F_0 is diagonal. Thus, Theorem B.10 implies the estimate

$$||\mathcal{E}_1(t,s,\xi)|| \lesssim \frac{\lambda(s)}{\lambda(t)}, \qquad t \geq s \geq t_\xi. \tag{3.3.14}$$

Using that the matrices M and N_1 are uniformly bounded with uniformly bounded inverses on the hyperbolic zone $Z_{hyp}(N)$ for sufficiently large zone constant N, this estimate transfers to the fundamental solution $\mathcal{E}(t,s,\xi)$ and thus together with the results from the dissipative zone we can conclude the following theorem.

[3] Thus we understand it as a usual identity after multiplying with a C^1 vector function from the right-hand side.

Theorem 3.11. *Assume (A1) – (A3) together with (C1). Then the fundamental solution $\mathcal{E}(t,s,\xi)$ satisfies for all $t \geq s \geq 0$ and uniform in ξ the estimate*

$$||\mathcal{E}(t,s,\xi)|| \lesssim \frac{\lambda(s)}{\lambda(t)}.$$

Assumption (C1) is only used for the estimate in the dissipative zone. If (C1) is violated the estimate in $Z_{diss}(N)$ may dominate the estimate in $Z_{hyp}(N)$ and the result of Theorem 3.11 is only valid uniform in ξ with $(s,\xi) \in Z_{hyp}(N)$.

3.3.3 The hyperbolic zone: high regularity theory

Now we replace Assumptions (A2) and (A3) by Assumption (A4) and consider the smaller hyperbolic zone

$$Z_{hyp}(N) = \{\ (t,\xi)\ |\ (1+t)|\xi| \geq N\ \}$$

with suitably chosen zone constant N. The aim of this section is to prove a stronger variant of Theorem 3.11, which allows the application of stationary phase method to deduce L^p–L^q estimates.

Diagonalization. The difference to Section 3.3.2 is that we perform more diagonalization steps. We use the special symbol classes defined by Definition 3.2. Remark that it holds $|\xi|\phi_{hyp,N} \in S_N\{1,0\}$ and by Assumption (A4)$_\ell$ also $b(t)\phi_{hyp,N} \in S_N^{\ell,\infty}\{0,1\}$. For the further calculations we omit the cut-off function $\phi_{hyp,N}$.

Step 1. Again, we consider $U^{(0)} = M^{-1}U$ and get the system

$$\mathrm{D}_t U^{(0)} = \big(\mathcal{D}(\xi) + R(t)\big)U^{(0)}$$

with coefficient matrices $\mathcal{D} \in S_N\{1,0\}$ and $R \in S_N^{\ell,\infty}\{0,1\}$ given by (3.3.11).

Step $k+1$. We construct recursively the diagonalizer $N_k(t,\xi)$ of order k. Let

$$N_k(t,\xi) = \sum_{j=0}^{k} N^{(j)}(t,\xi), \qquad F_k(t,\xi) = \sum_{j=0}^{k} F^{(j)}(t,\xi),$$

where $N^{(0)} = I$, $B^{(0)} = R(t)$ and $F^{(0)} = \operatorname{diag} B^{(0)} = F_0(t)$.

The construction goes along the following scheme. Note, that F_0 is a multiple of I. Then we set

$$\begin{aligned} F^{(j)} &= \operatorname{diag} B^{(j)}, \\ N^{(j+1)} &= \begin{pmatrix} & -B_{12}^{(j)} \,/\, 2|\xi| \\ B_{21}^{(j)} \,/\, 2|\xi| & \end{pmatrix}, \\ B^{(j+1)} &= (\mathrm{D}_t - \mathcal{D} - R)N_{j+1} - N_{j+1}(\mathrm{D}_t - \mathcal{D} - F_j). \end{aligned}$$

Now we prove by induction that $N^{(j)} \in S_N^{\ell-j+1,\infty}\{-j,j\}$ and $B^{(j)} \in S_N^{\ell-j,\infty}\{-j,j+1\}$.

For $j=0$ we know

$$F^{(0)} \in S_N^{\ell,\infty}\{0,1\}, \quad N^{(1)} \in S_N^{\ell,\infty}\{-1,1\}, \quad B^{(1)} \in S_N^{\ell-1,\infty}\{-1,2\},$$

which follows directly from the representation in Step 2 of the previous section.

For $j \geq 1$ we apply an inductive argument. Assume, we know $B^{(j)} \in S_N^{\ell-j,\infty}\{-j, j+1\}$. Then, by definition of $N^{(j+1)}$, we have from $|\xi|^{-1} \in S_N\{-1, 0\}$, that $N^{(j+1)} \in S_N^{\ell-j,\infty}\{-j-1, j+1\}$ and $F^{(j)} \in S_N^{\ell-j,\infty}\{-j, j+1\}$. Moreover,

$$\begin{aligned} B^{(j+1)} &= (\mathrm{D}_t - \mathcal{D} - R)(\sum_{\nu=0}^{j+1} N^{(\nu)}) - (\sum_{\nu=0}^{j+1} N^{(\nu)})(\mathrm{D}_t - \mathcal{D} - \sum_{\nu=0}^{j} F^{(j)}) \\ &= B^{(j)} + [N^{(j+1)}, \mathcal{D}] - F^{(j)} + \mathrm{D}_t N^{(j+1)} + RN^{(j+1)} \\ &\qquad\qquad + N^{(j+1)} \sum_{\nu=0}^{j} F^{(\nu)} - (\sum_{\nu=1}^{j+1} N^{(\nu)}) F^{(j)}. \end{aligned}$$

Now $B^{(j)} + [N^{(j+1)}, \mathcal{D}] - F^{(j)} = 0$ for all j. The sum of the remaining terms belongs to the symbol class $S_N^{\ell-j-1,\infty}\{-j-1, j+2\}$. Hence $B^{(j+1)} \in S_N^{\ell-j-1,\infty}\{-j-1, j+2\}$.

Now the definition of $B^{(k)}$ implies the operator identity

$$\big(\mathrm{D}_t - \mathcal{D}(\xi) - R(t)\big) N_k(t,\xi) = N_k(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(\xi) - F_{k-1}(t,\xi)\big) \qquad \mod S_N^{\ell-k,\infty}\{-k, k+1\}. \tag{3.3.15}$$

Thus, we have constructed the desired diagonalizer, if we can show that the matrix $N_k(t,\xi)$ is invertible on $Z_{hyp}(N)$ with uniformly bounded inverse. But this follows from $N_k - I \in S_N^{\ell-k+1,\infty}\{-1, 1\}$ by the choice of a sufficiently large zone constant N. Indeed, we have

$$||N_k - I|| \leq C \frac{1}{|\xi|} b(t) \leq C' \frac{1}{|\xi|(1+t)} \leq \frac{C'}{N} \to 0 \qquad \text{as } N \to \infty.$$

Thus, with the notation $R_k(t,\xi) = -N_k^{-1}(t,\xi) B^{(k)}(t,\xi)$ we have proven the following lemma.

Lemma 3.12. *Assume (A1) and (A4)$_\ell$.*

For each $1 \leq k \leq \ell$ there exists a zone constant N and matrix valued symbols

- $N_k(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,0\}$ *invertible for all* $(t,\xi) \in Z_{hyp}(N)$ *and with* $N_k^{-1}(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,0\}$
- $F_{k-1}(t,\xi) \in S_N^{\ell-k+1,\infty}\{0,1\}$ *diagonal with* $F_{k-1}(t,\xi) - \frac{ib(t)}{2} I \in S_N^{\ell-k+1,\infty}\{-1,2\}$
- $R_k(t,\xi) \in S_N^{\ell-k,\infty}\{-k, k+1\}$,

such that the (operator) identity

$$\big(\mathrm{D}_t - \mathcal{D}(\xi) - R(t)\big) N_k(t,\xi) = N_k(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(\xi) - F_{k-1}(t,\xi) - R_k(t,\xi)\big) \tag{3.3.16}$$

holds for all $(t,\xi) \in Z_{hyp}(N)$.

Remarks on perfect diagonalization. Lemma 3.12 can be understood as perfect diagonalization of the original system. If we define $F(t,\xi)$ as *asymptotic sum* of the $F^{(k)}(t,\xi)$,

$$F(t,\xi) \sim \sum_{k=0}^{\infty} F^{(k)}(t,\xi), \tag{3.3.17}$$

this means, we require $F(t,\xi) - F_k(t,\xi) \in S_N\{-k-1, k+2\}$ for all $k \in \mathbb{N}$, and similarly

$$N(t,\xi) \sim \sum_{k=0}^{\infty} N^{(k)}(t,\xi), \tag{3.3.18}$$

which can be chosen to be invertible, equation (3.3.15) implies

$$\big(\mathrm{D}_t - \mathcal{D}(\xi) - R(t)\big)N(t,\xi) - N(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(\xi) - F(t,\xi)\big) \in \bigcap_{k\in\mathbb{N}} S_N\{-k, k+1\}. \tag{3.3.19}$$

Thus if we define the *residual symbol classes*

$$\mathcal{H}\{m\} := \bigcap_{m_1+m_2=m} S_N\{m_1, m_2\}, \tag{3.3.20}$$

we can find $P_\infty(t,\xi) \in \mathcal{H}\{1\}$ such that

$$\big(\mathrm{D}_t - \mathcal{D}(\xi) - R(t)\big)N(t,\xi) = N(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(\xi) - F(t,\xi) - P_\infty(t,\xi)\big). \tag{3.3.21}$$

The classes $\mathcal{H}\{m\}$ are invariant under multiplication by $\exp(\pm it|\xi|)$. This explains why we perform more than one diagonalization step. Multiplication by $e^{\pm it|\xi|}$ is not a well defined operation on the symbol classes $S_N\{m_1, m_2\}$, it destroys the symbol estimates according to the following proposition. It is closely related to the geometry of the hyperbolic zone.

Proposition 3.13.

1. *$e^{\pm it|\xi|}S_N^{\ell_1,\ell_2}\{m_1, m_2\} \hookrightarrow S_N^{\ell_1,\ell_2}\{m_1+\ell, m_2-\ell\}$ with $\ell = \ell_1+\ell_2$,*
2. *$e^{\pm it|\xi|}\mathcal{H}\{m\} \hookrightarrow \mathcal{H}\{m\}$.*

Proof. It suffices to prove the first statement. It holds for $a \in S_N\{m_1, m_2\}$

$$\begin{aligned}
\mathrm{D}_t^k \mathrm{D}_{|\xi|}^\alpha e^{it|\xi|} a(t,\xi) &= \sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C_{k_1,k_2,\alpha_1,\alpha_2} |\xi|^{k_1} t^{\alpha_1} e^{it|\xi|} \mathrm{D}_t^{k_2} \mathrm{D}_{|\xi|}^{\alpha_2} a(t,\xi) \\
&\le \sum_{k_1+k_2=k}\sum_{\alpha_1+\alpha_2=\alpha} C'_{k_1,k_2,\alpha_1,\alpha_2} |\xi|^{m_1-\alpha_2+k_1} \left(\frac{1}{1+t}\right)^{m_2+k_2-\alpha_1} \\
&\le C_{k,\alpha} |\xi|^{m_1+\ell-\alpha} \left(\frac{1}{1+t}\right)^{m_2-\ell+k}
\end{aligned}$$

for $k \le \ell_1$, $\alpha \le \ell_2$ using Leibniz rule and the definition of the hyperbolic zone. ∎

Fundamental solution of the diagonalized system. After performing several diagonalization steps, we want to construct the fundamental solution of the transformed system

$$\big(\mathrm{D}_t - \mathcal{D}(\xi) - F_{k-1}(t,\xi) - R_k(t,\xi)\big)\mathcal{E}_k(t,s,\xi) = 0, \qquad \mathcal{E}_k(s,s,\xi) = I \in \mathbb{C}^{2\times 2} \tag{3.3.22}$$

and to obtain structural properties of it. The construction goes along the following steps:

- the fundamental solution $\mathcal{E}_0(t,s,\xi)$ to $\mathrm{D}_t - \mathcal{D}(\xi)$,
- influence of the main term $F^{(0)}(t,\xi)$ of $F_{k-1}(t,\xi)$,

- influence of $F_k(t,\xi) - F^{(0)}(t,\xi)$ and $R_k(t,\xi)$.

The fundamental solution $\mathcal{E}_0(t,s,\xi)$ describes a *phase function* of a Fourier integral operator, i.e. the oscillatory behaviour of the solution multiplier. The main term $F^{(0)}(t,\xi)$ describes the energy decay. Together with the other terms it constitutes a Fourier multiplier which behaves as symbol with restricted smoothness. The number k of diagonalization steps is directly connected to the smoothness properties of this symbol.

Step 1. Let

$$\mathcal{E}_0(t,s,\xi) = \exp\{i(t-s)\mathcal{D}(\xi)\} = \begin{pmatrix} e^{i(t-s)|\xi|} & \\ & e^{-i(t-s)|\xi|} \end{pmatrix}, \tag{3.3.23}$$

such that for $\widetilde{\mathcal{E}_0}(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\mathcal{E}_0(t,s,\xi)$ the equation

$$\mathrm{D}_t\widetilde{\mathcal{E}_0}(t,s,\xi) = \big(\mathcal{D}(\xi) + F^{(0)}(t,\xi)\big)\widetilde{\mathcal{E}_0}(t,s,\xi) \tag{3.3.24}$$

is satisfied. Thus $\widetilde{\mathcal{E}_0}$ describes the influence of the main diagonal terms.

Step 2. By the aid of $\widetilde{\mathcal{E}_0}(t,s,\xi)$ we define

$$\begin{aligned} \mathcal{R}_k(t,s,\xi) &= \widetilde{\mathcal{E}_0}(s,t,\xi)\big(F_{k-1}(t,\xi) + R_k(t,\xi) - F^{(0)}(t,\xi)\big)\widetilde{\mathcal{E}_0}(t,s,\xi), \\ &= F_{k-1}(t,\xi) + \mathcal{E}_0(s,t,\xi)R_k(t,\xi)\mathcal{E}_0(t,s,\xi) - F^{(0)}(t,\xi), \end{aligned} \tag{3.3.25}$$

such that, by the aid of the solution $\mathcal{Q}_k(t,s,\xi)$ to

$$\mathrm{D}_t\mathcal{Q}_k(t,s,\xi) = \mathcal{R}_k(t,s,\xi)\mathcal{Q}_k(t,s,\xi), \qquad \mathcal{Q}_k(s,s,\xi) = I \in \mathbb{C}^{2\times 2}, \tag{3.3.26}$$

the matrix $\mathcal{E}_k(t,s,\xi)$ can be represented as

$$\mathcal{E}_k(t,s,\xi) = \widetilde{\mathcal{E}_0}(t,s,\xi)\mathcal{Q}_k(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\mathcal{E}_0(t,s,\xi)\mathcal{Q}_k(t,s,\xi). \tag{3.3.27}$$

The solution to (3.3.26) is given by the Peano-Baker formula, Theorem B.5, as

$$\mathcal{Q}_k(t,s,\xi) = I + \sum_{\ell=1}^{\infty} i^\ell \int_s^t \mathcal{R}_k(t_1,s,\xi)\int_s^{t_1}\mathcal{R}_k(t_2,s,\xi)\ldots\int_s^{t_{\ell-1}}\mathcal{R}_k(t_\ell,s,\xi)\mathrm{d}t_\ell\ldots\mathrm{d}t_1. \tag{3.3.28}$$

Step 3. The series representation (3.3.28) for $\mathcal{Q}_k(t,s,\xi)$ can be used to deduce estimates. From the unitarity of $\mathcal{E}_0(t,s,\xi)$ it follows that

$$||\mathcal{R}_k(t,s,\xi)|| = ||R_k(t,\xi)|| \lesssim \frac{1}{(1+t)^2|\xi|}$$

and thus, using

$$\int_{t_\xi}^\infty \frac{\mathrm{d}\tau}{(1+\tau)^2|\xi|} = \frac{1}{(1+t_\xi)|\xi|} = \frac{1}{N}$$

together with Corollary B.7, it follows that

$$||\mathcal{Q}_k(t,s,\xi)|| \lesssim 1.$$

This gives the counterpart of Theorem 3.11 for the high regularity theory and works for all $k \geq 1$. In a second step we want to estimate ξ-derivatives of $\mathcal{Q}_k(t,s,\xi)$. Proposition 3.13 yields from $R_k(t,\xi) \in S_N^{\ell-k,\infty}\{-k,k+1\}$ under the Assumption (A4)$_\ell$ and with

$$k-1 \leq \ell - k \tag{3.3.29}$$

that $\mathcal{R}_k(t,s,\xi) \in S_N^{k-1,k-1}\{-1,2\}$ uniform in the variable s and derivations with respect to s behave like multiplications by $|\xi|$. Therefore, we set $\ell = 2k-1$ from now on.

Proposition 3.14. *Assume $a \in S_N^{k,k}\{-1,2\}$. Then*

$$b(t,s,\xi) = 1 + \sum_{j=1}^{\infty} \int_s^t a(t_1,\xi) \int_s^{t_1} a(t_2,\xi) \dots \int_s^{t_{j-1}} a(t_j,\xi) \mathrm{d}t_j \dots \mathrm{d}t_1$$

defines a symbol from $S_N^{k,k}\{0,0\}$ uniform in $s \geq t_\xi$.

Proof. We use Proposition B.6 to estimate this series. This yields in a first step (without taking derivatives)

$$\begin{aligned} |b(t,s,\xi)| &\lesssim 1 + \sum_{j=1}^{\infty} \int_s^t \frac{1}{|\xi|(1+t_1)^2} \int_s^{t_1} \frac{1}{|\xi|(1+t_2)^2} \cdots \int_s^{t_{j-1}} \frac{1}{|\xi|(1+t_j)^2} \mathrm{d}t_j \cdots \mathrm{d}t_1 \\ &\lesssim \exp\left\{ \int_{t_\xi}^t \frac{\mathrm{d}\tau}{|\xi|(1+\tau)^2} \right\} \lesssim 1 \end{aligned}$$

and taking α derivatives with respect to ξ yields in each summand further factors $|\xi|^{-|\alpha|}$ according to Leibniz rule. Furthermore, time-derivatives can be estimated from $\partial_t b(t,s,\xi) = a(t,\xi)b(t,s,\xi)$ together with $a \in S_N^{k,k}\{-1,2\} \subseteq S_N^{k,k}\{0,1\}$. $\square$

An almost immediate consequence of this proposition is the following structural representation of the fundamental solution.

Theorem 3.15. *Assume (A1) and (A4)$_{2k-1}$, $k \geq 1$. Then the fundamental solution $\mathcal{E}_k(t,s,\xi)$ of the transformed system* (3.3.22) *can be represented in the hyperbolic zone as*

$$\mathcal{E}_k(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)} \mathcal{E}_0(t,s,\xi) \mathcal{Q}_k(t,s,\xi) \qquad t,s \geq t_\xi$$

with a symbol $\mathcal{Q}_k(t,s,\xi)$ of restricted smoothness subject to the symbol estimates

$$\left\| \mathrm{D}_t^{\ell_1} \mathrm{D}_s^{\ell_2} \mathrm{D}_\xi^{\alpha} \mathcal{Q}_k(t,s,\xi) \right\| \leq C_{\ell,\alpha} |\xi|^{\ell_2 - |\alpha|} \left(\frac{1}{1+t} \right)^{\ell_1} \qquad t \geq s \geq t_\xi$$

for all multi-indices $|\alpha| \leq k-1$, all $\ell_1 \leq k-1$ and all $\ell_2 \in \mathbb{N}_0$.

Of special interest is $\mathcal{E}_k(t,t_\xi,\xi)$. The estimate of the previous lemma together with the properties of the derivatives of t_ξ from equation (3.3.6) imply

Corollary 3.16. *Assume (A1) and (A4)$_{2k-1}$, $k \geq 1$. Then*

$$\mathcal{Q}_k(t,t_\xi,\xi) \in S_N^{k-1,k-1}\{0,0\}$$

for $t \geq t_\xi$ and $|\xi| \leq N$.

Similar to the consideration in Section 3.1, the matrix $\mathcal{Q}_k(t,s,\xi)$ converges for $t \to \infty$ to a well-defined limit. This limit will be used in Section 3.5 to conclude the sharpness of our results.

Theorem 3.17. *Assume (A1) and (A4)$_{2k-1}$, $k \geq 1$. The limit*

$$\mathcal{Q}_k(\infty,s,\xi) = \lim_{t\to\infty} \mathcal{Q}_k(t,s,\xi)$$

exists uniform in ξ for $|\xi| > \xi_s$. Furthermore,

$$||\mathrm{D}_\xi^\alpha \mathcal{Q}_k(\infty,t_\xi,\xi)|| \leq C_\alpha |\xi|^{-|\alpha|}$$

for all multi-indices $|\alpha| \leq k-1$ and all $\xi \neq 0$.

Proof. We fix the starting value s and consider only $|\xi| \geq \xi_s$ (i.e. $s \geq t_\xi$). Taking the difference $\mathcal{Q}_k(t,s,\xi) - \mathcal{Q}_k(\tilde{t},s,\xi)$ in the series representation (3.3.28) yields

$$\mathcal{Q}_k(t,s,\xi) - \mathcal{Q}_k(\tilde{t},s,\xi) = \sum_{j=1}^\infty \int_{\tilde{t}}^t \mathcal{R}_k(t_1,s,\xi) \int_s^{t_1} \mathcal{R}_k(t_2,s,\xi) \dots \int_s^{t_{\ell-1}} \mathcal{R}_k(t_\ell,s,\xi)\mathrm{d}t_\ell \dots \mathrm{d}t_1,$$

such that with Proposition B.6

$$\begin{aligned} ||\mathcal{Q}_k(t,s,\xi) - \mathcal{Q}_k(\tilde{t},s,\xi)||_{L^\infty\{|\xi|\geq\xi_s\}} &\leq \int_{\tilde{t}}^t ||R(t_1,\xi)|| \exp\left\{\int_{s_\xi}^\infty ||R(\tau,\xi)||\mathrm{d}\tau\right\} \mathrm{d}t_1 \\ &\to 0, \qquad t,\tilde{t} \to \infty. \end{aligned}$$

Similarly one obtains for $|\alpha| \leq k-1$

$$||\mathrm{D}_\xi^\alpha \mathcal{Q}_k(t,s,\xi) - \mathrm{D}_\xi^\alpha \mathcal{Q}_k(\tilde{t},s,\xi)|| \lesssim |\xi|^{-\alpha} \int_{\tilde{t}}^t \frac{\mathrm{d}\tau}{|\xi|(1+\tau)^2} \to 0, \qquad t,\tilde{t} \to \infty$$

uniform in $|\xi| \geq \xi_s$. Now the second statement follows from the estimates of t_ξ, formula (3.3.6). $\square$

We have proved even more. The limit exists in the homogeneous symbol class $\dot{S}^0_{(k-1)}$ of restricted smoothness. Proposition B.6 may also be used to estimate the formal representation of $\mathcal{Q}_k(\infty,s,\xi)$ as symbol in (s,ξ).

Corollary 3.18. *The series representation*

$$\mathcal{Q}_k(\infty,s,\xi) = I + \sum_{j=1}^\infty i^j \int_s^\infty \mathcal{R}_k(t_1,s,\xi) \int_s^{t_1} \mathcal{R}_k(t_2,s,\xi) \dots \int_s^{t_{\ell-1}} \mathcal{R}_k(t_\ell,s,\xi)\mathrm{d}t_\ell \dots \mathrm{d}t_1$$

gives an asymptotic expansion of $\mathcal{Q}_k(\infty,s,\xi)$ in $S_N^{0,k-1}\{0,0\}$, i.e. the j-th term belongs to $S_N^{0,k-1}\{-j,j\}$.

Step 4. As in the proof of Theorem 3.1, Step 7, the transpose of the inverse of $\mathcal{Q}_k$ satisfies the related equation

$$\mathrm{D}_t \mathcal{Q}_k^{-T}(t,s,\xi) + \mathcal{R}_k^T(t,s,\xi)\mathcal{Q}_k^{-T}(t,s,\xi) = 0, \qquad \mathcal{Q}_k^{-T}(s,s,\xi) = I \in \mathbb{C}^{2\times 2}. \tag{3.3.30}$$

The matrix $\mathcal{R}_k^T(t,s,\xi)$ satisfies the same estimates like $\mathcal{R}_k(t,s,\xi)$ and therefore the reasoning of the previous step holds in the same way for $\mathcal{Q}_k^{-T}(t,s,\xi)$. In particular the matrix $\mathcal{Q}_k(t,s,\xi)$ is invertible in the hyperbolic zone and $\mathcal{Q}_k^{-1}(\infty,s,\xi)$ exists.

Corollary 3.19. *Assume (A1) and (A4)$_{2k-1}$, $k \geq 1$. Then the limit*

$$\mathcal{Q}_k^{-1}(\infty,s,\xi) = \lim_{t\to\infty} \mathcal{Q}_k^{-1}(t,s,\xi)$$

exists uniform in ξ for $|\xi| \geq \xi_s$.

Transforming back to the original problem. After constructing the fundamental solution $\mathcal{E}_k(t,s,\xi)$, we transform back to the original problem and get in the hyperbolic zone the representation

$$\mathcal{E}(t,s,\xi) = MN_k(t,\xi)\mathcal{E}_k(t,s,\xi)N_k^{-1}(s,\xi)M^{-1} \tag{3.3.31}$$

with uniformly bounded coefficient matrices $N_k, N_k^{-1} \in S_N^{k,\infty}\{0,0\}$. We combine this representation with the representation obtained in the dissipative zone. This yields for $0 \le s \le t_\xi \le t$ the expression

$$\mathcal{E}(t,s,\xi) = \frac{1}{\lambda(t)} MN_k(t,\xi)\mathcal{E}_0(t,t_\xi,\xi)\mathcal{Q}(t,t_\xi,\xi)N_k^{-1}(t_\xi,\xi)M^{-1}\lambda(t_\xi)\mathcal{E}(t_\xi,s,\xi). \tag{3.3.32}$$

Together with the definition of the micro-energy $U(t,\xi)$ from (3.2.8) this formula may be used to express also the previously introduced multipliers $\Phi_1(t,s,\xi)$ and $\Phi_2(t,s,\xi)$.

3.4 Estimates

The representations of solutions obtained so far allow us to conclude estimates for the asymptotic behaviour. This section is devoted to the study of estimates, which are directly related to our micro-energy (3.2.8), i.e. estimates for the fundamental solution $\mathcal{E}(t,s,\mathrm{D})$ or to the closely related energy operator $\mathbb{E}(t,\mathrm{D})$.

Estimates for the solution itself are postponed to Chapter 5.

3.4.1 L^2–L^2 estimates

The aim of this section is to give energy estimates following from the low regularity theory. The first result is an immediate consequence of Theorem 3.11.

Theorem 3.20. *Assume (A1) – (A3) and (C1). Then the L^2–L^2 estimate*

$$||\mathcal{E}(t,s,\mathrm{D})||_{2\to 2} \lesssim \frac{\lambda(s)}{\lambda(t)}$$

holds.

Using the definition of the micro-energy (3.2.8) we can reformulate this estimate in terms of the energy operator $\mathbb{E}(t,\mathrm{D})$. For convenience we recall the relation between the multiplier $\mathcal{E}(t,s,\xi)$ and $\mathbb{E}(t,\xi)$. They are a direct consequence of the definition of our micro-energy (3.2.8).

Proposition 3.21. 1. *It holds $\mathbb{E}(t,\xi) = \mathcal{E}(t,s,\xi)\mathbb{E}(s,\xi)$ for $s \ge t_\xi$.*

2. *The multiplier of the energy operator is related to the fundamental solution $\mathcal{E}(t,s,\xi)$ by*

$$\mathbb{E}(t,\xi)\begin{pmatrix} \frac{h(0,\xi)}{\langle\xi\rangle} & \\ & 1 \end{pmatrix} = \begin{pmatrix} \frac{|\xi|}{h(t,\xi)} & \\ & 1 \end{pmatrix}\mathcal{E}(t,0,\xi).$$

3. *The multiplier $|\xi|/h(t,\xi)$ induces a uniformly bounded family of operators on L^p, $p \in (1,\infty)$ converging strongly to the identity for $t \to \infty$.*

Corollary 3.22. *Assumptions (A1) – (A3), (C1) imply*

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2} \lesssim \frac{1}{\lambda(t)}.$$

We conclude this section with examples.

Example 3.4. Let

$$b(t) = \frac{\mu}{1+t}, \qquad \mu \in (0,1).$$

Then the Assumptions (A1) – (A3) and (C1) are satisfied and the above corollary gives again the known estimate $||\mathbb{E}(t,\mathrm{D})||^2_{2\to 2} \lesssim (1+t)^{-\mu}$ from Chapter 2.

For $\mu \in (1,2)$ Assumption (C2) is satisfied and can be used to deduce the same decay rate. Thus except for the value $\mu = 1$ we can reproduce the result of Theorem 2.8 for all sub-critical values of μ.

Example 3.5. Let $\mu > 0$ and $m \geq 1$. Then we consider

$$b(t) = \frac{\mu}{(e^{[m]}+t)\log(e^{[m]}+t)\cdots\log^{[m]}(e^{[m]}+t)}.$$

Again the assumptions are satisfied and we obtain

$$\lambda(t) = \left(\log^{[m]}(e^{[m]}+t)\right)^{\frac{\mu}{2}},$$

and the energy decay rate

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2} \lesssim \left(\log^{[m]}(e^{[m]}+t)\right)^{-\frac{\mu}{2}}$$

may become arbitrary small in the scale of iterated logarithms. This example is related to the paper of K. Mochizuki and H. Nakazawa, [MN96].

Example 3.6. If we consider $b(t) = \frac{\mu}{1+t}$ with $\mu > 2$, the decay rate in the dissipative zone dominates the one from the hyperbolic zone and, analogously to the above stated theorem, we obtain $||\mathbb{E}(t,\mathrm{D})||_{2\to 2} \lesssim (1+t)^{-1}$. This coincides with the estimate of Theorem 2.8.

3.4.2 L^p–L^q estimates

This section is devoted to the results of the high regularity theory. The basic estimate is given in the following theorem, it restates estimate (1.3.3) in the language of our operators. Although the proof is contained in the proof of Theorem 2.6 for $\rho = \frac{1}{2}$ and $\rho = -\frac{1}{2}$, respectively, we give it for convenience of the reader in a simplified form.

Theorem 3.23. *It holds*

$$||\mathcal{E}_0(t,0,\mathrm{D})||_{p,r\to q} \leq C_{p,q}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

for dual indices p and q, $p \in (1,2]$ and with regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Proof. The matrix $\mathcal{E}_0(t,0,\xi)$ has entries, which are linear combinations of the terms $e^{\pm it|\xi|}$. Therefore, it suffices to consider only these terms. We use a full dyadic decomposition of the phase space, $\phi_j(t,\xi) = \chi(2^{-j}t|\xi|)$ for $j \in \mathbb{Z}$, $\sum \phi_j(t,\xi) = 1$ for $\xi \neq 0$ to split the operator into components. Following the paper of P. Brenner, [Bre75], and the proof of Theorem 2.6 we obtain

$$\begin{aligned} I_j &= \left|\left|\mathcal{F}^{-1}\left[\phi_j(t,\xi)e^{\pm it|\xi|}\right]\right|\right|_\infty = 2^{jn}\left|\left|\mathcal{F}^{-1}\left[\chi(\eta)e^{\pm i2^j\eta}\right]\right|\right|_\infty \\ &\leq C2^{jn}(1+2^jt)^{-\frac{n-1}{2}}\sum_{|\alpha|\leq M}||\mathrm{D}^\alpha\chi(\eta)||_\infty \leq C2^{jn}(1+t)^{-\frac{n-1}{2}}, \end{aligned}$$

substituting $2^{-j}t|\xi| = \eta$ and using Lemma B.3, and

$$\tilde{I}_j = \left|\left|\phi_j(t,\xi)e^{\pm it|\xi|}\right|\right|_\infty \sim 1.$$

Interpolation yields for the dyadic components

$$||\mathcal{F}^{-1}[\phi_j(t,\xi)e^{\pm it|\xi|}\hat{u}(\xi)]||_q \le C2^{jn\left(\frac{1}{p}-\frac{1}{q}\right)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}||u||_p,$$

which implies for the corresponding operators the mapping property

$$e^{\pm it|\mathrm{D}|} : \dot{B}^r_{p,2} \to L^q$$

with regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$. Finally, the embedding relation[4] $L_{p,r} \hookrightarrow B^r_{p,2} = \dot{B}^r_{p,2} \cap L^p$ for $r > 0$ and $p \in (1,\infty)$ yields the desired result. $\square$

By the aid of this estimate we deduce from our representation a corresponding estimate for the dissipative Cauchy problem (3.0.1).

Theorem 3.24. *Assume (A1), (A4) and (C1). Then the operator $\mathcal{E}(t,s,\mathrm{D})$ satisfies for dual indices p and q, $p \in (1,2]$, $pq = p+q$, the norm estimate*

$$||\mathcal{E}(t,0,\mathrm{D})||_{p,r\to q} \lesssim \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

with regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Proof. We decompose the proof in two parts. First, we consider $\mathcal{E}(t,0,\mathrm{D})\phi_{diss,N}(t,\mathrm{D})$. Using the estimate $||\mathcal{E}(t,0,\xi)\phi_{diss,N}(t,\xi)|| \lesssim \frac{1}{\lambda^2(t)}$ together with the definition of the zone we get

$$||\mathcal{E}(t,0,\mathrm{D})\phi_{diss,N}(t,\mathrm{D})||_{p,q} \lesssim \frac{1}{\lambda^2(t)}(1+t)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)},$$

which is a stronger decay rate than the one given in the theorem.

In a second step we consider the hyperbolic part. For small frequencies we use the representation

$$\mathcal{E}(t,0,\xi)\phi_{hyp,N}(t,\xi) = \frac{1}{\lambda(t)}\underbrace{MN_k(t,\xi)}_{q\to q}\underbrace{\mathcal{E}_0(t-t_\xi,\xi)}_{p,r\to q}\underbrace{\mathcal{Q}_k(t,t_\xi,\xi)}_{p,r\to p,r}\underbrace{N_k^{-1}(t_\xi,\xi)M^{-1}\lambda(t_\xi)\mathcal{E}(t_\xi,0,\xi)}_{p,r\to p,r}\phi_{hyp,N}(t,\xi)$$

together with the mapping properties of the multipliers marked with a brace. They are a direct consequence of the estimates of Lemma 3.10, Lemma 3.12 and Theorem 3.15 in connection with the Marcinkiewicz multiplier theorem, Theorem B.2, and Theorem 3.23. It is essential, that $k-1 \ge \lceil \frac{n}{2} \rceil$. The operator $\mathcal{E}_0(t-t_\xi,\xi)$ brings the hyperbolic decay rate, the others are uniformly bounded.

For large frequencies the representation simplifies to

$$\mathcal{E}(t,0,\xi)\phi_{hyp,N}(t,\xi) = \frac{1}{\lambda(t)}\underbrace{MN_k(t,\xi)}_{q\to q}\underbrace{\mathcal{E}_0(t,\xi)}_{p,r\to q}\underbrace{\mathcal{Q}_k(t,0,\xi)}_{p,r\to p,r}\underbrace{N_k^{-1}(0,\xi)M^{-1}}_{p,r\to p,r}\phi_{hyp,N}(t,\xi),$$

the argumentation remains the same. $\square$

[4] For details on Besov spaces we refer to the treatment in the book of Th. Runst and W. Sickel, [RS96]. The above used embedding relation follows from Proposition 3, Section 2.6.2, together with the known relations for L^p spaces. The conclusion of the mapping property itself is analogous to the case of inhomogeneous spaces and uses the argument of P. Brenner, [Bre75, Lemma 2].

Example 3.7. If we use $b(t) = \frac{\mu}{1+t}$ with $\mu \in (0,1)$, we obtain the same L^p–L^q decay estimate as in Chapter 2, Theorem 2.8. In particular, we understand the structure of the estimate; it splits into the factor $(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$ coming from the free wave equation and $\lambda^{-1}(t) = (1+t)^{-\frac{\mu}{2}}$ coming from the dissipation itself.

Example 3.8. If $b(t) \in L^1(\mathbb{R}_+)$, the obtained estimate coincides with the estimate for free waves. This is also natural in view of Section 3.1 and especially the L^q-result of Corollary 3.4.

Example 3.9. We have not assumed monotonicity of the coefficient $b = b(t)$ and therefore we will give one non-monotonous example. Let

$$b(t) = \frac{2 + \cos(\alpha \log(1+t))}{4+4t}.$$

Then Assumptions (A1), (A4) and Condition (C1) are satisfied. Furthermore for α sufficiently large the function is not monotonous. It holds

$$\int b(t)\mathrm{d}t = \frac{1}{2}\log(e+t) + \frac{1}{4\alpha}\sin(\alpha\log(1+t)),$$

and thus $\lambda(t) \sim (1+t)^{\frac{1}{4}}$. Application of Theorem 3.24 yields the L^p–L^q estimate

$$||\mathcal{E}(t,0,\mathrm{D})||_{p,r\to q} \lesssim (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{4}},$$

which is independent ofso far the choice of α and gives the same decay order as the monotonous coefficient $\tilde{b}(t) = \frac{\mu}{1+t}$ with $\mu = \frac{1}{2}$.

Minimal regularity for the L^p–L^q estimate. With the notation

$$\ell_n = 2\left\lceil \frac{n}{2} \right\rceil + 1 = \begin{cases} n+1, & n \text{ even}, \\ n+2, & n \text{ odd}, \end{cases}$$

we can prove the above given L^p–L^q decay estimate under the weaker Assumption (A4)$_{\ell_n}$ on the coefficient function. If we use this regularity of the coefficient and perform $k = \lceil \frac{n}{2} \rceil$ diagonalization steps, we obtain $N_k(t,\xi) \in S_N^{0,\infty}\{0,0\}$ and $\mathcal{Q}_k(t,s,\xi)$ is uniformly in $t \geq s \geq t_\xi$ a symbol of smoothness $\lceil \frac{n}{2} \rceil$. Thus, $N_k\phi_{hyp}$, $N_k^{-1}\phi_{hyp}$ and $\mathcal{Q}_k\phi_{hyp}$ define operators $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ with uniformly bounded operator norm in $t \geq s$.

3.5 Sharpness

Finally, we want to prove the sharpness of the above given energy decay estimates. Our constructive approach enables us to formulate the question of sharpness as a *modified scattering result.* The basic idea is as follows:

- we relate the energy operator $\mathbb{E}(t,D)$ to the corresponding unitary operator $\mathbb{E}_0(t,\mathrm{D})$ for free waves, defined by $\mathbb{E}_0(t,\mathrm{D}) = M\mathcal{E}_0(t,0,\mathrm{D})M^{-1}$, and multiplied by the decay rate,

- this relation defines a Møller wave operator $W_+(\mathrm{D})$ defining appropriate data to the free wave equation with the same asymptotic properties (up to the factor $\lambda(t)$),

- furthermore, we need to know the mapping properties of the Møller wave operator,
- and the convergence defining the wave operator has to be understood.

A first observation follows immediately from Liouville theorem, Theorem B.8, and gives an expression for the determinant of $\mathbb{E}(t,\xi)$.

Lemma 3.25. *It holds* $\det \mathbb{E}(t,\xi) = \frac{1}{\lambda^2(t)}[\xi]$ *with* $[\xi] = \frac{|\xi|}{\langle\xi\rangle}$.

After these introductory remarks we can state the first theorem. It holds

Theorem 3.26. *Assume (A1), (A4)$_\ell$ with $\ell \geq 1$ and (C1). Then the limit*

$$W_+(\mathrm{D}) = \operatorname*{s-lim}_{t\to\infty} \lambda(t)(\mathbb{E}_0(t,\mathrm{D}))^{-1}\mathbb{E}(t,\mathrm{D})$$

exists as strong limit in $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and defines the modified Møller wave operator W_+. It satisfies

$$W_+(\xi) = (\mathbb{E}_0(t_\xi,\xi))^{-1} M \mathcal{Q}_k(\infty,t_\xi,\xi) N_k^{-1}(t_\xi,\xi) M^{-1}\lambda(t_\xi)\mathbb{E}(t_\xi,\xi)$$

for all $1 \leq k \leq \ell$.

Note, that t_ξ depends on the zone constant and this constant is chosen after diagonalizing k steps. Thus,

$$\mathcal{Q}_k(\infty,t_\xi,\xi)N_k^{-1}(t_\xi,\xi)$$

is independent of $1 \leq k \leq \ell$ for a sufficiently large zone constant N depending on ℓ. Note, further, that in order to define the matrix $\mathcal{Q}_k(t,s,\xi)$ and without estimating derivatives with respect to ξ we only need Assumption (A4)$_k$. The regularity (A4)$_{2k-1}$ was necessary to estimate $\mathcal{Q}_k(t,s,\xi)$ as symbol, cf. the consideration on page 55 leading to formula (3.3.29).

Proof. The proof consists of three steps.
Step 1. With the notation

$$V_c = \{U \in L^2(\mathbb{R}^n) \,|\, \operatorname{dist}(0, \operatorname{supp} \hat{U}) \geq c\},$$

we can construct the dense subspace $M = \bigcup_{c>0} V_c$ of $L^2(\mathbb{R}^n)$. Now Theorem 3.17 together with the representation $\mathbb{E}(t,\xi) = \mathcal{E}(t,t_\xi,\xi)\mathbb{E}(t_\xi,\xi)$ implies the existence of the limit

$$\lim_{t\to\infty} \lambda(t)\mathbb{E}_0(t,\mathrm{D})^{-1}\mathbb{E}(t,\mathrm{D})$$

as limit in the operator norm in $V_c \to V_c$ for all $c > 0$ as the following calculation shows. It holds

$$\begin{aligned}\lambda(t)\mathbb{E}_0(t,\xi)^{-1}\mathbb{E}(t,\xi) &= \lambda(t)\mathbb{E}_0(t_\xi,\xi)^{-1}M\mathcal{E}_0(t_\xi - t,\xi)N_k(t,\xi)\frac{\lambda(t_\xi)}{\lambda(t)}\mathcal{E}_0(t-t_\xi,\xi)\\ &\qquad \mathcal{Q}_k(t,t_\xi,\xi)N_k^{-1}(t_\xi,\xi)M^{-1}\mathbb{E}(t_\xi,\xi)\\ &= \mathbb{E}_0(t_\xi,\xi)^{-1}M\mathcal{E}_0(t_\xi-t,\xi)N_k(t,\xi)\mathcal{E}_0(t-t_\xi,\xi)\\ &\qquad \mathcal{Q}_k(t,t_\xi,\xi)N_k^{-1}(t_\xi,\xi)M^{-1}\lambda(t_\xi)\mathbb{E}(t_\xi,\xi),\end{aligned}$$

where $\mathcal{Q}_k(t,t_\xi,\xi) \to \mathcal{Q}_k(\infty,t_\xi,\xi)$ uniformly on $|\xi| \geq c$ by Theorem 3.17 and

$$\mathcal{E}_0(t_\xi - t,\xi)N_k(t,\xi)\mathcal{E}_0(t-t_\xi,\xi) = I + \mathcal{E}_0(t_\xi - t,\xi)\big(N_k(t,\xi) - I\big)\mathcal{E}_0(t-t_\xi,\xi)$$

and the second summand tends to zero like $(1+t)^{-1}$ uniformly on $|\xi| \geq c$. Thus, the limit exists pointwise on M.

Step 2. The energy estimate, Corollary 3.22, implies that $\lambda(t)\mathbb{E}_0(t,\mathrm{D})^{-1}\mathbb{E}(t,\mathrm{D})$ is uniformly bounded in $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Thus, the theorem of Banach-Steinhaus implies the existence of the strong limit and defines W_+.

Step 3. The previously defined operator W_+ is given on each subspace V_c as Fourier multiplier with symbol

$$W_+(\xi) = (\mathbb{E}_0(t_\xi,\xi))^{-1}\mathcal{Q}_k(\infty,t_\xi,\xi)N_k^{-1}(t_\xi,\xi)M^{-1}\lambda(t_\xi)\mathbb{E}(t_\xi,\xi),$$

which is independent of c. Thus, the representation holds on M and using the boundedness of W_+ on the whole space. □

Note, that there holds a corresponding result in the low regularity theory under Assumptions (A1) – (A3), (C1).

Corollary 3.27. *It holds under (A1), (A4)$_1$ and (C1) that*

$$\det W_+(\xi) = \lim_{t\to\infty} \lambda^2(t)\mathbb{E}(t,\xi) = [\xi]$$

and therefore $\operatorname{Ker} W_+(\mathrm{D}) = \{0\}$.

The representation of $W_+(\xi)$ allows us to conclude also estimates for derivatives with respect to ξ.

Corollary 3.28. *Under Assumptions (A1), (A4)$_{2k-1}$, $k \geq 1$, and (C1) it holds*

$$W_+(\xi) \in \dot{S}^0_{(k-1)}.$$

The proof of Theorem 3.26 gives no information about the convergence of the ξ-derivatives. The problematic term is $\mathcal{E}_0(t_\xi - t,\xi)N_k(t,\xi)\mathcal{E}_0(t-t_\xi,\xi)$, where the ξ-derivatives of $\mathcal{E}_0(t_\xi - t,\xi)$ behave as multiplications with $|\xi|$ and do not fit into the symbol estimates.

Interpretation of the result. What have we obtained so far? Theorem 3.26 may be used to construct for any data $(\langle \mathrm{D}\rangle u_1, u_2) \in L^2(\mathbb{R}^n)$ to Cauchy problem (3.0.1) corresponding data $(\langle \mathrm{D}\rangle \tilde{u}_1, \tilde{u}_2)^T = W_+(\mathrm{D})(\langle \mathrm{D}\rangle u_1, u_2)^T$ to the free wave equation $\Box\tilde{u} = 0$, such that the solutions are asymptotically equivalent up to the decay factor $\lambda^{-1}(t)$, i.e. it holds

$$\left|\left|\mathbb{E}_0(t,\mathrm{D})(\langle \mathrm{D}\rangle \tilde{u}_1, \tilde{u}_2)^T - \lambda(t)\mathbb{E}(t,\mathrm{D})(\langle \mathrm{D}\rangle u_1, u_2)^T\right|\right|_2 \to 0 \qquad \text{as } t\to\infty. \tag{3.5.1}$$

This is a direct consequence of the property of $\mathbb{E}_0(t,\xi)$ to be unitary. It implies that the above given L^2–L^2 estimates are indeed sharp and describe for all nonzero initial data the exact decay rate.

Remark concerning the supercritical case. A review of the proof of Theorem 3.26 implies the following observation. Condition (C1) is used only to give the uniform bound in the Banach-Steinhaus argument. Thus, Step 1 of the previous proof is valid under more general assumptions.

Corollary 3.29. *Assume (A1), (A4)$_1$. On $V_c \to V_c$ the limits*

$$W_+(\mathrm{D}) = \lim_{t\to\infty} \lambda(t)\mathbb{E}_0(-t,\mathrm{D})\mathbb{E}(t,\mathrm{D}), \qquad W_+^{-1}(\mathrm{D}) = \lim_{t\to\infty} \frac{1}{\lambda(t)}\mathbb{E}^{-1}(t,\mathrm{D})\mathbb{E}_0(t,\mathrm{D}),$$

exist as limits in the operator norm.

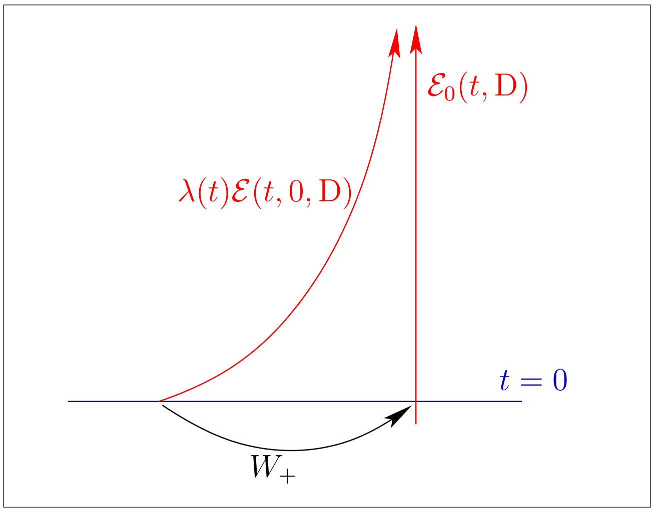

Figure 3.3: Modified scattering theory.

Example 3.10. If we restrict ourself to the example from Chapter 2, $b(t) = \frac{\mu}{1+t}$, then for all $\mu \geq 0$ and data $0 \neq (u_1, u_2)$ with $0 \notin \operatorname{supp}(\hat{u}_1, \hat{u}_2)$ the energy decays like

$$E(u;t) \sim \frac{1}{\lambda^2(t)} \sim (1+t)^{-\mu}.$$

For $\mu > 2$ the decay rate is not uniform in the norm of the data, the occurring constants depend on the distance of 0 from the support of $(\hat{u}_1, \hat{u}_2)$.

3.6 Summary

We will draw several conclusions from the considerations in this chapter. The main points can be summarised to be

- the hyperbolic zone determines the decay rate (under condition (C1)) and the necessary regularity of the data, cf. Figure 3.4,
- the dissipative zone is subordinate to the hyperbolic one,
- the dissipative term $b(t)u_t$ yields an energy decay of $\lambda^{-1}(t)$ in both components,
- solutions behave like free waves multiplied by the decay function.

In Table 3.1 we give an overview on the used assumptions related to the zones. It turns out, that the positivity of the coefficient function is used only in the dissipative zone.

Large frequencies. If we restrict considerations to the hyperbolic zone, which can be achieved by taking initial data with $0 \notin \operatorname{supp}(\hat{u}_1, \hat{u}_2)$, we can drop Assumption (A1) and Condition (C1) and derive estimates for the solutions under Assumption (A4)$_\ell$ alone. We give only one example.

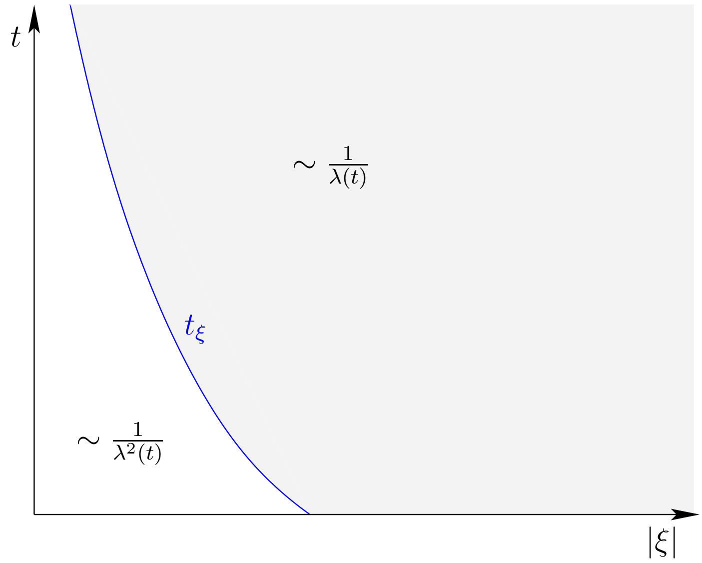

Figure 3.4: The hyperbolic zone determines the decay rate under Condition (C1).

Example 3.11. Let

$$b(t) = \frac{\cos \log(1+t)}{1+t}$$

and assume $0 \notin \operatorname{supp}(\hat{u}_1, \hat{u}_2)$. Then the solution $u(t,x)$ to (3.0.1) satisfies

$$E(u;t) \sim E_0(u;t) = E(u;0)$$

(with constants depending on the distance of 0 to the support of $(\hat{u}_1, \hat{u}_2)$). This estimate follows by application of Theorem 3.15 and

$$\lambda(t) = \exp\left\{\frac{1}{2}\sin\log(1+t)\right\} \sim 1,$$

together with Corollary 3.29.

Example 3.12. If we consider $b(t) = \frac{\sin(1+t)}{1+t}$, the above given conclusion cannot be drawn. In this case, Assumption (A4)$_1$ is not valid, therefore the diagonalization scheme brings no improvement for the remainders. It is an open question, whether for this coefficient function and under the above condition on the data, $0 \notin \operatorname{supp}(\hat{u}_1, \hat{u}_2)$, the energy decays to zero, remains bounded or even tends to infinity. The auxiliary function $\lambda(t)$ behaves as a constant, $\lambda(t) \sim 1$, in this case.

Zone	Assumptions on $b(t)$
dissipative zone	• we used only (A1), i.e. $b(t) \geq 0$, together with the technical condition $tb(t) \neq 1$ for large t • Assumption (C1) guarantees that the hyperbolic zone determines the final decay rate • smoothness properties of $b(t)$, (A4)$_\ell$, are used to estimate $\lambda^2(t_\xi)\mathcal{E}(t_\xi, 0, \xi)$ as symbol in ξ
hyperbolic zone, low regularity theory	• Assumptions (A1), (A2) are used to define the zone • Assumption (A3) allows estimates after diagonalizing one step • we need differentiability properties of $b(t)$ in order to diagonalize
hyperbolic zone, high regularity theory	• Assumption (A4)$_\ell$ allows as many diagonalization steps as we want • smoothness of $b(t)$ transfers to smoothness properties of the symbol $\mathcal{Q}_k$ in the covariable ξ • the sign of the coefficient function $b = b(t)$ does not matter in this part of the phase space

Table 3.1: Assumptions used in the zones.

4 Effective dissipation

In the previous chapter our main concern was to understand the influence of *small* dissipation terms. We observed a close relation of the solutions to free waves and the main contributions come from the behaviour of large frequencies. The non-effective dissipation term $b(t)u_t$ was asymptotically subordinate to the principal part.

This chapter is devoted to the study of dissipation terms which are *large*; thus the approach of the previous chapter has to be modified and the dissipation term should be included in the symbolic calculus (and therefore included in the 'phase function' for the WKB-representation of the solutions).

Main results are the representations of Theorems 4.6 and 4.11 and the resulting L^2–L^2 and L^p–L^q estimates of Theorems 4.21 and 4.25.

Finally, in Theorem 4.27 we explain what happens, if the dissipation becomes to strong. This we call the case of over-damping.

4.1 Strategies

4.1.1 Transformation of the problem

Our main strategy is to apply a transformation of the Cauchy problem. Following A. Matsumura, [Mat76], and for variable coefficient dissipation M. Reissig, [Rei01], we transform the dissipative equation to a wave equation with time-dependent potential or, as we will call it, a Klein-Gordon type equation. Therefore, we consider the new function

$$v(t,x) = \lambda(t)u(t,x), \tag{4.1.1}$$

where $\lambda(t) = \exp\left\{\frac{1}{2}\int_0^t b(\tau)\mathrm{d}\tau\right\}$ is the auxiliary function arising in the calculations of Chapter 3, such that

$$\Box v = \lambda''(t)u + 2\lambda'(t)u_t + \lambda(t)u_{tt} - \lambda(t)\Delta u = \left(\frac{1}{4}b^2(t) + \frac{1}{2}b'(t)\right)v.$$

Thus, after applying the partial Fourier transform, we have to solve the parameter-dependent differential equation

$$\hat{v}_{tt} + m(t,\xi)\hat{v} = 0 \tag{4.1.2}$$

with coefficient (micro-local mass term)

$$m(t,\xi) = |\xi|^2 - \frac{1}{4}b^2(t) - \frac{1}{2}b'(t). \tag{4.1.3}$$

For the behaviour of the solutions to this ordinary differential equation the sign of the coefficient $m = m(t,\xi)$ is important. While obviously $|\xi| \geq 0$ and $b^2(t) > 0$, the derivative $b'(t)$ may be negative.

Under the Assumptions (A1) – (A3) of *non-effective dissipation*, cf. Definition 3.1, the coefficient $b = b(t)$ is decaying and $-b'(t)$ dominates $b^2(t)$. Under the further assumption $b^2(t) = o(-b'(t))$

as $t \to \infty$ we have obviously $m(t,\xi) > 0$ for all ξ and $t \geq t_0$. Because we are interested in the time-asymptotics we may assume $t_0 = 0$. The positivity of the coefficient enables us to consider the micro-energy

$$V = (\sqrt{m(t,\xi)}\hat{v}, \mathrm{D}_t\hat{v})^T$$

and write the equation as system,

$$\mathrm{D}_t V = \begin{pmatrix} \frac{\mathrm{D}_t\sqrt{m(t,\xi)}}{\sqrt{m(t,\xi)}} & \sqrt{m(t,\xi)} \\ \sqrt{m(t,\xi)} & \end{pmatrix} V = AV.$$

Now we see, that the coefficient matrix consists of the self-adjoint anti-diagonal part and the skew diagonal entry. Thus, we obtain for the L^2-norm of V

$$\partial_t ||V||_2^2 = -2\ \mathrm{Im}\ (V, AV) = \int 2(\partial_t\sqrt{m(t,\xi)})\sqrt{m(t,\xi)}\hat{v}^2 \mathrm{d}\xi \leq 2 \left|\left| \frac{\partial_t\sqrt{m(t,\xi)}}{\sqrt{m(t,\xi)}} \right|\right|_\infty ||V||_2^2$$

and, therefore,

$$||V|| \lesssim 1$$

in the case that $(\partial_t\sqrt{m(t,\xi)})/\sqrt{m(t,\xi)} \in L^\infty_\xi L^1_t(\mathbb{R}_+ \times \mathbb{R}^n)$. A simple calculation shows that this is the case under the assumption $\liminf_{t\to\infty} b''/(-bb') > 0$.[1] This calculation gives the same energy decay rate like in Chapter 3, but under the more restrictive assumptions $b \in C^2$, $b^2 = o(-b')$ and $\liminf_{t\to\infty} b''/(-bb') > 0$. But, nevertheless, it gives an alternative interpretation of Assumption (A3).

It is also interesting to consider $m(t,\xi)$ in the case of *scale-invariant weak dissipation* treated in Chapter 2.

Example 4.1. Let $b = b(t) = \frac{\mu}{1+t}$. Then it holds

$$m(t,\xi) = |\xi|^2 + \frac{\mu(2-\mu)}{4(1+t)^2} \tag{4.1.4}$$

and, therefore, we get for $\mu \in (0,2)$ the positivity $m(t,\xi) > 0$ for all (t,ξ), while for $\mu = 2$ we get in correspondence to the free wave equation, $m(t,\xi) = |\xi|^2 > 0$ if $\xi \neq 0$, and finally for $\mu > 2$ there exists a part of the phase space with negative $m(t,\xi)$. This part is given by $(1+t)|\xi| \leq \frac{1}{2}\sqrt{\mu(2-\mu)}$ and corresponds in the consideration of Chapter 2 to the zone Z_3. The occurrence of this negative coefficient coincides with the 'take-over' of the zone Z_3 in the estimate for the hyperbolic energy, cf. Theorem 2.8.

Inspired by this example and the previous motivation we state that the case of *effective dissipation* will be characterised by the occurrence of a region of the extended phase space $\mathbb{R}_t \times \mathbb{R}^n_\xi$, where $m(t,\xi)$ becomes negative. This leads naturally to the definition of a so-called *separating curve* Γ, which dissects the phase space into the part corresponding to a positive micro-local mass term and the part corresponding to a negative micro-local mass term, cf. Figure 4.1.

To achieve this decomposition of the extended phase space we make the following assumptions:

[1]Indeed, we have

$$\frac{\partial_t\sqrt{m(t,\xi)}}{\sqrt{m(t,\xi)}} = \frac{1}{2}\frac{-\frac{1}{2}(bb'+b'')}{|\xi|^2 - \frac{1}{4}b^2 - \frac{1}{2}b'} \leq \frac{1}{2}\frac{\frac{1}{2}(bb'+b'')}{\frac{1}{4}b^2 + \frac{1}{2}b'}$$

and the denominator is a primitive of the numerator. So the assumption guarantees that the integrand does not change its sign for large t.

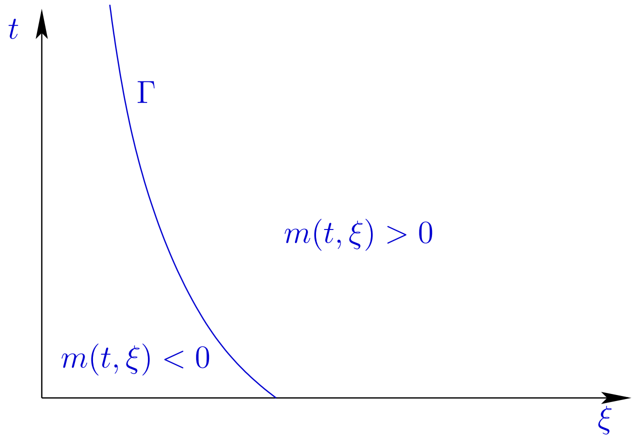

Figure 4.1: Definition of the separating curve Γ.

(B1) positivity $b(t) > 0$,

(B2) monotonicity, i.e. $b'(t)$ does not change its sign,

(B3) $|b'(t)| = o(b^2(t))$ as $t \to \infty$.

Later on we will include further symbol-like estimates for derivatives of $b = b(t)$.

Definition 4.1. *We call the dissipation term $b(t)u_t$ in equation* (1.2.1) *effective, if $b(t)$ satisfies Assumptions (B1) – (B3).*

Assumption (B3) allows us to understand $b'(t)$ as negligible term and to define the separating curve Γ in terms of the monotonous coefficient function $b = b(t)$,

$$\Gamma \ : \ |\xi| = \frac{1}{2} b(t). \tag{4.1.5}$$

By Assumption (B2) all vertical lines $|\xi| = \mathrm{const}$ cross the separating curve at most once.

Remark 4.2. The separating curve has to be distinguished from the zone boundary t_ξ used in Chapter 3 or used later on in this chapter. A zone boundary can be moved in the plane, the defining zone constant N can be chosen almost freely (with some technical restrictions, if we require invertibility of symbols). The separating curve is fixed, at least from its asymptotic behaviour. The choice of the constant $\frac{1}{2}$ in the above formula is directly related to the micro-local mass term $m(t, \xi)$.

Assumptions (B1) – (B3) imply that the coefficient $b = b(t)$ can not tend to zero to fast. The monotonicity of $b'(t)$ implies from (B3) for decaying $b = b(t)$ that

$$-\frac{b'}{b^2} = o(1), \qquad \text{and therefore} \qquad o(t) = -\int_0^t \frac{b'(\tau)}{b^2(\tau)} \mathrm{d}\tau = \frac{1}{b(t)} - \frac{1}{b(0)}$$

and hence

(B3)' $tb(t) \to \infty$ as $t \to \infty$

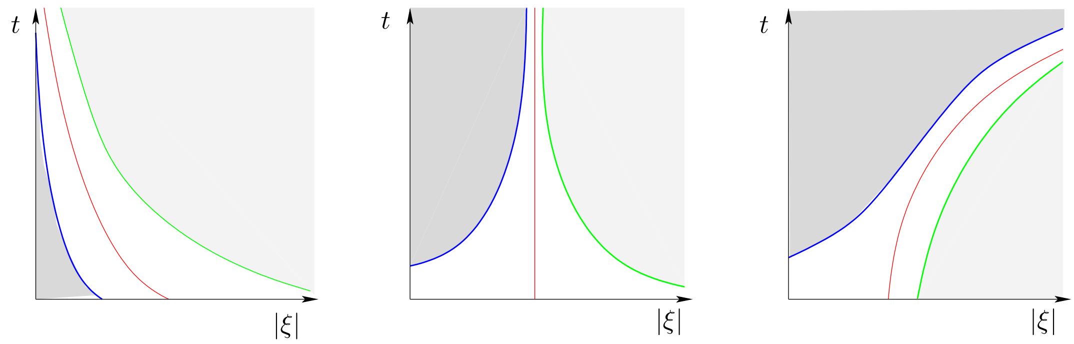

Figure 4.2: Effective weak and effective strong dissipation.

in contrast to (3.2.1) in the non-effective case.

For effective dissipation we introduce in Definition 4.5 the separating curve Γ in an abstract way. It is used to describe the behaviour of the function $m(t,\xi)$. It distinguishes for fixed time-level t between small and large frequencies. We will fix a notation.

Definition 4.2. *The two parts of the phase space separated by Γ are called hyperbolic part, containing large frequencies, and elliptic part, containing the small frequencies.*

The aim of this chapter is to achieve bounds on the solution in these two parts of the phase space in order to obtain decay estimates after transforming back to the original problem.

4.1.2 Effective weak and strong dissipation

If we apply the idea of the previous section, we have to distinguish between three (topologically) different cases. On the one hand, if $b(t)$ tends to zero as t goes to infinity, the separating curve approaches the t-axis and the hyperbolic part lies on top of the elliptic part.

Definition 4.3. *An effective dissipation term $b(t)u_t$ is called a effective weak dissipation, if the corresponding separating curve Γ approaches the t-axis as t tends to infinity.*

The situation changes if b tends to a finite limit. In this case we can use $\Gamma = \{|\xi| = \mathrm{const}\}$ and the elliptic and the hyperbolic part are independent of each other. This is the situation in [Rei01]. The third case arises for unbounded $b(t)$. Under this assumption the elliptic part lies on top of the hyperbolic part.

Definition 4.4. *An effective dissipation term $b(t)u_t$ is called a strong dissipation, if there exists a frequency $\xi_0 \neq 0$ such that the line $\{\xi = \xi_0\}$ belongs to the elliptic part for all $t \gg 1$.*

We will see that these three cases do not differ in the approach. The achieved representations of solutions coincide in their structure. Basic example for a strong dissipation is the damped wave equation $\Box u + u_t = 0$ with separating curve $\{|\xi| = \frac{1}{2}\}$ or, more generally, wave equations with dissipation terms bounded from below.

In Figure 4.2 the different cases are sketched.

4.1.3 Notation and basic tools

The separating curve, parts and zones. We formulate the previously discussed strategy in an abstract way starting from the separating curve and defining appropriate symbol classes related to it. Later on, these symbol classes explain what kind of assumptions we have to make for the coefficient function; this difference is essential in order to understand also non-monotonous coefficients.

Definition 4.5. *We call the function* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *admissible, if it satisfies the following assumptions:*

(Γ**1)** $\gamma \in C^1[0,\infty)$, $\gamma(t) > 0$, *monotonous,*

(Γ**2)** $t\gamma(t) \to \infty$ *as* $t \to \infty$.

Furthermore, for an admissible function $\gamma(t)$ *we define the corresponding separating curve*

$$\Gamma = \{|\xi| = \gamma(t)\},$$

together with the auxiliary symbol

$$\langle\xi\rangle_{\gamma(t)} := \sqrt{\big||\xi|^2 - \gamma^2(t)\big|}. \tag{4.1.6}$$

This symbol measures the *distance* of a point in the extended phase plane from the separating curve and will be used in the definition of symbol classes. It replaces $|\xi|$ from the approach of Chapter 3. The parts are denoted as

$$\begin{aligned} \Pi_{hyp} &= \{|\xi| > \gamma(t)\} && \text{for the } \textit{hyperbolic part} \text{ and} \\ \Pi_{ell} &= \{|\xi| < \gamma(t)\} && \text{for the } \textit{elliptic part}\,. \end{aligned}$$

Inside these parts the auxiliary symbol $\langle\xi\rangle_{\gamma(t)}$ is differentiable and satisfies the following proposition.

Proposition 4.1. *It holds*

$$\partial_t \langle\xi\rangle_{\gamma(t)} = \pm\frac{\gamma(t)\gamma'(t)}{\langle\xi\rangle_{\gamma(t)}}, \qquad \partial_{|\xi|} \langle\xi\rangle_{\gamma(t)} = \mp\frac{|\xi|}{\langle\xi\rangle_{\gamma(t)}},$$

where the upper sign is taken in the elliptic part.

Both parts of the phase space will be decomposed into zones,

$$\begin{aligned} Z_{hyp}(N) &= \{\langle\xi\rangle_{\gamma(t)} \geq N\gamma(t)\} \cap \Pi_{hyp} && \textit{hyperbolic zone,} \\ Z_{pd}(N,\epsilon) &= \{\epsilon\gamma(t) \leq \langle\xi\rangle_{\gamma(t)} \leq N\gamma(t)\} \cap \Pi_{hyp} && \textit{pseudo-differential zone,} \\ Z_{diss}(c_0) &= \{(1+t)|\xi| \leq c_0\} && \textit{dissipative zone}, \\ Z_{ell}(c_0,\epsilon) &= \{(1+t)|\xi| \geq c_0\} \cap \{\langle\xi\rangle_{\gamma(t)} \geq \epsilon\gamma(t)\} \cap \Pi_{ell} && \textit{elliptic zone,} \\ Z_{red}(\epsilon) &= \{\langle\xi\rangle_{\gamma(t)} \leq \epsilon\gamma(t)\} && \textit{reduced zone.} \end{aligned} \tag{4.1.7}$$

In the elliptic and in the hyperbolic zone we define symbol classes and perform later on a diagonalization procedure to extract the leading terms. In the remaining smaller zones the solutions can be estimated directly ($Z_{red}(\epsilon)$) or known estimates will be used ($Z_{diss}(c_0)$, estimate of Lemma 3.9). The dissipative zone can be skipped under the further assumption

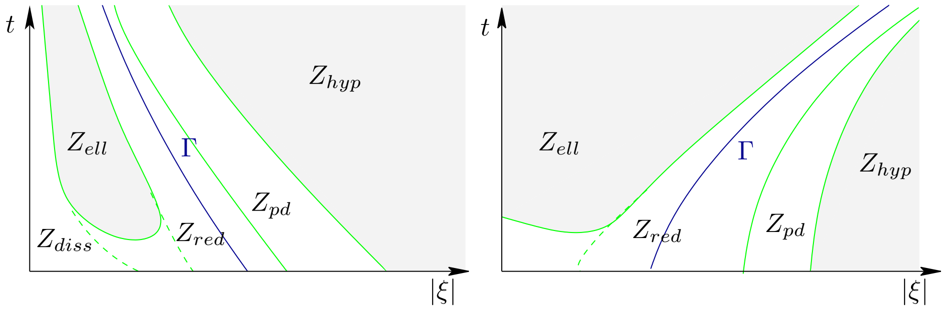

Figure 4.3: Zones used in the approach.

(Γ3) $\frac{1}{\gamma(t)(1+t)^2} \in L^1(\mathbb{R}_+)$

on the admissible function $\gamma = \gamma(t)$, i.e. if we are far away from the critical case of Chapter 2. If (Γ3) is valid, we define $Z_{ell}(\epsilon) := Z_{ell}(0, \epsilon)$.

The constants ϵ and N are determined later. The choice of N is related to the number of diagonalization steps and the existence of a suitable one is guaranteed by Lemma 4.5. It turns out, that in case of effective dissipation the first diagonalization step is sufficient to conclude the desired decay estimates and we may use the estimate of Lemma 4.4 together with an arbitrary small positive N. The constant ϵ is chosen later, all obtained estimates are independent of ϵ for sufficiently small value ϵ.

Symbols in Π_{hyp}. The hyperbolic symbol classes are directly related to the ones of Definition 3.2, except that we introduce one further weight $\gamma(t)$. Remark, that it holds

$$\langle\xi\rangle_{\gamma(t)} \sim |\xi| \qquad \text{uniformly on } Z_{hyp}(N). \tag{4.1.8}$$

Definition 4.6. *The time-dependent Fourier multiplier $a(t,\xi)$ belongs to the hyperbolic symbol class $S^{\ell_1,\ell_2}_{hyp,N}\{m_1,m_2,m_3\}$ with restricted smoothness ℓ_1,ℓ_2, if it satisfies the estimate*

$$\left|\mathrm{D}_t^k \mathrm{D}_\xi^\alpha a(t,\xi)\right| \le C_{k,\alpha} \langle\xi\rangle_{\gamma(t)}^{m_1-|\alpha|} \gamma^{m_2}(t) \left(\frac{1}{1+t}\right)^{m_3+k} \tag{4.1.9}$$

for all $(t,\xi) \in Z_{hyp}(N)$ and all $k \le \ell_1$, $|\alpha| \le \ell_2$.

Furthermore, we fix the notation $S_{hyp,N}\{m_1,m_2,m_3\}$ for $S^{\infty,\infty}_{hyp,N}\{m_1,m_2,m_3\}$. The rules for the symbolic calculus follow Proposition 3.5. It holds

Proposition 4.2.
1. *$S^{\ell_1,\ell_2}_{hyp,N}\{m_1,m_2,m_3\}$ is a vector space,*
2. *$S^{\ell_1,\ell_2}_{hyp,N}\{m_1-k,m_2,m_3+\ell\} \hookrightarrow S^{\ell_1,\ell_2}_{hyp,N}\{m_1,m_2-k,m_3+\ell\} \hookrightarrow S^{\ell_1,\ell_2}_{hyp,N}\{m_1,m_2,m_3\}$ for $\ell \ge k \ge 0$,*
3. *$S^{\ell_1,\ell_2}_{hyp,N}\{m_1,m_2,m_3\} \cdot S^{\ell_1,\ell_2}_{hyp,N}\{m_1',m_2',m_3'\} \hookrightarrow S^{\ell_1,\ell_2}_{hyp,N}\{m_1+m_1',m_2+m_2',m_3+m_3'\}$,*
4. *$\mathrm{D}_t^k \mathrm{D}_\xi^\alpha S^{\ell_1,\ell_2}_{hyp,N}\{m_1,m_2,m_3\} \hookrightarrow S^{\ell_1-k,\ell_2-|\alpha|}_{hyp,N}\{m_1-|\alpha|,m_2,m_3+k\}$,*
5. *$S^{0,0}_{hyp,N}\{-1,0,2\} \hookrightarrow L^\infty_\xi L^1_t(Z_{hyp})$.*

Proof. Statements 1 and 4 follow immediately from the definition. Statement 3 is a direct consequence of Leibniz rule. For statement 2 we use the definition of the zone, $|\xi| \sim \langle\xi\rangle_{\gamma(t)} \gtrsim \gamma(t)$, together with Assumption ($\Gamma2$). Statement 5 follows from

$$\int_{t_0}^{t_1} \frac{1}{|\xi|(1+t)^2}\mathrm{d}t = \frac{1}{|\xi|(1+t_0)} - \frac{1}{|\xi|(1+t_1)},$$

which is uniformly bounded on the hyperbolic zone by ($\Gamma2$). □

Symbols in Π_{ell}. The elliptic symbols are constructed in a similar manner. The main difference is that in the elliptic zone the auxiliary symbol $\langle\xi\rangle_t$ can be estimated

$$\langle\xi\rangle_{\gamma(t)} \sim \gamma(t) \qquad \text{uniformly on } Z_{ell}(c_0,\epsilon). \tag{4.1.10}$$

Now the definition of the symbol class reads as follows.

Definition 4.7. *The time-dependent Fourier multiplier $a(t,\xi)$ belongs to the elliptic symbol class $S^{\ell_1,\ell_2}_{ell,\epsilon}\{m_1,m_2,m_3\}$ of restricted smoothness ℓ_1,ℓ_2, if it satisfies the estimate*

$$\left|\mathrm{D}_t^k\mathrm{D}_\xi^\alpha a(t,\xi)\right| \le C_{k,\alpha}\,\langle\xi\rangle_{\gamma(t)}^{m_1-|\alpha|}\,|\xi|^{m_2}\left(\frac{1}{1+t}\right)^{m_3+k} \tag{4.1.11}$$

for all $(t,\xi)\in Z_{ell}(c_0,\epsilon)$ and all $k\le\ell_1$, $|\alpha|\le\ell_2$.

Again we fix the notation $S_{ell,\epsilon}\{m_1,m_2,m_3\}$ for $S^{\infty,\infty}_{ell,\epsilon}\{m_1,m_2,m_3\}$. The rules for the symbolic calculus are similar to that from the hyperbolic part.

Proposition 4.3. *1. $S^{\ell_1,\ell_2}_{ell,\epsilon}\{m_1,m_2,m_3\}$ is a vector space,*

2. $S^{\ell_1,\ell_2}_{ell,\epsilon}\{m_1-k,m_2,m_3+\ell\} \hookrightarrow S^{\ell_1,\ell_2}_{ell,\epsilon}\{m_1,m_2,m_3\}$ for $\ell\ge k\ge 0$,

3. $S^{\ell_1,\ell_2}_{ell,\epsilon}\{m_1,m_2,m_3\}\cdot S^{\ell_1,\ell_2}_{ell,\epsilon}\{m_1',m_2',m_3'\} \hookrightarrow S^{\ell_1,\ell_2}_{ell,\epsilon}\{m_1+m_1',m_2+m_2',m_3+m_3'\}$,

4. $\mathrm{D}_t^k\mathrm{D}_\xi^\alpha S^{\ell_1,\ell_2}_{ell,\epsilon}\{m_1,m_2,m_3\} \hookrightarrow S^{\ell_1-k,\ell_2-|\alpha|}_{ell,\epsilon}\{m_1-|\alpha|,m_2,m_3+k\}$,

5. $S^{0,0}_{ell,\epsilon}\{-1,0,2\} \hookrightarrow L^\infty_\xi L^1_t(Z_{ell})$.

Proof. Again we prove only the integrability statement. Under Assumption ($\Gamma3$) the statement follows immediately. Assume now that $\gamma(t)$ is monotonically decreasing and the dissipative zone is introduced. Then it holds

$$\int_{t_{\xi_1}}^{t_{\xi_2}} \frac{\mathrm{d}\tau}{\gamma(\tau)(1+\tau)^2} \le \frac{1}{\gamma(t_{\xi_2})}\int_{t_{\xi_1}}^{t_{\xi_2}} \frac{\mathrm{d}\tau}{(1+\tau)^2} \le \frac{1}{\gamma(t_{\xi_2})(1+t_{\xi_1})} \sim 1,$$

where t_{ξ_1} and t_{ξ_2} denotes the lower and the upper boundary of the elliptic zone, respectively. Furthermore the definition of the elliptic zone implies that the quotient is constant in ξ. □

4.2 Representation of solutions

We start with the case, where the dissipative term $b(t)$ satisfies the assumptions (B1) – (B3)'. Thus $\gamma(t) = \frac{1}{2}b(t)$ is admissible in the sense of Definition 4.5. If we assume further the symbol-like estimate

(B4)$_\ell$

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}t^k}b(t)\right| \le C_k b(t)\left(\frac{1}{1+t}\right)^k \qquad \text{for } k = 1, 2, \dots, \ell,$$

we obtain

$$b(t) \in S^{\ell,\infty}_{hyp,N}\{0,1,0\} \cap S^{\ell,\infty}_{ell,\epsilon}\{0,1,0\}.$$

Thus, for this definition of the separating curve we obtain

$$\langle\xi\rangle_{\gamma(t)} \in S^{\ell,\infty}_{hyp,N}\{1,0,0\} \cap S^{\ell,\infty}_{ell,\epsilon}\{1,0,0\},$$

and by the aid of (B1) – (B3)

$$\langle\xi\rangle^{-1}_{\gamma(t)} \in S^{\ell,\infty}_{hyp,N}\{-1,0,0\} \cap S^{\ell,\infty}_{ell,\epsilon}\{-1,0,0\}.$$

Similar to the notation in Chapter 3 we denote (B4)$_\infty$ shortly by (B4).

In Sections 4.2.1 to 4.2.3 we construct the main terms of the representation of solutions for equation (4.1.2) under Assumptions (B1) to (B3) and (B4)$_2$ using $\gamma(t) = \frac{1}{2}b(t)$. Later on in Section 4.2.4 we will discuss the more general case of non-monotonous coefficients related to a given separating curve Γ.

4.2.1 The hyperbolic part

Consideration in the hyperbolic zone. We consider the micro-energy

$$V = (\langle\xi\rangle_{\gamma(t)}\,\hat{v}, \mathrm{D}_t\hat{v})^T. \tag{4.2.1}$$

Then it holds

$$\mathrm{D}_t V = \left[\begin{pmatrix} & \langle\xi\rangle_{\gamma(t)} \\ \langle\xi\rangle_{\gamma(t)} & \end{pmatrix} + \begin{pmatrix} \frac{\mathrm{D}_t\langle\xi\rangle_{\gamma(t)}}{\langle\xi\rangle_{\gamma(t)}} & 0 \\ \frac{b'(t)}{2\langle\xi\rangle_{\gamma(t)}} & 0 \end{pmatrix}\right] V. \tag{4.2.2}$$

The entries of the second matrix are uniformly integrable over the hyperbolic zone. The function $\partial_t\langle\xi\rangle_{\gamma(t)}$ does not change its sign and, therefore, for $(t_1,\xi), (t_2,\xi) \in Z_{hyp}(N)$

$$\int_{t_1}^{t_2}\left|\frac{\partial_t\langle\xi\rangle_{\gamma(t)}}{\langle\xi\rangle_{\gamma(t)}}\right|\mathrm{d}t = \left|\int_{t_1}^{t_2}\frac{\mathrm{d}\,\langle\xi\rangle_{\gamma(t)}}{\langle\xi\rangle_{\gamma(t)}}\right| = \left|\log\frac{\langle\xi\rangle_{\gamma(t_2)}}{\langle\xi\rangle_{\gamma(t_1)}}\right|$$

and $\langle\xi\rangle_{\gamma(t)} \sim |\xi|$ uniformly in the hyperbolic zone. Furthermore,

$$\int_{t_1}^{t_2}\left|\frac{b'(t)}{\langle\xi\rangle_{\gamma(t)}}\right|\mathrm{d}t = \left|\int_{t_1}^{t_2}\frac{\mathrm{d}b(t)}{\sqrt{|\xi|^2 - b^2(t)/4}}\right| = \left|\arcsin\frac{b(t_2)}{2|\xi|} - \arcsin\frac{b(t_1)}{2|\xi|}\right|$$

and $|\xi| \ge b(t)/2$ in the hyperbolic zone.

Therefore, the following lemma holds for the fundamental solution $\mathcal{E}_V(t,s,\xi)$ to (4.2.2).

Lemma 4.4. *Assume (B1), (B2) and (B3)'. Then in the hyperbolic zone $Z_{hyp}(N)$ the estimate*

$$||\mathcal{E}_V(t,s,\xi)|| \lesssim 1$$

is valid.

Because this lemma is true for all $N > 0$ we may set $N = \epsilon$ and the pseudo-differential zone introduced by (4.1.7) reduces to the empty set.

Diagonalization. To prove the previous lemma we used the special structure of the lower order terms. It is also possible to use the previously introduced symbol classes to perform several steps of perfect diagonalization to conclude more structural properties of $\mathcal{E}_V(t,s,\xi)$.

We use the matrices M and M^{-1} from Section 3.3.2 and consider $V^{(0)} = M^{-1}V$ such that

$$\mathrm{D}_T V^{(0)} = [\mathcal{D}(t,\xi) + R_1(t,\xi)]V^{(0)}$$

with $\mathcal{D}(t,\xi) = \operatorname{diag}(\langle\xi\rangle_{\gamma(t)}, -\langle\xi\rangle_{\gamma(t)})$ and $R_1(t,\xi) \in S^{\ell,\infty}_{hyp,N}\{0,0,1\}$ under Assumption (B4)$_{\ell+1}$. By the above argument we see that the entries of $R_1(t,\xi)$ are uniformly integrable over the hyperbolic zone.

Using $\langle\xi\rangle^{-1}_{\gamma(t)} \in S^{\ell,\infty}_{hyp,N}\{-1,0,0\}$ we can perform several steps of diagonalization and prove the following lemma. Note that we need one derivative more than in Chapter 3 for the definition of $R_1(t,\xi)$.

Lemma 4.5. *Assume $b = b(t)$ satisfies (B1), (B2), (B3)' and (B4)$_{\ell+1}$. Then for all $k \le \ell$ there exists a zone constant N and matrix-valued symbols*

- $N_k(t,\xi) \in S^{\ell-k+1,\infty}_{hyp,N}\{0,0,0\}$,
  invertible for all $(t,\xi) \in Z_{hyp}(N)$ with $N^{-1}(t,\xi) \in S^{\ell-k+1,\infty}_{hyp,N}\{0,0,0\}$,
- $F_{k-1}(t,\xi) \in S^{\ell-k+1,\infty}_{hyp,N}\{-1,0,2\}$ *diagonal and uniformly integrable over $Z_{hyp}(N)$,*
- $R_k(t,\xi) \in S^{\ell-k,\infty}_{hyp,N}\{-k,0,k+1\}$,

such that the operator identity

$$\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - R(t)\big)N_k(t,\xi) = N_k(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(\xi) - F_{k-1}(t,\xi) - R_k(t,\xi)\big)$$

holds.

To construct a representation for the fundamental solution $\mathcal{E}_{V,k}(t,s,\xi)$ to the transformed system

$$\big(\mathrm{D}_t - \mathcal{D}(\xi) - F_{k-1}(t,\xi) - R_k(t,\xi)\big)\mathcal{E}_{V,k}(t,s,\xi) = 0, \qquad \mathcal{E}_{V,k}(s,s,\xi) = I \in \mathbb{C}^{2\times 2} \tag{4.2.3}$$

we can follow the lines of Section 3.3.3. In particular, Proposition 3.13 and Proposition 3.14 have a counterpart within our new hyperbolic symbol classes. Again we need $\ell \ge 2k-1$ to obtain estimates for ξ-derivatives up to order $k-1$.

Theorem 4.6. *Assume (B1), (B2) and (B4)$_{2k}$. Then the fundamental solution $\mathcal{E}_{V,k}(t,s,\xi)$ of (4.2.3) can be represented as*

$$\mathcal{E}_{V,k}(t,s,\xi) = \tilde{\mathcal{E}}(t,s,\xi)\mathcal{Q}_{hyp,k}(t,s,\xi), \tag{4.2.4}$$

where

$$\tilde{\mathcal{E}}(t,s,\xi) = \exp\left\{ i\int_s^t \mathcal{D}(\tau,\xi)\mathrm{d}\tau \right\} = \operatorname{diag}\left(e^{i\int_s^t \langle\xi\rangle_{\gamma(\tau)}\mathrm{d}\tau}, e^{-i\int_s^t \langle\xi\rangle_{\gamma(\tau)}\mathrm{d}\tau} \right) \tag{4.2.5}$$

and the symbol $\mathcal{Q}_{hyp,k}(t,s,\xi)$ *satisfies*

$$\left|\left| \mathrm{D}_t^{\ell_1}\mathrm{D}_s^{\ell_2}\mathrm{D}_\xi^{\alpha}\mathcal{Q}_{hyp,k}(t,s,\xi) \right|\right| \le C_{\ell,\alpha}\,\langle\xi\rangle_{\gamma(t)}^{\ell_2-|\alpha|}\left(\frac{1}{1+t}\right)^{\ell_1}, \qquad (t,\xi),(s,\xi)\in Z_{hyp}(N), \quad t\ge s$$

for all multi-indices $|\alpha|\le k-1$, $\ell_1\le k-1$ *and* $\ell_2\le 2k+1$.

In the case of effective weak dissipation or more generally if $\gamma(t)$ remains bounded, i.e. if there exists an interval $(\gamma(\infty),\infty)$ such that for all ξ_0 with $|\xi_0|\in(\gamma(\infty),\infty)$ the line $\{\xi=\xi_0\}$ ends up in the hyperbolic part, cf. Figure 4.2, we may ask the question whether the limits $\lim_{t\to\infty}\mathcal{Q}_{hyp,k}(t,s,\xi)$ exist or not. Like in Chapter 3 this is the case, the reasoning is exactly the same. The restriction on the number of s-derivatives comes from the new phase function given in (4.2.5).

Corollary 4.7. *In the case that* $\gamma(t)$ *is bounded the limit*

$$\lim_{t\to\infty}\mathcal{Q}_{hyp,k}(t,s,\xi) = \mathcal{Q}_{hyp,k}(\infty,s,\xi)$$

exists uniform in ξ *for* $|\xi|\ge c>\gamma(\infty)\sqrt{N^2+1}$ *and satisfies for* $|\alpha|\le k-1$ *and* $\ell\le 2k+1$

$$||\mathrm{D}_s^{\ell}\mathrm{D}_\xi^{\alpha}\mathcal{Q}_{hyp,k}(\infty,s,\xi)|| \le C_\alpha|\xi|^{\ell-|\alpha|}.$$

Similar to Chapter 3 the inverse of the matrix $\mathcal{Q}_{hyp,k}(t,s,\xi)$ exists and satisfies the same statement, it is uniformly bounded and converges for $t\to\infty$ to $\mathcal{Q}_{hyp,k}^{-1}(\infty,s,\xi)$.

These further diagonalization steps bring no further improvements on decay estimates as we will see later (due to the fact that the further decay factor $\lambda(t)^{-1}$ occurs), but they may be used to deduce sharp results for the regularity of solutions and data.

Remarks on the pseudo-differential zone. If we diagonalize the terms of lower order in the hyperbolic zone we chose N sufficiently large and the pseudo-differential zone may be non-empty. Inside the pseudo-differential zone it is sufficient to use the estimate

$$||\mathcal{E}_V(t,s,\xi)|| \lesssim 1$$

from Lemma 4.4.

4.2.2 The elliptic part

In this part of the phase space we consider again the micro-energy

$$V = (\langle\xi\rangle_{\gamma(t)}\,\hat{v}, \mathrm{D}_t\hat{v})^T, \tag{4.2.6}$$

which leads to the system

$$\mathrm{D}_tV = \left[\begin{pmatrix} & \langle\xi\rangle_{\gamma(t)} \\ -\langle\xi\rangle_{\gamma(t)} & \end{pmatrix} + \begin{pmatrix} \frac{\mathrm{D}_t\langle\xi\rangle_{\gamma(t)}}{\langle\xi\rangle_{\gamma(t)}} & 0 \\ -\frac{b'(t)}{2\langle\xi\rangle_{\gamma(t)}} & 0 \end{pmatrix} \right] V. \tag{4.2.7}$$

The main difference to the consideration in the hyperbolic zone is, that the first matrix is not self-adjoint any more. Thus, it does not lead to a unitary fundamental solution. We apply two steps of diagonalization inside the elliptic zone.

Step 1. In a first step we use the diagonalizer of the first matrix, which has to be understood as the principal part in this zone,

$$M = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \qquad M^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \tag{4.2.8}$$

such that for $V^{(0)} = M^{-1}V$ the equation

$$\mathrm{D}_t V^{(0)} = \big[\mathcal{D}(t,\xi) + R(t,\xi)\big] V^{(0)} \tag{4.2.9}$$

holds with coefficient matrices

$$\mathcal{D}(t,\xi) = \begin{pmatrix} -i\,\langle\xi\rangle_{\gamma(t)} & \\ & i\,\langle\xi\rangle_{\gamma(t)} \end{pmatrix} \in S^{\ell,\infty}_{ell,\epsilon}\{1,0,0\}, \tag{4.2.10}$$

$$R(t,\xi) = \begin{pmatrix} \frac{\mathrm{D}_t\langle\xi\rangle_{\gamma(t)}}{2\langle\xi\rangle_{\gamma(t)}} - i\frac{b'(t)}{4\langle\xi\rangle_{\gamma(t)}} & -\frac{\mathrm{D}_t\langle\xi\rangle_{\gamma(t)}}{2\langle\xi\rangle_{\gamma(t)}} + i\frac{b'(t)}{4\langle\xi\rangle_{\gamma(t)}} \\ -\frac{\mathrm{D}_t\langle\xi\rangle_{\gamma(t)}}{2\langle\xi\rangle_{\gamma(t)}} - i\frac{b'(t)}{4\langle\xi\rangle_{\gamma(t)}} & \frac{\mathrm{D}_t\langle\xi\rangle_{\gamma(t)}}{2\langle\xi\rangle_{\gamma(t)}} + i\frac{b'(t)}{4\langle\xi\rangle_{\gamma(t)}} \end{pmatrix} \in S^{\ell,\infty}_{ell,\epsilon}\{0,0,1\} \tag{4.2.11}$$

under Assumption (B4)$_{\ell+1}$. The matrix $F_0(t,\xi) = \operatorname{diag} R(t,\xi)$ is no multiple of the identity. In the diagonalization scheme from in Sections 3.3.3 or 4.2.1 we used essentially that $F_0(t,\xi)$ commutes with all occurring matrices. This property is not valid any more and thus we will include the entries of F_0 in the diagonalization scheme for the next diagonalization step(s).

Step 2. The difference of the entries of $\mathcal{D}(t,\xi) + F_0(t,\xi)$ satisfies

$$i\delta(t,\xi) = 2\,\langle\xi\rangle_{\gamma(t)} + \frac{b'(t)}{2\,\langle\xi\rangle_{\gamma(t)}} \sim \langle\xi\rangle_{\gamma(t)} \tag{4.2.12}$$

for t sufficiently large by Assumption (B3). The derivatives of $\delta(t,\xi)$ satisfy similar estimates, such that the following lemma holds.

Lemma 4.8. *Under Assumptions (B1) – (B3) and (B4)$_{\ell+1}$ it holds*

$$\delta^{-1}(t,\xi) \in S^{\ell,\infty}_{ell,\epsilon}\{-1,0,0\}.$$

Now we can follow the usual procedure to diagonalize further steps. Let

$$\begin{aligned} N^{(1)}(t,\xi) &= \begin{pmatrix} & -R_{12}/\delta \\ R_{21}/\delta & \end{pmatrix} \in S^{\ell,\infty}_{ell,\epsilon}\{-1,0,1\}, \\ B^{(1)}(t,\xi) &= \mathrm{D}_t N^{(1)}(t,\xi) - \big(R(t,\xi) - F_0(t,\xi)\big) N^{(1)}(t,\xi) \in S^{\ell-1,\infty}_{ell,\epsilon}\{-1,0,2\}, \\ N_1(t,\xi) &= I + N^{(1)}(t,\xi) \in S^{\ell,\infty}_{ell,\epsilon}\{0,0,0\}. \end{aligned}$$

For sufficiently large time $t \geq t_0$ the matrix $N_1(t,\xi)$ is invertible with uniformly bounded inverse $N_1^{-1}(t,\xi)$. Now with $R_1(t,\xi) = -N_1^{-1}(t,\xi)B^{(1)}(t,\xi)$ the operator identity

$$\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - R(t,\xi)\big)N_1(t,\xi) = N_1(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - F_0(t,\xi) - R_1(t,\xi)\big) \tag{4.2.13}$$

holds. Furthermore, symbols from $S^{0,0}_{ell,\epsilon}\{-1,0,2\}$ are integrable over the elliptic zone by Proposition 4.3. This diagonalization step will be sufficient to obtain structural properties and estimates of the solution representation.

Lemma 4.9. *Assume (B1) – (B3) and (B4)$_2$. Then there exists a starting time t_0 such that in $Z_{ell}(c_0,\epsilon)\cap\{t\geq t_0\}$ there exist symbols*

- $N_1(t,\xi)\in S^{1,\infty}_{ell,\epsilon}\{0,0,0\}$, *invertible with* $N_1^{-1}(t,\xi)\in S^{1,\infty}_{ell,\epsilon}\{0,0,0\}$,
- $F_0(t,\xi)=\operatorname{diag}\left(\frac{\mathrm{D}_t\langle\xi\rangle_{\gamma(t)}}{2\langle\xi\rangle_{\gamma(t)}}-i\frac{b'(t)}{4\langle\xi\rangle_{\gamma(t)}},\frac{\mathrm{D}_t\langle\xi\rangle_{\gamma(t)}}{2\langle\xi\rangle_{\gamma(t)}}+i\frac{b'(t)}{4\langle\xi\rangle_{\gamma(t)}}\right)\in S^{1,\infty}_{ell,\epsilon}\{0,0,1\}$,
- $R_1(t,\xi)\in S^{0,\infty}_{ell,\epsilon}\{-1,0,2\}$,

which satisfy the operator identity

$$\big(\mathrm{D}_t-\mathcal{D}(t,\xi)-R(t,\xi)\big)N_1(t,\xi)=N_1(t,\xi)\big(\mathrm{D}_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-R_1(t,\xi)\big).$$

It is possible to apply ℓ diagonalization steps under Assumption (B4)$_{\ell+1}$.

Step 3. Construction of the fundamental solution. We can not follow the consideration from the theory of the hyperbolic zone (cf. Section 3.3.3 and 4.2.1), the main diagonal entries are not real. The idea is to transform the diagonalized system to an integral equation with diagonal dominated kernel and application of Theorem B.10.

The definition of the auxiliary symbol $\langle\xi\rangle_{\gamma(t)}$ implies the following estimate:

Proposition 4.10. *It holds*

$$\frac{\sqrt{\langle\xi\rangle_{\gamma(t)}\,\gamma(t)+\langle\xi\rangle^2_{\gamma(t)}}}{\sqrt{\langle\xi\rangle_{\gamma(s)}\,\gamma(s)+\langle\xi\rangle^2_{\gamma(s)}}}\sim\frac{\gamma(t)}{\gamma(s)}\sim\frac{\langle\xi\rangle_{\gamma(t)}}{\langle\xi\rangle_{\gamma(s)}}$$

uniformly in $Z_{ell}(c_0,\epsilon)\cap\{t\geq t_0\}$. Furthermore, if $\gamma(\infty)\neq 0$ the quotient of two of the terms tends to a nonzero and continuous limit as $t\to\infty$ and for $|\xi|<\gamma(\infty)$.

This estimate is useful to understand and prove the following theorem, which states the main result within the elliptic zone.

Theorem 4.11. *Assume (B1) – (B3), (B4)$_2$. Then the fundamental solution $\mathcal{E}_{V,1}(t,s,\xi)$ of the transformed system $\mathrm{D}_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-R_1(t,\xi)$ can be represented as*

$$\mathcal{E}_{V,1}(t,s,\xi)=\frac{\langle\xi\rangle_{\gamma(t)}}{\langle\xi\rangle_{\gamma(s)}}\exp\left\{\int_s^t\langle\xi\rangle_{\gamma(\tau)}\,\mathrm{d}\tau\right\}\mathcal{Q}_{ell,1}(t,s,\xi)$$

for $(t,\xi),(s,\xi)\in Z_{ell}(c_0,\epsilon)\cap\{t\geq t_0\}$, $t\geq s$, and with a uniformly bounded matrix $\mathcal{Q}_{ell,1}(t,s,\xi)$.

In the case that $\gamma(t)\not\to 0$ as $t\to\infty$ and without introducing a dissipative zone the limit

$$\lim_{t\to\infty}\mathcal{Q}_{ell,1}(t,s,\xi)=\mathcal{Q}_{ell,1}(\infty,s,\xi)$$

exists uniformly on compact sets in $|\xi|\in[0,\gamma(\infty))$ and defines a continuous function $\mathcal{Q}_{ell,1}(\infty,s,\xi)$.

Proof. We transform the system for $\mathcal{E}_{V,1}(t,s,\xi)$ to an integral equation for $\mathcal{Q}_{ell,1}(t,s,\xi)$. If we differentiate

$$\exp\left\{i\int_s^t[\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)]\mathrm{d}\tau\right\}\mathcal{E}_{V,1}(t,s,\xi)$$

with respect to t the diagonal structure of $\mathcal{D} + F_0$ implies

$$\begin{aligned}
&\mathrm{D}_t\left[\exp\left\{i\int_s^t(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi))\mathrm{d}\tau\right\}\mathcal{E}_{V,1}(t,s,\xi)\right]\\
&\quad= -\big(\mathcal{D}(t,\xi)+F_0(t,\xi)\big)\exp\left\{i\int_s^t(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi))\mathrm{d}\tau\right\}\mathcal{E}_{V,1}(t,s,\xi)\\
&\qquad+\exp\left\{i\int_s^t(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi))\mathrm{d}\tau\right\}\big(\mathcal{D}(t,\xi)+F_0(t,\xi)+R_1(t,\xi)\big)\mathcal{E}_{V,1}(t,s,\xi)\\
&\quad= \exp\left\{i\int_s^t(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi))\mathrm{d}\tau\right\}R_1(t,\xi)\mathcal{E}_{V,1}(t,s,\xi),
\end{aligned}$$

such that by integration over the interval $[s,t]$ we obtain

$$\begin{aligned}
\mathcal{E}_{V,1}(t,s,\xi) &= \exp\left\{-i\int_s^t(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi))\mathrm{d}\tau\right\}\mathcal{E}_{V,1}(s,s,\xi)\\
&\quad+i\int_s^t\exp\left\{-i\int_\theta^t(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi))\mathrm{d}\tau\right\}R_1(\theta,\xi)\mathcal{E}_{V,1}(\theta,s,\xi)\mathrm{d}\theta. \qquad (4.2.14)
\end{aligned}$$

The exponential is not bounded. In order to compensate this bad behaviour we introduce a weight factor. Let therefore,

$$\mathcal{Q}_{ell,1}(t,s,\xi) = \exp\left\{-\int_s^t w(\tau,\xi)\mathrm{d}\tau\right\}\mathcal{E}_{V,1}(t,s,\xi)$$

with an appropriate weight $w(t,\xi)$. Then we obtain the integral equation

$$\begin{aligned}
\mathcal{Q}_{ell,1}(t,s,\xi) &= \exp\left\{\int_s^t(i\mathcal{D}(\tau,\xi)+iF_0(\tau,\xi)-w(\tau,\xi)I)\mathrm{d}\tau\right\}\\
&\quad+\int_s^t\exp\left\{\int_\theta^t(i\mathcal{D}(\tau,\xi)+iF_0(\tau,\xi)-w(\tau,\xi)I)\mathrm{d}\tau\right\}R_1(\theta,\xi)\mathcal{Q}_{ell,1}(\theta,s,\xi)\mathrm{d}\theta. \qquad (4.2.15)
\end{aligned}$$

We made essential use of the fact that the weight factor commutes with all matrices. This integral equation is well-posed in $L^\infty(\{(t,s,\xi)\,|\,(t,\xi),(s,\xi)\in Z_{ell},\ t\geq s\})$ for suitable weight $w(t,\xi)$. By Proposition 4.3 the matrix R_1 is uniformly integrable over the elliptic zone. It remains to see that the exponential function remains bounded and this is guaranteed by a sign condition on the exponent.

The entries of $i\mathcal{D}(t,\xi)+iF_0(t,\xi)$ are given by

$$\begin{aligned}
(I) &= \langle\xi\rangle_{\gamma(t)} + \frac{\partial_t\langle\xi\rangle_{\gamma(t)}}{2\langle\xi\rangle_{\gamma(t)}} + \frac{\gamma'(t)}{2\langle\xi\rangle_{\gamma(t)}},\\
(II) &= -\langle\xi\rangle_{\gamma(t)} + \frac{\partial_t\langle\xi\rangle_{\gamma(t)}}{2\langle\xi\rangle_{\gamma(t)}} - \frac{\gamma'(t)}{2\langle\xi\rangle_{\gamma(t)}}.
\end{aligned}$$

Now it follows that the first one is dominating. Inequality $(II)\leq(I)$ is equivalent to

$$\gamma^2(t)-|\xi|^2+\gamma'(t)\leq 0,$$

which is true in $Z_{ell}(c_0,\epsilon)$ for $t\geq t_0$ from $|b'(t)| = o(b^2(t))$, Assumption (B3). Thus choosing $w(t,\xi)=(I)$ gives the optimal weight function and Theorem B.9 implies the well-posedness in L^∞.

Now from

$$\int_s^t \left[\frac{\partial_t \langle\xi\rangle_{\gamma(\tau)}}{2\langle\xi\rangle_{\gamma(\tau)}} + \frac{\gamma'(\tau)}{2\langle\xi\rangle_{\gamma(\tau)}} \right] \mathrm{d}\tau = \frac{1}{2}\log\frac{\langle\xi\rangle_{\gamma(t)}}{\langle\xi\rangle_{\gamma(s)}} + \frac{1}{2}\log\frac{\gamma(t)+\langle\xi\rangle_{\gamma(t)}}{\gamma(s)+\langle\xi\rangle_{\gamma(s)}}$$
$$= \frac{1}{2}\log\frac{\langle\xi\rangle_{\gamma(t)}\gamma(t)+\langle\xi\rangle^2_{\gamma(t)}}{\langle\xi\rangle_{\gamma(s)}\gamma(s)+\langle\xi\rangle^2_{\gamma(s)}}$$

and Proposition 4.10 the representation for $\mathcal{E}_{V,1}(t,s,\xi)$ follows. Furthermore,

$$H(t,s,\xi) = \exp\left\{ \int_s^t (i\mathcal{D}(\tau,\xi) + iF_0(\tau,\xi) - w(t,\xi)I)\mathrm{d}\tau \right\}$$
$$= \operatorname{diag}\left(1, \frac{\gamma(s)+\langle\xi\rangle_{\gamma(s)}}{\gamma(t)+\langle\xi\rangle_{\gamma(t)}} \exp\left\{-2\int_s^t \langle\xi\rangle_{\gamma(\tau)}\,\mathrm{d}\tau\right\}\right) \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

for $t \to \infty$.

It remains to show the existence of the limit. Choosing t_0 sufficiently large the integral kernel (i.e. essentially $R_1(\theta,\xi)$) can be estimated uniformly on compact subsets of $[0,\gamma(\infty))$ by an L^1-function in θ and therefore the representation of $\mathcal{Q}_{ell,1}(t,s,\xi)$ by a Neumann series

$$\mathcal{Q}_{ell,1}(t,s,\xi) = H(t,s,\xi) + \sum_{k=1}^\infty i^k \int_s^t H(t,t_1,\xi)R_1(t_1,\xi) \int_s^{t_1} H(t_1,t_2,\xi)R_1(t_2,\xi)\cdots$$
$$\int_s^{t_{k-1}} H(t_{k-1},t_k,\xi)R_1(t_k,\xi)\mathrm{d}t_k\cdots\mathrm{d}t_2\mathrm{d}t_1 \quad (4.2.16)$$

converges. The existence of the limit follows by the same way as in the proofs for Theorems 3.17 or 3.1. Furthermore the uniform convergence on compact subsets implies continuity. □

Corollary 4.12. *Assume* $\frac{1}{(1+t)^2 b(t)} \in L^1(\mathbb{R}_+)$*, such that (*$\Gamma$*3) is satisfied. Then* $\mathcal{Q}_{ell,1}(\infty,s,0) \neq 0$*.*

Proof. We solve the ordinary differential equation $\hat{u}_{tt} + b(t)\hat{u}_t$ arising for $\xi = 0$ directly. It holds

$$\hat{u}(t,0) = \int_0^t \frac{\mathrm{d}\tau}{\lambda^2(\tau)}\hat{u}_t(0,0) + \hat{u}(0,0)$$

and from Assumption (B3) integrability of $1/\lambda^2$ follows

$$\hat{v}(t,0) \sim \lambda(t)\left[\hat{u}(0,0) + \hat{u}_t(0,0)\int_0^\infty \frac{\mathrm{d}\tau}{\lambda^2(\tau)}\right].$$

We can also represent $\hat{v}$ by the fundamental solution $\mathcal{E}_V$, which simplifies in this case to

$$\mathcal{E}_V(t,s,0) = \frac{\gamma(t)}{\gamma(s)}\lambda(t)N_1(t,0)\mathcal{Q}_{ell,1}(t,s,0)N_1^{-1}(s,0),$$

where the matrices N_1 and N_1^{-1} are uniformly bounded and tend to the identity for $t \to \infty$. If we compare the results, we see that the first row of $\mathcal{Q}_{ell,1}(\infty,s,0)$ can never be zero. □

In the case that $\gamma(t) \not\to 0$ we see that $\mathcal{Q}_{ell,1}(\infty, s, \xi)$ is different from zero in a neighbourhood of the frequency $\xi = 0$. We can even see more, both rows of the matrix $\mathcal{Q}_{ell,1}(t, s, \xi)$ behave differently. While the first one tends to a non-zero limit the second one satisfies a decay-estimate.

Corollary 4.13. *Assume that* $\gamma(t) \not\to 0$. *Then the limit*

$$\lim_{t\to\infty} H(s,t,\xi)\mathcal{Q}_{ell,1}(t,s,\xi)$$

exists uniformly on compact subsets of $[0, \gamma(\infty))$.

Proof. In the series representation (4.2.16) we can factor out $H(t, s, \xi)$ and get

$$H(s,t,\xi)\mathcal{Q}_{ell,1}(t,s,\xi) = I + \sum_{k=1}^{\infty} i^k \int_s^t H(t_1,s,\xi)R_1(t_1,\xi) \int_s^{t_1} H(t_1,t_2,\xi)R_1(t_2,\xi)\dots$$
$$\int_s^{t_{k-1}} H(t_{k-1},t_k,\xi)R_1(t_k,\xi)\mathrm{d}t_k \dots \mathrm{d}t_2\mathrm{d}t_1$$

which remains bounded and takes limits by the same argumentation as above. $\square$

Corollary 4.14. *The second row of the matrix* $\mathcal{Q}_{ell,1}(t, s, \xi)$ *satisfies in the elliptic zone the estimate*

$$||e_2^T \mathcal{Q}_{ell,1}(t,s,\xi)|| \lesssim \frac{\gamma(s)}{\gamma(t)} \exp\left\{-2\int_s^t \langle\xi\rangle_{\gamma(\tau)}\,\mathrm{d}\tau\right\} \lesssim \frac{\gamma(s)(1+s)}{\gamma(t)(1+t)}. \tag{4.2.17}$$

The different behaviour of the two rows of the matrix $\mathcal{Q}_{ell,1}(t, s, \xi)$ transfers by diagonalization to

$$\mathcal{Q}_{ell,0}(t,s,\xi) = N_1(t,\xi)\mathcal{Q}_{ell,1}(t,s,\xi)N_1^{-1}(s,\xi) \tag{4.2.18}$$

using that the principal part of N_1 is the identity matrix, that means $N_1(t, \xi) - I \in S^{1,\infty}_{ell,\epsilon}\{-1, 0, 1\}$. Thus after applying the diagonalizer matrices the first row will remain bounded, but the second one decays at least like symbols from $S^{0,0}_{ell,\epsilon}\{-1, 0, 1\}$. Thus the second estimate of Corollary 4.14 is also true for $\mathcal{Q}_{ell,0}(t, s, \xi)$.

The dissipative zone. In case that (Γ3) does not hold, i.e. if in our case $b(t)$ is 'close to' $\frac{1}{1+t}$, we introduced the dissipative zone to ensure integrability of $S^{0,0}_{ell,\epsilon}\{-1, 0, 2\}$ over $Z_{ell}(c_0, \epsilon)$. In these cases we apply Lemma 3.9 to estimate the fundamental solution to $U = (\frac{c_0}{1+t}\hat{u}, \mathrm{D}_t\hat{u})$ and relate this estimate to the corresponding one for $V = (\langle\xi\rangle_{\gamma(t)}\,\hat{v}, \mathrm{D}_t\hat{v})^T$.

4.2.3 The reduced zone

In the reduced zone we replace $\langle\xi\rangle_{\gamma(t)}$ by $\epsilon\gamma(t)$. Thus, we consider

$$V = (\epsilon\gamma(t)\hat{v}, \mathrm{D}_t\hat{v})^T, \tag{4.2.19}$$

such that

$$\mathrm{D}_t V = \begin{pmatrix} \frac{\mathrm{D}_t\gamma(t)}{\gamma(t)} & \epsilon\gamma(t) \\ \frac{|\xi|^2 - \frac{1}{4}b^2(t) - \frac{1}{2}b'(t)}{\epsilon\gamma(t)} & \end{pmatrix} V. \tag{4.2.20}$$

The lower left corner entry can be estimated by $\epsilon\gamma(t) - \frac{b'(t)}{\epsilon b(t)}$ and by Assumption (B3) the second term is dominated by the first one for all (fixed) ϵ. Thus, we can estimate the norm of the coefficient matrix by $2\epsilon\gamma(t)$ for sufficiently large t. Application of Corollary B.7 implies an estimate for the corresponding fundamental solution $\mathcal{E}_V(t, s, \xi)$ within the reduced zone. It holds

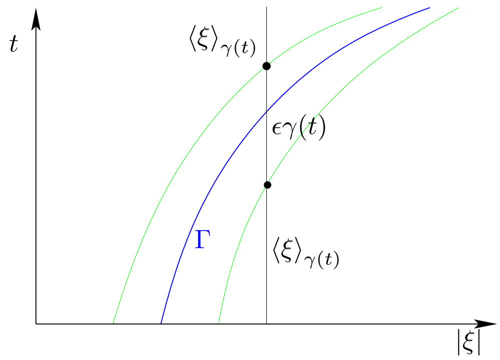

Figure 4.4: On the definition of V in the reduced zone. Note, that the defined energy is continuous and 'cuts out' the zero of $\langle\xi\rangle_{\gamma(t)}$ on the separating curve Γ.

Lemma 4.15. *Under Assumptions (B1)–(B3) the fundamental solution $\mathcal{E}_V(t,s,\xi)$ to* (4.2.20) *can be estimated by*

$$||\mathcal{E}_V(t,s,\xi)|| \le \exp\left\{\epsilon\int_s^t b(\tau)\mathrm{d}\tau\right\}$$

for $t \ge s \ge t_0$ with sufficiently large t_0 and $(t,\xi),(s,\xi) \in Z_{red}(\epsilon)$.

This estimate seems to be very rough. But we can make the reduced zone as small as we want and therefore we can control the constant ϵ. This (in general) exponential estimate is then dominated by $\lambda(t)$ and gives no contribution to the final energy and L^p–L^q estimates. This idea for the choice of ϵ is used e.g. in the proof of Theorem 4.21, where $\epsilon < \frac{1}{4}$ will be necessary.

4.2.4 Treatment of non-monotonous coefficients

The considerations done so far can be generalized to non-monotonous coefficients. Let therefore $\gamma(t)$ be an admissible function in the sense of Definition 4.5 and let the coefficient $b = b(t)$ satisfy Assumption (B1) and (B4)$_\ell$ together with the relation

(Bγ) $|b(t) - 2\gamma(t)| \le c\gamma(t)\left(\frac{1}{1+t}\right)$.

This implies for the micro-local mass term $m(t,\xi)$ defined by equation (4.1.3) the following estimates. Remark that including $b'(t)$ in this term implies the loss of one further derivative compared to Sections 4.2.1 – 4.2.3.

Proposition 4.16. 1. $m(t,\xi) \in S^{\ell-1,\infty}_{hyp,N}\{2,0,0\}. \cap S^{\ell-1,\infty}_{ell,\epsilon}\{2,0,0\}$,

2. *There exist a starting time t_0 and a zone constant N such that*

$$|m(t,\xi)| \gtrsim \gamma^2(t) \qquad \textit{in} \quad Z_{hyp}(N) \cup (Z_{ell}(c_0,\epsilon) \cap \{t \ge t_0\}).$$

Furthermore,

$$\sqrt{|m(t,\xi)|} \in S^{\ell-1,\infty}_{hyp,N}\{1,0,0\} \cap S^{\ell-1,\infty}_{ell,\epsilon}\{1,0,0\}$$

and

$$\frac{1}{\sqrt{|m(t,\xi)|}} \in S^{\ell-1,\infty}_{hyp,N}\{-1,0,0\} \cap S^{\ell-1,\infty}_{ell,\epsilon}\{-1,0,0\}.$$

We explain only the first estimate of the second statement in the elliptic zone, the other estimates are straightforward. It holds

$$
\begin{aligned}
-m(t,\xi) = \frac{1}{4}b^2(t) + \frac{1}{2}b'(t) - |\xi|^2 \geq \frac{1}{4}b^2(t) + \frac{1}{2}b'(t) - (1-\epsilon')\gamma^2(t) \\
\geq -C\gamma^2(t)\left(\frac{1}{1+t}\right) - C'\gamma(t)\left(\frac{1}{1+t}\right) + \epsilon'\gamma^2(t)
\end{aligned}
$$

using $|\xi|^2 \leq (1-\epsilon')\gamma^2(t)$ together with (Bγ) and after this (Γ2) to absorb the second term in the last one for sufficiently large t_0.

These estimates allow us to consider the micro-energy

$$V = (\sqrt{|m(t,\xi)|}\hat{v}, \mathrm{D}_t\hat{v})^T \tag{4.2.21}$$

in $Z_{hyp}(N) \cup (Z_{ell}(c_0,\epsilon) \cap \{t \geq t_0\})$ and a suitable continuous extension inside the remaining zones.[2] Proposition 4.16 allows us to diagonalize inside the elliptic and inside the hyperbolic zone. We will sketch this approach and the corresponding results. The system for V reads as

$$\mathrm{D}_t V = \begin{pmatrix} \frac{\mathrm{D}_t\sqrt{|m(t,\xi)|}}{\sqrt{|m(t,\xi)|}} & \sqrt{|m(t,\xi)|} \\ \sqrt{|m(t,\xi)|} & 0 \end{pmatrix} V, \tag{4.2.22}$$

leading after two steps of diagonalization to

$$\mathrm{D}_t V^{(1)} = \left(\mathcal{D}(t,\xi) + F_0(t,\xi) + R_1(t,\xi)\right) V^{(1)}, \tag{4.2.23}$$

where inside the hyperbolic zone

$$\mathcal{D}(t,\xi) = \mathrm{diag}\left(\sqrt{|m(t,\xi)|}, -\sqrt{|m(t,\xi)|}\right), \tag{4.2.24}$$

$$F_0(t,\xi) = \frac{\mathrm{D}_t\sqrt{|m(t,\xi)|}}{2\sqrt{|m(t,\xi)|}} I, \tag{4.2.25}$$

$$R_1(t,\xi) \in S^{\ell-2,\infty}_{hyp,N}\{-1,0,2\}; \tag{4.2.26}$$

while inside the elliptic zone we get

$$\mathcal{D}(t,\xi) = \mathrm{diag}\left(-i\sqrt{|m(t,\xi)|}, i\sqrt{|m(t,\xi)|}\right), \tag{4.2.27}$$

$$F_0(t,\xi) = \frac{\mathrm{D}_t\sqrt{|m(t,\xi)|}}{2\sqrt{|m(t,\xi)|}} I, \tag{4.2.28}$$

$$R_1(t,\xi) \in S^{\ell-2,\infty}_{ell,\epsilon}\{-1,0,2\}. \tag{4.2.29}$$

Now the construction of the fundamental solution follows the lines of the previous sections. In the hyperbolic zone $\mathcal{D}(t,\xi)$ is self-adjoint, thus the fundamental solution is unitary. Furthermore, R_1 is integrable and the integral over the diagonal term is uniformly bounded. Thus, we get

$$||\mathcal{E}_V(t,s,\xi)|| \lesssim 1 \tag{4.2.30}$$

like in Lemma 4.4. Inside the elliptic zone $\sqrt{|m(t,\xi)|}$ occurs in the exponential and the coefficient in front changes slightly. The result is closely related to the representations given by M.V. Fedoryuk, [Фед85, Глава 7, §2].

[2] There the precise structure of the micro-energy is not essential. The coefficient replacing $\sqrt{|m(t,\xi)|}$ in the reduced zone should be small.

Theorem 4.17. *Assume (B1), (B4)$_2$ and (Bγ). Then the fundamental solution $\mathcal{E}_{V,1}(t,s,\xi)$ to the transformed system $\mathrm{D}_t - \mathcal{D}(t,\xi) - F_0(t,\xi) - R_1(t,\xi)$ can be represented inside the elliptic zone as*

$$\mathcal{E}_{V,1}(t,s,\xi) = \left(\frac{m(t,\xi)}{m(s,\xi)}\right)^{\frac{1}{4}} \exp\left\{\int_s^t \sqrt{|m(\tau,\xi)|}\mathrm{d}\tau\right\} \mathcal{Q}_{ell,1}(t,s,\xi)$$

for $t \geq s$ and with a uniformly bounded matrix $\mathcal{Q}_{ell,1}(t,s,\xi)$ tending to a continuous limit as $t \to \infty$ in the case that $\gamma(t) \not\to 0$.

The treatment of the dissipative zone and the reduced zone remains the same as for monotonous coefficients. Inside the pseudo-differential zone we stop after the first diagonalization step and use the rough estimate

$$\int_s^t \left|\frac{\partial_t \sqrt{m(\tau,\xi)}}{\sqrt{m(\tau,\xi)}}\right| \mathrm{d}\tau \lesssim \int_s^t \frac{\mathrm{d}\tau}{1+\tau} \lesssim \log\frac{1+t}{1+s},$$

which leads to some polynomial loss of decay which can be compensated by the exponential estimates in the elliptic zone.

Example 4.3. To give one example of a coefficient function, which can be handled by this approach, we consider

$$\gamma(t) = (1+t)^\kappa$$

with $\kappa \in (-1,1)$ and define

$$b(t) = 2(1+t)^\kappa + \sin\big(\alpha\log(1+t)\big)(1+t)^{\kappa-1}$$

with $\alpha \in \mathbb{R}$. This coefficient satisfies (B1) and (B4) and obviously also (Bγ). At least for sufficiently large α this coefficient violates (B2).

4.3 Estimates

4.3.1 Relation to the energy operator and auxiliary estimates

We want to obtain estimates for the solution representation and for the energy operator $\mathbb{E}(t,\mathrm{D})$ to our original Cauchy problem. The representation of $\mathcal{E}_V(t,s,\xi)$ in the different zones constructed in Section 4.2 may be used to conclude a representation for the fundamental solution $\mathcal{E}(t,s,\xi)$ to the micro-energy (3.2.8) used in Chapter 3 and corresponding estimates for the operators $\mathcal{E}(t,s,\mathrm{D})$ and $\mathbb{E}(t,\mathrm{D})$. We restrict ourselves to the case of monotonous coefficient functions $b=b(t)$ satisfying (B1) to (B3) and (B4)$_2$ with the notation $2\gamma(t) = b(t)$.

Outside the reduced zone it holds

$$\mathcal{E}(t,s,\xi) = T(t,\xi)\mathcal{E}_V(t,s,\xi)T^{-1}(s,\xi), \tag{4.3.1}$$

where we used the matrix-valued function $T(t,\xi)$,

$$\begin{pmatrix} h(t,\xi)\hat{u} \\ \mathrm{D}_t\hat{u}\end{pmatrix} = \underbrace{\begin{pmatrix} \frac{h(t,\xi)}{\lambda(t)\langle\xi\rangle_{\gamma(t)}} & 0 \\ i\frac{\gamma(t)}{\lambda(t)\langle\xi\rangle_{\gamma(t)}} & \frac{1}{\lambda(t)}\end{pmatrix}}_{T(t,\xi)} \begin{pmatrix}\langle\xi\rangle_{\gamma(t)}\,\hat{v} \\ \mathrm{D}_t\hat{v}\end{pmatrix} \tag{4.3.2}$$

with inverse

$$T^{-1}(t,\xi) = \begin{pmatrix} \frac{\lambda(t)\langle\xi\rangle_{\gamma(t)}}{h(t,\xi)} & 0 \\ -i\frac{\gamma(t)\lambda(t)}{h(t,\xi)} & \lambda(t) \end{pmatrix}. \tag{4.3.3}$$

This relation follows directly from the definition of $v(t,x)$ in (4.1.1). Recall that inside the dissipative zone $h(t,\xi) = \frac{c_0}{1+t}$, while outside it is equal to $|\xi|$.

Inside the reduced zone, we have replaced $\langle\xi\rangle_{\gamma(t)}$ by $\epsilon\gamma(t)$ and, therefore, we replace in the definition of the matrix T the corresponding terms. This yields

$$T(t,\xi) = \begin{pmatrix} \frac{h(t,\xi)}{\epsilon\lambda(t)\gamma(t)} & 0 \\ i\frac{1}{\epsilon\lambda(t)} & \frac{1}{\lambda(t)} \end{pmatrix}, \qquad ||T(t,\xi)|| \sim \lambda^{-1}(t) \tag{4.3.4}$$

for all $(t,\xi) \in Z_{red}(\epsilon)$.

Auxiliary estimates. We continue this section with some auxiliary estimates, which are essentially used to obtain energy and L^p–L^q estimates later on.

Lemma 4.18. *Assume (B1) – (B3) and set* $2\gamma(t) = b(t)$ *and* $\lambda(t) = \exp\left\{\frac{1}{2}\int_0^t b(\tau)\mathrm{d}\tau\right\}$. *Then it holds:*

1. *The definition of* $\langle\xi\rangle_{\gamma(t)}$ *implies* $\langle\xi\rangle_{\gamma(t)} - \gamma(t) \le -\frac{|\xi|^2}{b(t)}$.
2. *It holds*
$$\frac{\lambda(s)}{\lambda(t)} \exp\left\{\int_s^t \langle\xi\rangle_{\gamma(\tau)}\,\mathrm{d}\tau\right\} \lesssim \exp\left\{-|\xi|^2\int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}.$$
3. *With* $(1+t_\xi)|\xi| \sim 1$ *it holds*
$$\exp\left\{-|\xi|^2\int_0^{t_\xi} \frac{\mathrm{d}\tau}{b(\tau)}\right\} \sim 1.$$
4. *It holds*
$$b^2(t)\int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \to \infty.$$
5. *For all* $\alpha \in \mathbb{R}$ *the function*
$$\left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^\alpha \lambda(t)$$
is monotonously increasing for large t.

Proof. The first statement is an elementary inequality and implies the second statement directly from the definition of $\lambda(t)$.

The third statement follows for decreasing $b(t)$ from

$$\int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \le \frac{t}{b(t)} \lesssim (1+t)^2,$$

using the monotonicity of $1/b(t)$ together with $tb(t) \to \infty$ from Assumption (B3).

The next one follows from the calculation

$$b^2(t)\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right) \gtrsim b^2(t) - b'(t)\int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \gtrsim b^2(t) - \int_0^t \frac{b'(\tau)}{b(\tau)}\mathrm{d}\tau \sim b^2(t) - \log b(t) \to \infty$$

in case $b(t) \to 0$. Otherwise the estimate is obvious by the monotonicity of b.

The last statement can be obtained by differentiating the expression. It holds

$$\begin{aligned}\partial_t\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\alpha}\lambda(t) &= \alpha\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\alpha-1}\frac{1}{b(t)}\lambda(t) + b(t)\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\alpha}\lambda(t)\\ &= \frac{1}{b(t)}\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\alpha-1}\lambda(t)\left(\alpha + b^2(t) + b^2(t)\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)\end{aligned}$$

and from statement 4 we find a sufficiently large time t_0 depending on α, such that the expression is positive for all $t \geq t_0$. □

Representation of $\mathcal{E}(t,s,\xi)$ in the elliptic zone. Because of its technicality, we separate the estimate of $\mathcal{E}(t,s,\xi)$ inside the elliptic zone from the proofs of the main results of this section. We will see that we have to combine the estimates of Section 4.2 with a new idea to get desired results.

Inside the elliptic zone, i.e. for $(t,\xi),(s,\xi) \in Z_{ell}(c_0,\epsilon)$, it holds

$$\mathcal{E}_V(t,s,\xi) \sim \frac{b(t)}{b(s)} \exp\left\{\int_s^t \langle\xi\rangle_{\gamma(\tau)}\,\mathrm{d}\tau\right\} \mathcal{Q}_{ell,0}(t,s,\xi),$$

where the matrix $\mathcal{Q}_{ell,0}$ defined in (4.2.18) is uniformly bounded for $s \leq t$. This yields in combination with (4.3.1) for the energy multiplier $\mathcal{E}(t,s,\xi)$ the estimate

$$\begin{aligned}|\mathcal{E}(t,s,\xi)| &\lesssim \exp\left\{\int_s^t \big(\langle\xi\rangle_{\gamma(\tau)} - \gamma(\tau)\big)\mathrm{d}\tau\right\}\begin{pmatrix}|\xi| & \\ b(t) & b(t)\end{pmatrix} |\mathcal{Q}_{ell,0}(t,s,\xi)| \begin{pmatrix}\frac{1}{|\xi|} & \\ \frac{1}{|\xi|} & \frac{1}{b(s)}\end{pmatrix}\\ &\lesssim \exp\left\{-|\xi|^2\int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}\begin{pmatrix}1 & \frac{|\xi|}{b(s)}\\ \frac{b(t)}{|\xi|} & \frac{b(t)}{b(s)}\end{pmatrix},\end{aligned}$$

where we used Lemma 4.18.1 . The estimate for the first row seems to be optimal, while the estimate obtained for the second row is not optimal in this form, because at least for increasing coefficient functions $b = b(t)$ it is increasing in t like $b(t)$ for fixed frequency ξ, which contradicts to our a priori knowledge that the energy itself decays.

As we will see later the reason for this behaviour is that during the transformation back to the energy $\mathcal{E}(t,s,\xi)$ further terms cancel inside the difference $\lambda(t)\mathrm{D}_t\hat{u} = \gamma(t)\hat{v} - \mathrm{D}_t\hat{v}$, which we estimated by the bounds for the two summands.

Our basic idea is to relate the entries of the above given estimate to the multipliers $\Phi_i(t,s,\xi)$ and use Duhamel's formula to improve the estimates for the second row using estimates from the first one. A comparison yields so far

$$|\Phi_1(t,s,\xi)| \lesssim \exp\left\{-|\xi|^2 \int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}, \tag{4.3.5}$$

$$|\Phi_2(t,s,\xi)| \lesssim \frac{1}{b(s)} \exp\left\{-|\xi|^2 \int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}, \tag{4.3.6}$$

$$|\partial_t\Phi_1(t,s,\xi)| \lesssim b(t) \exp\left\{-|\xi|^2 \int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\},$$

$$|\partial_t\Phi_2(t,s,\xi)| \lesssim \frac{b(t)}{b(s)} \exp\left\{-|\xi|^2 \int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}.$$

The multipliers Φ_i solve the second order equation $\ddot{\Phi}_i + |\xi|^2\Phi_i + b(t)\dot{\Phi}_i = 0$ and, therefore, we obtain for $\Psi_i(t,s,\xi) = \partial_t\Phi_i(t,s,\xi)$ the first order equation

$$\partial_t\Psi_i + b(t)\Psi_i = -|\xi|^2\Phi_i(t,s,\xi), \qquad \Psi_i(s,s,\xi) = i\delta_{i2}, \tag{4.3.7}$$

where the source term on the right hand side can be estimated with the above bounds. Duhamel's formula yields now

$$\begin{aligned}
\Psi_1(t,s,\xi) &= -|\xi|^2 \int_s^t \frac{\lambda^2(\tau)}{\lambda^2(t)} \Phi_1(\tau,s,\xi)\mathrm{d}\tau, \\
|\Psi_1(t,s,\xi)| &\lesssim \frac{|\xi|^2}{\lambda^2(t)} \int_s^t b(\tau)\lambda^2(\tau)\frac{1}{b(\tau)} \exp\left\{-|\xi|^2\int_s^\tau \frac{\mathrm{d}\theta}{b(\theta)}\right\}\mathrm{d}\tau \\
&\lesssim \frac{|\xi|^2}{\lambda^2(t)} \left[\lambda^2(\tau)\frac{1}{b(\tau)} \exp\left\{-|\xi|^2\int_s^\tau \frac{\mathrm{d}\theta}{b(\theta)}\right\}\right]_s^t \\
&\qquad - \frac{|\xi|^2}{\lambda^2(t)} \int_s^t \lambda^2(\tau)\left(\frac{|\xi|^2}{b^2(\tau)} - \frac{b'(\tau)}{b^2(\tau)}\right) \exp\left\{-|\xi|^2\int_s^\tau \frac{\mathrm{d}\theta}{b(\theta)}\right\}\mathrm{d}\tau \\
&\lesssim \frac{|\xi|^2}{b(t)} \exp\left\{-|\xi|^2\int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\} - \frac{|\xi|^2}{b(s)}\frac{\lambda^2(s)}{\lambda^2(t)},
\end{aligned}$$

using $\partial_t\lambda^2(t) = b(t)\lambda^2(t)$ and $|\xi|^2/b^2(t) \le 1/2$ from the definition of the elliptic part together with $b'(t)/b^2(t) = o(1)$ from Assumption (B3). The second summand is subordinate to the first one because

$$\frac{b(s)}{b(t)} \exp\left\{-|\xi|^2\int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\} \frac{\lambda^2(t)}{\lambda^2(s)} = \exp\left\{\int_s^t \underbrace{\left(b(\tau) - \frac{|\xi|^2}{b(\tau)} - \frac{b'(\tau)}{b(\tau)}\right)}_{>0, \quad \text{if } \tau > t_0}\mathrm{d}\tau\right\} > 1 \tag{4.3.8}$$

for $t \ge s \ge t_0$ with t_0 sufficiently large.

Similarly, one obtains for Ψ_2 the representation

$$\begin{aligned}
\Psi_2(t,s,\xi) &= i\frac{\lambda^2(s)}{\lambda^2(t)} - |\xi|^2 \int_s^t \frac{\lambda^2(\tau)}{\lambda^2(t)}\Phi_2(\tau,s,\xi)\mathrm{d}\tau, \\
|\Psi_2(t,s,\xi)| &\lesssim \frac{\lambda^2(s)}{\lambda^2(t)} + \frac{|\xi|^2}{\lambda^2(t)b(s)} \int_s^t \lambda(\tau)\exp\left\{-|\xi|^2\int_s^\tau \frac{\mathrm{d}\theta}{b(\theta)}\right\}\mathrm{d}\tau \\
&\lesssim \frac{\lambda^2(s)}{\lambda^2(t)} + \frac{|\xi|^2}{b(t)b(s)} \exp\left\{-|\xi|^2\int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}.
\end{aligned}$$

Thus, we have proven the following lemma.

Lemma 4.19. *Assume $(t,\xi),(s,\xi)\in Z_{ell}(c_0,\epsilon)$ with $t\ge s$. Then the multiplier $\mathcal{E}(t,s,\xi)$ satisfies the pointwise estimate*

$$|\mathcal{E}(t,s,\xi)|\lesssim \exp\left\{-|\xi|^2\int_s^t\frac{\mathrm{d}\tau}{b(\tau)}\right\}\begin{pmatrix}1 & \frac{|\xi|}{b(s)}\\ \frac{|\xi|}{b(t)} & \frac{|\xi|^2}{b(t)b(s)}\end{pmatrix}+\frac{\lambda^2(s)}{\lambda^2(t)}\begin{pmatrix}0&0\\0&1\end{pmatrix}. \tag{4.3.9}$$

There exist at least two important special cases. If we set $s=t_\xi$ to be the lower bound of the elliptic zone (for increasing $b=b(t)$) or $t=t_\xi$ the upper bound of the elliptic zone (for decreasing coefficients $b=b(t)$), then some of the b's cancel with $|\xi|$ and the estimate simplifies. We will employ Lemma 4.19 to prove the main results of this section.

Remark 4.4. The estimate of Lemma 4.19 for the second row is better than the estimate obtained directly by application of Theorem 4.11. This is related to the fact that we can not use the better behaviour of the second row of $\mathcal{Q}_{ell,0}(t,s,\xi)$ during the transformation to $\mathcal{E}(t,s,\xi)$ and to the fact that there arises some cancellation in the difference $\gamma(t)\hat{v}(t)-\mathrm{D}_t\hat{v}(t)$.

Remark 4.5. For small ξ and with $s=t_0$ fixed, the second summand in (4.3.9) is dominated by the first one. In case of strong dissipation and with $s=t_\xi$, the lower boundary of the elliptic zone for large frequencies, we can use $|\xi|\sim b(t_\xi)$ to deduce from (4.3.8) the following estimate.

Corollary 4.20. *It holds in the notation of Remark 4.5 that*

$$|\mathcal{E}(t,t_\xi,\xi)|\lesssim \exp\left\{-|\xi|^2\int_0^t\frac{\mathrm{d}\tau}{b(\tau)}\right\}\begin{pmatrix}1 & 1\\ \frac{|\xi|}{b(t)} & \frac{|\xi|}{b(t)}\end{pmatrix}.$$

4.3.2 L^2–L^2 estimates

We start with the formulation of the estimate in L^2-scale or equivalently with the estimate of the multiplier $\mathcal{E}(t,s,\xi)$ in L^∞-norm. It holds

Theorem 4.21. *Assume (B1)–(B3) and (B4)$_2$. Then the L^2–L^2 estimate*

$$||\mathcal{E}(t,0,\mathrm{D})||_{2\to2}\lesssim\left(1+\int_0^t\frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac12}$$

is valid.

Proof. It suffices to consider the zones separately. In the dissipative zone the estimate follows directly from Lemma 3.9 together with $tb(t)\to\infty$ as consequence of (B3).

Similarly, for the part of the hyperbolic zone contained in $|\xi|\ge c$ we obtain a decay of $\lambda^{-1}(t)$ from the transformation back to our original problem, cf. formulae (4.3.1) – (4.3.3).

It remains to understand the influence of the elliptic zone, of the reduced zone and the influence of hyperbolic zone for small frequencies.

The elliptic zone for small frequencies. We denote by t_ξ the upper boundary of the dissipative zone[3]. Then the multiplier $\mathcal{E}(t,0,\xi)$ can be represented as $\mathcal{E}(t,t_\xi,\xi)\mathcal{E}(t_\xi,0,\xi)$, where the first one satisfies

[3] If $c_0=0$ we set $t_\xi=0$.

the estimate of Lemma 4.19 and the second one can be estimated by Lemma 3.9. This yields for all $(t,\xi) \in Z_{ell}(c_0,\epsilon) \cap \{|\xi| \le c_0\}$

$$\begin{aligned} |\mathcal{E}(t,0,\xi)| &\lesssim |\mathcal{E}(t,t_\xi,\xi)|\,|\mathcal{E}(t_\xi,0,\xi)| \\ &\lesssim \exp\left\{-|\xi|^2\int_{t_\xi}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\} \begin{pmatrix} 1 & \frac{1}{(1+t_\xi)b(t_\xi)} \\ \frac{|\xi|}{b(t)} & \frac{|\xi|}{b(t)}\frac{1}{(1+t_\xi)b(t_\xi)} \end{pmatrix} \frac{1}{1+t_\xi} \\ &\lesssim \exp\left\{-|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right\} \begin{pmatrix} |\xi| & |\xi| \\ \frac{|\xi|^2}{b(t)} & \frac{|\xi|^2}{b(t)} \end{pmatrix}, \end{aligned}$$

where we used (B3)' to conclude $(1+t_\xi)b(t_\xi) \gtrsim 1$ and Lemma 4.18.3 to extend the above integral.

Now we distinguish between estimates for the first row and estimates for the second one. For the first row we obtain

$$|\xi| \exp\left\{-|\xi|^2\int_{t_{\xi_1}}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\} \lesssim \left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{1}{2}}$$

and therefore the desired estimate. For the second row we obtain

$$\frac{|\xi|^2}{b(t)} \exp\left\{-|\xi|^2\int_{t_{\xi_1}}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\} \lesssim \frac{1}{b(t)}\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-1},$$

which decays faster by Lemma 4.18.4. The maximum of the dominating function are taken along the lines

$$|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \sim \mathrm{const}$$

and using again Lemma 4.18.4 we see that these lines belong (at least for large values of t) to the interior of $Z_{ell}(c_0,\epsilon) \cap \{|\xi| \le c_0\}$. It remains to show, that the remaining parts of the phase space have a better behaviour. We distinguish between different cases related to the behaviour of the separating curve.

Case 1. Weak dissipation, $\gamma(t) \to 0$. In this case, for small frequencies the reduced zone and the hyperbolic zone lie on top of the elliptic one.

In the reduced zone we obtained the estimate of Lemma 4.15, which together with (4.3.4) and the above estimates yields for (t,ξ) inside this zone

$$|\mathcal{E}(t,0,\xi)| \lesssim |\xi| \exp\left\{-|\xi|^2\int_0^{t_\xi} \frac{\mathrm{d}\tau}{b(\tau)} + \int_{t_\xi}^t (\epsilon-\frac{1}{2})b(\tau)\mathrm{d}\tau\right\} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

where t_ξ denotes the upper boundary of the elliptic zone. Thus, using

$$|\xi|^2 \le (\frac{1}{2}-\epsilon)b^2(t)$$

for ϵ sufficiently small, we claim, that inside the reduced zone the multiplier satisfies the same estimates like in the elliptic zone. It remains to consider the hyperbolic zone, but there the maximum of the dominating function

$$\exp\left\{-|\xi|^2\int_0^{t_\xi} \frac{\mathrm{d}\tau}{b(\tau)} - \frac{1}{2}\int_{t_\xi}^t b(\tau)\mathrm{d}\tau\right\}$$

is taken at a point ξ independent on $t \geq t_\xi$ and, thus, it decays like $\lambda^{-1}(t)$, which is faster than the estimate from inside the elliptic zone.
Case 2. Strong dissipation, $\gamma(t) \to \infty$. In this case, the elliptic part lies on top of the hyperbolic one. It remains to consider large frequencies $|\xi|$. Inside the hyperbolic zone $\mathcal{E}(t,0,\xi)$ behaves like $\frac{1}{\lambda(t)}$, while in the reduced zone we have

$$||\mathcal{E}(t,0,\xi)|| \lesssim \exp\left\{\int_{t_{\xi_1}}^t (\epsilon - \frac{1}{2})b(\tau)\mathrm{d}\tau - \int_0^{t_{\xi_1}} \frac{1}{2}b(\tau)\mathrm{d}\tau\right\},$$

which is also decaying in t, but decaying less than in the hyperbolic part. Here t_{ξ_1} denotes the lower boundary of the reduced zone. Thus the essential supremum of our estimate of $||\mathcal{E}(t,0,\xi)||$ has to be taken inside the elliptic zone.

There it holds for large frequencies

$$\begin{aligned}
|\mathcal{E}(t,0,\xi)| &\lesssim |\mathcal{E}(t,t_{\xi_2},\xi)|\,|\mathcal{E}(t_{\xi_2},0,\xi)| \\
&\lesssim \exp\left\{-|\xi|^2 \int_{t_{\xi_2}}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\} \begin{pmatrix} 1 & 1 \\ \frac{|\xi|}{b(t)} & \frac{|\xi|}{b(t)} \end{pmatrix} \frac{1}{\lambda(t_{\xi_2})} \exp\left\{\epsilon \int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)\mathrm{d}\tau\right\} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
&\lesssim \exp\left\{-|\xi|^2 \int_{t_{\xi_2}}^t \frac{\mathrm{d}\tau}{b(\tau)} + (\epsilon - \frac{1}{2}) \int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)\mathrm{d}\tau - \frac{1}{2}\int_0^{t_{\xi_1}} b(\tau)\mathrm{d}\tau\right\} \begin{pmatrix} 1 & 1 \\ \frac{|\xi|}{b(t)} & \frac{|\xi|}{b(t)} \end{pmatrix}
\end{aligned}$$

with t_{ξ_2} the upper boundary of the reduced zone. Thus, using

$$\exp\left\{-|\xi|^2 \int_{t_{\xi_2}}^t \frac{\mathrm{d}\tau}{b(\tau)} + (\epsilon - \frac{1}{2}) \int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)\mathrm{d}\tau - \frac{1}{2}\int_0^{t_{\xi_1}} b(\tau)\mathrm{d}\tau\right\} \leq \exp\left\{-c_0^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}$$

for $c_0 < \frac{\sqrt{2}}{2}b(0)$ and ϵ sufficiently small we see that the maximum of the first row is taken for large t inside $|\xi| \leq c_0$.
Case $\gamma(t)$ tends to a finite limit. In this case essentially the same kinds of estimates are used to conclude that the essential supremum of the multiplier is taken inside the elliptic zone and therefore the resulting estimate follows. ☐

By the aid of Proposition 3.21 we conclude

Corollary 4.22. *Assumptions (B1) – (B3) and (B4)$_2$ imply*

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2} \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{1}{2}}.$$

Examples. We continue this section with examples for special coefficient functions.

Example 4.6. Let

$$b(t) = \mu(1+t)^\kappa, \qquad \mu > 0, \kappa \in (-1,1).$$

For $\kappa = 0$ the energy estimates are given by A. Matsumura, [Mat76]. We obtain for all $\kappa \in (-1,1)$

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2} \lesssim (1+t)^{\frac{\kappa-1}{2}}.$$

This estimate is slightly better than the estimate given by H. Uesaka, [Ues80].

Example 4.7. If we set $b(t)=\frac{\mu}{1+t}$ with $\mu\geq 2$ we have from the consideration of Chapter 2

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2}\lesssim (1+t)^{-1},$$

which coincides formally with the estimate from Example 4.6.

Example 4.8. Let

$$b(t)=\frac{\log^{[m]}(e^{[m]}+t)}{1+t},\qquad m\geq 1.$$

Then Assumptions (B1) – (B4) are satisfied and Corollary 4.22 implies from

$$\int_0^t\frac{\mathrm{d}\tau}{b(\tau)}\sim\frac{(1+t)^2}{\log^{[m]}(e^{[m]}+t)}$$

as $t\to\infty$ the estimate

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2}\lesssim (1+t)^{-1}\big(\log^{[m]}(e^{[m]}+t)\big)^{\frac{1}{2}},$$

which comes close to the one of Example 4.7.

Example 4.9. Let $b(t)=\mu t$. Then Corollary 4.22 implies

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2}\lesssim\big(\log(e+t)\big)^{-\frac{1}{2}}.$$

Example 4.10. The previous example can be improved in the following way. For

$$b(t)=\mu(e^{[m]}+t)\log(e^{[m]}+t)\dots\log^{[m-1]}(e^{[m]}+t)$$

it holds

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2}\lesssim\big(\log^{[m]}(e^{[m]}+t)\big)^{-\frac{1}{2}}.$$

This can be understood as counterpart to the Example 3.5. Again the energy decay rate becomes arbitrary small (within the scale of iterated logarithms).

If we make the further assumption,

(B5) $\int\limits_0^\infty\frac{\mathrm{d}t}{b(t)}=\infty,$

the estimate of Corollary 4.22 implies the decay of the energy to zero.

Corollary 4.23. *Assume (B1) – (B3), (B4)$_2$ and (B5). Then*

$$\lim_{t\to\infty}\mathbb{E}(t,\mathrm{D})=0$$

in $L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$.

Example 4.11. Assume $\frac{1}{b(t)}\in L^1(\mathbb{R}_+)$. Then Corollary 4.22 gives only the obvious estimate

$$||\mathbb{E}(t,\mathrm{D})||_{2\to 2}\lesssim 1,$$

following from (B1) alone. We will investigate this case in Section 4.4 and show that this estimate is sharp. This situation will be called *over-damping*.

4.3.3 L^p–L^q estimates

We start with an auxiliary lemma.

Lemma 4.24. *Under Assumption (B1) it holds*

$$\left|\left| \, |\xi|^\ell \, \exp\left\{ -|\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right\} \right|\right|_p \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right)^{-\frac{\ell}{2}-\frac{n}{2p}}.$$

Proof. The proof follows by direct calculation. Then it holds

$$\begin{aligned}
\left|\left| \, |\xi|^\ell \, \exp\left\{ -|\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right\} \right|\right|_p &= \int_0^\infty \left(|\xi|^\ell \exp\left\{ -|\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right\} \right)^p |\xi|^{n-1} \mathrm{d}|\xi| \\
&= \int_0^\infty |\xi|^{p\ell+n-1} \exp\left\{ -p|\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right\} \mathrm{d}|\xi| \\
&= \frac{1}{2} \int_0^\infty \eta^{\frac{n+\ell p}{2}-1} e^{-p\eta} \mathrm{d}\eta \left(\int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right)^{\frac{n+\ell p}{2}}
\end{aligned}$$

by setting $\eta = |\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}$ with $\mathrm{d}\eta = 2|\xi| \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \mathrm{d}|\xi|$. ☐

Contrary to the non-effective case, we did not perform more diagonalization steps to obtain L^p–L^q decay estimates. This is related to the fact that the asymptotic properties are mainly described by the elliptic part Π_{ell} while the hyperbolic part Π_{hyp} brings in the factor $\lambda^{-1}(t)$. Instead we estimate the L^1-norm of the Fourier multiplier to deduce the L^1–L^∞ estimate and interpolation with the previously proven L^2–L^2 results yields the following statement.

Theorem 4.25. *Assume (B1) – (B3) and (B4)$_2$. Then for dual indices $p \in [1,2]$, $pq = p + q$ and with regularity $r > n\left(\frac{1}{p} - \frac{1}{q}\right)$ the estimate*

$$||\mathcal{E}(t,0,\mathrm{D})||_{p,r\to q} \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

holds.

Proof. It suffices to prove the corresponding statement for $p = 1$ and $q = \infty$. Then the general statement follows by the Riesz-Thorin interpolation theorem for L^p-spaces, [SW71], together with Theorem 4.21. Let therefore $r > n$. Then an estimate of the L^1–L^∞ decay rate is given by the L^1-norm of the multiplier, i.e. by the function

$$|| \langle\xi\rangle^{-r} \, \mathcal{E}(t,0,\xi)||_1.$$

We will estimate this L^1-norm in the different zones separately. Inside the dissipative zone this yields the rate $(1+t)^{-1-n}$, which is much stronger than the desired result. It remains to consider the remaining part of the phase space.
Part $\Pi_{ell} \cap \{|\xi| \le c_0\}$. In this part we have $\langle\xi\rangle \sim 1$ and all components of $\mathcal{E}(t,0,\xi)$ can be estimated by

$$|\xi| \exp\left\{ -|\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right\},$$

such that the desired L^1-norm can be estimated by Lemma 4.24 as

$$||\mathcal{E}(t,0,\xi)||_{L^1(\Pi_{ell}\cap\{|\xi|\le c_0\})} \lesssim \left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{n}{2}-\frac{1}{2}},$$

which is the desired estimate.

Part $\Pi_{hyp}\cap\{|\xi|\le c_0\}$. If $\gamma(t)\to 0$ as $t\to\infty$ the hyperbolic part lies on top of the elliptic one. There, we obtain with ξ_t the inverse function of t_ξ

$$\begin{aligned}
||\mathcal{E}(t,0,\xi)||_{L^1(\Pi_{hyp}\cap\{|\xi|\le c_0\})} &\lesssim \int_{\xi_t}^{c_0} |\xi|^n \exp\left\{-|\xi|^2\int_0^{t_\xi}\frac{\mathrm{d}\tau}{b(\tau)}\right\}\frac{\lambda(t_\xi)}{\lambda(t)}\mathrm{d}|\xi| \\
&\lesssim \int_{\xi_t}^{c_0}\left(|\xi|^2\int_0^{t_\xi}\frac{\mathrm{d}\tau}{b(\tau)}\right)^{\frac{n}{2}}\exp\left\{-|\xi|^2\int_0^{t_\xi}\frac{\mathrm{d}\tau}{b(\tau)}\right\} \\
&\qquad\qquad \left(1+\int_0^{t_\xi}\frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{n+1}{2}}\frac{\lambda(t_\xi)}{\lambda(t)}\mathrm{d}\left(|\xi|\left(\int_0^{t_\xi}\frac{\mathrm{d}\tau}{b(\tau)}\right)^{\frac{1}{2}}\right) \\
&\lesssim \left(1+\int_0^t\frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{n+1}{2}}
\end{aligned}$$

using the monotonicity of the function

$$\left(1+\int_0^t\frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{n+1}{2}}\lambda(t)$$

following from Lemma 4.18.5.

Large frequencies. For $\Pi_{hyp}\cap\{|\xi|\ge c\}$ we have the uniform decay rate $\lambda^{-1}(t)$ of the multiplier $\mathcal{E}(t,0,\xi)$, while for $\Pi_{ell}\cap\{|\xi|\ge c\}$ the multiplier decays at least like $\exp\{-c_0^2\int_{t_\xi}^t\frac{\mathrm{d}\tau}{b(\tau)}\}$. Together with $||\langle\xi\rangle^{-r}||_1\lesssim 1$ this gives a much stronger decay rate then obtained in the elliptic part for small frequencies. $\square$

Remark 4.12. Further diagonalization steps in the hyperbolic zone can only be used to improve the estimate in the used regularity for $p\in(1,2]$ from $r>n\left(\frac{1}{p}-\frac{1}{q}\right)$ to $r=n\left(\frac{1}{p}-\frac{1}{q}\right)$. Note that for $p=1$ the given regularity is sharp within the scale of Bessel potential spaces.

Corollary 4.26. *Under the same assumptions as in Theorem 4.25 it holds*

$$||\mathbb{E}(t,\mathrm{D})||_{p,r\to q}\lesssim\left(1+\int_0^t\frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

Examples. We review the examples of Section 4.3.2 and give the corresponding L^p–L^q estimates. Except for the case of constant dissipation, where estimates are given by A. Matsumura, [Mat76], these estimates are new.

Example 4.13. Let

$$b(t)=\mu(1+t)^\kappa,\qquad\qquad \mu>0.$$

Then we obtain for all $\kappa\in(-1,1)$

$$||\mathbb{E}(t,\mathrm{D})||_{p,r\to q}\lesssim(1+t)^{(\kappa-1)\left(\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{2}\right)},$$

under the conditions on p, q and r from Theorem 4.25.

Example 4.14. If we set $b(t) = \frac{\mu}{1+t}$ with $\mu \geq n+3$ we have from the consideration of Chapter 2

$$||\mathbb{E}(t,\mathrm{D})||_{p,r\to q} \lesssim (1+t)^{-n\left(\frac{1}{p}-\frac{1}{p}\right)-1},$$

which coincides formally with the estimate from Example 4.13.

Example 4.15. Let

$$b(t) = \frac{\log^{[m]}(e^{[m]}+t)}{1+t}, \qquad m \geq 1.$$

Then we get

$$||\mathbb{E}(t,\mathrm{D})||_{p,r\to q} \lesssim (1+t)^{-n\left(\frac{1}{p}-\frac{1}{p}\right)-1}\big(\log^{[m]}(e^{[m]}+t)\big)^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{p}\right)+\frac{1}{2}}.$$

Example 4.16. Let $b(t) = \mu t$. Then it holds

$$||\mathbb{E}(t,\mathrm{D})||_{p,r\to q} \lesssim \big(\log(e+t)\big)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{p}\right)-\frac{1}{2}}.$$

Example 4.17. The previous example can be improved in the following way. For

$$b(t) = \mu(e^{[m]}+t)\log(e^{[m]}+t)\dots\log^{[m-1]}(e^{[m]}+t)$$

it holds

$$||\mathbb{E}(t,\mathrm{D})||_{p,r\to q} \lesssim \big(\log^{[m]}(e^{[m]}+t)\big)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{p}\right)-\frac{1}{2}}.$$

Example 4.18. Assume $\frac{1}{b(t)} \in L^1(\mathbb{R}_+)$. Then we get

$$||\mathbb{E}(t,\mathrm{D})||_{p,r\to q} \lesssim 1,$$

that means, we obtained *no decay* to zero at all. In Section 4.4 we will show that this estimate is sharp.

4.4 How to interpret over-damping?

We conclude this chapter on effective dissipation with further results in the special case of an increasing dissipation term. If we assume that (B5) is violated, i.e. if

(OD) the inverse of the coefficient becomes integrable,

$$\int_0^\infty \frac{\mathrm{d}\tau}{b(\tau)} < \infty,$$

the decay estimates proven in Section 4.3 trivialise in the sense, that we obtain no energy and no L^p–L^q decay to zero any more. We want to make this result more precise. The first main result is the following consequence of the representation of solutions within the elliptic part. Remark that (OD) together with the other assumptions implies (B3).

Theorem 4.27. *Assume (B1), (B2), (B4)$_2$ and (OD). Then for $(u_1,u_2) \in L^2(\mathbb{R}^n)\times H^{-1}(\mathbb{R}^n)$ the limit*

$$u(\infty,x) = \lim_{t\to\infty} u(t,x)$$

exists in $L^2(\mathbb{R}^n)$ and is different from zero for non-zero data. Furthermore, if the data is more regular, $(u_1,u_2) \in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$, then it holds

$$||u(t,\cdot)-u(\infty,\cdot)||_2 = \mathcal{O}(t/b(t)).$$

The proof of this statement is based on the representation of solutions proven in Section 4.2 together with the following estimate. We denote by t_ξ the lower boundary of the elliptic zone.

Proposition 4.28. *Assume that $\gamma^{-1}(t) \in L^1$. Then*

$$-|\xi| t_\xi \le \int_{t_\xi}^{\infty} [\langle\xi\rangle_{\gamma(t)} - \gamma(t)] \mathrm{d}t \le 0.$$

Proof. It holds

$$\gamma(t) - \frac{|\xi|^2}{\gamma(t)} \le \sqrt{\gamma^2(t) - |\xi|^2} \le \gamma(t) - \frac{|\xi|^2}{2\gamma(t)}$$

for all $|\xi| \le \gamma(t)$ and, therefore, the desired statement follows from

$$\int_t^\infty \frac{\mathrm{d}\tau}{\gamma(\tau)} = \frac{t}{\gamma(t)} + \int_t^\infty \frac{\tau\gamma'(\tau)}{\gamma(\tau)} \mathrm{d}\tau \ge \frac{t}{\gamma(t)}$$

with $\gamma'(t) > 0$ in the over-damping case. $\square$

Using this statement, we can prove that in the elliptic zone the (weighted) limit of the solution representation $\mathcal{E}_V(t,s,\xi)$ for $t \to \infty$ exists. We set $e_1^T = (1,0)$ and extract $\hat{u}(t,\xi)$ from $V = \big(\langle\xi\rangle_{\gamma(t)}\, \lambda(t)\hat{u}, \mathrm{D}_t(\lambda(t)\hat{u})\big)^T$.

Lemma 4.29. *Assume (B1),(B2),(B4)$_2$ and (OD). Then the limit*

$$S(s,\xi) = e_1^T \lim_{t\to\infty} \frac{1}{\lambda(t)\,\langle\xi\rangle_{\gamma(t)}} \mathcal{E}_V(t,s,\xi)$$

exists uniformly on compact sets in ξ and is different from zero.

Proof. The over-damping condition implies that $\gamma(t) \to \infty$ and therefore it suffices to consider the elliptic part Π_{ell} of the phase space and to use the representation of Theorem 4.11 together with the fact that the diagonalizer $N(t,\xi) \to I$ as $t \to \infty$. This yields

$$\frac{1}{\lambda(t)\,\langle\xi\rangle_{\gamma(t)}} \mathcal{E}_V(t,s,\xi) \sim \frac{1}{\lambda(s)\,\langle\xi\rangle_{\gamma(s)}} \exp\left\{ \int_s^t [\langle\xi\rangle_{\gamma(\tau)} - \gamma(\tau)] \mathrm{d}\tau \right\} \mathcal{Q}_{ell,0}(t,s,\xi).$$

Now the exponential converges by Proposition 4.28. Furthermore, Corollary 4.13 implies the convergence $\mathcal{Q}_{ell,0}(t,s,\xi) \to \mathcal{Q}_{ell,0}(\infty,s,\xi)$ and this matrix is non-zero in the first row:

The representation of $\mathcal{Q}_{ell,0}(\infty,s,\xi)$ by a Neumann series for $s \ge t_0$ following from (4.2.16) converges for $s \to \infty$ to the matrix $\mathrm{diag}(1,0)$ and, therefore, the upper left corner entry of $\mathcal{Q}_{ell,0}(\infty,s,\xi)$ is different from zero for large s. Thus, at least the first element of the row $S(s,\xi)$ is nonzero for large s. Now the obvious relation $S(s,\xi) = S(s_1,\xi)\mathcal{E}_V(s_1,s,\xi)$ implies from the invertibility of $\mathcal{E}_V(s_1,s,\xi)$ that $S(s,\xi)$ can never be zero for any choice of s and ξ. $\square$

Of special interest is the case $s = 0$. The multiplier $S(0,\xi)$ takes the Cauchy data in the form $V(0,\xi) = (\langle\xi\rangle_{\gamma(0)}\, \hat{u}_1, \hat{u}_2 - i\frac{1}{2}b(0)\hat{u}_1)^T$ and maps it to the asymptotic state $\hat{u}(\infty,\xi)$. This will be used in the following proof.

Proof. (Theorem 4.27) The first part of Theorem 4.27 follows from the observation that

$$\hat{u}(\infty,\xi) = S(0,\xi)V(0,\xi). \tag{4.4.1}$$

The convergence follows at least for data having compact support on the Fourier level, thus on a dense subset of L^2-space. Together with an a priori bound of the solution we conclude that the limit exists for all data from $L^2(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$. This a priori bound follows the same way we have proven the L^2–L^2 estimate.

It is possible to obtain a better description of this limit. From the solution representation we know that $||\hat{u}_t(t,\cdot)||_2 = \mathcal{O}(1/b(t))$, similarly we can obtain $||\hat{u}_{tt}(t,\cdot)||_2 = \mathcal{O}(1/tb(t))$. The proofs of these estimates of higher order are postponed to Chapter 5. We want to draw only one consequence from the latter one. Using the differential equation

$$\hat{u}_{tt} + |\xi|^2\hat{u} + b(t)\hat{u}_t = 0 \tag{4.4.2}$$

for all t together with the existence of the limit for the second summand, we get from $u_{tt} \to 0$ in $L^2(\mathbb{R}^n)$ under regularity assumptions on the data, $(u_1,u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, that

$$\lim_{t\to\infty} b(t)u_t(t,x) = \Delta u(\infty,x) \tag{4.4.3}$$

converges in $L^2(\mathbb{R}^n)$. Furthermore,

$$||u(\infty,\cdot) - u(t,\cdot)||_2 \le \int_t^\infty ||u_t(\tau,\cdot)||_2 \mathrm{d}\tau \le C\big(||u_1||_{H^2} + ||u_2||_{H^1}\big)\int_t^\infty \frac{\mathrm{d}\tau}{b(\tau)} = \mathcal{O}\left(t/b(t)\right), \tag{4.4.4}$$

and the second statement of Theorem 4.27 follows. □

Remark 4.19. Using the commutation properties of Fourier multipliers, we see that regularity of the initial data transfers directly to convergence in Sobolov spaces. So the proof of Theorem 4.27 implies directly a corresponding result for data $(u_1,u_2) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$. In this case $u(t,\cdot) \to u(\infty,\cdot)$ in $H^s(\mathbb{R}^n)$.

Interpretation. At least for sufficiently high regularity of the data we have seen, that the term $|\xi|^2\hat{u} + b(t)\hat{u}_t$ overrules the influence of the second time-derivative $\hat{u}_{tt}$ in the equation. Thus, it seems to be natural to consider the parabolic problem

$$w_t = \frac{1}{b(t)}\Delta w, \qquad w(0,\cdot) = w_0 \in L^2(\mathbb{R}^n), \tag{4.4.5}$$

as related differential equation describing the asymptotic behaviour. Its solution is given by

$$\hat{w}(t,\xi) = \exp\left\{-|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}\hat{w}(0,\xi). \tag{4.4.6}$$

For $t \to \infty$ it takes in $L^2(\mathbb{R}^n)$ the limit

$$w(\infty,x) = e^{\beta\Delta}w_0(x) \tag{4.4.7}$$

with $\beta = ||1/b(\cdot)||_1$. The operator $e^{\beta\Delta}$ is smoothing, it maps the space of L^2-functions into $H^\infty(\mathbb{R}^n)$.

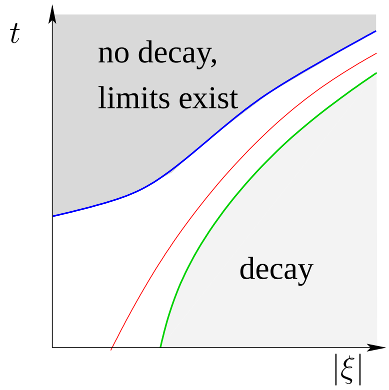

Figure 4.5: Time-asymptotic properties of the multiplier $\mathcal{E}(t,s,\xi)$ in the case of over-damping.

Regularity of the asymptotic profile. Lemma 4.29 states that the solution operator, mapping the Cauchy data to the solution at the time level t, tends strongly to an operator associating an asymptotic profile $u(\infty,x)$. This operator is represented by a Fourier multiplier and, therefore, it is natural to ask for estimates of this Fourier multiplier with respect to the frequency variable ξ. It turns out that the operator is smoothing in the following sense.

Theorem 4.30. *Assume (B1), (B2), (B4)$_2$ and (OD). Then the multiplier $S(s,\xi)$ satisfies the estimate*

$$||S(s,\xi)|| \lesssim \langle\xi\rangle^{-1}\ e^{-c|\xi|t_\xi}, \qquad |\xi|\geq\xi_s$$

uniform in s,ξ.

Proof. The representation used in the proof of Lemma 4.29 implies together with Proposition 4.28, that

$$||S(t_\xi,\xi)|| \lesssim \frac{1}{\lambda(t_\xi)\,\langle\xi\rangle},$$

and, together with the estimates coming from the hyperbolic zone, Lemma 4.4, and the reduced zone, Lemma 4.15,

$$||\mathcal{E}_V(t_\xi,s,\xi)|| \lesssim \exp\{\epsilon|\xi|t_\xi\},$$

we can use the representation $S(s,\xi)=S(t_\xi,\xi)\mathcal{E}_V(t_\xi,s,\xi)$ to conclude the desired estimate. To estimate $\lambda(t_\xi)$, we use, that by $b'(t)>0$ and Assumption (B4)$_1$

$$b(t)\leq b(t)+tb'(t)\leq cb(t)$$

and, therefore, after integration

$$\int_0^t b(\tau)\mathrm{d}\tau \sim tb(t)$$

holds. $\square$

An almost immediate consequence of this estimate for the multiplier $S(s,\xi)$ is, that the asymptotic state $u(\infty,\cdot)=S(0,\mathrm{D})V_0$ for Cauchy data $u_1\in H^1(\mathbb{R}^n)$ and $u^2\in L^2(\mathbb{R}^n)$, and thus $V_0\in L^2(\mathbb{R}^n)$, is a very smooth function.

Theorem 4.31. *Assume (B1), (B2), (B4)$_2$ and (OD). Then for data $u_1 \in H^1(\mathbb{R}^n)$ and $u^2 \in L^2(\mathbb{R}^n)$ the asymptotic state $u(\infty,\cdot)$ defined by Theorem 4.27 belongs to the space $B^\infty(\mathbb{R}^n)$ and satisfies the estimates*

$$||\mathrm{D}^\alpha u(\infty,\cdot)||_\infty \le C_1 C_2^{|\alpha|}\alpha! \tag{4.4.8}$$

for all multi-indices $\alpha \in \mathbb{N}_0^n$ and with constants independent of α.

Proof. It suffices to prove the following statement: Let $g \in L^2(\mathbb{R}^n)$ and $\hat{f}(\xi) = e^{-c|\xi|t_\xi}\hat{g}(\xi)$. Then f satisfies (4.4.8).

Using the boundedness of the Fourier transform as map $L^1 \to L^\infty$ this statement follows from

$$||\xi^\alpha \hat{f}(\xi)||_1 = \Big|\Big|\underbrace{\xi^\alpha e^{-c|\xi|}}_{\in L^\infty}\underbrace{e^{c|\xi|(1-t_\xi)}}_{||\cdot||_{L^2}\le C}\underbrace{e^{c|\xi|t_\xi}\hat{g}(\xi)}_{||\cdot||_{L^2}\le C}\Big|\Big|_1 \le C_1 C_2^{|\alpha|}\alpha!$$

by using

$$\sup_\xi \left|\xi^\alpha e^{-c|\xi|}\right| \le c^{-|\alpha|}\sup_{|\xi|}(c|\xi|)^{|\alpha|}e^{-c|\xi|} \le c^{-|\alpha|}|\alpha|^{|\alpha|}e^{-|\alpha|} \le c^{-|\alpha|}\alpha!,$$

because the maximum of $z^s e^{-z}$ for $z > 0$ is taken at the point $s = z$ for each $s > 0$ and by Stirling's formula $|\alpha|^{|\alpha|} \le \alpha! e^{|\alpha|}$. □

Corollary 4.32. *It holds $S(t,\mathrm{D}) : L^2(\mathbb{R}^n) \to A(\mathbb{R}^n)$, the asymptotic profile is a real-analytic function.*

4.5 Summary

In the case of effective dissipation, we have used a reduction to a Klein-Gordon type equation with a negative mass term to deduce the representation of solutions. We have seen especially, that

- the asymptotic behaviour is described by the elliptic zone, i.e. by small frequencies,
- related to this fact, the solutions satisfy asymptotic estimates of a type known for parabolic problems,
- the oscillatory behaviour of the multiplier inside the hyperbolic zone has no influence on decay estimates.

This relation to parabolic problems is known for wave equations with constant dissipation as pointed out in Section 1.3.2. There arises the question, whether it is possible to obtain similar results at least in the case of $\gamma(t) \not\to 0$ for $t \to \infty$. Section 5.4 is devoted to the study of this question based on the representation of solutions from Section 4.2. In Table 4.1 we sketch the relation between the zones and the used assumptions on the coefficient function.

Under the weaker assumptions (B1), (B4)$_\ell$ and (Bγ) related to a given admissible function $\gamma = \gamma(t)$ subject to (Γ1) and (Γ2) we sketched in Section 4.2.4 an approach to diagonalize and to deduce main terms of the representation of solutions.

Zones	Assumptions on $b(t)$
dissipative zone	• we used (B1), i.e. $b(t) \geq 0$, together with (B3)', $tb(t) \to \infty$, which implies condition (C2) • smoothness of $b = b(t)$ plays no role
elliptic and hyperbolic zone	• Assumption (B2) is used to define the separating curve • Assumptions (B1), (B3) are used to replace $m(t,\xi)$ by $\langle\xi\rangle^2_{\gamma(t)}$ inside these zones • Assumption (B4)$_\ell$ allows as many diagonalization steps as we want • the sign of the coefficient $b = b(t)$ is essential inside the elliptic zone
reduced zone	• (B1) and (B2) are used to define this zone • (B3) is used to make the mass-term small in this zone

Table 4.1: Assumptions used in the different zones/parts.

Modified scattering for (effective) weak dissipation. We want to conclude this chapter with some remarks concerning the asymptotic properties of the solution in the hyperbolic zone for weak dissipation.

If the dissipation term is non-effective we have seen from Theorem 3.17 and in consequence from the modified scattering theory of Section 3.5 that inside the hyperbolic zone $Z_{hyp}(N)$ and for $t \to \infty$ the fundamental solution $\mathcal{E}(t,s,\xi)$ behaves like

$$\frac{\lambda(s)}{\lambda(t)} M \mathcal{E}_0(t,s,\xi) M^{-1} \qquad \text{with} \qquad \mathcal{E}_0(t,s,\xi) = \operatorname{diag}\left(e^{i(t-s)|\xi|}, e^{i(s-t)|\xi|}\right). \tag{4.5.1}$$

$\mathcal{E}_0(t,s,\xi)$ coming from the propagation of free waves. Now in the case of effective weak dissipation, Corollary 4.7 in combination with (4.3.2) yields for $\mathcal{E}(t,s,\xi)$ the related behaviour like

$$\frac{\lambda(s)}{\lambda(t)} M \tilde{\mathcal{E}}(t,s,\xi) M^{-1}, \tag{4.5.2}$$

where the unitary matrix $\tilde{\mathcal{E}}(t,s,\xi)$ is given by (4.2.5). This modified hyperbolic behaviour is related to the representation for large frequencies given by T. Narazaki in [Nar04] for damped wave equations and also by K. Nishihara in [Nis03] in the special case of three-dimensional space.

If we restrict our considerations to *large frequencies*, i.e. if we consider data (u_1, u_2) with $0 \notin \operatorname{supp}(\hat{u}_1, \hat{u}_2)$ we obtain for the hyperbolic energy the two-sided estimate

$$E(u;t) \sim \frac{1}{\lambda^2(t)}, \tag{4.5.3}$$

which is much stronger than the decay rate obtained for arbitrary data. In general, this modified scattering behaviour is overruled by the influence of the elliptic part.

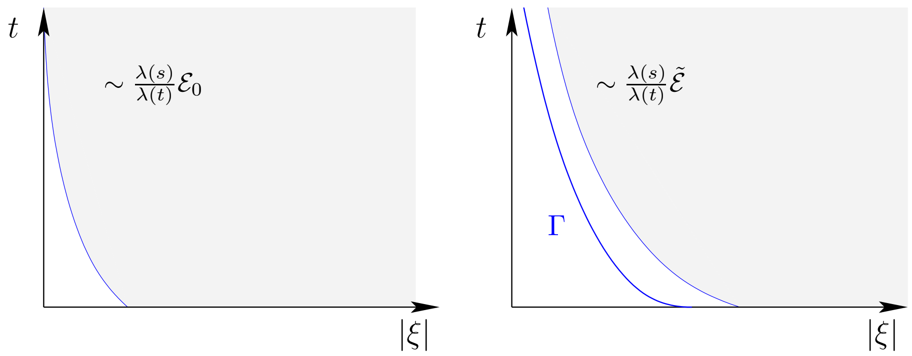

Figure 4.6: Modified scattering theory for non-effective dissipation (left) and a related description for effective weak dissipation (right).

Example 4.20. If we consider $b(t) = \mu(1+t)^{-\kappa}$ with $\kappa \in (0,1)$ and $\mu > 0$, this yields for data $u_1 \in H^1$ and $u_2 \in L^2$ with $0 \notin \operatorname{supp}(\hat{u}_1, \hat{u}_2)$ the estimate for the hyperbolic energy

$$E(u;t) \sim \exp\left(\frac{\mu}{1-\kappa} t^{1-\kappa}\right),$$

which is closely related to Example 3.10.

5 Further results

5.1 New directions

For non-effective weak dissipation the modified scattering theory of Theorem 3.26 implies, that for all non-zero data the energy decay rate is given by the auxiliary function $\lambda(t)$,

$$E(u;t) \sim \frac{1}{\lambda^2(t)}.$$

In the case of effective dissipation such a result is not valid. There exists a relation between further conditions on the data and corresponding improved energy decay rates. The aim of this section is to give an outline of several results in this direction related to our representations of Chapter 4. We restrict ourselves to the estimates in L^2 scale, the proofs can be generalized to the case $L_{p,r}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, where we need $q < \infty$ for the improved results in the strong topology.

5.1.1 Norm estimates versus estimates in the strong topology

The estimates proven so far are estimates for the norm of the so-called energy operator. As already pointed out in Remark 2.2 of the second chapter F. Hirosawa and H. Nakazawa recently obtained a faster decay of the energy. In order to understand this estimate better we replace the estimate in the norm-topology by a convergence in the strong operator topology. Similar to the proof of Theorem 3.26, the key tool to understand such estimates is the theorem of Banach-Steinhaus. Condition (B5) is essential for this result.

Theorem 5.1. *Assume (B1) – (B3), (B4)$_2$ and (B5). Then the strong limit*

$$\operatorname*{s-lim}_{t\to\infty} \sqrt{\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}}\, \mathbb{E}(t,\mathrm{D}) = 0$$

is taken in $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Proof. We employ the Banach-Steinhaus argument for the dense subspace $M = \bigcup_{c>0} V_c$ of $L^2(\mathbb{R}^n)$, where

$$V_c = \{U \in L^2(\mathbb{R}^n) |\, \mathrm{dist}(0, \operatorname{supp} \hat{U}) \geq c\}.$$

The energy estimate implies the uniform bound

$$\sqrt{\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}}\, \mathbb{E}(t,\mathrm{D}) \lesssim 1.$$

On the other hand, for data $(u_1, u_2)^T \in V_c$ the representation of $\mathbb{E}(t,\xi)$ obtained in Chapter 4 shows that

Case 1: if $b(t) \to 0$ as $t \to \infty$ or $\operatorname{supp}(u_1, u_2)$ contains only frequencies ending up in the hyperbolic part for $b(t) \le c$, the energy satisfies the estimate (cf. formula (4.5.3))

$$||\mathbb{E}(t, \mathrm{D})(u_1, u_2)||_2 \lesssim \frac{1}{\lambda(t)},$$

tending to zero faster than the above given estimate, and
Case 2: if $b(t) \not\to 0$ and $\operatorname{supp}(\hat{u}_1, \hat{u}_2)$ contains frequencies which 'end up' in the elliptic part, we use that the dominating function (cf. Corollary4.20)

$$|\xi| \exp\left\{ -|\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right\}$$

can be estimated by

$$\exp\left\{ -c^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \right\}$$

on V_c. This estimate again decays faster.

Thus, in both cases the theorem of Banach-Steinhaus implies the desired result. □

Example 5.1. We sketch one application of Theorem 5.1. For the special coefficient function $b(t) = \mu(1+t)^\kappa$ with exponent $\kappa \in (-1, 1)$ and $\mu > 0$ we obtain by Corollary 4.22

$$E(u; t) = \mathcal{O}(t^{\kappa-1}), \qquad t \to \infty,$$

while now we have

$$E(u; t) = o(t^{\kappa-1}), \qquad t \to \infty.$$

The main difference is, that the last estimate is not uniform in the data.

Remark 5.2. The theorem of Banach-Steinhaus implies also, that both estimates, the norm-estimate and this estimate in the strong operator topology, are sharp at the same time.

Remark 5.3. In [HN03] F. Hirosawa and H. Nakazawa obtained this result for the special case $b(t) = \mu(1+t)^\kappa$ with $\mu > 2$ for the range $\kappa \in [-1, -\frac{1}{2})$. The above theorem extends this result to $\mu > 0$ for $\kappa \in (-1, 1]$.

5.1.2 Exceptional behaviour of the frequency $\xi = 0$

The reason for the improvement of the decay rate in Theorem 5.1 is the exceptional behaviour of the frequency $\xi = 0$. Under Assumption (B5) the L^∞-norm of the multiplier $\mathcal{E}(t, 0, \xi)$ is determined by the neighbourhood of the line

$$|\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} = \mathrm{const}\,. \tag{5.1.1}$$

This curve approaches the t-axis as $t \to \infty$ if (B5) holds. If we assume, that the data can be estimated in their Fourier image for small frequencies by $|\xi|^\mu$ with some $\mu > 0$, this implies a further decay along this line and therefore improves the estimates.

Our strategy is as follows. We do not consider the energy operator $\mathbb{E}(t, \mathrm{D})$ as operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ but as operator from a *smaller space with stronger topology* to $L^2(\mathbb{R}^n)$. For $\mu \ge 0$ this space is given by

$$[\mathrm{D}]^\mu L^2(\mathbb{R}^n) = \{\ u \in L^2(\mathbb{R}^n) \mid [\xi]^{-\mu} \hat{u} \in L^2(\mathbb{R}^n)\ \} \tag{5.1.2}$$

with $[\xi] = |\xi| / \langle \xi \rangle$ and endowed with the induced norm

$$||u||_{[\mathrm{D}]^\mu L^2} = ||[\xi]^{-\mu} \hat{u}||_2. \tag{5.1.3}$$

Note, that by Plancherel's theorem $[\mathrm{D}]^0 L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

Theorem 5.2. *Assume (B1) – (B3), (B4)$_2$ together with (B5). Then*

$$\operatorname*{s-lim}_{t\to\infty} \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\frac{\mu+1}{2}} \mathbb{E}(t, \mathrm{D}) = 0.$$

holds as strong limit in $[\mathrm{D}]^\mu L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$*, furthermore,*

$$||\mathbb{E}(t, \mathrm{D})||_{[\mathrm{D}]^\mu L^2 \to L^2} \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{\mu+1}{2}}.$$

Proof. (Sketch) The proof follows essentially the proof of the L^2-estimate, Theorem 4.21, and the improvement from Theorem 5.1, taking into account, that we can use a further factor $|\xi|^\mu$ in the multiplier. ☐

This estimate is the basic tool for several improvement results. The condition $[\xi]^{-\mu} \hat{u} \in L^2(\mathbb{R}^n)$ does not mean a zero behaviour in the usual sense, it can be understood as a *regularity* statement of $\hat{u}$ in $\xi = 0$. This is related to the following version of Sobolev-Hardy inequality, which is by itself a consequence of a result of R.S. Strichartz, [Str67, Theorem 3.6].

Lemma 5.3. *Assume* $0 \le s < \frac{n}{2}$*. Then*

$$||\, |x|^{-s} f||_2 \le C ||f||_{H^s}$$

for all $f \in H^s(\mathbb{R}^n)$.

Applying this lemma in the Fourier image yields a continuous embedding of the weighted L^2-space $\langle x \rangle^{-s} L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n, \langle x \rangle^{2s}\, \mathrm{d}x)$, whose Fourier image is $H^s(\mathbb{R}^n)$, into $[\mathrm{D}]^s L^2(\mathbb{R}^n)$. Indeed we have with a smooth cut-off function $\chi \in C_0^\infty$, $\chi(\xi) = 1$ in a neighbourhood of $\xi = 0$,

$$\begin{aligned} ||f||_{[\mathrm{D}]^s L^2} &= ||[\xi]^s \hat{f}||_2 \sim ||\, |\xi|^s \chi(\xi) \hat{f}||_2 + ||(1 - \chi(\xi))\hat{f}||_2 \\ &\lesssim ||\chi(\xi)\hat{f}||_{H^s} + ||\hat{f}||_2 \le ||\hat{f}||_{H^s} = ||f||_{\langle x \rangle^{-s} L^2}. \end{aligned}$$

Thus, Theorem 5.2 implies

Corollary 5.4. *Assume (B1) – (B3), (B4)$_2$ and (B5).*
Then for data $(\langle \mathrm{D} \rangle u_1, u_2) \in \langle x \rangle^{-s} L^2(\mathbb{R}^n)$, $0 \le s < \frac{n}{2}$*, it follows*

$$||\mathbb{E}(t, \mathrm{D})||_{\langle x \rangle^{-s} L^2 \to L^2} \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{1+s}{2}}$$

and, furthermore, the improved estimate

$$\operatorname*{s-lim}_{t\to\infty} \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\frac{1+s}{2}} \mathbb{E}(t, \mathrm{D}) = 0$$

is taken in $\langle x \rangle^{-s} L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Example 5.4. Set $b(t) = (1+t)^\kappa$, $\kappa \in (-1,1)$. Then for data $(\langle D \rangle u_1, u_2)$ from $\langle x \rangle^s L^2(\mathbb{R}^n)$ with $0 \le s < \frac{n}{2}$ we have

$$E(u;t) = o(t^{(\kappa-1)(s+1)}), \qquad t \to \infty.$$

This result is closely related to a result of R. Ikehata and A. Saeki, [SI00, Theorem 1.3]. They proved for $b(t) = 1$ and $s = 1$ the energy estimate

$$E(u;t) \le C/(1+t)^2$$

with a constant C depending on $||u_1||_{H^1}$, $||u_2||_2$ and $|| \langle x \rangle (u_1 + u_2)||_2$ and for space dimension $n > 2$. For $n = 2$, which is also critical in our approach, they used the weighted space $\langle x \rangle^{-1} \log(1 + \langle x \rangle)^{-1} L^2(\mathbb{R}^2)$ to recover the same estimate. The log-term occurs from a corresponding logarithmic Sobolev-Hardy inequality valid in this case, see e.g. W. Dan and Y. Shibata, [DS95].

Remark 5.5. Corollary 5.4 shows, that weight conditions on the data may be used to improve the decay rate for the hyperbolic energy. The possible improvement is limited by the space dimension. This can be understood by the relation to Theorem 5.2 and the application of Sobolev-Hardy inequality in the Fourier image. If we assume more decay for the data, we have continuity in the Fourier image and we need further moment conditions on the data to ensure a zero behaviour in the frequency $\xi = 0$ of sufficiently high order.

Remark 5.6. In the case of the damped wave equation R. Ikehata obtained in [Ike03d] and [Ike03c] for data from $L^2(\mathbb{R}^n) \cap \langle x \rangle^\mu L^1(\mathbb{R}^n)$ with large μ improved decay rates assuming further moment conditions.

5.1.3 Data from $H^s \cap L^p$, $p \in [1,2)$.

Like in the classical paper of A. Matsumura, [Mat76], the assumption of a further L^p-regularity, $p \in [1,2)$, for the data allows an improvement of the energy decay rate. Lemma 4.24 can be used to obtain corresponding estimates within the elliptic part. It follows

Theorem 5.5. *Assume (B1) – (B3), (B4)$_2$ and (B5). Then for data* $(\langle \mathrm{D} \rangle u_1, u_2) \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $p \in [1,2]$, *it follows*

$$||\mathbb{E}(t,\mathrm{D})||_{L^2 \cap L^p \to L^2} \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}}.$$

Furthermore, for $p \in (1,2]$ *the decay rate can be improved to*

$$\operatorname*{s-lim}_{t\to\infty} \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\frac{1}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \mathbb{E}(t,\mathrm{D}) = 0$$

in $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Remark 5.7. For $p = 1$ the subspace $M = \bigcup_{c>0} V_c$ is not dense in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. This is the reason to exclude this case in the second statement. In this case, the improved decay rate follows for the closed subspace of data with vanishing mean, which is the closure of M in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Remark 5.8. Under certain geometric conditions this vanishing mean condition follows immediately. If we consider the initial boundary value problem on the half-space with Dirichlet boundary conditions and data from $L^1(\mathbb{R}^n_+) \cap L^2(\mathbb{R}^n_+)$, the usual odd continuation to a Cauchy problem satisfies this moment condition. In the case $b(t) = 1$ this was used by R. Ikehata in [Ike03a], [Ike03b] and [Ike04].

Example 5.9. If we set $b(t) = 1$ and $p = 1$ we get for data $(\langle \mathrm{D} \rangle u_1, u_2) \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ the energy decay rate

$$E(u;t) = \mathcal{O}(t^{-\frac{n}{2}-1}), \qquad t \to \infty,$$

like in [Mat76].

Example 5.10. More generally, for $b(t) = (1+t)^\kappa$ with $\kappa \in (-1,1)$ we get for $(\langle \mathrm{D} \rangle u_1, u_2) \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ the energy decay rate

$$E(u;t) = \mathcal{O}(t^{-(1-\kappa)\frac{n+2}{2}}), \qquad t \to \infty.$$

Example 5.11. For $b(t) = 1 + t$ we get under the same assumptions on the data

$$E(u;t) = \mathcal{O}((\log(e+t))^{-\frac{n+2}{2}}), \qquad t \to \infty.$$

5.2 Estimates for the solution itself

The estimates for the solution follow basically from the proven estimates for the micro-energy $U(t,\xi) = (h(t,\xi)\hat{u}, \mathrm{D}_t \hat{u})$. In both cases, for weak dissipation in Chapter 3 and for effective dissipation in Chapter 4 we have constructed explicit Fourier multiplier representations. Using

$$\frac{1}{h(t,\xi)} \lesssim 1 + t,$$

one can recover estimates for u from energy estimates, one can even obtain better estimates if one uses the representation of the solutions in the different zones.

Similar to the treatment in Chapter 2 we consider the *solution operator*

$$\mathbb{S}(t,\mathrm{D}) \ : \ (u_1, \langle D \rangle^{-1} u_2) \mapsto u(t,\cdot),$$

normalised in such a way, that $\mathbb{S}(t,\mathrm{D}) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and formulate estimates for its asymptotic behaviour.

5.2.1 Remarks on free waves

For the sake of completeness we start with the case of free waves, $\Box u = 0$. It is well-known that in this case the solution to the Cauchy problem can be represented as

$$\hat{u}(t,\xi) = \cos(t|\xi|)\hat{u}_1 + it \operatorname{sinc}(t|\xi|)\hat{u}_2,$$

where $\operatorname{sinc} x = \sin x / x$. Using the boundedness of $\operatorname{sinc} x$ one obtains an increasing behaviour of the solution. It can also be seen, that the two multipliers constituting $\mathbb{S}(t,\xi)$ behave differently.

Lemma 5.6. *For the case of free waves the limit*

$$\operatorname*{s-lim}_{t\to\infty} (1+t)^{-1} \mathbb{S}(t,\mathrm{D}) = 0$$

exists as strong limit in $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Proof. The uniform bound $||\mathbb{S}(t,\mathrm{D})||_{2\to 2} \lesssim 1+t$ can be seen immediately from the representation of u. To obtain the existence of the limit we again employ the Banach-Steinhaus argument for the dense subset $M = \bigcup_{c>0} V_c$, $V_c = \{\operatorname{dist}(0, \operatorname{supp} U) \geq c\}$. For each of the subspaces V_c we obtain for the second multiplier

$$\operatorname{sinc}(t|\xi|) \leq \langle t|\xi|\rangle^{-1} \to 0, \qquad t\to\infty$$

uniform on $|\xi| \geq c$. □

Remark 5.12. On the smaller space V_c we have even more. If $0 \notin \operatorname{supp}(\hat{u}_1, \hat{u}_2)$, we get boundedness of the solution, i.e.

$$||u(t,\cdot)||_2 \lesssim 1,$$

where occurring constants depend on the norm of the data and the distance of 0 to the Fourier support of them.

5.2.2 Non-effective weak dissipation

The representation of Chapter 3, Lemma 3.8 and Theorem 3.15, implies immediately the following theorem.

Theorem 5.7. *Assume (A1), (A4)$_1$ and (C1). Then the solution operator satisfies the L^2–L^2 estimate*

$$||\mathbb{S}(t,\mathrm{D})||_{2\to 2} \lesssim \frac{1+t}{\lambda^2(t)}.$$

Proof. In the hyperbolic zone and for $|\xi| \geq c$ we estimated $|\xi|\hat{u}$ by $\lambda^{-1}(t)$ and we can just divide by $|\xi|$ to get an estimate for u by

$$\frac{1}{\lambda(t)} \lesssim \frac{1+t}{\lambda^2(t)}.$$

For small $|\xi|$ we have to take into account that dividing by $h(t,\xi)$ brings a further factor $(1+t)$. Thus we obtain inside the dissipative zone $(1+t)/\lambda^2(t)$ and in the hyperbolic zone for small frequencies

$$\frac{1}{|\xi|\lambda(t)\lambda(t_\xi)} \sim \frac{1+t_\xi}{\lambda(t)\lambda(t_\xi)} \lesssim \frac{1+t}{\lambda^2(t)},$$

using the monotonicity of $t/\lambda(t)$ for large t following from condition (C1). □

This result coincides for the case $b(t) = \frac{\mu}{1+t}$, $\mu < 1$, with the estimate from Theorem 2.7. The following observation is important to understand the essential difference between the solution estimate and the estimate for the energy.

> *In the non-effective dissipative case the estimate for the solution operator $\mathbb{S}(t,\mathrm{D})$ comes from properties of the dissipative zone.*

This coincides with the L^2–L^2 estimate of Theorem 2.7.1 and the case of effective dissipation. Note, that energy estimates depend on large frequencies.

We can improve the decay rate of the above theorem. For $|\xi| > c$ we end up in the hyperbolic zone and, thus, we can estimate the multiplier uniformly in $|\xi| > c$ by

$$\frac{\lambda^2(t)}{1+t}||\mathbb{S}(t,\xi)|| \lesssim \frac{\lambda(t)}{1+t} \lesssim \frac{1}{\lambda(t)} \to 0$$

using Proposition 3.7 for $\lambda(t)\to\infty$. Otherwise, if $\lambda(t)$ remains bounded, $\lambda(t)(1+t)^{-1} \to 0$.

Corollary 5.8. *In $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ it holds*

$$\operatorname*{s-lim}_{t\to\infty} \frac{\lambda^2(t)}{1+t} \mathbb{S}(t, \mathrm{D}) = 0.$$

Example 5.13. If $b(t) \in L^1(\mathbb{R}_+)$ this corollary implies

$$\operatorname*{s-lim}_{t\to\infty} (1+t)^{-1} \mathbb{S}(t, \mathrm{D}) = 0,$$

like in the case of free waves.

Example 5.14. If we set

$$b(t) = \frac{\mu}{(e^{[m]}+t)\log(e^{[m]}+t)\cdots\log^{[m]}(e^{[m]}+t)}$$

like in Example 3.5, we get

$$\operatorname*{s-lim}_{t\to\infty} (1+t)^{-1}\big(\log^{[m]}(e^{[m]}+t)\big)^{\mu} \mathbb{S}(t, \mathrm{D}) = 0.$$

Example 5.15. For $b(t) = \frac{\mu}{1+t}$ and with $\mu \in (0,1)$ we obtain

$$\operatorname*{s-lim}_{t\to\infty} (1+t)^{\mu-1} \mathbb{S}(t, \mathrm{D}) = 0,$$

related to the statement of Theorem 2.7.1.

A review of the proof of Theorem 3.24 together with the structure of the multiplier $\mathbb{S}(t,\xi)$ in the zones yield L^p–L^q decay estimates for the solution operator. Note that, similar to the treatment of Chapter 2, for p and q near to 2 the dissipative zone determines the decay rate.

Theorem 5.9. *Assume (A1), (A4) together with (C1). Then the solution operator $\mathbb{S}(t,\mathrm{D})$ satisfies the L^p–L^q estimate*

$$||\mathbb{S}(t,\mathrm{D})||_{p,r\to q} \lesssim \begin{cases} \frac{1}{\lambda^2(t)}(1+t)^{1-n\left(\frac{1}{p}-\frac{1}{q}\right)}, & p \ge p^*, \\ \frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, & p < p^*, \end{cases}$$

for dual indices $p \in (1,2]$, $pq = p+q$ and with regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$. The critical value p^* is chosen in such a way that

$$(1+t)^{1-\frac{n+1}{2}\left(\frac{2}{p^*}-1\right)} \gtrsim \lambda(t).$$

Remark 5.16. For $b(t) = \frac{\mu}{1+t}$ with $\mu \in [0,1)$, this estimate coincides with the corresponding one from Theorem 2.7.

Remark 5.17. In case that $tb(t) \to 0$, the critical value p^* is given by $p^* = \frac{2n+2}{n+3}$.

5.2.3 Effective dissipation

Also in the case of effective dissipation the representation of solutions from Chapter 4, Theorems 4.6, 4.11 and 4.15, together with the definition of the micro-energy, implies estimates for the solution itself. Without taking time or spatial derivatives, we have nothing to cancel the non-decay of the factor

$$\exp\left\{-|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}$$

arising from the representation of Lemma 4.11. On the other hand, this factor decays for non-integrable $b^{-1}(t)$ uniformly on $|\xi| \ge c > 0$. Thus, together with Theorem 4.27 one obtains the following statement.

Theorem 5.10. *Assume (B1) – (B3), (B4)$_2$. Then the estimate*

$$||\mathbb{S}(t,\mathrm{D})||_{2\to 2} \lesssim 1$$

holds uniformly in t. Furthermore,

$$\operatorname*{s-lim}_{t\to\infty} \mathbb{S}(t,\mathrm{D}) = \mathbb{S}(\infty,\mathrm{D})$$

exists on $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\mathbb{S}(\infty,\mathrm{D}) = 0$ iff (B5) holds.

Furthermore, a review of the proof of Theorem 4.25 yields for $\mathbb{S}(t,\mathrm{D})$ a corresponding L^p–L^q result.

Theorem 5.11. *Assume (B1) – (B3), (B4)$_2$. Then the L^p–L^q estimate*

$$||\mathbb{S}(t,\mathrm{D})||_{p,r\to q} \lesssim \left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

holds for dual indices $p \in [1,2]$, $pq = p+q$ and with regularity $r > n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Example 5.18. If we consider the special case $b(t) = (1+t)^\kappa$ with $\kappa \in (-1,1)$, we get for the solution $u = u(t,x)$ to (1.2.1) the decay estimate

$$||u(t,x)||_q \leq C(1+t)^{-(1-\kappa)\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(||u_1||_{L_{p,r}} + ||u_2||_{L_{p,r-1}}\big).$$

Example 5.19. If we set $b(t) = 1+t$, we obtain the estimate

$$||u(t,x)||_q \leq C\big(\log(e+t)\big)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\big(||u_1||_{L_{p,r}} + ||u_2||_{L_{p,r-1}}\big).$$

Example 5.20. For $b(t) = \frac{\mu}{1+t}$ with sufficiently large μ we get from the consideration of Chapter 2 the corresponding decay results. Furthermore, the stronger decay in Z_1 (related to estimates in the subspace V_c) implies, that for all $\mu > 1$ the statement

$$\operatorname*{s-lim}_{t\to\infty} \mathbb{S}(t,\mathrm{D}) = 0$$

is also valid. For $\mu = 1$ the logarithmic term in the estimate of Theorem 2.7.1. comes from the treatment of the zones Z_2 and Z_3 and, therefore, we obtain

$$\operatorname*{s-lim}_{t\to\infty} \log(e+t)\, \mathbb{S}(t,\mathrm{D}) = 0$$

in this case.

Like the corresponding estimate for the energy this result may be improved under further assumptions on the data. We restrict our consideration to a zero-behaviour for $\xi = 0$ in $L^2(\mathbb{R}^n)$ and obtain (similar to Theorem 5.2) the following result. Note, that Assumption (B5) guarantees, that the maximum of the multiplier is taken on a line approaching $\xi = 0$.

Theorem 5.12. *Assume (B1) – (B3), (B4)$_2$ and (B5). Then for $\mu \geq 0$ the strong limit*

$$\operatorname*{s-lim}_{t\to\infty} \left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\mu} \mathbb{S}(t,\mathrm{D}) = 0$$

is taken in $[\mathrm{D}]^{2\mu} L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Now the argument of Section 5.1.2 transfers this result to weighted initial data and data satisfying further moment conditions.

Example 5.21. For $b(t) = (1+t)^\kappa$ with $\kappa \in (-1,1)$ we obtained

$$||\mathbb{S}(t,\mathrm{D})||_{[\mathrm{D}]^\mu L^2 \to L^2} \lesssim (1+t)^{\frac{1-\kappa}{2}\mu}.$$

Example 5.22. For $b(t) = 1+t$ the corresponding estimate reads as

$$||\mathbb{S}(t,\mathrm{D})||_{[\mathrm{D}]^\mu L^2 \to L^2} \lesssim \big(\log(e+t)\big)^{-\frac{\mu}{2}}.$$

Example 5.23. For $b(t) = (1+t)^\kappa$ with $\kappa > 1$ no improvement is possible by choosing data from $[\mathrm{D}]^\mu L^2(\mathbb{R}^n)$. This is a direct consequence of Lemma 4.29.

5.3 Energy estimates of higher order

Higher order energy estimates are a natural tool to investigate parabolic problems, while for the wave equation they bring no profit. This is different for damped wave equations. The estimates of A. Matsumura, [Mat76], yield the same (parabolic) improvement by taking higher order derivatives like for the heat equation.

We recall the L^2–L^2 estimates from that paper. It holds for a solution $u(t,x)$ of the damped wave equation $\Box u + u_t = 0$ to Cauchy data u_1 and u_2 (cf. the overview given in Section 1.3.2)

$$||\mathrm{D}_t^\ell \mathrm{D}_x^\alpha u(t,\cdot)||_{L^2} \le C(1+t)^{-\ell-\frac{|\alpha|}{2}} \left\{ ||u_1||_{H^{\ell+|\alpha|}} + ||u_2||_{H^{\ell+|\alpha|-1}} \right\}, \tag{5.3.1}$$

similar to the estimates for the heat equation, $w_t = \Delta w$ with Cauchy data w_0,

$$||\mathrm{D}_t^\ell \mathrm{D}_x^\alpha w(t,\cdot)||_{L^2} \le C(1+t)^{-\ell-\frac{|\alpha|}{2}} \, ||w_0||_{H^{\ell+|\alpha|}}. \tag{5.3.2}$$

The aim of this section is to underline, that corresponding results are also valid for time-dependent dissipation terms. In detail, we show that

- in case of *non-effective dissipation*, higher order energies satisfy the same decay estimates like the usual hyperbolic (first order) energy,
- in case of *effective dissipation* and under Assumption (B5), derivatives improve the decay rates and the improvement differs between time and spatial derivatives and
- in case of *over-damping*, only time derivatives give improvements.

In order to make this more precise we define in analogy to the energy operator $\mathbb{E}(t,\mathrm{D})$ *energy operators of higher order.* They are given by

$$\mathbb{E}_\ell^k(t,\mathrm{D}) : (\langle \mathrm{D}\rangle^k u_1, \langle \mathrm{D}\rangle^{k-1} u_2) \mapsto |\mathrm{D}|^{k-\ell}\mathrm{D}_t^\ell u(t,\cdot) \tag{5.3.3}$$

for $\ell \le k$ and with $k \ge 1$ and describe the behaviour of $k-\ell$ spatial and ℓ time derivatives. The number k gives the total number of derivatives and stands for the order of the energy. The energy operator itself is given by $\mathbb{E}(t,\mathrm{D}) = (\mathbb{E}_0^1(t,\mathrm{D}), \mathbb{E}_1^1(t,\mathrm{D}))^T$. We exclude the case $k = 0$ because of its exceptional behaviour. It was considered in Section 5.2.

The main task is to provide estimates for time derivatives. The spatial derivatives can be considered later on using the following relation.

Proposition 5.13. *It holds* $\mathbb{E}_\ell^k(t,\xi) = [\xi]^{k-\ell}\mathbb{E}_\ell^\ell(t,\xi)$.

5.3.1 Non-effective weak dissipation

In the case of non-effective dissipation, we simply apply the differential equation in system form to the known estimates[1] for the micro-energy $U = (h(t,\xi)\hat{u}, \mathrm{D}_t\hat{u})^T$, which yields by Leibniz rule

$$\mathrm{D}_t U = A(t,\xi)U, \qquad \mathrm{D}_t^\ell U = \sum_{m=0}^{\ell-1} \binom{\ell-1}{m} (\mathrm{D}_t^m A(t,\xi)) \mathrm{D}_t^{\ell-1-m} U. \tag{5.3.4}$$

Now we can prove by induction over ℓ the following statement for the fundamental solution $\mathcal{E}(t,0,\xi)$, which brings no improvement by taking higher order derivatives in t.

Lemma 5.14. *Assume (A1), (A4)$_\ell$ together with (C1). Then the estimate*

$$||\mathrm{D}_t^\ell \mathcal{E}(t,0,\xi)|| \lesssim \frac{1}{\lambda(t)} \sum_{m=0}^{\ell} (h(t,\xi)+b(t))^m \left(\frac{1}{1+t}\right)^{\ell-m} \lesssim \frac{1}{\lambda(t)} \langle\xi\rangle^\ell$$

holds.

We obtain even more. Applying (5.3.4) recursively, we get

$$\mathrm{D}_t^\ell \mathcal{E}(t,0,\xi) = \sum_{m=0}^{\ell-1} \binom{\ell-1}{m} (\mathrm{D}_t^m A(t,\xi)) \mathrm{D}_t^{\ell-1-m} \mathcal{E}(t,0,\xi) = \cdots = B(t,\xi)\mathcal{E}(t,0,\xi),$$

where $B(t,\xi) = F(A, \mathrm{D}_t A, \cdots, \mathrm{D}_t^\ell A)$ defines a symbol of order ℓ uniform in the variable t. Thus, we obtain not only the resulting L^2–L^2 estimate for $\mathrm{D}_t^\ell \mathcal{E}(t,0,\xi)$, but also an L^p–L^q estimate.

Theorem 5.15. *Assume (A1), (A4)$_\ell$ together with (C1). Then the L^p–L^q estimate*

$$||\mathrm{D}_t^\ell \mathcal{E}(t,0,\mathrm{D})||_{p,(r+\ell)\to q} \lesssim \frac{1}{\lambda(t)} (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

holds for dual indices $pq = p+q$, $p \in (1,2]$ and with regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Using, that the main contribution of $B(t,\xi)$ comes from $A(t,\xi)^\ell$,

$$\begin{aligned} B(t,\xi) &= (A(t,\xi))^\ell + \mathcal{O}\left(\sum_{m=1}^{\ell-1} \left(\frac{1}{1+t}\right)^{1+m} |\xi|^{\ell-1-m}\right), \qquad (1+t)|\xi| \to \infty, \\ M^{-1}B(t,\xi)M &= M^{-1} \begin{pmatrix} |\xi|^\ell & \\ & (-1)^\ell |\xi|^\ell \end{pmatrix} M + \mathcal{O}(|\xi|^{\ell-1}(1+t)^{-1}), \end{aligned}$$

and, therefore, the modified scattering results of Theorem 3.26 imply the sharpness of the above given estimate. Using the previously introduced notation for the higher order energy operators $\mathbb{E}_\ell^k(t,\mathrm{D})$, we obtained the following (sharp) estimate, which is independent on k and ℓ.

Corollary 5.16. *For all $k, \ell \in \mathbb{N}_0$, $k \geq \max\{\ell, 1\}$ it holds*

$$||\mathbb{E}_\ell^k(t,\mathrm{D})||_{p,r\to q} \lesssim \frac{1}{\lambda(t)} (1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

for dual indices $pq = p+q$, $p \in (1,2]$ and with regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

[1] We use ϕ_{diss} and ϕ_{hyp} as smooth functions in order to differentiate the coefficient matrix, cf. the definition of the micro-energy and the introduction of these functions on page 44.

Thus, as already mentioned, for the case of non-effective weak dissipation higher order derivatives have the same time-asymptotic behaviour as the energy. This is closely related to the asymptotic relation to free waves and the modified scattering theory introduced in Section 3.5.

5.3.2 Effective dissipation

In the case of effective dissipation, we can use essentially the same arguments within the hyperbolic part for large frequencies, $\Pi_{hyp} \cap \{|\xi| \geq c\}$, like in the case of non-effective weak dissipation. The crucial point is to estimate the small frequencies in $\Pi_{ell} \cap \{|\xi| \leq c\}$. For this, we follow the idea of Lemma 4.19 and recall, that the entries of $\mathbb{E}_\ell^\ell(t,\xi)$ are multiples of $\partial_t^\ell \Phi_i(t,0,\xi)$. Let t_ξ denote the lower boundary of the elliptic zone, i.e. $b(t_\xi) \sim |\xi|$ for large ξ in the case of strong dissipation.

Lemma 5.17. *Assume (B1) – (B3) and (B4)$_{\ell+1}$. Then for all $(t,\xi) \in Z_{ell}(c_0,\epsilon) \cap \{t \geq t_0\}$ the pointwise estimate*

$$|\partial_t^\ell \Phi_i(t,t_\xi,\xi)| \lesssim \sum_{\substack{\mu+\nu=\ell \\ \mu\geq 1}} \left(\frac{|\xi|^2}{b(t)}\right)^\mu \left(\frac{1}{1+t}\right)^\nu \exp\left\{-|\xi|^2 \int_{t_\xi}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}$$

holds.

Proof. The proof goes by induction over ℓ. For $\ell = 1$ the statement is contained in Corollary 4.20. For $\ell > 1$ we assume the induction hypothesis

$$|\partial_t^{\ell'} \Phi_i(t,t_\xi,\xi)| \lesssim \sum_{\substack{\mu+\nu=\ell' \\ \mu\geq 1}} \left(\frac{|\xi|^2}{b(t)}\right)^\mu \left(\frac{1}{1+t}\right)^\nu \exp\left\{-|\xi|^2 \int_{t_\xi}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}$$

for all $\ell' \leq \ell$. Differentiating the equation

$$\ddot\Phi_i + |\xi|^2 \Phi_i + b(t)\dot\Phi_i = 0$$

ℓ times with respect to t yields

$$\partial_t\left(\partial_t^{\ell+1}\Phi_i\right) + b(t)\left(\partial_t^{\ell+1}\Phi_i\right) = -|\xi|^2 \partial_t^\ell \Phi_i - \sum_{k=1}^{\ell} \binom{\ell}{k} \left(\partial_t^k b(t)\right)\left(\partial_t^{\ell-k+1}\Phi_i\right) =: \Psi_\ell$$

with new right-hand side $\Psi_\ell(t,t_\xi,\xi)$. Its solution is given by

$$\partial_t^{\ell+1}\Phi_i(t,t_\xi,\xi) = \frac{\lambda^2(t_\xi)}{\lambda^2(t)} \partial_t^{\ell+1}\Phi_i(t_\xi,t_\xi,\xi) + \int_{t_\xi}^t \frac{\lambda^2(\tau)}{\lambda^2(t)} \Psi_\ell(\tau,t_\xi,\xi)\mathrm{d}\tau, \tag{5.3.5}$$

which can be estimated by the induction hypothesis together with the estimate of the initial values

$$|\partial_t^k \Phi_i(t_\xi,t_\xi,\xi)| \lesssim \langle\xi\rangle^{\delta(k)}, \qquad \delta(k) \in \mathbb{N},$$

following directly by applying the equation to the initial values $\Phi_i(t_\xi,t_\xi,\xi)$ and $\partial_t\Phi_i(t_\xi,t_\xi,\xi)$ given in (2.1.7). Now the integral equation (5.3.5) together with these initial values and the induction hypothesis gives

$$
\begin{aligned}
\left|\partial_t^{\ell+1}\Phi_i(t,t_\xi,\xi)\right| \lesssim & \frac{\lambda^2(t_\xi)}{\lambda^2(t)}\langle\xi\rangle^{\delta(k)} \\
& + |\xi|^2\int_{t_\xi}^t \frac{\lambda^2(\tau)}{\lambda^2(t)} \sum_{\substack{\mu+\nu=\ell\\ \mu\ge 1}} \left(\frac{|\xi|^2}{b(\tau)}\right)^\mu \left(\frac{1}{1+\tau}\right)^\nu \exp\left\{-|\xi|^2\int_{t_\xi}^\tau \frac{\mathrm{d}\theta}{b(\theta)}\right\}\mathrm{d}\tau \\
& + \int_{t_\xi}^t \frac{\lambda^2(\tau)}{\lambda^2(t)} \sum_{k=1}^{\ell} b(\tau)\left(\frac{1}{1+t}\right)^k \\
& \qquad \sum_{\ell'=1}^{\ell-k+1} \sum_{\substack{\mu+\nu=\ell'\\ \mu\ge 1}} \left(\frac{|\xi|^2}{b(\tau)}\right)^\mu \left(\frac{1}{1+\tau}\right)^\nu \exp\left\{-|\xi|^2\int_{t_\xi}^\tau \frac{\mathrm{d}\theta}{b(\theta)}\right\}\mathrm{d}\tau \\
\lesssim & \frac{\lambda^2(t_\xi)}{\lambda^2(t)}\langle\xi\rangle^{\delta(k)} + \sum_{\substack{\mu+\nu=\ell+1\\ \mu\ge 1}} \left(\frac{|\xi|^2}{b(t)}\right)^\mu \left(\frac{1}{1+t}\right)^\nu \exp\left\{-|\xi|^2\int_{t_\xi}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}.
\end{aligned}
$$

Furthermore, the first summand is subordinate to the second one. It holds

$$
\frac{\lambda^2(t_\xi)}{\lambda^2(t)}\langle\xi\rangle^{\delta(k)} \lesssim \left(|\xi|\frac{b(t_\xi)}{b(t)}\right)^{\ell+1} \exp\left\{-|\xi|^2\int_{t_\xi}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}
$$

from

$$
\begin{aligned}
1 &\lesssim \exp\left\{\int_{t_\xi}^t \left(-|\xi|^2\frac{1}{b(\tau)} + b(\tau) - (\ell+1)\frac{b'(\tau)}{b(\tau)} + (\ell+1)\log|\xi| - \delta(k)\log\langle\xi\rangle\right)\mathrm{d}\tau\right\} \\
&\lesssim \exp\left\{\int_{t_\xi}^t \frac{1}{b(\tau)}\Big(\underbrace{b^2(\tau) - |\xi|^2 - (\ell+1)b'(\tau) + c_k b(\tau)\log|\xi|}_{\ge\frac{1}{2}b^2(\tau)\ge 0,\qquad \tau\ge t_0}\Big)\mathrm{d}\tau\right\}
\end{aligned}
$$

for large $|\xi|$, while for small $|\xi|$ the estimate is obvious. □

In order to apply the previous lemma, we remark that each term of the form

$$
\left(\frac{|\xi|^2}{b(t)}\right)^\mu \exp\left\{-|\xi|^2\int_{t_\xi}^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}
$$

brings a decay rate of order

$$
\frac{1}{b^\mu(t)}\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\mu}.
$$

To understand the influence of the different summands, we distinguish between the case of effective weak and the case of strong dissipation.

Proposition 5.18. *Assume (B1) – (B3). Then it holds*

$$
b(t)\int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \lesssim t,
$$

if $b'(t) \le 0$*, and*

$$b(t) \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} \gtrsim t,$$

if $b'(t) \ge 0$.

Proof. Integration by parts implies

$$b(t) \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} = t + b(t) \int_0^t \frac{\tau b'(\tau)}{b^2(\tau)} \mathrm{d}\tau$$

and the statement follows immediately. $\square$

Example 5.24. With $b(t) = (1+t)^\kappa$ with $\kappa \in (-1, 1)$, we can calculate the expression explicitly and get

$$b(t) \int_0^t \frac{\mathrm{d}\tau}{b(\tau)} = \frac{1}{1-\kappa}(1+t)^\kappa t^{1-\kappa} \sim t.$$

The estimates of Proposition 5.18 are both valid. If $b(t) = 1 + t$, thus if we take $\kappa = 1$, only the second estimate

$$(1+t) \int_0^t \frac{\mathrm{d}\tau}{1+\tau} = (1+t)\log(1+t) \gtrsim 1 + t$$

holds.

The first theorem corresponds mainly to the case of effective weak dissipation.

Theorem 5.19. *Assume (B1) – (B3), (B4)*$_{\ell+1}$ *together with* $b'(t) \le 0$*. Then the* L^2*–*L^2 *estimate*

$$||\mathbb{E}_\ell^k(t, \mathrm{D})||_{2\to 2} \lesssim \frac{1}{b^\ell(t)} \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{k+\ell}{2}}$$

holds for all $k \ge \ell$*. Furthermore, the strong limit*

$$\operatorname*{s-lim}_{t\to\infty} b^\ell(t) \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\frac{k+\ell}{2}} \mathbb{E}_\ell^k(t, \mathrm{D}) = 0$$

is taken in $L^2 \to L^2$*.*

For strong dissipation terms we use the second inequality of Proposition 5.18 to get the following theorem. We distinguish between estimates containing time derivatives and such estimates containing only spatial derivatives. The latter ones are special cases of the solution estimate of Theorem 5.12.

Theorem 5.20. *Assume (B1) – (B3), (B4)*$_{\ell+1}$ *together with* $b'(t) \ge 0$*.*

1. *For* $\ell = 0$ *the* L^2*–*L^2 *estimate*

$$||\mathbb{E}_0^k(t, \mathrm{D})||_{2\to 2} \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{k}{2}}$$

holds for all $k \in \mathbb{N}_0$*. Furthermore, if (B5) is satisfied, the strong limit*

$$\operatorname*{s-lim}_{t\to\infty} \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\frac{k}{2}} \mathbb{E}_0^k(t, \mathrm{D}) = 0$$

is taken in $L^2 \to L^2$*.*

2. *For $\ell \geq 1$ the L^2–L^2 estimate*

$$||\mathbb{E}^k_\ell(t,\mathrm{D})||_{2\to 2} \lesssim \frac{1}{b(t)}\left(\frac{1}{1+t}\right)^{\ell-1}\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{k-\ell}{2}-1}$$

holds for all $k \geq \ell \geq 1$. Furthermore, if (B5) is satisfied, the strong limit

$$\operatorname*{s-lim}_{t\to\infty} b(t)(1+t)^{\ell-1}\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{\frac{k-\ell}{2}+1}\mathbb{E}^k_\ell(t,\mathrm{D}) = 0$$

is taken in $L^2 \to L^2$.

We proceed with some examples.

Example 5.25. For damped waves, i.e. $b(t)=1$, we obtain the estimate

$$||\mathbb{E}^k_\ell(t,\mathrm{D})||_{2\to 2} \lesssim \left(\frac{1}{1+t}\right)^{\ell+\frac{k-\ell}{2}}.$$

In view of the definition of the higher order energy operator (5.3.3) we control $k-\ell$ spatial and ℓ time derivatives and therefore the estimate coincides with the estimates given by A. Matsumura, cf. formula (5.3.1) with $|\alpha| = k-\ell$.

Example 5.26. For $b(t) = \frac{\mu}{1+t}$ with sufficiently large μ related estimates for higher order derivatives were proven in Theorem 2.9. These estimates fit to the estimates of the above given theorems. In this case, the improvement by spatial and time derivatives is the same.

Example 5.27. For $b(t) = (1+t)^\kappa$ with $\kappa \in (-1,1)$ we obtain

$$||\mathbb{E}^k_\ell(t,\mathrm{D})||_{2\to 2} \lesssim (1+t)^{-\frac{\kappa+1}{2}\ell-\frac{1-\kappa}{2}k}.$$

The improvement rates for time and spatial derivatives are different for all $\kappa > -1$. Time derivatives bring more improvements on the decay order than spatial derivatives.

Example 5.28. For $b(t) = (1+t)$ time derivatives improve the decay rate by one order, while spatial derivatives give logarithmic orders. It holds for $\ell > 1$

$$||\mathbb{E}^k_\ell(t,\mathrm{D})||_{2\to 2} \lesssim (1+t)^{-\ell}\big(\log(e+t)\big)^{-\frac{k-\ell}{2}-1},$$

while for $\ell = 0$

$$||\mathbb{E}^k_0(t,\mathrm{D})||_{2\to 2} \lesssim \big(\log(e+t)\big)^{-\frac{k}{2}}.$$

Example 5.29. If we set $b(t) = (1+t)\log(1+t)\cdots\log^{[m]}(e^{[m]}+t)$ spatial derivatives give improvements by the rate $\big(\log^{[m+1]}(e^{m+1})\big)^{-1}$. It holds

$$||\mathbb{E}^k_\ell(t,\mathrm{D})||_{2\to 2} \lesssim \big(\log(1+t)\cdots\log^{[m]}(e^{[m]}+t)\big)^{-1}(1+t)^{-\ell}\big(\log^{[m+1]}(e^{[m+1]}+t)\big)^{-\frac{k-\ell}{2}-1}$$

for $\ell > 0$ and

$$||\mathbb{E}^k_0(t,\mathrm{D})||_{2\to 2} \lesssim \big(\log^{[m+1]}(e^{[m+1]}+t)\big)^{-\frac{k}{2}}$$

Example 5.30. In case, that the over-damping condition (OD) is satisfied, i.e. if (B5) is violated, the solution tends to a non-zero limit and, therefore, spatial derivatives give no improvement of the decay at all. In this case we obtain

$$||\mathbb{E}_\ell^k(t,\mathrm{D})||_{2\to 2} \lesssim \frac{1}{b(t)}(1+t)^{1-\ell}$$

for all $k \geq \ell \geq 1$.

Similar to the case of non-effective dissipation the expressions obtained for $\partial_t^\ell \Phi_i(t,t_\xi,\xi)$ in combination with the estimates for derivatives of $\mathcal{E}(t,0,\xi)$ in Π_{hyp},

$$||\partial_t^\ell \mathcal{E}(t,0,\xi)|| \lesssim \frac{1}{\lambda(t)} \langle\xi\rangle^\ell,$$

allow us to conclude also L^p–L^q estimates for higher order energies. In the case of effective dissipation these estimates are determined by the behaviour of small frequencies inside the elliptic part.

The following theorem is a consequence of Lemma 4.24 together with the idea of the proof of Theorem 4.25.

Theorem 5.21. *Assume (B1) – (B3) and (B4)$_{\ell+1}$. Then for $p \in [1,2]$ and with q the corresponding dual index the L^p–L^q decay estimate*

$$||\mathbb{E}_\ell^k(t,\mathrm{D})||_{p,r\to q} \lesssim \begin{cases} \frac{1}{b^\ell(t)}\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{k+\ell}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, & b'(t)\leq 0,\\ \left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{k}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, & b'(t)\geq 0,\ \ell=0,\\ \frac{1}{b^\ell(t)}\left(\frac{1}{1+t}\right)^{\ell-1}\left(1+\int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-\frac{k-\ell}{2}-1-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, & b'(t)\geq 0,\ \ell\geq 1, \end{cases}$$

holds for $r > n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Remark 5.31. While for $\ell = 0$ and $\ell = 1$ the constructed representations may be used to claim optimality for the estimates as norm-estimates, for $\ell > 1$ we can not ensure the existence of a corresponding term in the representation and, therefore, the optimality is not guaranteed.

Furthermore, for $p \in (1,2]$ more diagonalization steps in the hyperbolic part together with a higher regularity of the coefficient function may be used to obtain the sharp regularity $r = n\left(\frac{1}{p}-\frac{1}{q}\right)$. For $p = 1$ the above used regularity is sharp within the scale of Bessel potential spaces.

5.4 The diffusion phenomenon for effective dissipation

The estimates of higher order energies hint to an underlying parabolic structure of damped waves and wave equations with effective dissipation. As already pointed out in the introduction, Section 1.3.2, this underlying parabolic structure can be expressed in terms of the so-called diffusion phenomenon and the asymptotics of the solutions to the damped wave equation are related to corresponding solutions of the heat equation.

The aim of this section is to extend this result to the case of time-dependent coefficients. We relate the solutions to our Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = 0,\\ u(0,\cdot) = u_1, \quad \mathrm{D}_t u(0,\cdot) = u_2, \end{cases} \tag{5.4.1}$$

to corresponding solutions of the parabolic problem

$$\begin{cases} w_t = \frac{1}{b(t)} \Delta w, \\ w(0, \cdot) = w_0, \end{cases} \tag{5.4.2}$$

with related data w_0. There arise two main difficulties in this section. On the one hand we have to find an expression for the related datum w_0. In the case of the damped wave equation $w_0 = u_1 + iu_2$ is used. On the other hand, we have to give a precise meaning to the asymptotic relation between these two Cauchy problems.

We will distinguish between two cases. A review of the estimates related to the diffusion phenomenon for damped waves given by T. Narazaki in [Nar04] shows, that the relation to the heat equation holds in the neighbourhood of the exceptional frequency $\xi = 0$, [Nar04, Theorem 1.1], while for data u_1 and u_2 with $0 \notin \operatorname{supp}(\hat{u}_2, \hat{u}_2)$ exponential decay rates occur and the behaviour is a modified hyperbolic one, [Nar04, Theorem 1.2]. In the latter case, the decay rate is stronger than the decay for the parabolic problem. In our discussion of the diffusion phenomenon we will speak of (frequency-) *local diffusion*, if this relation to one exceptional frequency occurs. Contrary to this in Section 4.4 we observed that in case of an over-damping both, the solution of the dissipative wave equation and the solution of the corresponding parabolic equation tend to a (in general nonzero) limit as $t \to \infty$. We will speak of *global diffusion*, if a relation to the corresponding parabolic problem takes place for all frequencies.

5.4.1 The local diffusion phenomenon

The treatment of this section follows in some ideas the paper of T. Narazaki, [Nar04]. Assumptions on the coefficient are (B1) – (B3) together with $(\text{B4})_2$ and

(LD) $\displaystyle\int_0^\infty \frac{\mathrm{d}\tau}{b^3(\tau)} = \infty,$

to characterise the local diffusion phenomenon.

Under these assumptions, Theorem 4.11 implies with formula (4.2.18) and $\gamma(t) = \frac{1}{2}b(t)$ the representation

$$\mathcal{E}_V(t, t_0, \xi) = \frac{\langle\xi\rangle_{\gamma(t)}}{\langle\xi\rangle_{\gamma(t_0)}} \exp\left\{\int_{t_0}^t \langle\xi\rangle_{\gamma(\tau)}\,\mathrm{d}\tau\right\} \mathcal{Q}_{ell,0}(t, t_0, \xi) \tag{5.4.3}$$

for the fundamental solution $\mathcal{E}_V(t, t_0, \xi)$ corresponding to the micro-energy V. Furthermore, we have $\mathcal{Q}_{ell,0}(t, t_0, \xi) \to \mathcal{Q}_{ell,0}(\infty, t_0, \xi)$ uniformly on compact subsets of $\{|\xi| \le c\}$ in the case of strong dissipation.

The transformation back to $\hat{u}(t, \xi)$ yields (assuming that the choice $t_0 = 0$ is possible)

$$\mathbb{S}(t, \xi) = \exp\left\{\int_0^t \big(\langle\xi\rangle_{\gamma(\tau)} - \gamma(\tau)\big)\mathrm{d}\tau\right\} e_1^T \mathcal{Q}_{ell,0}(t, 0, \xi)$$

for $|\xi| \le c$ and the proof to Corollary 4.12 implies for $\mathbb{S}(\infty, 0) = e_1^T \mathcal{Q}_{ell,0}(\infty, 0, 0)$ the representation

$$e_1^T Q_{ell,0}(\infty, 0, 0) = \left(1, i\int_0^\infty \frac{\mathrm{d}\tau}{\lambda^2(\tau)}\right).$$

Compared to the representation of solutions to the corresponding parabolic problem,

$$\hat{w}(t, \xi) = S_P(t, \xi)\hat{w}_0(\xi) = \exp\left\{-|\xi|^2 \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right\} \hat{w}_0(\xi), \tag{5.4.4}$$

and therefore $\hat{w}(t,0) = \hat{w}_0(0)$, this can be used as choice for the initial datum w_0. We set

$$w_0 = u_1 + iu_2 \int_0^\infty \frac{\mathrm{d}\tau}{\lambda^2(\tau)} = \tilde{W}(\mathrm{D})\,(u_1, \langle \mathrm{D}\rangle^{-1} u_2)^T, \tag{5.4.5}$$

and thus $\tilde{W}(\xi) = (1, i\int_0^\infty \frac{\mathrm{d}\tau}{\lambda^2(\tau)} \langle\xi\rangle)$.

Example 5.32. For the special case $b(t) \equiv 1$ this gives $w_0 = u_1 + iu_2$ like it is used by T. Narazaki in [Nar04] or K. Nishihara in [Nis03]. For the general constant dissipation $b(t) = b_0$ the representation would be $w_0 = u_1 + \frac{i}{b_0} u_2$.

Main result of this section is the following comparison of damped waves and solutions of the parabolic problem. Note, that $||S_P(t,\mathrm{D})||_{2\to 2} = 1$.

Theorem 5.22. *Assume (B1) – (B3), (B4)$_2$ and (LD). Then in the case of strong dissipation*

$$||\mathbb{S}(t,\mathrm{D}) - S_P(t,\mathrm{D})\tilde{W}(\mathrm{D})||_{2\to 2} \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{\gamma(\tau)}\right)^{-1},$$

while in the case of effective weak dissipation and under the further assumption $\frac{1}{b(t)(1+t)^2} \in L^1(\mathbb{R}_+)$

$$||\mathbb{S}(t,\mathrm{D}) - S_P(t,\mathrm{D})\tilde{W}(\mathrm{D})||_{2\to 2} \lesssim \frac{1+t}{b^3(t)} \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-2}.$$

Remark 5.33. If the data satisfies $0 \notin \operatorname{supp}(\hat{u}_1, \hat{u}_2)$ (and equivalently $0 \notin \operatorname{supp} \hat{w}_0$) both terms, $\mathbb{S}(t,\mathrm{D})(u_1, \langle\mathrm{D}\rangle^{-1} u_2)^T$ and $S_P(t,\mathrm{D})\tilde{W}(u_1, \langle\mathrm{D}\rangle^{-1} u_2)^T$ tend to zero under Assumption (B5) (which follows from (LD)). The statement is only of interest for neighbourhoods of the exceptional frequency $\xi = 0$.

Proof. Let $c > 0$. In view of Remark 5.33, it suffices to consider

$$||\mathbb{S}(t,\xi) - S_P(t,\xi)\tilde{W}||_{L^\infty\{|\xi|\le c\}}.$$

We distinguish between the two cases of effective weak and strong dissipation.
Case 1: Strong dissipation. In the first case we assume that $b(t) \geq 2c > 0$. Under this assumption only the elliptic zone is of interest and we may use the representation (with $\gamma(t) = \frac{1}{2} b(t)$)

$$\mathbb{S}(t,\xi) = \exp\left\{\int_0^t \big(\langle\xi\rangle_{\gamma(\tau)} - \gamma(\tau)\big)\mathrm{d}\tau\right\} e_1^T \mathcal{Q}_{ell,0}(t,0,\xi),$$

where the matrix function $e_1^T \mathcal{Q}_{ell,0}(t,0,\xi)$ tends uniformly to a non-zero limit as $t \to \infty$. This is a direct consequence of Theorem 4.11 in connection with Corollary 4.12.

Furthermore, we know from Corollary 4.12, that

$$e_1^T \mathcal{Q}_{ell,0}(\infty,0,0) = \left(1, i\int_0^\infty \frac{\mathrm{d}\tau}{\lambda^2(\tau)}\right).$$

In order to understand the behaviour of $e_1^T \mathcal{Q}_{ell,0}(\infty,0,\xi)$ in a neighbourhood of $\xi = 0$, we differentiate the integral equation (4.2.15) with respect to ξ. This yields for $\mathrm{D}_\xi^\alpha \mathcal{Q}_{ell,1}(t,s,\xi)$ with $|\alpha| = 1$ a corresponding integral equation with the same kernel and a further source term

$$\int_s^t \exp\left\{\int_\theta^t (i\mathcal{D}(\tau,\xi) + iF_0(\tau,\xi) - w(\tau,\xi)I)\mathrm{d}\tau\right\} \tilde{R}_1(\theta,\xi)\mathcal{Q}_{ell,1}(\theta,s,\xi)\mathrm{d}\theta,$$

$\tilde{R}_1 \in S^{1,\infty}_{ell,\epsilon}\{-1,0,2\} + S^{1,\infty}_{ell,\epsilon}\{-2,0,2\} \subseteq S^{1,\infty}_{ell,\epsilon}\{-1,0,2\}$ ($\gamma(t)$ is bounded from below!) related to the derivative of R_1 and the derivative of the exponent, which itself is a symbol of order $\{0,0,0\}$. Thus, the boundedness of $\mathcal{Q}_{ell,1}(t,s,\xi)$ together with the reasoning used for the proof of Theorem 4.11 implies continuity of the derivative and its uniform convergence. Together with the smoothness of the diagonalizer $N_1(t,\xi)$ and its inverse we obtain smoothness of $e_1^T\mathcal{Q}_{ell,0}(\infty,0,\xi)$ for small ξ. Higher derivatives can be handled exactly the same way. Thus, together with the dependence of $\mathcal{Q}_{ell,0}$ on $|\xi|$ by rotational symmetry of the problem, we obtain

$$e_1^T\mathcal{Q}_{ell,0}(\infty,0,\xi) - e_1^T\mathcal{Q}_{ell,0}(\infty,0,0) = \mathcal{O}(|\xi|^2), \qquad \xi \to 0.$$

Furthermore, $e_1^T\mathcal{Q}_{ell,0}(\infty,0,\xi) - e_1^T\mathcal{Q}_{ell,0}(t,0,\xi) = \mathcal{O}(\frac{1}{(1+t)\gamma(t)})$ as $t \to \infty$ uniformly on compact sets in ξ from the representation (4.2.16) and $N_1(t,\xi) - I \in S^{1,\infty}_{ell,\epsilon}\{-1,0,1\}$.

The main tool to get the desired estimate is the comparison of the leading terms of the representations. It holds

$$\begin{aligned} 0 &\le \exp\left\{-|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\} - \exp\left\{\int_0^t \big(\langle\xi\rangle_{\gamma(\tau)} - \gamma(\tau)\big)\mathrm{d}\tau\right\} \\ &= \exp\left\{-|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\} \underbrace{\left(1 - \exp\left\{\int_0^t \left(\langle\xi\rangle_{\gamma(\tau)} - \gamma(\tau) + \frac{|\xi|^2}{2\gamma(\tau)}\right)\mathrm{d}\tau\right\}\right)}_{\sim\int_0^t \frac{|\xi|^4}{\gamma^3(\tau)}\mathrm{d}\tau \sim \frac{t|\xi|^4}{\gamma^3(t)}} \end{aligned}$$

for small $\frac{t|\xi|^4}{\gamma^3(t)}$, while for large ones the second exponential becomes small. Thus one obtains under this smallness assumption

$$\begin{aligned} &\lesssim \frac{t}{\gamma^3(t)}|\xi|^4 \exp\left\{-|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\} \qquad (*) \\ &\lesssim \frac{t}{\gamma^3(t)}\left(1+\int_0^t \frac{\mathrm{d}\tau}{2\gamma(\tau)}\right)^{-2} \lesssim \frac{1}{\gamma(t)(1+t)} \end{aligned}$$

by Proposition 5.18. The maximum of expression (*) is taken on a line with $|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{2\gamma(\tau)} \sim 1$ and again by Proposition 5.18 the above used smallness assumption follows. If the smallness assumption is violated the estimate follows directly

$$\exp\left\{-|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\} \le \exp\left\{-C\frac{\sqrt{\gamma(t)}}{\sqrt{1+t}}\gamma(t)\int_0^t \frac{\mathrm{d}\tau}{\gamma(t)}\right\} \le \exp\left\{-C'\sqrt{\gamma(t)(1+t)}\right\}$$

using Proposition 5.18.

Combining all the estimates yields

$$\begin{aligned} &||\mathbb{S}(t,\xi) - S_P(t,\xi)\tilde{W}(\xi)||_{L^\infty\{|\xi|\le c\}} \\ &\quad \le \sup_{|\xi|\le c}\left|\left|\exp\left\{\int_0^t \big(\langle\xi\rangle_{\gamma(\tau)} - \gamma(\tau)\big)\mathrm{d}\tau\right\} e_1^T\mathcal{Q}_{ell,0}(t,0,\xi)\right.\right. \\ &\qquad\qquad \left.\left. - \exp\left\{-|\xi|^2\int_0^t \frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\} e_1^T\mathcal{Q}_{ell,0}(\infty,0,0)\right|\right| \end{aligned}$$

and

$$\begin{aligned}
&\exp\left\{\int_0^t \big(\langle\xi\rangle_{\gamma(\tau)}-\gamma(\tau)\big)\mathrm{d}\tau\right\} e_1^T\mathcal{Q}_{ell,0}(t,0,\xi)-\exp\left\{-|\xi|^2\int_0^t\frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\}e_1^T\mathcal{Q}_{ell,0}(\infty,0,0)\\
&=\left(\exp\left\{\int_0^t \big(\langle\xi\rangle_{\gamma(\tau)}-\gamma(\tau)\big)\mathrm{d}\tau\right\}-\exp\left\{-|\xi|^2\int_0^t\frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\}\right)e_1^T\underbrace{\mathcal{Q}_{ell,0}(t,0,\xi)}_{\lesssim 1}\\
&\quad+\exp\left\{-|\xi|^2\int_0^t\frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\}\big(e_1^T\mathcal{Q}_{ell,0}(t,0,\xi)-e_1^T\mathcal{Q}_{ell,0}(\infty,0,\xi)\big)\\
&\quad+\exp\left\{-|\xi|^2\int_0^t\frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\}\underbrace{\big(e_1^T\mathcal{Q}_{ell,0}(\infty,0,\xi)-e_1^T\mathcal{Q}_{ell,0}(\infty,0,0)\big)}_{\lesssim|\xi|^2}\\
&\lesssim\frac{1}{(1+t)\gamma(t)}+\left(1+\int_0^t\frac{\mathrm{d}\tau}{\gamma(\tau)}\right)^{-1}\lesssim\left(1+\int_0^t\frac{\mathrm{d}\tau}{\gamma(\tau)}\right)^{-1}.
\end{aligned}$$

The estimate is determined by the behaviour of $\mathcal{Q}_{ell,0}(\infty,0,\xi)$ near $\xi=0$.
Case 2: Effective weak dissipation. If $b(t)\to 0$ as $t\to\infty$, we can not take the limit of $\mathcal{Q}_{ell,0}(t,0,\xi)$ for $t\to\infty$. Nevertheless, the leading terms of the representation cancel. Using (Γ3) we can set $c_0=0$ in the definition of the elliptic zone. Let $\phi(t,\xi)$ be the characteristic function of the elliptic zone. Then, similar to the treatment in the first case, we obtain $e_1^T\mathcal{Q}_{ell,0}(t,0,\xi)-e_1^T\tilde{W}(0)=\mathcal{O}(\frac{1}{(1+t)\gamma(t)})$ for $(t,\xi)\in Z_{ell}(0,\epsilon)$. Furthermore, the difference of the exponentials satisfies

$$0\le\exp\left\{-|\xi|^2\int_0^t\frac{\mathrm{d}\tau}{2\gamma(\tau)}\right\}-\exp\left\{\int_0^t\big(\langle\xi\rangle_{\gamma(\tau)}-\gamma(\tau)\big)\mathrm{d}\tau\right\}\lesssim\frac{1+t}{\gamma^3(t)}\left(1+\int_0^t\frac{\mathrm{d}\tau}{\gamma(\tau)}\right)^{-2},$$

similar to the treatment in the first case. Using Proposition 5.18 it follows

$$\frac{1}{(1+t)\gamma(t)}\lesssim\frac{1+t}{\gamma^3(t)}\left(1+\int_0^t\frac{\mathrm{d}\tau}{\gamma(\tau)}\right)^{-2}$$

and, therefore,

$$||(\mathbb{S}(t,\xi)-S_P(t,\xi)\tilde{W}(\xi))\phi(t,\xi)||\lesssim\frac{1+t}{b^3(t)}\left(1+\int_0^t\frac{\mathrm{d}\tau}{b(\tau)}\right)^{-2},$$

which is determined by the difference of the exponentials in this case. Note, that the line $|\xi|^2\int_0^t\frac{\mathrm{d}\tau}{\gamma(\tau)}\sim 1$, where the maximum of the difference of the exponentials is taken, lies inside the elliptic zone for large t.

It remains to show, that the difference decays faster outside the elliptic zone. For $|\xi|\ge\gamma(t)$ we get

$$\begin{aligned}
||S_P(t,\xi)||&\lesssim\exp\left\{-b^2(t)\int_0^t\frac{\mathrm{d}\tau}{b(\tau)}\right\}\lesssim\exp\{-Cb(t)t\},\\
||\mathbb{S}(t,\xi)||&\lesssim\exp\{-Cb(t_\xi)t_\xi\}\frac{\lambda(t_\xi)}{\lambda(t)}\lesssim\exp\{-C'b(t)t\},
\end{aligned}$$

and, thus, for both terms the bound $\exp\{-Ct\gamma(t)\}$ follows, which decays faster. $\square$

Remark 5.34. Assumption (LD) was necessary to get control, if $t|\xi|^4$ is not dominated by $\gamma^3(t)$. So Condition (LD) guarantees, that the diffusion phenomenon of this form takes place only in neighbourhoods of the exceptional frequency $\xi = 0$. At least for effective weak dissipation we know, that for $|\xi| \geq c > 0$ the hyperbolic zone is essential and yields a close relation to a modified hyperbolic representation, cf. Figure 4.6.

Theorem 5.22 states an asymptotic equivalence for solutions of the dissipative wave equation and the corresponding parabolic problem. A similar equivalence can be stated for spatial derivatives, where we use in the proof a further factor $|\xi|^s$ for small $|\xi|$.

Corollary 5.23. *Under the assumptions of Theorem 5.22 and strong dissipation it holds*

$$||\mathbb{S}(t,\mathrm{D}) - S_P(t,\mathrm{D})\tilde{W}(\mathrm{D})||_{H^s\to\dot{H}^s} \lesssim \left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-1-\frac{s}{2}},$$

while in the case of effective weak dissipation

$$||\mathbb{S}(t,\mathrm{D}) - S_P(t,\mathrm{D})\tilde{W}(\mathrm{D})||_{H^s\to\dot{H}^s} \lesssim \frac{1+t}{b^3(t)}\left(1 + \int_0^t \frac{\mathrm{d}\tau}{b(\tau)}\right)^{-2-\frac{s}{2}}.$$

5.4.2 The global diffusion phenomenon

The treatment of this section is closely related to the discussion of the case of over-damping. If we replace Assumption (LD) by

(GD) $\displaystyle\int_0^\infty \frac{\mathrm{d}\tau}{b^3(\tau)} < \infty,$

the exponentials

$$\exp\left\{\int_s^t \big(\langle\xi\rangle_{\gamma(\tau)} - \frac{1}{2}b(\tau)\big)\mathrm{d}\tau\right\} \qquad \text{and} \qquad \exp\left\{-|\xi|^2\int_s^t \frac{\mathrm{d}\tau}{b(\tau)}\right\}$$

behave inside the elliptic part asymptotically equivalent for $t \to \infty$. This is a direct consequence of

$$\langle\xi\rangle_{\gamma(\tau)} - \frac{1}{2}b(\tau) + \frac{|\xi|^2}{b(\tau)} \sim \frac{|\xi|^4}{b^3(\tau)}. \tag{5.4.6}$$

We can formulate this equivalence in form of the existence of a corresponding limit.

Theorem 5.24. *Assume (B1), (B2), (B4)$_2$ together with (SD). Then the limit*

$$W(\xi) = \lim_{t\to\infty} S_p(t,\xi)^{-1}\mathbb{S}(t,\xi)$$

exists locally uniform in ξ. Furthermore, $W(\xi) \neq 0$.

Proof. For $S_p(t,\xi)^{-1}\mathbb{S}(t,\xi)$ we obtain the representation

$$\begin{aligned} S_p(t,\xi)^{-1}\mathbb{S}(t,\xi) = &\exp\left\{\int_{t_\xi}^t \left(\langle\xi\rangle_{\gamma(\tau)} - \gamma(\tau) + |\xi|^2\frac{1}{2\gamma(\tau)}\right)\mathrm{d}\tau\right\} e_1^T \mathcal{Q}_{ell,0}(t,t_\xi,\xi)\\ &\exp\left\{\int_0^{t_\xi} \left(|\xi|^2\frac{1}{2\gamma(\tau)} - \gamma(\tau)\right)\mathrm{d}\tau\right\}\mathcal{E}(t_\xi,0,\xi)\end{aligned}$$

and, due to the Assumption (SD), the exponential converges for $t\to\infty$ to a non-zero limit, while, due to Theorem 4.11, the limit of $e_1^T\mathcal{Q}_{ell,0}(t,t_\xi,\xi)$ as $t\to\infty$ exists. Furthermore, this limit is also non-zero. ∎

Thus, if we take data (u_1, u_2) with compact Fourier support, by the aid of Theorem 5.24 we can construct data $w_0 = W(\mathrm{D})(u_1, \langle \mathrm{D} \rangle^{-1} u_2)^T$ to the corresponding parabolic problem, such that the solutions coincide asymptotically.

It turns out, that the function $W(\xi)$ increases exponentially in ξ. Therefore, $W(\mathrm{D})$ induces an operator mapping L^2-functions to Gevrey distributions. The importance of Theorem 5.24 lies in the asymptotic equivalence of the multipliers localized to the elliptic part.

6 Further developments and open problems

In this short concluding chapter we give an overview on related questions arising in connection with the considerations of this thesis. The list is not complete in any sense, it should only give some hints of possible generalisations, applications and also parallel developments.

Influence of oscillations. In the language of the results of M. Reissig and co-authors, [RY00], [Rei01], [RS03], the coefficient function $b = b(t)$ in our approach is allowed to have certain very slow oscillations. For a variable speed of propagation the influence of oscillations is well understood and, if we allow weaker assumptions on the derivatives, i.e. if we allow slow or even fast oscillations, there occurs a loss of decay for the energy. By means of an approach related to Floquet theory it can even be shown that for arbitrary oscillations in the propagation speed no L^p–L^q decay occurs. In the case of a periodic coefficient this is treated by K. Yagdjian in [Yag00].

In case of oscillations which do not influence the principal part directly, oscillations may also have deteriorating influences on the resulting estimates. Till now there exists no description of their precise influence, even if we remain in the cases introduced in this thesis. For a coefficient function $b = b(t)$ oscillating around the critical case $b(t) = \frac{1}{1+t}$ the influence of the decay estimate for the solution may even be worse.

Domains. Throughout the thesis treated the Cauchy problem for a wave equation with time-dependent dissipation. It seems to be natural to ask for results on more general domains. If one treats the Dirichlet problem on a domain with sufficiently smooth boundary representations of the solution and the energy operator can be obtained in terms of a spectral calculus of the Dirichlet extension of the Laplacian. This leads to representations in the Hilbert space $L^2(\Omega)$ and can be used to deduce estimates in L^2-scale.

The representations obtained in this thesis can be used. With the notation $\mathrm{D} = \sqrt{-\Delta}$ the energy operator is given as $\mathbb{E}(t, \mathrm{D})$ and the solution operator as $\mathbb{S}(t, \mathrm{D})$ as analytic functions of the Dirichlet Laplacian.

It turns out that one has to distinguish different cases,

- bounded domains (where $-\Delta$ has a pure point spectrum),
- unbounded domains without Poincaré inequality (i.e. $0 \in \sigma(-\Delta)$),
- unbounded domains with Poincaré inequality (i.e. $-\Delta$ is strictly positive).

In the first case only estimates in L^2-scale are of interest and one has only to distinguish between weak and strong dissipation. For the latter cases also L^p–L^q estimates are worth to consider and at least for exterior domains the behaviour may be guessed to be closely related to that of the Cauchy problem.

M. Yamaguchi treated in [Yam80] semi-linear perturbations of the damped wave equation and the Euler-Poisson-Darboux equation on bounded domains.

Exterior domains are considered in the papers of A. Matsumura, [Mat77], H. Uesaka, [Ues80] and F. Hirosawa and H. Nakazawa, [HN03] to deduce energy estimates. Furthermore, in [Ike02] R. Ikehata has proven the diffusive structure of damped waves in exterior domains.

Results for domains which are neither interior nor exterior are rather seldom in the literature. Recently P. Lesky and R. Racke, [LR03], obtained L^p–L^q decay estimates for the wave and the Klein-Gordon equation in so-called wave guides. These are domains of the structure $\Omega \times \mathbb{R}^m$ with a bounded domain Ω with smooth boundary, such that Poincaré inequality is valid in them.

Abstract problems of the form

$$u_{tt} + Au + b(t)u_t = 0$$

for a function $u(t)$ taking values in a Hilbert space H and with a positive closed operator $A : H \supseteq D(A) \to H$ can be treated by the same arguments in terms of a spectral calculus for the operator A.

For the corresponding damped problem with $b(t) \equiv 1$ R. Ikehata and K. Nishihara investigated in [IN03] a corresponding diffusion phenomenon towards an abstract parabolic problem.

A scattering theory for abstract Cauchy problems with time-dependent operator $A(t)$ was developed by A. Arosio in [Aro84]. The treatment is closely related to our approach of Section 3.1.

Coefficients depending on both variables seem to be a closely related problem. Nevertheless, there arise essential problems in dealing with

$$u_{tt} - \Delta u + b(t,x)u_t = 0, \qquad u(0,\cdot) = u_1, \quad \mathrm{D}_t u(0,\cdot) = u_2..$$

The main point is that one has to control all frequencies in order to deduce sharp operator estimates. By means of the pseudo-differential calculus and a diagonalization/decoupling procedure J. Rauch and M. Taylor obtained in [RT75] estimates of the solution and the energy in the Calkin algebra $\mathcal{L}(L^2)/_{\mathcal{K}(L^2)}$ of bounded modulo compact operators.

The obtained pseudo-differential representations are closely related to our results restricted to the hyperbolic part. In case of non-effective dissipation their results transfer to estimates in the operator algebra. Our considerations show that for effective dissipation terms essential properties of the solutions are lost in this way.

A different approach to handle coefficients depending on t and x are so-called weighted energy inequalities. By means of this technique the cited results of A. Matsumura, [Mat77], H. Uesaka, [Ues80], K. Mochizuki, [Moc77], [MN96] and F. Hirosawa / H. Nakazawa, [HN03], are obtained. All these results are estimates in L^2-scale and provide no further structural information on the representation of solutions.

For coefficients depending on x only and under the strong effectivity assumption

$$b(x) \geq c_0 > 0 \quad \text{for large values of } |x|,$$

M. Nakao, [Nak01], has proven L^p–L^q estimates related to damped waves. His approach works on general exterior domains with further effectivity assumptions near parts of the boundary and is based on L^2-estimates for the local energy.

It is an interesting question to weaken the above effectivity assumption for large x and to consider coefficients estimated from below like

$$b(x) \geq c_0 \langle x \rangle^{-\alpha}$$

for some $\alpha \in (0,1)$. For the upper estimate $|b(x)| \leq \langle x \rangle^{-1-\epsilon}$ it is known from the scattering results of K. Mochizuki, [Moc77], that the solutions are asymptotically free. One may conjecture that in this case the same L^p–L^q estimates like for the free wave equation are valid.

Recently T. Matsuyama, [Mat], considered coefficients depending on t and x, but supported only in a set of the form $\{(t,x) \mid |x| \leq C(1+t)^{\alpha}\}$ with $\alpha \in (0, \frac{1}{2})$. Under this assumption the dissipation term is not effective and he obtained L^p–L^q estimates related to free waves and also a scattering result, both based on local energy estimates.

Appendices

A Notation – Guide to the reader

A.1 Preliminaries

We use *slanted* text style for notions, we define or introduce in the text, while *italics* is used to emphasise words and phrases and to underline links to other fields of research. Not all parts are completely split into theorems and their proofs, at several passages a step by step derivation of results is preferred and theorems are formulated as conclusions of these calculations.

In formulas, the brackets [, (, { are used without special meaning in order to underline several levels. Furthermore, $\{\cdot\}$ is also used to denote sets. Bracket symbols with special meaning are

$\langle\cdot\rangle$	which stands for $\langle x\rangle = \sqrt{1+\lvert x\rvert^2}$,
$\lvert\cdot\rvert$	denotes the absolute value of a scalar expression and for a matrix the matrix of the absolute values of its entries,
$[\xi]$	with definition $[\xi] = \lvert\xi\rvert / \langle\xi\rangle$,
$\lceil\cdot\rceil$	denotes the smallest integer larger then a given number, $\lceil x\rceil = \min\{m\in\mathbb{Z} \mid x\le m\}$,
$\lfloor\cdot\rfloor$	corresponds to $\lfloor x\rfloor = \max\{m\in\mathbb{Z} \mid x\ge m\}$,
$\lvert\lvert\cdot\rvert\rvert$	for a vector or a matrix denotes a sub-multiplicative matrix norm. We use the row sum norm in applications.

The matrix norm has to be distinguished from norms in certain function spaces or operator norms. The corresponding space is used as index of this norm. Exceptions are the frequently used Lebesgue and Bessel potential spaces, where we set

$$||\cdot||_p = ||\cdot||_{L^p} \qquad \text{and} \qquad ||\cdot||_{p,r} = ||\cdot||_{L_{p,r}}.$$

Operator norms and operator spaces are denoted by an intuitive arrow notation, e.g.

$L^p \to L^q$	for $\mathcal{L}(L^p, L^q)$, endowed with the norm topology, and
$\lvert\lvert\cdot\rvert\rvert_{p,r\to q}$	for the operator norm in $L_{p,r} \to L^q$.

Furthermore, the asymptotic relations

$f \lesssim g$	if there exists a constant $C > 0$, such that for all arguments $f \le Cg$ holds,
$f \gtrsim g$	if $g \lesssim f$,
$f \sim g$	if $f \lesssim g$ and $g \lesssim f$

for nonnegative functions f and g are used frequently. We use $f \approx g$, if we need a stronger equivalence of functions, the notion may vary from occurrence to occurrence and will be explained there. In these cases the quotient f/g is more regular than just bounded from below and from above.

A.2 Frequently used function spaces

We collect some of the function spaces occurring in this thesis together with a short definition:

$L^p(\mathbb{R}^n)$	Lebesgue spaces, $1 \le p \le \infty$,
$L^pL^r(\mathbb{R}^n \times \mathbb{R}^m)$	mixed space, $L^pL^r(\mathbb{R}^n \times \mathbb{R}^m) = L^p(\mathbb{R}^n, L^r(\mathbb{R}^m))$,
$L_{p,\alpha(\mathbb{R}^n)}$	Bessel potential spaces, $L_{p,\alpha}(\mathbb{R}^n) = \langle \mathrm{D}\rangle^{-\alpha} L^p(\mathbb{R}^n)$,
$\dot{L}_{p,\alpha}(\mathbb{R}^n)$	Riesz potential spaces, $\dot{L}_{p,\alpha}(\mathbb{R}^n) = \lvert\mathrm{D}\rvert^{-\alpha} L^p(\mathbb{R}^n) \subseteq \mathcal{S}'_{\mathcal{P}}(\mathbb{R}^n)$,
$H^s(\mathbb{R}^n)$, $\dot{H}^s(\mathbb{R}^n)$	special case for $p = 2$: $H^s(\mathbb{R}^n) = L_{2,s}(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n) = \dot{L}_{2,s}(\mathbb{R}^n)$,
$H^{\pm\infty}(\mathbb{R}^n)$	projective and inductive limit, $H^{\infty}(\mathbb{R}^n) = \bigcap_s H^s(\mathbb{R}^n)$ and $H^{-\infty}(\mathbb{R}^n) = \bigcup_s H^s(\mathbb{R}^n) = \big(H^{\infty}\big)'$,
$C^k(\mathbb{R}^n)$	space of k-times continuously differentiable functions,
$C^\infty(\mathbb{R}^n)$	inductive limit $C^\infty(\mathbb{R}^n) = \bigcap_k C^k(\mathbb{R}^n)$,
$B^k(\mathbb{R}^n)$	$B^k(\mathbb{R}^n) = \{f \in C^k(\mathbb{R}^n) \mid \mathrm{D}_x^\alpha f \in L^\infty \ \forall \lvert\alpha\rvert \le k\}$,
$B^\infty(\mathbb{R}^n)$	inductive limit $B^\infty(\mathbb{R}^n) = \bigcap_k B^k(\mathbb{R}^n)$,
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space of rapidly decaying functions, $\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid x^\alpha \mathrm{D}_x^\beta f(x) \in L^\infty \ \forall \alpha, \beta\}$,
$\mathcal{S}'(\mathbb{R}^n)$	space of tempered distributions,
$\mathcal{S}_0(\mathbb{R}^n)$	space of Schwartz functions satisfying moment conditions of all orders, $\mathcal{S}_0(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) \mid \mathrm{D}_\xi^\alpha \hat{f}(0) = 0, \ \forall \alpha\}$, (sometimes also denoted as $\mathcal{Z}(\mathbb{R}^n)$, cf. [RS96, 2.6]),
$\mathcal{S}'_{\mathcal{P}}(\mathbb{R}^n)$	dual space of $\mathcal{S}_0(\mathbb{R}^n)$, tempered distributions modulo polynomials,
$B^s_{p,q}(\mathbb{R}^n)$	Besov space $B^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \mid 2^{sj}\phi_j(\mathrm{D})f \in \ell^q_j(L^p_x)\}$, where $\{\phi_j(\xi)\}_{j\in\mathbb{N}_0}$ is dyadic partition of unity, [RS96],
$F^s_{p,q}(\mathbb{R}^n)$	Lizorkin-Triebel space, defined as $F^s_{p,q}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \mid 2^{sj}\phi_j(\mathrm{D})f \in L^p_x(\ell^q_j)\}$,
$\dot{B}^s_{p,q}(\mathbb{R}^n)$, $\dot{F}^s_{p,q}(\mathbb{R}^n)$	corresponding homogeneous spaces (i.e. using a full dyadic decomposition $\{\phi_j(\xi)\}_{j\in\mathbb{Z}}$ and replacing $\mathcal{S}'$ by $\mathcal{S}'_{\mathcal{P}}$),
$A(\mathbb{R}^n)$	space of real analytic functions,

Besov and Lizorkin-Triebel spaces are independent of the chosen dyadic decomposition and the resulting norms are equivalent. Due to its importance for the subject of the thesis, we mention also the following multiplier spaces:

$M^q_p(\mathbb{R}^n)$	space of multipliers inducing bounded translation invariant operators $L^p \to L^q$, [Hör60],
$S^k_{(\ell)}$, S^k	set of multipliers with symbol-like estimates of order k with restricted smoothness ℓ, $S^k_{(\ell)} = \{\, m(\xi) \in C^\ell(\mathbb{R}^n) \mid \lvert \mathrm{D}_\xi^\alpha m(\xi)\rvert \le C_\alpha \langle\xi\rangle^{k-\lvert\alpha\rvert}, \lvert\alpha\rvert \le \ell \,\}$, endowed with the induced locally-convex topology, $S^k = S^k_{(\infty)}$,
$\dot{S}^k_{(\ell)}$, $\dot{S}^k$	corresponding spaces with homogeneous estimates, $\dot{S}^k_{(\ell)} = \{\, m(\xi) \in C^\ell(\mathbb{R}^n \setminus \{0\}) \mid \lvert \mathrm{D}_\xi^\alpha m(\xi)\rvert \le C_\alpha \lvert\xi\rvert^{k-\lvert\alpha\rvert}, \lvert\alpha\rvert \le \ell \,\}$, $\dot{S}^k = \dot{S}^k_{(\infty)}$.

A.3 Symbols used throughout the thesis

Some of the symbols are used in all chapters of the thesis and for convenience of the reader we will collect them here. The following list can also be seen as a list of definitions for these auxiliary functions. If the symbols are related to a particular chapter, we give also the corresponding reference.

Our aim is the investigation of the Cauchy problem

$$\Box u + b(t)u_t = 0, \qquad u(0,\cdot) = u_1, \quad \mathrm{D}_t u(0,\cdot) = u_2 \tag{A.3.1}$$

with time-dependent dissipation term $b(t)u_t$. Related to it we use

$E(u;t)$	hyperbolic energy, $E(u;t) = \frac{1}{2}\int(\lvert\nabla u\rvert^2 + \lvert u_t\rvert^2)\mathrm{d}x$,
$\Phi_j(t,\xi)$, $j=1,2$	Fourier multiplier of the solution representation, i.e. $\hat{u}(t,\xi) = \sum_{j=1,2}\Phi_j(t,\xi)\hat{u}_j(\xi)$; it is a fundamental system of solutions to the ordinary differential equation $\hat{u}_{tt} + \lvert\xi\rvert^2\hat{u} + b(t)\hat{u}_t = 0$,
$h(t,\xi)$	$h(t,\xi) = \frac{N}{1+t}\phi_{diss,N}(t,\xi) + \lvert\xi\rvert\phi_{hyp,N}(t,\xi)$ with the characteristic functions $\phi_{diss,N}$ and $\phi_{hyp,N}$ of the zones used in the case of non-effective dissipation,
$U(t,\xi)$	micro-energy, $U = (h(t,\xi)\hat{u}, \mathrm{D}_t\hat{u})^T$, satisfies $\mathrm{D}_t U = A(t,\xi)U$,
$\mathcal{E}(t,s,\xi)$	fundamental solution to $\mathrm{D}_t - A(t,\xi)$, i.e. the matrix-valued solution to $\mathrm{D}_t\mathcal{E} = A(t,\xi)\mathcal{E}$, $\mathcal{E}(s,s,\xi) = I \in \mathbb{C}^{2\times 2}$,
$\mathcal{E}_0(t,s,\xi)$	$\mathcal{E}_0(t,s,\xi) = \operatorname{diag}\big(\exp((t-s)\lvert\xi\rvert), \exp((s-t)\lvert\xi\rvert)\big)$,
$\mathcal{E}_k(t,s,\xi)$	fundamental solution of the system after k steps of diagonalization, $k \geq 1$, used in Chapter 3,
$\lambda(t)$	$\lambda(t) = \exp\left\{\frac{1}{2}\int_0^t b(\tau)\mathrm{d}\tau\right\}$,
$W_+(\xi)$	multiplier corresponding to the Møller wave operator (Section 3.1) or the modified wave operator (Section 3.5),
$\mathbb{E}(t,\xi)$	multiplier corresponding to the energy operator $\mathbb{E}(t,\mathrm{D}) : (\langle\mathrm{D}\rangle u_1, u_2)^T \mapsto (\lvert\mathrm{D}\rvert u, \mathrm{D}_t u)$, it consists of the columns $\mathbb{E}_0^1(t,\xi)$ and $\mathbb{E}_1^1(t,\xi)$,
$\mathbb{E}_\ell^k(t,\xi)$	multiplier corresponding to the operator $(\langle\mathrm{D}\rangle^k u_1, \langle\mathrm{D}\rangle^{k-1} u_2)^T \mapsto \lvert\mathrm{D}\rvert^{k-\ell}\mathrm{D}_t^\ell u$, Section 5.3,
$\mathbb{S}(t,\xi)$	multiplier corresponding to the solution operator $\mathbb{S}(t,\mathrm{D}) : (u_1, \langle\mathrm{D}\rangle^{-1} u_2)^T \mapsto u$, Section 5.2,
$v(t,x)$	$v(t,x) = \lambda(t)u(t,x)$, used in Chapter 4,
$\gamma(t)$, Γ	admissible function $\gamma(t)$, used in Chapter 4 to define the separating curve $\Gamma = \{\gamma(t) = \lvert\xi\rvert\}$,
$\langle\xi\rangle_{\gamma(t)}$	$\langle\xi\rangle_{\gamma(t)} = \sqrt{\lvert\lvert\xi\rvert^2 - \gamma^2(t)\rvert}$ (Chapter 4),
$V(t,\xi)$	micro-energy, $V(t,\xi) = (\langle\xi\rangle_{\gamma(t)}\hat{v}, \mathrm{D}_t\hat{v})^T$,
$\mathcal{E}_V(t,s,\xi)$	fundamental solution of the system for $V(t,\xi)$,
$\mathcal{E}_{V,k}(t,s,\xi)$	fundamental solution after k steps of diagonalization in the hyperbolic and in the elliptic zone, Section 4.2.

B Basic tools

The purpose of this appendix is, to collect several basic tools, which are essential for the results of this thesis. They are well-known and, only if necessary and possible, we sketch the main ideas of the proof.

B.1 Bessel's differential equation and Bessel functions

In Chapter 2 we used a reduction of our partial differential equation to Bessel's equation in order to represent solutions explicitly. Following the treatise of G.N. Watson, [Wat22], we collect some of the most important formulae used throughout the calculations of Chapter 2.

There are several ways to define the Bessel functions. We will use the power series expansion

$$\Lambda_\nu(z) = z^{-\nu}\mathcal{J}_\nu(z) = \sum_{k=1}^{\infty}(-1)^k \frac{z^{2k}}{2^{\nu+2k}\Gamma(k+1)\Gamma(k+\nu+1)} \tag{B.1.1}$$

to define the *Bessel function of first kind and order* $\nu \in \mathbb{R}$, $\mathcal{J}_\nu(z)$. These functions satisfy the ordinary differential equation

$$z^2 w'' + zw' + (z^2 - \nu^2)w = 0. \tag{B.1.2}$$

For non-integral values of ν the functions $\mathcal{J}_\nu(z)$ and $\mathcal{J}_{-\nu}(z)$ are linearly independent and therefore they form a fundamental system of solutions. For integral values of ν one has to find a suitable replacement. Due to H.M. Weber (and in this form L. Schläfli) one defines

$$\mathcal{Y}_\nu(z) = \frac{\mathcal{J}_\nu(z)\cos\nu\pi - \mathcal{J}_{-\nu}(z)}{\sin\nu\pi} \tag{B.1.3}$$

analytically continued to $\nu \in \mathbb{C}$, and calls $\mathcal{Y}_\nu(z)$ the Bessel function of second kind and order ν or, shortly, *Weber function* of this order. To understand their properties for small arguments, one may use the relation

$$\begin{aligned} 2\left(\gamma + \log\frac{z}{2}\right)\mathcal{J}_\nu(z) - \pi\mathcal{Y}_\nu(z) = &\sum_{r=0}^{\nu-1}\frac{(\nu-r-1)!}{r!}\left(\frac{z}{2}\right)^{\nu-2r} \\ &+ \sum_{r=0}^{\infty}(-1)^r\frac{\psi(r+\nu+1)-\psi(r+1)+2\gamma}{r!(\nu+r)!}\left(\frac{z}{2}\right)^{\nu+2r} \end{aligned} \tag{B.1.4}$$

for integral values of ν. In this formula γ stands for the Euler-Mascheroni constant, $\gamma = -\Gamma'(1)$, and ψ denotes the Gaussian ψ-function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{s=1}^{z-1}\frac{1}{s}, \tag{B.1.5}$$

the last expression for $z = 1, 2, \ldots$. The treatment of large real arguments is simpler, if one uses the Bessel functions of third kind or *Hankel functions*. They are defined due to N. Nielsen as

$$\mathcal{H}_\nu^{\pm}(z) = \mathcal{J}_\nu(z) \pm i\mathcal{Y}_\nu(z) \tag{B.1.6}$$

and, contrary to the functions of first and second kind, they are complex-valued for real arguments. All three kinds of Bessel functions satisfy the same kind of recurrence relations. We will write down them only for the Hankel functions. It holds, [Wat22, §3.61],

$$\mathcal{H}^{\pm}_{\nu-1}(z) + \mathcal{H}^{\pm}_{\nu+1}(z) = \frac{2\nu}{z}\mathcal{H}^{\pm}_{\nu}(z), \tag{B.1.7a}$$

$$\mathcal{H}^{\pm}_{\nu-1}(z) - \mathcal{H}^{\pm}_{\nu+1}(z) = 2(\mathcal{H}^{\pm}_{\nu})'(z), \tag{B.1.7b}$$

$$\nu\mathcal{H}^{\pm}_{\nu}(z) + z(\mathcal{H}^{\pm}_{\nu})'(z) = z\mathcal{H}^{\pm}_{\nu-1}(z), \tag{B.1.7c}$$

$$\nu\mathcal{H}^{\pm}_{\nu}(z) - z(\mathcal{H}^{\pm}_{\nu})'(z) = z\mathcal{H}^{\pm}_{\nu+1}(z). \tag{B.1.7d}$$

To understand these fundamental systems of solutions, it is of aid to know its Wronskian. Following the treatment of G.N. Watson, [Wat22, §3.63], it holds

$$\mathcal{W}(\mathcal{J}_\nu(z), \mathcal{Y}_\nu(z)) = \frac{2}{\pi z} \tag{B.1.8}$$

and, therefore, also

$$\mathcal{W}(\mathcal{H}^{+}_{\nu}(z), \mathcal{H}^{-}_{\nu}(z)) = -2i\mathcal{W}(\mathcal{J}_\nu(z), \mathcal{Y}_\nu(z)) = -\frac{4i}{\pi z}. \tag{B.1.9}$$

B.2 Fourier multiplier and multiplier spaces

By the aid of the *Fourier transform*

$$\hat{f}(\xi) = \mathcal{F}_{x\to\xi}[f] = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\mathrm{d}x \tag{B.2.1}$$

with inverse $\mathcal{F}^{-1}_{\xi\to x} = \mathcal{F}^{*}_{x\to\xi}$, extendible to isomorphisms $\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, we can use the description of translation invariant operators by so-called *Fourier multipliers*. In our notation we write

$$m(\mathrm{D})f := \mathcal{F}^{-1}_{\xi\to x}\big[m(\xi)\mathcal{F}_{x\to\xi}[f]\big] \tag{B.2.2}$$

for a suitably regular function or distribution $m(\xi)$, the *multiplier*. For details on operators of this kind we refer to the treatment in the paper of L. Hörmander, [Hör60].

Basic facts follow directly from the mapping properties of the Fourier transform,

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad \text{unitary}, \tag{B.2.3}$$

$$\mathcal{F} : L^1(\mathbb{R}^n) \to C_\infty(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n), \qquad C_\infty(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) \mid \lim_{|x|\to\infty} f(x) = 0\}, \tag{B.2.4}$$

together with Hölder's inequality. We denote for $p \le q$

$$M^q_p(\mathbb{R}^n) := \{m(\xi) \mid m(\mathrm{D}) : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)\}, \tag{B.2.5}$$

the so-called *multiplier space*[1]. It is a Banach space endowed with the corresponding operator norm. It holds

Proposition B.1. *1.* $M^2_2(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$,

[1] For $p > q$ there exist no bounded translation invariant operators except the trivial one.

2. $M_p^p(\mathbb{R}^n) \subseteq M_2^2(\mathbb{R}^n)$ *for all* $p \in [1,\infty]$,
3. $M_p^p(\mathbb{R}^n) = M_q^q(\mathbb{R}^n)$ *for dual* p *and* q,
4. $L^1(\mathbb{R}^n) \subseteq M_1^\infty(\mathbb{R}^n)$,
   the same for the space of bounded complex measures, $\mathbb{M}_b(\mathbb{R}^n) \subseteq M_1^\infty(\mathbb{R}^n)$,
5. $M_1^\infty(\mathbb{R}^n) \cap M_2^2(\mathbb{R}^n) \subseteq M_p^q(\mathbb{R}^n)$ *for arbitrary dual* p *and* q.

Under regularity assumptions, the concatenation of such operators corresponds to the multiplication of the multipliers. This may be used to deduce mapping properties of given multipliers in connection with the following characterisation.

Theorem B.2 (Marcinkiewicz multiplier theorem). *Assume* $m(\xi) \in C^k(\mathbb{R}^n \setminus \{0\})$ *for* $k = \lceil \frac{n}{2} \rceil$ *and*

$$\left|\mathrm{D}_\xi^\alpha m(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|}$$

for all $|\alpha| \leq k$.[2] *Then* $m \in M_p^p(\mathbb{R}^n)$ *for all* $p \in (1,\infty)$.

A proof can be found in the book of E.M. Stein on singular integrals, [Ste70, Chapter IV.3].

Example B.1. The multiplier $R_i(\xi) = \xi_i/|\xi|$ satisfies the assumptions of Theorem B.2 and defines, therefore, a bounded linear operator

$$R_i(\mathrm{D}) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

for all $p \in (1,\infty)$, the so-called i-th *Riesz transform*. Together with the formal unitarity

$$\sum_{i=1}^n R_i(\mathrm{D}) R_i^*(\mathrm{D}) = I,$$

the operator $R = (R_1, \ldots, R_n)^T$ forms an isomorphism $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, \mathbb{R}^n)$. Furthermore, together with

$$R(\mathrm{D})\,|\mathrm{D}| f = \nabla f,$$

we deduce, that for all $p \in (1,\infty)$ the norms $||\,|\mathrm{D}| f||_p$ and $||\nabla f||_p$ are equivalent.

The cases $p = 1$ and $p = \infty$ are exceptional.

To deduce M_p^q-properties, the oscillatory behaviour of the multiplier may be of importance. In Chapter 2 we use a dyadic decomposition and mapping properties in Besov spaces combined with the stationary phase method. Basic tool is the following version of Littman's lemma taken from the paper of P. Brenner, [Bre75, Lemma 4].

Lemma B.3. *Let* P *be a real and smooth function in the neighbourhood of* $\operatorname{supp}\phi$, $\phi \in C_0^\infty$. *Assume further, that the rank of the Hessian* $H_P(\xi) = (\partial^2 P/\partial_{\xi_i}\partial_{\xi_j})$ *is at least* ρ *on* $\operatorname{supp}\phi$. *Then, there exists an integer* M, *depending on the space dimension, and a constant* $C > 0$, *depending on bounds of derivatives of* P *on* $\operatorname{supp}\phi$, *such that*

$$||\mathcal{F}^{-1}[e^{itP(\xi)}\phi(\xi)]||_\infty \leq C\,\langle t\rangle^{-\frac{\rho}{2}} \sum_{|\alpha|\leq M} ||\mathrm{D}^\alpha \phi||_1$$

holds.

[2] Thus, with the notation introduced on page 49 we need $m \in \dot{S}_k^0$.

For completeness, we give also a lemma which explains how to sample the estimates for the dyadic components. It is a combination of [Bre75, Lemma 1] and [Bre75, Lemma 2]. Let, therefore, $\chi_j(\xi)$ be a dyadic decomposition satisfying (2.2.7). Basic idea of the proof are embedding relations between Lebesgue and Besov spaces.

Lemma B.4. *Let $a \in L^\infty(\mathbb{R}^n)$ and assume that*

$$||\mathcal{F}^{-1}[a(\xi)\chi_j(\xi)\hat{v}]||_q \leq C||v||_p$$

holds uniform for all j and a dual pair $p \in (1,2]$, $pq = p+q$. Then for a constant A independent of a it follows

$$||\mathcal{F}^{-1}[a(\xi)\hat{v}]||_q \leq AC||v||_p.$$

B.3 The Peano-Baker formula

First order systems of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}u = A(t)u, \qquad u(0) = u_0 \in \mathbb{C}^n \tag{B.3.1}$$

are solved in terms of the fundamental solution $\mathcal{E}(t,s)$ as $u(t) = \mathcal{E}(t,0)u_0$. The matrix function $\mathcal{E}(t,s)$ is the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,s) = A(t)\mathcal{E}(t,s), \qquad \mathcal{E}(s,s) = I \in \mathbb{C}^{n\times n}. \tag{B.3.2}$$

It is well known, that for a constant matrix this fundamental solution can be expressed in terms of the matrix exponential,

$$\mathcal{E}(t,s) = \exp((t-s)A), \qquad \exp(A) = I + \sum_{k=1}^\infty \frac{1}{k!}A^k. \tag{B.3.3}$$

For variable coefficients this representation is not valid any more. For the sake of completeness, we give the representation used several times throughout our calculations.

Theorem B.5. *Let $A \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental solution $\mathcal{E}(t,s)$ to $\partial_t - A(t)$ is given by the Peano-Baker formula*

$$\mathcal{E}(t,s) = I + \sum_{k=1}^\infty \int_s^t A(t_1) \int_s^{t_1} A(t_2) \cdots \int_s^{t_{k-1}} A(t_k)\mathrm{d}t_k \cdots \mathrm{d}t_2 \mathrm{d}t_1.$$

The proof follows by differentiating the series term by term. To prove the convergence of the series and its formal derivative one uses the domination by the exponential series following from Proposition B.6.

Proposition B.6. *Assume $r \in L^1_{loc}(\mathbb{R})$. Then*

$$\left| \int_s^t r(t_1) \int_s^{t_1} r(t_2) \ldots \int_s^{t_{k-1}} r(t_k)\mathrm{d}t_k \ldots \mathrm{d}t_1 \right| \leq \frac{1}{k!}\left(\int_s^t |r(\tau)|\mathrm{d}\tau \right)^k \tag{B.3.4}$$

for all $k \in \mathbb{N}$.

The proof follows by induction over k.

Corollary B.7. *Let $A \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental matrix $\mathcal{E}(t,s)$ satisfies*

$$||\mathcal{E}(t,s)|| \le \exp\left\{\int_s^t ||A(\tau)|| \mathrm{d}\tau\right\}.$$

In several applications we need not only the estimates for the fundamental solution, but also statements about its asymptotic behaviour and invertibility. It is convenient to use the Theorem of Liouville in the following form, a proof may be found in standard text-books on differential equations like the one of V.I. Arnold, [Arn01], or M.V. Fedoryuk, [Фед85].

Theorem B.8. *Let $A \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{n\times n})$. Then the fundamental solution $\mathcal{E}(t,s)$ satisfies*

$$\det \mathcal{E}(t,s) = \exp \int_s^t \operatorname{tr} A(\tau) \mathrm{d}\tau.$$

B.4 Remarks on Volterra integral equations

The estimate of Corollary B.7 is in general not sharp, to obtain better estimates, we are interested in solutions to the Volterra equation

$$f(t,p) + \int_0^t k(t,\tau,p) f(\tau,p) \mathrm{d}\tau = \psi(t,p) \tag{B.4.1}$$

with kernel $k = k(t,\tau,p)$ and right-hand side $\psi(t,p)$, both depending on some parameter $p \in P \subseteq \mathbb{R}^n$.

Theorem B.9. *Assume $\psi \in L^\infty(\mathbb{R}_+ \times P)$, $k \in L^\infty(\mathbb{R}_+^2 \times P)$ and*

$$\int_0^t ||k(\cdot,\tau,p)||_\infty \mathrm{d}\tau \in L^\infty(\mathbb{R}_+ \times P).$$

Then, there exists a (unique) solution $f(t,p)$ of (B.4.1) *in $L^\infty(\mathbb{R}_+ \times P)$,*

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}_+,\, p\in P} |f(t,p)| < \infty.$$

Sketch of the proof. Uniqueness of the solution follows for small t by the contraction mapping principle. It remains to show the global bound on the solution.

We may represent the solutions to this integral equation by the Neumann series

$$f(t,p) = \psi(t,p) + \sum_{k=1}^\infty (-1)^k \int_0^t k(t,t_1,p) \int_0^{t_1} k(t_1,t_2,p) \\ \cdots \int_0^{t_{k-1}} k(t_{k-1},t_k,p)\psi(t_k,p) \mathrm{d}t_k \cdots \mathrm{d}t_2 \mathrm{d}t_1$$

and use Proposition B.6 to conclude

$$\begin{aligned} ||f(t,p)||_\infty &\le ||\psi||_\infty \left(1 + \sum_{k=1}^\infty \int_0^t ||k(\cdot,t_1,p)||_\infty \int_0^{t_1} ||k(\cdot,t_2,p)||_\infty \cdots \mathrm{d}t_1\right) \\ &\le ||\psi||_\infty \exp\left\{\int_0^t ||k(\cdot,\tau,p)||_\infty\right\}. \end{aligned}$$

□

For results under weaker assumptions on the integral kernel we refer to the treatment of G. Gripenberg, S.-O. Londen and O. Staffans, [GLS90].

For the applications we may take also domains for the parameter p depending on both variables t and τ. In this case one can trivially extent the kernel function $k(t,\tau,p)$ by zero to a larger common parameter domain without changing the solution. This will be the case in most of the applications.

Due to its importance for the understanding of the results in Chapter 4 we give one auxiliary application of this theorem.

Theorem B.10. *Assume*

$$A(t,p) \in L^\infty(P, L^1_{loc}(\mathbb{R}_+, \mathbb{C}^{n\times n})), \quad \textit{diagonal,} \quad \operatorname{Re} A(t,p) \le a(t,p) I$$
$$B(t,p) \in L^\infty(P, L^1(\mathbb{R}_+, \mathbb{C}^{n\times n})).$$

Then the fundamental solution $\mathcal{E}(t,s,p)$ to $\partial_t - A(t,p) - B(t,p)$ satisfies

$$||\mathcal{E}(t,s,p)|| \lesssim \exp\left\{ \int_s^t a(\tau,p)\mathrm{d}\tau \right\}.$$

Sketch of proof. In order to prove this, we consider the fundamental solution $\mathcal{E}_0(t,s,p)$ to the system $\partial_t - A(t,p)$ and conclude from

$$\mathcal{E}(t,s,p) = \exp\left\{ \int_s^t A(\tau,p)\mathrm{d}\tau \right\}, \qquad \partial_t \mathcal{E}_0^{-1}(t,s,p) = -\mathcal{E}_0^{-1}(t,s,p)A(t,p),$$

that

$$\partial_t(\mathcal{E}_0^{-1}(t,s,p)\mathcal{E}(t,s,p)) = \mathcal{E}_0^{-1}(t,s,p)B(t,p)\mathcal{E}(t,s,p).$$

Thus, we obtain the integral equation

$$\begin{aligned}\mathcal{E}(t,s,p) &= I + \mathcal{E}_0(t,s,p)\int_s^t \mathcal{E}_0^{-1}(\tau,s,p)B(\tau,p)\mathcal{E}(\tau,s,p)\mathrm{d}\tau \\ &= I + \int_s^t \mathcal{E}_0(t,\tau,p)B(\tau,p)\mathcal{E}(\tau,s,p)\mathrm{d}\tau,\end{aligned}$$

which can be transformed to

$$\begin{aligned}\exp\left\{ -\int_s^t a(\tau,p)\mathrm{d}\tau \right\}\mathcal{E}(t,s,p) =& \exp\left\{ -\int_s^t a(\tau,p)\mathrm{d}\tau \right\} \\ &+ \int_s^t \exp\left\{ \int_s^t [A(\tau,p) - a(\tau,p)I]\mathrm{d}\tau \right\} B(\tau,p)\mathcal{E}(\tau,s,p)\mathrm{d}\tau.\end{aligned}$$

Now the exponential term is bounded by I and the assumptions on $B(t,p)$ an be used to conclude the boundedness of $\exp\{-\int_s^t a(\tau,p)\mathrm{d}\tau\}\mathcal{E}(t,s,p)$ by Theorem B.9. □

Remark B.2. If the parameter domain P is compact and $B(t,p) \in C(P, L^1(\mathbb{R}_+, \mathbb{C}^{n\times n}))$, the second condition on B is vacuous and follows from the L^1-property.

B.5 Potential spaces

Under the notion *potential space over L^p*, we understand a (in most cases dense) subspace of L^p, which is representable as image of a Fourier multiplier and endowed with the induced norm. Thus, for given $\phi(\xi) \in M_p^p(\mathbb{R}^n)$, we consider

$$\phi(\mathrm{D})L^p(\mathbb{R}^n) = \{ \ \phi(\mathrm{D})f \mid f \in L^p(\mathbb{R}^n) \ \}$$

and define the norm in this space by

$$||g||_{\phi(\mathrm{D})L^p} = \inf_{g=\phi(\mathrm{D})f} ||f||_{L^p}.$$

Using the reflexivity of $L^p(\mathbb{R}^n)$, $p \in (1,\infty)$, one obtains, that the infimum is really taken. The vector space $\phi(\mathrm{D})L^p(\mathbb{R}^n)$ with the norm $||\cdot||_{\phi(\mathrm{D})L^p}$ forms a Banach space.

Special examples of such potential spaces are the well-known Bessel potential spaces $L_{p,r}(\mathbb{R}^n) = \langle \mathrm{D} \rangle^{-r} L^p(\mathbb{R}^n)$ used in this thesis as representations of fractional order Sobolev spaces over $\mathbb{R}^n$ or the sets $[\mathrm{D}]^k L^p(\mathbb{R}^n)$ of functions having a zero in the frequency $\xi = 0$ of order k.[3]

Note, that for the definition of potential spaces only the residue class of ϕ modulo invertible elements in $M_p^p(\mathbb{R}^n)$ is of interest. Translation invariant operators between such potential spaces can be characterised using the classes $M_p^q(\mathbb{R}^n)$. It holds (if the symbols are sufficiently regular functions such that the multiplication is well defined in the multiplier space, [Hör60])

$$m(\mathrm{D}) : \phi_1(\mathrm{D})L^p(\mathbb{R}^n) \to \phi_2(\mathrm{D})L^q(\mathbb{R}^n) \qquad \text{iff} \qquad \phi_2^{-1}(\xi)m(\xi)\phi_1(\xi) \in M_p^q(\mathbb{R}^n).$$

For us the situation $p = q = 2$ is of special interest, where $M_2^2(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

[3] There exists a relation to decay assumptions, cf. Lemma 5.3 and the discussion in Section 5.1.

Bibliography

[AMS63] N. Aronszajn, F. Mulla, and P. Szeptycki. On spaces of potentials connected with L^p classes. *Ann. Inst. Fourier (Grenoble)*, 13:211–306, 1963.

[Arn01] V.I. Arnol'd. *Gewöhnliche Differentialgleichungen. Übersetzt aus dem Russischen von Tobias Damm. (Ordinary differential equations). 2. Aufl.* Springer-Lehrbuch. Berlin: Springer. xii, 344 S. DM 49.90, 2001.

[Aro84] Alberto Arosio. Linear second order differential equations in Hilbert spaces. The Cauchy problem and asymptotic behaviour for large time. *Arch. Ration. Mech. Anal.*, 86:147–180, 1984.

[AS61] N. Aronszajn and K. T. Smith. Theory of Bessel potentials. I. *Ann. Inst. Fourier (Grenoble)*, 11:385–475, 1961.

[Asg36] Leifur Asgeirsson. Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten. *Math. Ann.*, 113:321–346, 1936.

[Bre75] Philip Brenner. On $L_p - L_{p'}$ estimates for the wave-equation. *Math. Z.*, 145(3):251–254, 1975.

[CS76] Robert Wayne Carroll and Ralph E. Showalter. *Singular and degenerate Cauchy problems*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976.

[Dar89] Gaston Darboux. *Leçons sur la théorie générale des surfaces et les applications geométriques du calcul infinitésimal.* Gauthier-Villars et Fils, Paris, 1889.

[DS95] Wakako Dan and Yoshihiro Shibata. On a local energy decay of solutions of a dissipative wave equation. *Funkc. Ekvacioj, Ser. Int.*, 38(3):545–568, 1995.

[Eul70] Leonhard Euler. *Institutiones calculi integralis, Volumen tertium.* Petropolis, 1770.

[Фед85] Михаил В. Федорюк. Обыкновенные дифференциальные уравнения. Учебное пособие. Избранные Главы Высшей Математики для Инженеров и Студентов Вузов. Москва: "Наука". Главная Редакция Физико-Математической Литературы, 1985.

[Fed93] Mikhail V. Fedoryuk. *Asymptotic analysis: linear ordinary differential equations. Translated from the Russian by Andrew Rodick.* Berlin: Springer-Verlag. viii, 363 p., 1993.

[Fuj66] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci., Univ. Tokyo, Sect. I*, 13:109–124, 1966.

[Gal03] Anahit Galstian. L_p–L_q decay estimates for the equations with exponentially growing speed of propagation. *Appl. Anal.*, 82(3):197–214, 2003.

[GLS90] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations*, volume 34 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990.

[GR98] Th. Gallay and G. Raugel. Scaling variables and asymptotic expansions in damped wave equations. *J. Differ. Equations*, 150(1):42–97, 1998.

[HL92] Ling Hsiao and Tai-Ping Liu. Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping. *Commun. Math. Phys.*, 143(3):599–605, 1992.

[HN01] Fumihiko Hirosawa and Hideo Nakazawa. Precise energy decay estimates for the dissipative hyperbolic equations in an exterior domain. Preprint 2000-008, University of Tsukuba, Institute of Mathematics, 2001.

[HN03] Fumihiko Hirosawa and Hideo Nakazawa. Rapid decay of the total energy for dissipative wave equations. *Tsukuba J. Math.*, 27(2):217–232, 2003.

[Hör60] Lars Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104:93–140, 1960.

[HR03] Fumihiko Hirosawa and Michael Reissig. From wave to Klein-Gordon type decay rates. In *Albeverio, Sergio (ed.) et al., Nonlinear hyperbolic equations, spectral theory, and wavelet transformations. A volume of Advances in Partial Differential Equations. Basel: Birkhäuser. Oper. Theory, Adv. Appl. 145, 95-155* . 2003.

[Ike02] Ryo Ikehata. Diffusion phenomenon for linear dissipative wave equations in an exterior domain. *J. Differential Equations*, 186(2):633–651, 2002.

[Ike03a] Ryo Ikehata. A remark on a critical exponent for the semilinear dissipative wave equation in the one dimensional half space. *Differ. Integral Equ.*, 16(6):727–736, 2003.

[Ike03b] Ryo Ikehata. Critical exponent for semilinear damped wave equations in the N-dimensional half space. *J. Math. Anal. Appl.*, 288(2):803–818, 2003.

[Ike03c] Ryo Ikehata. Decay estimates by moments and masses of initial data for linear damped wave equations. *Int. J. Pure Appl. Math.*, 5(1):77–94, 2003.

[Ike03d] Ryo Ikehata. Improved decay rates for solutions to one-dimensional linear and semilinear dissipative wave equations in all space. *J. Math. Anal. Appl.*, 277(2):555–570, 2003.

[Ike03e] Ryo Ikehata. L^2-convergence results for linear dissipative wave equations in unbounded domains. *Asymptotic Anal.*, 36(1):63–74, 2003.

[Ike04] Ryo Ikehata. New decay estimates for linear damped wave equations and its application to nonlinear problem. *Math. Meth. Appl. Sci.*, 27:865–889, 2004.

[IM01] Ryo Ikehata and Tokio Matsuyama. L^2-behavior of solutions to the linear heat and wave equations in exterior domains. *Sci. Math. Jpn.*, 55(1):33–42, 2001.

[IN03] Ryo Ikehata and Kenji Nishihara. Diffusion phenomenon for second order linear evolution equations. *Stud. Math.*, 158(2):153–161, 2003.

[Kat66] T. Kato. Wave operators and similarity for some non-selfadjoint operators. *Math. Ann.*, 162:258–279, 1966.

[Kla85] Sergiu Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.

[Lit73] Walter Littman. $L^p - L^q$-estimates for singular integral operators arising from hyperbolic equations. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pages 479–481. Amer. Math. Soc., Providence, R.I., 1973.

[LP73] P.D. Lax and R.S. Phillips. Scattering theory for dissipative hyperbolic systems. *J. Funct. Anal.*, 14:172–235, 1973.

[LR03] Peter H. Lesky and Reinhard Racke. Nonlinear wave equations in infinite waveguides. *Commun. Partial Differ. Equations*, 28(7-8):1265–1301, 2003.

[Mat] Tokio Matsuyama. L^p–estimates for the wave equation with time-dependent dissipation. *to appear*.

[Mat76] Akitaka Matsumura. On the asymptotic behavior of solutions of semi-linear wave equations. *Publ. Res. Inst. Math. Sci.*, 12(1):169–189, 1976.

[Mat77] Akitaka Matsumura. Energy decay of solutions of dissipative wave equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 53(7):232–236, 1977.

[Mel95] Richard B. Melrose. *Geometric scattering theory*. Stanford Lectures. Cambridge University Press, Cambridge, 1995.

[MN96] Kiyoshi Mochizuki and Hideo Nakazawa. Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation. *Publ. Res. Inst. Math. Sci.*, 32(3):401–414, 1996.

[MN03] Pierangelo Marcati and Kenji Nishihara. The L^p–L^q estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media. *J. Differ. Equations*, 191(2):445–469, 2003.

[Moc77] Kiyoshi Mochizuki. Scattering theory for wave equations with dissipative terms. *Publ. Res. Inst. Math. Sci.*, 12(2):383–390, 1976/77.

[Nak] Hideo Nakazawa. Scattering theory for nonconservative wave equations in $\mathbb{R}^n$. *to appear*.

[Nak01] Mitsuhiro Nakao. L^p estimates for the linear wave equation and global existence for semilinear wave equations in exterior domains. *Math. Ann.*, 320(1):11–31, 2001.

[Nar04] Takashi Narazaki. L^p–L^q estimates for damped wave equations and their applications to semi-linear problem. *J. Math. Soc. Japan*, 56(2):585–626, 2004.

[Nei85] Hagen Neidhardt. A nuclear dissipative scattering theory. *J. Oper. Theory*, 14:57–66, 1985.

[Nei89] Hagen Neidhardt. On the dissipative Lax-Phillips scattering theory. *J. Math. Anal. Appl.*, 144(1):75–88, 1989.

[Nis97] Kenji Nishihara. Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping. *J. Differential Equations*, 137(2):384–395, 1997.

[Nis03] Kenji Nishihara. L^p-L^q estimates of solutions to the damped wave equation in 3-dimensional space and their application. *Math. Z.*, 244(3):631–649, 2003.

[Pec76] Hartmut Pecher. L^p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen. I. *Math. Z.*, 150(2):159–183, 1976.

[Pet89] Vesselin Petkov. *Scattering theory for hyperbolic operators.* Studies in Mathematics and Its Applications, 21. Amsterdam etc.: North- Holland. xiv, 373 p. Dfl 190.00; $ 92.75, 1989.

[Pon85] Gustavo Ponce. Global existence of small solutions to a class of nonlinear evolution equations. *Nonlinear Anal., Theory Methods Appl.*, 9:399–418, 1985.

[Rac92] Reinhard Racke. *Lectures on nonlinear evolution equations*. Aspects of Mathematics, E19. Friedr. Vieweg & Sohn, Braunschweig, 1992.

[Rei97] Michael Reissig. On L_p-L_q estimates for solutions of a special weakly hyperbolic equation. In *Proceedings of the Conference on Nonlinear Evolution Equations and Infinite-dimensional Dynamical Systems (Shanghai, 1995)*, pages 153–164, River Edge, NJ, 1997. World Sci. Publishing.

[Rei01] Michael Reissig. Klein-Gordon type decay rates for wave equations with a time-dependent dissipation. *Adv. Math. Sci. Appl.*, 11(2):859–891, 2001.

[RS96] Thomas Runst and Winfried Sickel. *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations.* de Gruyter Series in Nonlinear Analysis and Applications. 3. Berlin: de Gruyter. x, 547 p. DM 268.00 , 1996.

[RS03] Michael Reissig and James Smith. L^p–L^q estimate for wave equation with bounded time-dependent coefficient. Technical Report 08, TU Bergakademie Freiberg, Fakultät für Mathematik und Informatik, 2003.

[RT74] Jeffrey Rauch and Michael Taylor. Exponential decay of solutions to hyperbolic equations in bounded domains. *Indiana Univ. Math. J.*, 24:79–86, 1974.

[RT75] J. Rauch and M. Taylor. Decay of solutions to nondissipative hyperbolic systems on compact manifolds. *Commun. Pure Appl. Math.*, 28:501–523, 1975.

[RY00] Michael Reissig and Karen Yagdjian. L_p-L_q decay estimates for the solutions of strictly hyperbolic equations of second order with increasing in time coefficients. *Math. Nachr.*, 214:71–104, 2000.

[SI00] Akinobu Saeki and Ryo Ikehata. Remarks on the decay rate for the energy of the dissipative linear wave equations in exterior domains. *SUT J. Math.*, 36(2):267–277, 2000.

[Ste70] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[Ste93] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[Str67] R.S. Strichartz. Multipliers on fractional Sobolev spaces. *J. Math. Mech.*, 16:1031–1060, 1967.

[Str70] Robert S. Strichartz. A priori estimates for the wave equation and some applications. *J. Functional Analysis*, 5:218–235, 1970.

[SW71] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971.

[TT80] Kazuo Taniguchi and Yoshiharu Tozaki. A hyperbolic equation with double characteristics which has a solution with branching singularities. *Math. Japon.*, 25(3):279–300, 1980.

[TY01] Grozdena Todorova and Borislav Yordanov. Critical exponent for a nonlinear wave equation with damping. *J. Differential Equations*, 174(2):464–489, 2001.

[Ues80] Hiroshi Uesaka. The total energy decay of solutions for the wave equation with a dissipative term. *J. Math. Kyoto Univ.*, 20(1):57–65, 1980.

[vW71] Wolf von Wahl. L^p-decay rates for homogeneous wave-equations. *Math. Z.*, 120:93–106, 1971.

[Wat22] George N. Watson. *A treatise on the theory of Bessel functions*. Cambridge University Press, 1922.

[Wei54] Alexander Weinstein. On the wave equation and the equation of Euler-Poisson. In *Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory*, pages 137–147. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[Wir02] Jens Wirth. On the existence of the Møller wave operator for wave equations with small dissipative terms. Preprint, arXiv:math.AP/0210098, 2002.

[Wir04] Jens Wirth. Solution representations for a wave equation with weak dissipation. *Math. Meth. Appl. Sc.*, 27(1):101–124, 2004.

[Yag97] Karen Yagdjian. *The Cauchy problem for hyperbolic operators*, volume 12 of *Mathematical Topics*. Akademie Verlag, Berlin, 1997.

[Yag00] Karen Yagdjian. Parametric resonance and nonexistence of global solution to nonlinear hyperbolic equations. In *Partial differential equations (Torino, 2000)*, pages 157–170. Univ. Torino, Turin, 2000.

[Ягд89] Карен Ягдджян. Исследование задачи Коши для слабо гиперболических уравнений. Диссертация, Академия Наук Армянской ССР, Институт Математики, 1989.

[Yam80] Masaru Yamaguchi. Existence and stability of global bounded classical solutions of initial boundary value problem for semilinear wave equations. *Funkc. Ekvacioj, Ser. Int.*, 23:289–308, 1980.

[YM00] Han Yang and Albert Milani. On the diffusion phenomenon of quasilinear hyperbolic waves. *Bull. Sci. Math.*, 124(5):415–433, 2000.

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