ASYMPTOTIC PROPERTIES OF THE LSE IN A REGRESSION MODEL WITH LONG-MEMORY STATIONARY ERRORS

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We consider asymptotic properties of the least squares estimator (LSE) in a regression model with long-memory stationary errors. First we derive a necessary and sufficient condition that the LSE be asymptotically efficient relative to the best linear unbiased estimator (BLUE). Then we derive the asymptotic distribution of the LSE under a condition on the higher-order cumulants of the white-noise process of the errors.

1. Introduction. Let the observed process $\{y_t\}$ follow the regression model of the form

$$y_t = X_t'\beta + \varepsilon_t,$$

where $X_t = (x_{t1}, \ldots, x_{tk})'$ is a k-vector of nonstochastic regressors and $\{\varepsilon_t\}$, the sequence of errors, is a stationary process with mean 0 and spectral density $f(\lambda)$, and $\beta = (\beta_1, \ldots, \beta_k)'$ is a vector of unknown regression parameters. Throughout this paper $f(\lambda)$ is assumed to have the form

$$(1) \qquad f(\lambda) = f^*(\lambda)/|1 - e^{i\lambda}|^{2d}, \qquad 0 < d < 1/2,$$

where $f^*(\lambda)$ is a positive continuous function. Since $f(\lambda)$ diverges to ∞ as $\lambda \to 0$, $\{\varepsilon_t\}$ is a strongly dependent process and its autocorrelations $\gamma_h = E\varepsilon_t\varepsilon_{t+h}$, $h = 0, 1, 2 \ldots$, are not absolutely summable. The $f(\lambda)$ of (1) includes the spectral densities of both a fractional Gaussian noise model [Mandelbrot and Van Ness (1968)] and a fractional ARIMA model [Granger and Joyeux (1980) and Hosking (1981)], two popular long-memory models [Geweke and Porter-Hudak (1983), Theorem 1].

Here we discuss the asymptotic efficiency and the distribution of the LSE for β. These problems have been investigated widely. However, the case that $\{\varepsilon_t\}$ is a stationary process with the spectral density $f(\lambda)$ of (1) has not been clarified fully yet since this process causes considerable mathematical difficulties.

In Section 2 we consider the asymptotic efficiency of the LSE relative to the BLUE. The LSE is not identical with the BLUE unless $\{\varepsilon_t\}$ is an uncorrelated process. However, the LSE is used frequently since the covariance matrix of $\{\varepsilon_t\}$ must be known for the computation of the BLUE while the LSE is always available and is an unbiased estimator. Hence it is important to compare the LSE with the BLUE for various correlation structures of $\{\varepsilon_t\}$.

Grenander (1954) derived a necessary and sufficient condition on X_t that the LSE be asymptotically efficient relative to the BLUE when $f(\lambda)$ is positive and continuous. On the other hand, only a few results seem to be known when $\{\varepsilon_t\}$ follows a long-memory model. Adenstedt (1974), Beran and Künsch (1985) and Samarov and Taqqu (1988) discussed properties of the sample mean, $\bar{y}_T = \sum_{t=1}^{T} y_t/T$ for the case $x_{t1} = 1$, $k = 1$. Beran and Künsch (1985) also considered so-called M-estimators [Huber (1981)]. Yajima (1988) considered the case $x_{ti} = t^{i-1}$, $k \geq 1$, a polynomial regression. One of the remarkable results is that the LSE is no longer asymptotically efficient, which differs from what occurs when $f(\lambda)$ is positive and continuous. Here we evaluate the asymptotic covariance matrices of the LSE and the BLUE for more general regressors. Next we extend Grenander's result to our model.

In Section 3 we discuss an asymptotic distribution for the LSE. Yajima (1989) derived a central limit theorem for finite Fourier transforms, which is equivalent to $x_{ij} = e^{i\lambda_j t}$. Here we consider more general regressors. Eicker (1967) and Hannan (1979) discussed the same problem under the condition that the white-noise process of $\{\varepsilon_t\}$ is independently distributed or a martingale difference. Here we impose a different condition on higher-order cumulants of the white-noise process and evaluate the asymptotic covariance matrix in detail.

2. The asymptotic efficiency of the LSE.
First we introduce so-called Grenander's conditions on X_t [Grenander (1954), Grenander and Rosenblatt (1957) and Anderson (1971)]. Let

$$a_{ij}^T(h) = \sum_{t=1}^{T-h} x_{t+h,i} x_{tj}, \qquad h = 0, 1, \ldots,$$

$$= \sum_{t=1-h}^{T} x_{t+h,i} x_{tj}, \qquad h = 0, -1, \ldots.$$

(G.1) $\qquad a_{ii}^T(0) \to \infty \quad \text{as } T \to \infty, \quad i = 1, \ldots, k.$

(G.2) $\qquad \lim_{T \to \infty} x_{T+1,i}^2/a_{ii}^T(0) = 0, \quad i = 1, \ldots, k.$

(G.3) The limit of

$$a_{ij}^T(h)/\{a_{ii}^T(0)a_{jj}^T(0)\}^{1/2} = r_{ij}^T(h)$$

as $T \to \infty$ exists for every i, j and h, $i, j = 1, \ldots, k$, and $h = 0, \pm 1, \ldots$.

Let

$$\lim_{T \to \infty} r_{ij}^T(h) = \rho_{ij}(h), \qquad i, j = 1, 2, \ldots, k, \qquad h = 0, \pm 1, \ldots,$$

and let $R(h) = [\rho_{ij}(h)]$.

(G.4) $R(0)$ is nonsingular.

Then there is a Hermitian matrix function $M(\lambda)$ with positive semidefinite

increments such that

$$R(h) = \int_{-\pi}^{\pi} e^{ih\lambda} \, dM(\lambda).$$

Grenander (1954) derived a necessary and sufficient condition that the LSE be asymptotically efficient under (G.1)–(G.4) when $f(\lambda)$ is positive and continuous. We present his result to make the comparison between it and our result clear [cf. Grenander (1954), Grenander and Rosenblatt (1957) and Anderson (1971)]. Let $\hat{\beta}_T$ and $\tilde{\beta}_T$ be the LSE and the BLUE, respectively. Then

$$\hat{\beta}_T = \left(\tilde{X}_T' \tilde{X}_T \right)^{-1} \tilde{X}_T' Y_T,$$

$$\tilde{\beta}_T = \left(\tilde{X}_T' \Sigma_T^{-1} \tilde{X}_T \right)^{-1} \tilde{X}_T' \Sigma_T^{-1} Y_T,$$

where $Y_T = (y_1, \ldots, y_T)'$, $\tilde{X}_T = (x_{ti})$, a $T \times k$ matrix of rank k ($\leq T$) and $\Sigma_T = (\sigma_{ij})$ with $\sigma_{ij} = \gamma_{i-j}$ is the $T \times T$ covariance matrix of $\tilde{\varepsilon}_T = (\varepsilon_1, \ldots, \varepsilon_T)'$.

THEOREM (Grenander).   *Let*

$$D_T = \mathrm{diag}(\|x_1\|_T, \ldots, \|x_k\|_T),$$

where $\|x_i\|_T = (a_{ii}^T(0))^{1/2}$.

   (i) *Assume that $f(\lambda)$ is a continuous function in $[-\pi, \pi]$. Then under (G.1)–(G.4),*

$$(2) \quad \lim_{T \to \infty} D_T E\{(\hat{\beta}_T - \beta)(\hat{\beta}_T - \beta)'\} D_T = 2\pi R(0)^{-1} \int_{-\pi}^{\pi} f(\lambda) \, dM(\lambda) R(0)^{-1}.$$

   (ii) *Assume that $f(\lambda)$ is positive and continuous in $[-\pi, \pi]$. Then under (G.1)–(G.4),*

$$(3) \quad \lim_{T \to \infty} D_T E\{(\tilde{\beta}_T - \beta)(\tilde{\beta}_T - \beta)'\} D_T = \left[(2\pi)^{-1} \int_{-\pi}^{\pi} f(\lambda)^{-1} \, dM(\lambda) \right]^{-1}.$$

   (iii) *A necessary and sufficient condition under (G.1)–(G.4) that the LSE be asymptotically efficient for $\{\varepsilon_t\}$ with a positive and continuous spectral density is that $M(\lambda)$ increase at not more than k values of λ, $0 \leq \lambda \leq \pi$, and the sum of the ranks of the increases in $M(\lambda)$ is k.*

Now we turn to the case that $f(\lambda)$ is expressed of the form (1). Let $M_{pq}(\lambda)$ be the (p, q)th entry of $M(\lambda)$. And let

$$m_{pq}^T(\lambda) = \left(\sum_{t=1}^{T} x_{tp} e^{-it\lambda} \right) \left(\sum_{t=1}^{T} x_{tq} e^{it\lambda} \right) \Big/ \left(2\pi \|x_p\|_T \|x_q\|_T \right)$$

and

$$M_{pq}^T(\lambda) = \int_{-\pi}^{\lambda} m_{pq}^T(w) \, dw$$

and $M^T(\lambda) = [M_{pq}^T(\lambda)]$. If we regard $M^T(\lambda)$ and $M(\lambda)$ as matrix measures

in $[-\pi, \pi]$, $R^T(h) = [r_{ij}^T(h)]$ and $R(h)$ are their characteristic functions, respectively. Then (G.3) implies

(4)
$$M^T(\lambda) \to_w M(\lambda)$$

as $T \to \infty$, where $\to_w$ means that $M^T(\lambda)$ converges weakly to $M(\lambda)$. [See Eicker (1967), page 68, Kawata (1972), Theorem 9.2.1, and Ibragimov and Rozanov (1978), Section 7.4.] That is, for any continuous function $\phi(\lambda)$ in $[-\pi, \pi]$,

(5)
$$\lim_{T \to \infty} \int_{-\pi}^{\pi} \phi(\lambda) \, dM^T(\lambda) = \int_{-\pi}^{\pi} \phi(\lambda) \, dM(\lambda).$$

This fact is the key to developing the following arguments. Asymptotic properties of the LSE and the BLUE depend heavily on the behavior of $M_{ii}(\lambda)$ near $\lambda = 0$. Hence hereafter we assume, if necessary, by changing the numbering of x_{ti}, $i = 1, \ldots, k$, that

(6)
$$M_{ii}(0+) - M_{ii}(0) > 0, \quad i = 1, \ldots, m, \quad 0 \le m \le k,$$
$$M_{ii}(0+) - M_{ii}(0) = 0, \quad i = m+1, \ldots, k.$$

Also, N stands for general constants being independent of T but is not always the same one in each context. And α is a k-component vector and $\|\alpha\|$ implies the Euclidean norm. First we evaluate the asymptotic covariance matrix of the LSE.

THEOREM 2.1. *Let $\{\varepsilon_t\}$ be a stationary process with spectral density $f(\lambda)$ as in (1). Let conditions (G.1)–(G.4) be satisfied.*

(i) *Suppose that $m = 0$ in (6). Then relation (2) holds and every entry of the right-hand side matrix is a finite value if and only if for any δ (> 0) there exist c such that*

(7)
$$\int_{|\lambda| \le c} f(\lambda) \, dM_{ii}^T(\lambda) < \delta, \quad i = m+1, \ldots, k,$$

for every T.

(ii) *Suppose that $m > 0$ in (6) and condition (7) holds. Then the limit of*

$$\int_{-\pi}^{\pi} f(\lambda) \, dM_{ij}^T(\lambda)/T^{2d}, \quad 1 \le i, j \le m,$$

as $T \to \infty$ exists if and only if the limit of

$$\int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM_{ij}^T(\lambda)/T^{2d}, \quad 1 \le i, j \le m,$$

as $T \to \infty$ exists.

Next if we define

$$b_{ij}^{(1)} = f^*(0) \lim_{T \to \infty} \int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM_{ij}^T(\lambda)/T^{2d}, \quad 1 \le i, j \le m,$$

then

$$\lim_{T \to \infty} \hat{D}_T^{-1} \big(\check{X}_T' \check{X}_T\big) E\{(\check{\beta}_T - \beta)(\check{\beta}_T - \beta)'\}\big(\check{X}_T' \check{X}_T\big) \hat{D}_T^{-1} = 2\pi B,$$

where

$$\hat{D}_T = \text{diag}\big(\|x_1\|_T T^d, \ldots, \|x_m\|_T T^d, \|x_{m+1}\|_T, \ldots, \|x_k\|_T\big)$$

and

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

and $B_1 = [b_{ij}^{(1)}]$, a $m \times m$ matrix and B_2 is a $(k - m) \times (k - m)$ matrix whose (i, j)th entry is $\int_{-\pi}^{\pi} f(\lambda) \, dM_{i+m, j+m}(\lambda)$.

PROOF. (i) First assume that relation (2) holds. We have

$$D_T E\{(\hat{\beta}_T - \beta)(\hat{\beta}_T - \beta)'\} D_T$$
$$= \big(D_T^{-1} \check{X}_T' \check{X}_T D_T^{-1}\big)^{-1} \big(D_T^{-1} \check{X}_T' \Sigma_T \check{X}_T D_T^{-1}\big) \big(D_T^{-1} \check{X}_T' \check{X}_T D_T^{-1}\big)^{-1}.$$

(G.3) implies

$$\lim_{T \to \infty} D_T^{-1} \check{X}_T' \check{X}_T D_T^{-1} = R(0).$$

Hence

$$\lim_{T \to \infty} D_T^{-1} \check{X}_T' \Sigma_T \check{X}_T D_T^{-1} = \lim_{T \to \infty} 2\pi \int_{-\pi}^{\pi} f(\lambda) \, dM^T(\lambda)$$
$$= 2\pi \int_{-\pi}^{\pi} f(\lambda) \, dM(\lambda).$$

Then if we note that for any $\delta \, (> 0)$ there exists c such that

$$\int_{|\lambda| \le c} f(\lambda) \, dM_{ii}(\lambda) < \delta, \qquad i = m + 1, \ldots, k,$$

[Halmos (1974), page 125, Theorem B] and that $f(\lambda)$ is continuous in $[-\pi, -c]$ and $[c, \pi]$ for any $c \, (> 0)$, condition (7) immediately follows from (4) by the argument of Loéve (1977), page 185, Theorem A, replacing $\int_{|\lambda| \ge c}$ in his proof by $\int_{|\lambda| \le c}$ and vice versa. The converse can be shown in the same way as in Loéve (1977) if we note that

$$\left| \int_A f(\lambda) \, dM_{ij}(\lambda) \right| \le \int_A f(\lambda) \, dM_{ii}(\lambda) + \int_A f(\lambda) \, dM_{jj}(\lambda)$$

for any set $A \subset [-\pi, \pi]$.

(ii) We have

$$\hat{D}_T^{-1} \big(\check{X}_T' \check{X}_T\big) E\{(\hat{\beta}_T - \beta)(\hat{\beta}_T - \beta)'\}\big(\check{X}_T' \check{X}_T\big) \hat{D}_T^{-1} = \hat{D}_T^{-1} \check{X}_T' \Sigma_T \check{X}_T \hat{D}_T^{-1}.$$

First we show that $\int_{-\pi}^{\pi} f(\lambda) \, dM_{ij}^T(\lambda)/T^{2d}$ and $\int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM_{ij}^T(\lambda)/T^{2d}$, $1 \le i, j \le m$, are bounded with respect to T. Since noting

$$m_{ii}^T(\lambda) \le T/2\pi$$

and

$$\int_{-\pi}^{\pi} m_{ii}^{T}(\lambda) \, d\lambda = 1,$$

we have

$$\int_{-\pi}^{\pi} f(\lambda) \, dM_{ii}(\lambda)/T^{2d} \leq \max_{|\lambda| \leq \pi} f(\lambda)^{*} \int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM_{ii}^{T}(\lambda)/T^{2d}$$

$$\leq (2\pi)^{-1} T^{1-2d} \max_{|\lambda| \leq \pi} f(\lambda)^{*} \int_{-\pi/T}^{\pi/T} |1 - e^{i\lambda}|^{-2d} \, d\lambda \leq N.$$

Next it follows from (4) that

$$(8) \qquad \lim_{T \to \infty} \int_{|\lambda| \geq c} f(\lambda) \, dM_{ij}^{T}(\lambda)/T^{2d} = 0$$

and

$$(9) \qquad \lim_{T \to \infty} \int_{|\lambda| \geq c} |1 - e^{i\lambda}|^{-2d} \, dM_{ij}(\lambda)/T^{2d} = 0$$

for $1 \leq i \leq m$, $m + 1 \leq j \leq k$. Then relation (10) follows from (7) and (8). Hence the proof is complete. $\square$

$$\int_{-\pi}^{\pi} f(\lambda) \, dM_{ij}^{T}(\lambda)/T^{2d}, \qquad 1 \leq i, j \leq m,$$

as $T \to \infty$ exists. Then noting (8) and (9), we have that

$$\min_{|\lambda| \leq c} f^{*}(\lambda) \limsup_{T \to \infty} \alpha' \left[\int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM^{T}(\lambda) \right] \alpha/T^{2d}$$

$$\leq \lim_{T \to \infty} \alpha' \left[\int_{-\pi}^{\pi} f(\lambda) \, dM^{T}(\lambda) \right] \alpha/T^{2d}$$

$$\leq \max_{|\lambda| \leq c} f^{*}(\lambda) \liminf_{T \to \infty} \alpha' \left[\int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM^{T}(\lambda) \right] \alpha/T^{2d}$$

for any k-component vector α. Noting that $f^{*}(\lambda)$ is continuous and, next, letting c go to 0, we see that the limit of $\alpha' [\int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM^{T}(\lambda)] \alpha$ as $T \to \infty$ exists. The converse can be shown similarly if we note that

$$\min_{|\lambda| \leq c} f^{*}(\lambda) \lim_{T \to \infty} \alpha' \left[\int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM^{T}(\lambda) \right] \alpha/T^{2d}$$

$$\leq \liminf_{T \to \infty} \alpha' \left[\int_{-\pi}^{\pi} f(\lambda) \, dM^{T}(\lambda) \right] \alpha/T^{2d}$$

$$\leq \limsup_{T \to \infty} \alpha' \left[\int_{-\pi}^{\pi} f(\lambda) \, dM^{T}(\lambda) \right] \alpha/T^{2d}$$

$$\leq \max_{|\lambda| \leq c} f^{*}(\lambda) \lim_{T \to \infty} \alpha' \left[\int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{-2d} \, dM^{T}(\lambda) \right] \alpha/T^{2d}.$$

Next we shall prove the second assertion. We have already proved

$$\lim_{T \to \infty} \int_{-\pi}^{\pi} f(\lambda) \, dM_{ij}^T(\lambda)/T^{2d} = b_{ij}^{(1)}, \qquad i \le i, j \le m.$$

Hence it suffices to show that

(10) $\lim_{T \to \infty} \int_{-\pi}^{\pi} f(\lambda) \, dM_{ij}^T(\lambda)/T^d = 0, \qquad 1 \le i \le m, \qquad m + 1 \le j \le k.$

We have that

$$\left| \int_{-\pi}^{\pi} f(\lambda) \, dM_{ij}^T(\lambda) \right| / T^d$$

$$\le \left\{ \int_{|\lambda| \le c} f(\lambda) \, dM_{ii}^T(\lambda)/T^{2d} \right\}^{1/2} \left\{ \int_{|\lambda| \le c} f(\lambda) \, dM_{jj}^T(\lambda) \right\}^{1/2}$$

$$+ \left\{ \int_{|\lambda| > c} f(\lambda) \, dM_{ii}^T(\lambda)/T^{2d} \right\}^{1/2} \left\{ \int_{|\lambda| > c} f(\lambda) \, dM_{jj}^T(\lambda) \right\}^{1/2}$$

for $1 \le i \le m$, $m + 1 \le j \le k$. Then relation (10) follows from (7) and (8). Hence the proof is complete. $\square$

REMARK 2.1.   (i) If $\max_{|\lambda| \le c} m_{ii}^T(\lambda)$, $i = m + 1, \ldots, k$, is bounded with respect to T for some c, then condition (7) is obviously satisfied. If $x_{ti} = \cos \nu_i t$, $\sin \nu_i t$, $\nu_i \ne 0$, or a periodic function, a stronger condition holds: $\max_{|\lambda| \le c} m_{ii}^T(\lambda)$ converges to 0 as $T \to \infty$ for some c.

(ii) For example let

$$x_{ti} = t^{n(i)}, \qquad i = 1, \ldots, m,$$

with $-1/2 < n(i)$ and $n(i) \ne n(j)$, $i \ne j$. Then it is shown in the same way as in Theorem 2.2 of Yajima (1988) that

$$b_{ij}^{(1)} = f^*(0) \Big[\{(2n(i) + 1)(2n(j) + 1)\}^{1/2} \Gamma(1 - 2d)/\{\Gamma(d)\Gamma(1 - d)\} \Big]$$

$$\times \int_0^1 \int_0^1 x^{n(i)} y^{n(j)} |x - y|^{2d-1} \, dx \, dy.$$

If $n(i) = -1/2$ for some i, the evaluation of the asymptotic covariance matrix in Theorem 2.1(ii) can be sharpened and a direct calculation shows that we have to replace $\|x_i\| T^d$ in $\hat{D}_T$ by $\|x_i\| T^d/(\log T)^{1/2}$ and $(2n(i) + 1)$ in $b_{ij}^{(1)}$ by 1, respectively.

Now we consider the asymptotic covariance matrix of the BLUE. Here we impose a new condition on x_{ti} besides (G.1)–(G.4).

(G.5) $\max_{1 \le t \le T} x_{ti}^2/a_{ii}^T(0) = o(1/T^\delta), \qquad i = 1, \ldots, k,$

for some $\delta > 1 - 2d$.

THEOREM 2.2. *Let $\{\varepsilon_t\}$ be a stationary process with the spectral density of (1). Assume that*

(11) $$0 < M_{ii}(0+) - M_{ii}(0) < 1, \qquad i = 1, \dots, m,$$

and the right-hand side matrix of (3) is well defined. Then under (G.1)–(G.5), relation (3) holds.

PROOF. We have

$$D_T E\{(\tilde\beta_T - \beta)(\tilde\beta_T - \beta)'\} D_T = \left(D_T^{-1} \tilde X_T' \Sigma_T^{-1} \tilde X_T D_T^{-1} \right)^{-1}.$$

Hence it suffices to show that

$$\lim_{T \to \infty} D_T^{-1} \tilde X_T' \Sigma_T^{-1} \tilde X_T D_T^{-1} = (2\pi)^{-1} \int_{-\pi}^{\pi} f(\lambda)^{-1} \, dM(\lambda).$$

We shall prove the result successively for the three cases. (i) $f^*(\lambda)$ a constant. (ii) $f^*(\lambda) = c/|\phi(e^{i\lambda})|^2$ with a constant c, where $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$ with $\phi_p \neq 0$ and $\phi(z) \neq 0$, $|z| \leq 1$, and all the roots of $\phi(z) = 0$ are distinct. (iii) $f^*(\lambda)$ is a general positive continuous function.

CASE (i). We can put $f^*(\lambda) = 1/2\pi$ without loss of generality. Let $\{\eta_t\}$ be the white-noise process of $\{\varepsilon_t\}$. Since $\{\varepsilon_t\}$ is a fractional ARIMA$(0, d, 0)$ process, $\{\varepsilon_t\}$ has the infinite autoregressive representation

$$\sum_{j=0}^{\infty} \pi_j \varepsilon_{t-j} = \eta_t,$$

where

$$\pi_j = \Gamma(j - d)/\{\Gamma(j+1)\Gamma(-d)\}.$$

[See Hosking (1981), Theorem 1.] Then let A_T be the $T \times T$ lower triangular matrix with (i, j)th entry π_{i-j} and let Σ_T^* be the $T \times T$ covariance matrix associated with a stationary process with the spectral density $1/\{(2\pi)^2 f(\lambda)\}$. By noting that $f(\lambda)^{-1}$ is a continuous function in $[-\pi, \pi]$, it follows from (5) that

$$\lim_{T \to \infty} D_T^{-1} \tilde X_T' \Sigma_T^* \tilde X_T D_T^{-1} = \lim_{T \to \infty} (2\pi)^{-1} \int_{-\pi}^{\pi} f(\lambda)^{-1} \, dM^T(\lambda)$$

$$= (2\pi)^{-1} \int_{-\pi}^{\pi} f(\lambda)^{-1} \, dM(\lambda).$$

Hence we have only to show that

(12) $$\lim_{T \to \infty} D_T^{-1} \tilde X_T' \left(\Sigma_T^* - A_T' A_T \right) \tilde X_T D_T^{-1} = 0$$

and

(13) $$\lim_{T \to \infty} D_T^{-1} \tilde X_T' \left(A_T' A_T - \Sigma_T^{-1} \right) \tilde X_T D_T^{-1} = 0.$$

First we shall show relation (12). Put

$$\mu_T = \max_{1 \le t \le T,\, 1 \le i \le k} x_{ti}^2 / a_{ii}^T(0).$$

Then noting that

$$\Sigma_T^* - A_T' A_T = \left[\sum_{j=\min(T-s+1,\, T-t+1)}^{\infty} \pi_j \pi_{j+|s-t|} \right]$$

and

(14) $$|\pi_j| \le N j^{-d-1},$$

we find that the absolute value of every entry of $D_T^{-1} \tilde{X}_T' (\Sigma_T^* - A_T' A_T) \tilde{X}_T D_T^{-1}$ is bounded by

$$2\mu_T \sum_{s=1}^{T} \sum_{t=1}^{s} \left(\sum_{j=T-s+1}^{\infty} \pi_j^2 \right)^{1/2} \left(\sum_{j=T-s+1}^{\infty} \pi_{j+s-t}^2 \right)^{1/2} \le N T^{1-2d} \mu_T.$$

Hence we have relation (12). Now we prove (13). Let

$$\phi_{i,j}(d) = -\binom{i}{j} \Gamma(j-d) \Gamma(i-j-d+1) / \{ \Gamma(-d) \Gamma(i-d+1) \}$$

and

$$\sigma_t^2(d) = \Gamma(t+1) \Gamma(t+1-2d) / \{ \Gamma(t+1-d) \}^2.$$

Let B_T be the $T \times T$ lower triangular matrix with (i,j)th entry $-\phi_{i-1,i-j}(d)$ and let

$$C_T = \operatorname{diag}\big(\sigma_0^2(d), \sigma_1^2(d), \ldots, \sigma_{T-1}^2(d) \big).$$

Then

$$\Sigma_T^{-1} = B_T' C_T^{-1} B_T$$

[Hosking (1981), Theorem 1, and Yajima (1985), Lemma 3.2]. First we show

(15) $$\lim_{T \to \infty} D_T^{-1} \tilde{X}_T' \big(A_T' A_T - B_T' B_T \big) \tilde{X}_T D_T^{-1} = 0.$$

Let $\tilde{w}_{i,T}$ and $\tilde{z}_{i,T}$ be the ith column of $A_T \tilde{X}_T D_T^{-1}$ and $B_T \tilde{X}_T D_T^{-1}$, respectively. Since

(16) $$\Sigma_T^* \ge A_T' A_T \ge \Sigma_T^{-1}$$

[Shaman (1976), Corollary 2.1 and Theorem 2.2], where the inequality is defined in positive semidefinite sense, $\|\tilde{w}_{i,T}\|$ is bounded over all T. Hence it suffices to show

$$\lim_{T \to \infty} \|\tilde{w}_{i,T} - \tilde{z}_{i,T}\| = 0, \qquad i = 1, \ldots, k.$$

Define $\lambda_{t,i}(d)$ by

$$\lambda_{t,i}(d) = 1 + \phi_{t,i}(d) / \pi_i.$$

Then

$$\lambda_{t,i}(d) = 1 - \{ \Gamma(t+1) \Gamma(t+1-i-d) \} / \{ \Gamma(t+1-d) \Gamma(t+1-i) \}.$$

We evaluate $\lambda_{t,i}(d)$ by

$$|\lambda_{t,i}(d)| \le N\{i/(t-i)\}, \qquad i \le t^\xi,$$
$$\le N\{t/(t-i)\}^d, \qquad t^\xi < i \le t,$$

with ξ, $0 < \xi < 1$, being specified later. [See Yajima (1985), page 311.] Then

$$\|\tilde{w}_{i,T} - \tilde{z}_{i,T}\|^2 \le \mu_T \sum_{i=2}^{T} \left(\sum_{j=1}^{i-1} |\pi_j| \, |\lambda_{i-1,j}(d)| \right)^2$$

$$= \mu_T \sum_{i=2}^{T} \left(\sum_{j=1}^{i^\xi} + \sum_{j=i^\xi+1}^{i-1} \right)^2$$

$$\le N\mu_T (T^{2\xi(1-d)-1} + T^{2-\xi(2d+1)}).$$

By taking ξ such that $2 - \xi(2d+1) - \delta < 0$, the result is obtained. Finally we show

(17)
$$\lim_{T \to \infty} D_T \tilde{X}_T' \big(B_T' B_T - \Sigma_T^{-1} \big) \tilde{X}_T D_T = 0.$$

Let z_{ijT} be the (i, j)th entry of $B_T \tilde{X}_T D_T^{-1}$. Then if we note that

$$\lim_{t \to \infty} \sigma_i^2(d) = 1$$

[Yajima (1985), Lemma 3.2], and

$$\lim_{T \to \infty} z_{ijT} = 0,$$

with i, j being fixed, it follows from Toeplitz's limit theorem that

$$\lim_{T \to \infty} \sum_{j=1}^{T} \big(1 - 1/\sigma_{j-1}(d) \big)^2 z_{ijT}^2 = 0, \qquad i = 1, \ldots, k,$$

which implies (17). Then relation (13) follows from (15) and (17).

CASE (ii). From (5) and (16),

$$\limsup_{T \to \infty} \alpha' \big(D_T^{-1} \tilde{X}_T' \Sigma_T^{-1} \tilde{X}_T D_T^{-1} \big) \alpha \le \alpha' \left[(2\pi)^{-1} \int_{-\pi}^{\pi} f(\lambda)^{-1} \, dM(\lambda) \right] \alpha.$$

Hence we have only to show that

(18)
$$\liminf_{T \to \infty} \alpha' \big(D_T^{-1} \tilde{X}_T' \Sigma_T^{-1} \tilde{X}_T D_T^{-1} \big) \alpha \ge \alpha' \left[(2\pi)^{-1} \int_{-\pi}^{\pi} f(\lambda)^{-1} \, dM(\lambda) \right] \alpha.$$

The assertion is shown straightforward for a general $\phi(e^{i\lambda})$. Hence to make the proof clearer, we consider the case that $c = 1/2\pi$ and $\phi(e^{i\lambda}) = 1 - \phi_1 e^{i\lambda}$. Then $\{\varepsilon_t\}$ is a fractional ARIMA$(1, d, 0)$ process. Let I_T be the $T \times T$ identity matrix and L_T be the $T \times T$ matrix with 1's on the diagonal directly below the main diagonal and 0's elsewhere. Then let

$$F_T = I_T - \phi_1 L_T.$$

Define $\hat{\boldsymbol{\varepsilon}}_T = (\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_T)$ by

$$\hat{\boldsymbol{\varepsilon}}_T = F_T \tilde{\boldsymbol{\varepsilon}}_T.$$

Let $\hat{\Sigma}_T$ be the covariance matrix of $\hat{\boldsymbol{\varepsilon}}_T$. We partition $\hat{\Sigma}_T$ and F_T into

$$\hat{\Sigma}_T = \begin{pmatrix} \hat{\sigma}_1^2 & \tilde{\sigma}_T' \\ \tilde{\sigma}_T & \hat{\Sigma}_{2T} \end{pmatrix} \quad \text{and} \quad F_T = \begin{pmatrix} \tilde{f}_{1T}' \\ F_{2T}' \end{pmatrix},$$

respectively, with $\hat{\sigma}_1^2 = E\hat{\varepsilon}_1^2 \, (= E\varepsilon_1^2)$, $\tilde{\sigma}_T$, a $(T-1)$-component column vector, $\hat{\Sigma}_{2T}$, a $(T-1) \times (T-1)$ matrix, $\tilde{f}_{1T} = (1, 0, \ldots, 0)'$ and F_{2T} a $T \times (T-1)$ matrix. Then

$$\begin{aligned}
\Sigma_T^{-1} &= F_T' \hat{\Sigma}_T^{-1} F_T \\
&= \left(\tilde{f}_{1T} \tilde{f}_{1T}' - \tilde{f}_{1T} \tilde{\sigma}_T' \hat{\Sigma}_{2T}^{-1} F_{2T}' - F_{2T} \hat{\Sigma}_{2T}^{-1} \tilde{\sigma}_T \tilde{f}_{1T}' + F_{2T} \hat{\Sigma}_{2T}^{-1} \tilde{\sigma}_T \tilde{\sigma}_T' \hat{\Sigma}_{2T}^{-1} F_{2T}' \right) / \sigma_{1T}^* \\
&\quad + F_{2T} \hat{\Sigma}_{2T}^{-1} F_{2T}',
\end{aligned}$$

where

$$\sigma_{1T}^* = \hat{\sigma}_1^2 - \tilde{\sigma}_T' \hat{\Sigma}_{2T}^{-1} \tilde{\sigma}_T.$$

Lemma A.1 in the Appendix assures that σ_{1T}^* converges to a positive value as $T \to \infty$. And it follows from (G.5) and Lemma A.1 that

$$\lim_{T \to \infty} \alpha' D_T^{-1} \tilde{X}_T \tilde{f}_{1T} \tilde{f}_{1T}' \tilde{X}_T D_T^{-1} \alpha = 0$$

and

$$\lim_{T \to \infty} \alpha' D_T^{-1} \tilde{X}_T' \tilde{f}_{1T} \tilde{\sigma}_T' \hat{\Sigma}_{2T}^{-1} F_{2T}' \tilde{X}_T D_T^{-1} \alpha = 0.$$

Then if we note that $\hat{\Sigma}_{2T}$ is identical with the $(T-1) \times (T-1)$ covariance matrix of the case that $\{\varepsilon_t\}$ is a fractional ARIMA$(0, d, 0)$ process and $F_{2T} \hat{\Sigma}_{2T}^{-1} \tilde{\sigma}_T \tilde{\sigma}_T' \hat{\Sigma}_{2T}^{-1} F_{2T}'$ is a positive semidefinite matrix, we obtain

$$\liminf_{T \to \infty} \alpha' D_T^{-1} \tilde{X}_T' \Sigma_T^{-1} \tilde{X}_T D_T^{-1} \alpha \geq \lim_{T \to \infty} \alpha' D_T^{-1} \tilde{X}_T' F_{2T} \hat{\Sigma}_{2T}^{-1} F_{2T}' \tilde{X}_T D_T^{-1} \alpha$$

$$= \alpha' \left[(2\pi)^{-1} \int_{-\pi}^{\pi} f(\lambda)^{-1} \, dM(\lambda) \right] \alpha.$$

We complete the proof of (18).

CASE (iii). There exist $\phi^{(1)}(z) = 1 - \sum_{j=1}^{p} \phi_j^{(1)} z^j$ and $\phi^{(2)}(z) = 1 - \sum_{j=1}^{p} \phi_j^{(2)} z^j$ such that $\phi^{(i)}(z) \neq 0$, $|z| \leq 1$, $i = 1, 2$, and all the roots of $\phi^{(i)}(z) = 0$, $i = 1, 2$, are distinct, and if we define

$$f_L^*(\lambda) = c_1 / \left| \phi^{(1)}(e^{i\lambda}) \right|^2 \quad \text{and} \quad f_U^*(\lambda) = c_2 / \left| \phi^{(2)}(e^{i\lambda}) \right|^2,$$

then

$$f_L^*(\lambda) \leq f^*(\lambda) \leq f_U^*(\lambda), \qquad -\pi \leq \lambda \leq \pi,$$

and

$$f_L^*(\lambda)^{-1} - f_U^*(\lambda)^{-1} \leq \varepsilon, \qquad -\pi \leq \lambda \leq \pi,$$

for any $\varepsilon \, (> 0)$. Then the proof is complete as in Grenander (1954), page 259.

$\square$

If condition (11) does not hold, then the right-hand side matrix of (3) is no longer well defined. A general result for this case has not been obtained yet. Here we consider a polynomial function as a typical example. If we assume that

$$(19) \qquad x_{ti} = t^{i-1}, \qquad i = 1, \ldots, p, \qquad 0 \le p \le m,$$

then

$$M_{ii}(0+) - M_{ii}(0) = 1, \qquad i = 1, \ldots, p,$$

and

$$(20) \qquad 0 < M_{ii}(0+) - M_{ii}(0) < 1, \qquad i = p+1, \ldots, m.$$

THEOREM 2.3. *Let conditions (19) and (20) be satisfied. Let*

$$\tilde{D}_T = \text{diag}\big(\|x_1\|_T/T^d, \ldots, \|x_p\|_T/T^d, \|x_{p+1}\|_T, \ldots, \|x_k\|_T\big).$$

Then under (G.1)–(G.5),

$$\lim_{T \to \infty} \tilde{D}_T E\{(\tilde{\beta}_T - \beta)(\tilde{\beta}_T - \beta)'\}\tilde{D}_T = 2\pi \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}^{-1},$$

where W_1 is a $p \times p$ matrix with (i,j)th entry,

$$\Gamma(i-d)\Gamma(j-d)\{(2i-1)(2j-1)\}^{1/2}$$
$$/\{f^*(0)\Gamma(i-2d)\Gamma(j-2d)(i+j-1-2d)\}$$

and W_2 is a $(k-p) \times (k-p)$ matrix with (i,j)th entry

$$\int_{-\pi}^{\pi} f(\lambda)^{-1} \, dM_{i+p, j+p}(\lambda).$$

PROOF. Let $\sigma^{ij}(T)$ and $\sigma_{ij}^*(T)$ be the (i,j)th entry of $\tilde{D}_T^{-1}\tilde{X}_T'\Sigma_T^{-1}\tilde{X}_T\tilde{D}_T^{-1}$ and $\tilde{D}_T^{-1}\tilde{X}_T'\Sigma_T^*\tilde{X}_T\tilde{D}_T^{-1}$, respectively. Then by Theorem 2.3 of Yajima (1988), it suffices to show that

$$\lim_{T \to \infty} \sigma^{ij}(T) = 0, \qquad 1 \le i \le p, \qquad p+1 \le j \le k.$$

From (16),

$$\big|\sigma_{ij}^*(T) - \sigma^{ij}(T)\big| \le \big(\sigma_{ii}^*(T) - \sigma^{ii}(T)\big)^{1/2}\big(\sigma_{jj}^*(T) - \sigma^{jj}(T)\big)^{1/2}.$$

Since $\pi m_{ii}^T(\lambda)$, $i = 1, \ldots, p$, has the same property as the Fejér kernel has and $|1 - e^{i\lambda}|^{2d}$ satisfies a Lipschitz condition of order $2d$, we have

$$\int_{-\pi}^{\pi} |1 - e^{i\lambda}|^{2d} \, dM_{ii}^T(\lambda) = O(T^{-2d}), \qquad i = 1, \ldots, p$$

[Zygmund (1959), Chapter 3, Theorem 3.15]. Hence the first right-hand side term in the previous inequality is bounded in T, while Theorem 2.2 assures that the second term converges to 0 as $T \to \infty$. Hence we have only to show that

$$\lim_{T \to \infty} \sigma_{ij}^*(T) = 0.$$

We have

$$\left| \sigma_{ij}^*(T) \right| \leq \left\{ T^{2d}(2\pi)^{-1} \int_{|\lambda| \leq c} f(\lambda)^{-1} \, dM_{ii}^T(\lambda) \right\}^{1/2}$$

$$\times \left\{ (2\pi)^{-1} \int_{|\lambda| \leq c} f(\lambda)^{-1} \, dM_{jj}^T(\lambda) \right\}^{1/2}$$

$$+ \left\{ T^{2d}(2\pi)^{-1} \int_{|\lambda| > c} f(\lambda)^{-1} \, dM_{ii}^T(\lambda) \right\}^{1/2}$$

$$\times \left\{ (2\pi)^{-1} \int_{|\lambda| > c} f(\lambda)^{-1} \, dM_{jj}^T(\lambda) \right\}^{1/2}.$$

Noting that $f(0)^{-1} = 0$, we see that the first right-hand side term can be made as small as desired uniformly in T by letting c go to 0. Next assumption (19) assures that the second term converges to 0 as $T \to \infty$, c (> 0) being fixed. Hence the proof is complete. $\square$

REMARK 2.2. (i) Shaman (1976) assumed that $f(\lambda)$ is positive and continuous to derive relation (16). However, as is seen from his proof, his result still holds for an invertible stationary process having an infinite autoregressive representation. A fractional ARIMA(p, d, q) process with $-1/2 < d < 1/2$ is invertible. [See Hosking (1981), Theorem 2, and Yajima (1985), Proposition 2.3.]

(ii) (G.1) and (G.2) imply

$$\lim_{T \to \infty} \max_{1 \leq t \leq T} x_{ti}^2 / a_{ii}^T(0) = 0.$$

[See Anderson (1971), Lemma 2.6.1.] (G.5) specifies the rate of convergence.

(iii) Clearly $\pi_j = 0$, $1 \leq j \leq \infty$, for $d = 0$ and $N = 0$ in inequality (14). Hence as is well known, (G.5) is unnecessary in this case.

Now we have a generalization of the result given by Grenander (1954).

THEOREM 2.4. *Assume that $m = 0$ in (6) and condition (7) holds. Then a necessary and sufficient condition under (G.1)–(G.5) that the LSE be asymptotically efficient for a stationary process $\{\varepsilon_t\}$ with the spectral density $f(\lambda)$ of (1) is that $M(\lambda)$ increase at not more than k values of λ, $0 < \lambda \leq \pi$, and the sum of the ranks of the increases in $M(\lambda)$ is k.*

PROOF. Combining Theorem 2.1(i) with Theorem 2.2 and following the same arguments developed by Anderson (1971), Section 10.2, we have the result. $\square$

The intuitive interpretation of Theorem 2.4 is the following. If $M(\lambda)$ increase at $\lambda = 0$, at which $f(\lambda)$ diverges to ∞, then the LSE, being con-

structed by neglecting the correlation structure of $\{\varepsilon_t\}$, cannot separate $M(\lambda)$ from $f(\lambda)$ so that it is not asymptotically efficient. However, if $M(\lambda)$ increases at λ, $0 < \lambda \le \pi$, where $f(\lambda)$ takes a finite value, then the LSE can discriminate between $M(\lambda)$ and $f(\lambda)$ and, hence, is asymptotically efficient.

EXAMPLE 2.1. We shall give an example which clarifies the implication of Theorems 2.1–2.4. Let

$$y_t = \beta_1 x_{t1} + \beta_2 x_{t2} + \varepsilon_t,$$

where

$$x_{t1} = \tau_{10} + \tau_{11} \cos \nu_1 t + \tau_{12} \cos \nu_2 t,$$
$$x_{t2} = \tau_{20} + \tau_{21} \cos \nu_1 t + \tau_{22} \cos \nu_2 t,$$

with $\nu_i \ne 0$, $i = 1, 2$, and $\nu_1 \ne \nu_2$. And let

$$\tilde{\tau}_i = (\tau_{1i}, \tau_{2i})', \qquad i = 0, 1, 2.$$

Then

$$R(h) = M_0 + \cos \nu_1 h M_1 + \cos \nu_2 h M_2,$$

where $M_0 = \Gamma \tilde{\tau}_0 \tilde{\tau}_0' \Gamma$, $M_i = (1/2)\Gamma \tilde{\tau}_i \tilde{\tau}_i' \Gamma$, $i = 1, 2$, and

$$\Gamma = \text{diag}\left(\left(\tau_{10}^2 + \left(\tau_{11}^2 + \tau_{12}^2\right)/2\right)^{-1/2}, \left(\tau_{20}^2 + \left(\tau_{21}^2 + \tau_{22}^2\right)/2\right)^{-1/2}\right).$$

[See Anderson (1971), page 581.] Define the relative efficiency of the LSE by

$$e(d) = \lim_{T \to \infty} \det\left[E(\tilde{\beta}_T - \beta)(\tilde{\beta}_T - \beta)'\right] / \det\left[E(\hat{\beta}_T - \beta_T)(\hat{\beta}_T - \beta)'\right].$$

Now we consider the following three cases in each of which $e(d)$ takes a different value.

(i) Let $\tilde{\tau}_0 = (0, 0)'$, $\tilde{\tau}_1 = (1, 0)'$ and $\tilde{\tau}_2 = (0, 1)'$. Then $m = 0$ in (6) and condition (7) holds. Since

$$\max_{1 \le t \le T} x_{ti}^2 / a_{ii}^T(0) = O(1/T), \qquad i = 1, 2,$$

(G.5) also holds. Then it follows from Theorems 2.1(i), 2.2 and 2.4 that the LSE is asymptotically efficient and, hence, $e(d) = 1$. Actually

$$\lim_{T \to \infty} D_T \{E(\hat{\beta}_T - \beta)(\hat{\beta}_T - \beta_t)'\} D_T = \lim_{T \to \infty} D_T \{E(\tilde{\beta}_T - \beta)(\tilde{\beta}_T - \beta)'\} D_T$$

$$= 2\pi \begin{pmatrix} f(\nu_1) & 0 \\ 0 & f(\nu_2) \end{pmatrix}.$$

(ii) Let $\tilde{\tau}_0 = (1, 0)'$, $\tilde{\tau}_1 = (1, 0)'$ and $\tilde{\tau}_2 = (0, 1)'$. Then $m = 1$ in (6) and conditions (7) and (11) are satisfied. It follows from Theorem 2.1(ii) and Theorem 2.2 of Yajima (1988) that

$$\lim_{T \to \infty} \hat{D}_T^{-1}\left(\check{X}_T' \check{X}_T\right) E\{(\hat{\beta}_T - \beta)(\hat{\beta}_T - \beta)'\}\left(\check{X}_T' \check{X}_T\right)\hat{D}_T^{-1}$$

$$= 2\pi \begin{pmatrix} b_{11}^{(1)} & 0 \\ 0 & f(\nu_2) \end{pmatrix},$$

where

$$b_{11}^{(1)} = (2/3) f^*(0) \Gamma(1 - 2d) / \{(2d + 1)\Gamma(1 - d)\Gamma(1 + d)\}.$$

On the other hand, it follows from Theorem 2.2 by noting

$$\lim_{T \to \infty} D_T^{-1}(\tilde{X}_T' \tilde{X}_T) D_T^{-1} = R(0) = I_2$$

that

$$\lim_{T \to \infty} D_T^{-1}(\tilde{X}_t' \tilde{X}_T) E\{(\tilde{\beta}_T - \beta)(\tilde{\beta}_T - \beta)'\}(\tilde{X}_T' \tilde{X}_T) D_T^{-1} = 2\pi \begin{pmatrix} 3f(\nu_1) & 0 \\ 0 & f(\nu_2) \end{pmatrix}.$$

Hence $e(d) = 0$.

(iii) Let $\tilde{\tau}_0 = (1, 0)'$, $\tilde{\tau}_1 = (0, 1)'$ and $\tilde{\tau}_2 = (0, 0)'$. Then $m = 1$ in (6) and $p = 1$ in (19). It follows now from Theorem 2.1(ii) by noting $\lim_{T \to \infty} \hat{D}_T^{-1}(\tilde{X}_T' \tilde{X}_T) \hat{D}_T^{-1} = I_2$ that

$$\lim_{T \to \infty} \check{D}_T \{E(\hat{\beta}_T - \beta)(\hat{\beta}_T - \beta)'\} \check{D}_T = 2\pi \begin{pmatrix} b_{11}^{(1)} & 0 \\ 0 & f(\nu_1) \end{pmatrix},$$

where

$$b_{11}^{(1)} = f^*(0) \Gamma(1 - 2d) / \{(2d + 1)\Gamma(1 - d)\Gamma(1 + d)\}.$$

Theorem 2.3 implies that

$$\lim_{T \to \infty} \check{D}_T E\{(\tilde{\beta}_T - \beta)(\tilde{\beta}_T - \beta)'\} \check{D}_T = 2\pi \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}^{-1},$$

where

$$w_1 = \Gamma(1 - d)^2 / \{ f^*(0)\Gamma(1 - 2d)\Gamma(2 - 2d)\},$$

and $w_2 = f(\nu_1)^{-1}$. Hence

$$0 < e(d) = (1 + 2d)\Gamma(1 + d)\Gamma(2 - 2d)/\Gamma(1 - d) < 1, \qquad 0 < d < 1/2.$$

The actual values of $e(d)$ are listed in Table 1 of Yajima (1988).

3. The asymptotic distribution of the LSE.

In this section let $\{\varepsilon_t\}$ be a strictly stationary process all of whose moments exist, which has the infinite moving average representation

$$\varepsilon_t = \sum_{j=0}^{\infty} b_j \eta_{t-j},$$

where $\sum_{j=0}^{\infty} b_j z^j \neq 0$, $|z| < 1$, so that

$$(21) \qquad 2\pi f(\lambda)/\sigma_\eta^2 = \left| \sum_{j=0}^{\infty} b_j e^{ij\lambda} \right|^2$$

with $\sigma_\eta^2 = E\eta_t^2$. Next let $c_r^\eta(t_1, \ldots, t_{r-1})$ be the rth-order cumulant of $\eta_t, \eta_{t+t_1}, \ldots, \eta_{t+t_{r-1}}$ and $f_r^\eta(\lambda_1, \ldots, \lambda_{r-1})$ be the rth-order cumulant spectral

density of $\{\eta_t\}$. Then

$$c_r^\eta(t_1, \ldots, t_{r-1}) = \int_{\Pi^{r-1}} \exp\left(i \sum_{j=1}^{r-1} \lambda_j t_j\right) f_r^\eta(\lambda_1, \ldots, \lambda_{r-1}) \, d\lambda_1 \ldots \lambda_{r-1}$$

with $\Pi = [-\pi, \pi]$. Assume

$$(22) \qquad \sum_{t_1, \ldots, t_{r-1} = -\infty}^{\infty} |c_r^\eta(t_1, \ldots, t_{r-1})| < \infty.$$

Then we have an asymptotic distribution of the LSE, which is a generalization of the result given by Yajima (1989) for finite Fourier transforms.

THEOREM 3.1. *Let c_r^η satisfy* (22). *Assume the same conditions as in Theorem* 2.1(ii). *Further assume that*

$$(23) \qquad \max_{|\lambda| \leq \delta} \left(m_{ii}^T(\lambda)\right)^{1/2} = o\left(1/T^{1/4 + d/2}\right), \qquad i = m+1, \ldots, k,$$

for some $\delta \ (> 0)$ and

$$(24) \qquad \int_{-\pi}^{\pi} \left(m_{ii}^T(\lambda)\right)^{1/2} d\lambda = o\left(1/T^{1/4 + d/2}\right), \qquad i = 1, \ldots, k.$$

Then $\hat{D}_T^{-1}(\bar{X}_T' \bar{X}_T)(\hat{\beta}_T - \beta)$ has a limiting normal distribution with means 0 and covariances given in Theorem 2.1(ii).

PROOF. It follows from Lemma A.2 in the Appendix that the cumulant of any order of the LSE converges to the corresponding cumulant of the normal distribution. Hence we have the result. □

REMARK 3.1. If $x_{ti} = t^{i-1}$, $1 \leq i \leq m$, and $x_{ti} = \cos \nu_i$, $\sin \nu_i$, $\nu_i \neq 0$, or a periodic function for $m+1 \leq i \leq k$, by noting that

$$\max_{|\lambda| \leq \delta} \left(m_{ii}^T(\lambda)\right)^{1/2} = O\left(1/T^{1/2}\right), \qquad i = m+1, \ldots, k,$$

for some $\delta \ (> 0)$ and

$$\int_{-\pi}^{\pi} \left(m_{ii}^T(\lambda)\right)^{1/2} d\lambda = O\left(\log T/T^{1/2}\right), \qquad i = 1, \ldots, k,$$

we see that conditions (23) and (24) hold.

APPENDIX

Define the backward shift operator B by $B\varepsilon_t = \varepsilon_{t-1}$. And let

$$\hat{\varepsilon}_t = \phi(B)\varepsilon_t, \qquad -\infty < t < \infty.$$

And let $H\{\hat{\varepsilon}, s, t\}$, $s \leq t$, be the Hilbert space generated by $\{\hat{\varepsilon}_s, \hat{\varepsilon}_{s+1}, \ldots, \hat{\varepsilon}_t\}$ and the inner product of x and y in $H\{\hat{\varepsilon}, s, t\}$ is defined by $(x, y) = Exy$. And let $P_{\{\hat{\varepsilon}, s, t\}} x$ be the projection of x into $H\{\hat{\varepsilon}, s, t\}$. Then we have the following result.

LEMMA A.1. *Let*

$$\eta^*_{t,T} = \varepsilon_t - P_{\{\hat\varepsilon,\,p+1,T\}}\varepsilon_t, \qquad t = 1,\ldots,p.$$

Then the covariance matrix of $\{\eta^*_{1,T},\ldots,\eta^*_{p,T}\}$ *converges to a positive definite matrix as* $T \to \infty$.

PROOF.   Since

$$\lim_{T\to\infty} \eta^*_{t,T} = \varepsilon_t - P_{\{\hat\varepsilon,\,p+1,\infty\}}\varepsilon_t$$
$$= \eta^*_t,$$

say, the covariance matrix of $\{\eta^*_{1,T},\ldots,\eta^*_{p,T}\}$ converges to that of $\{\eta^*_1,\ldots,\eta^*_p\}$. Define $\{\hat\eta^*_t\}$ by

$$\hat\eta^*_t = \hat\varepsilon_t - P_{\{\hat\varepsilon,\,t+1,\infty\}}\hat\varepsilon_t, \qquad -\infty < t < \infty.$$

And define $\{\omega_j\}$ and $\{\theta_j\}$, $0 \le j < \infty$, by

$$\sum_{j=0}^\infty \psi_j z^j = (1-z)^{-d} \quad \text{and} \quad \sum_{j=0}^\infty \theta_j z^j = \phi(z)^{-1},$$

respectively. Then since

$$\varepsilon_t = \sum_{j=0}^\infty \theta_j \hat\varepsilon_{t-j} \quad \text{and} \quad \hat\varepsilon_t = \sum_{j=0}^\infty \psi_j \hat\eta^*_{t+j},$$

we have

(A.1) $$\eta^*_i = \sum_{j=0}^\infty \left(\sum_{n=0}^\infty \theta_{i+j-p+n}\psi_n \right)\hat\eta^*_{p-j}, \qquad i = 1,\ldots,p,$$

with $\theta_n = 0$, $n < 0$. Let Q be the $p \times p$ matrix whose (i,j)th entry is the coefficient of $\hat\eta^*_{p+1-j}$ in η^*_{p+1-i} of (A.1). Then

$$Q = \left[\sum_{n=0}^\infty \theta_{n-i+j}\psi_n \right].$$

θ_j is expressed as

$$\theta_j = \sum_{i=1}^p c_i \nu_i^j, \qquad -p+1 \le j \le \infty,$$

where c_i, $i = 1,\ldots,p$, is a nonzero constant and $|\nu_i| < 1$, $i = 1,\ldots,p$, with $\nu_i \ne \nu_n$, $i \ne n$. Then Q is expressed as

$$Q = Q_1 Q_2 Q_3,$$

where Q_1 and Q_3 are $p \times p$ matrices with (i,j)th entry $c_j \nu_j^{-i+1}$ and ν_i^{j-1}, respectively, and

$$Q_2 = \text{diag}\left((1-\nu_1)^{-d}, \ldots, (1-\nu_p)^{-d} \right).$$

Hence Q is a nonsingular matrix. Since $\{\hat\eta^*_t\}$ is an uncorrelated process, $(\eta^*_1,\ldots,\eta^*_p\}$ are linearly independent, which means that the covariance matrix

of $\{\eta_1^*, \ldots, \eta_p^*\}$ is positive definite. □

LEMMA A.2.   *Assume (23) and (24).*

(i)

$$\text{cum}\left(\sum_{t(1)=1}^{T} x_{t(1), i(1)}\varepsilon_{t(1)}, \ldots, \sum_{t(r)=1}^{T} x_{t(r), i(r)}\varepsilon_{t(r)} \right) = o\left(T^{qd} \prod_{j=1}^{r} \|x_{i(j)}\|_T \right)$$

for $1 \le i(j) \le m$, $j = 1, \ldots, q$, and $m + 1 \le i(j) \le k$, $j = q + 1, \ldots, r$, $r \ge 3$, $q > 0$.

(ii)

$$\text{cum}\left(\sum_{t(1)=1}^{T} x_{t(1), i(1)}\varepsilon_{t(1)}, \ldots, \sum_{t(r)=1}^{T} x_{t(r), i(r)}\varepsilon_{t(r)} \right)$$

$$= o\left(\prod_{j=1}^{r} \|x_{i(j)}\|_T \right)$$

for $m + 1 \le i(j) \le k$, $j = 1, \ldots, r$, $r \ge 3$.

PROOF.   (i) We can prove the result as in Yajima (1989). Let

$$x_{Tj}^*(\lambda) = \sum_{t=1}^{T} x_{tj}e^{i\lambda t}$$

and

$$P_{Tj}(\delta) = \max_{|\lambda| \le \delta} |x_{Tj}^*(\lambda)|$$

and

$$Q_{Tj} = \int_{-\pi}^{\pi} |x_{Tj}^*(\lambda)| \, d\lambda.$$

And let

$$b(\lambda) = \sum_{j=0}^{\infty} b_j e^{ij\lambda}.$$

For notational simplicity we express $\int_{\Pi^{k-1}} d\lambda_1 \ldots d\lambda_{k-1}$ by $\int d\lambda$. Then it follows from Lemma 2 of Yajima (1989) that

$$\text{cum}\left(\sum_{t(1)=1}^{T} x_{t(1), i(1)}\varepsilon_{t(1)}, \ldots, \sum_{t(r)=1}^{T} x_{t(r), i(r)}\varepsilon_{t(r)} \right)$$

$$= \int x_{T, i(1)}^*\left(-\sum_{j=1}^{r-1} \lambda_j \right) \prod_{j=2}^{r} x_{T, i(j)}^*(\lambda_{j-1})$$

$$\times b\left(\sum_{j=1}^{r-1} \lambda_j \right) \prod_{j=1}^{r-1} b(-\lambda_j) f_r^{\eta}\left(\sum_{j=1}^{r-1} \lambda_j, \lambda_2, \ldots, \lambda_{r-1} \right) d\lambda$$

$$= \int g(\tilde{\lambda}) \, d\lambda,$$

say, where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{r-1})$. We choose δ so that δ satisfies (23). Since we can change the numbering of $x_{t,i(j)}$ and λ_j if necessary, we can restrict the domain of integration to

$$S = \left\{ \tilde{\lambda} \,\middle|\, |\lambda_j| < 1/T, 1 \le j \le n, |\lambda_j| \ge 1/T, n+1 \le j \le q-1, \right.$$
$$\left. |\lambda_{j+q-1}| > \delta, 1 \le j \le u, |\lambda_{j+q}| \le \delta, u \le j \le r-q-1 \right\}.$$

It follows from (1) and (21) that

(A.2)
$$|b(\lambda)| \le N/|\lambda|^d, \qquad -\pi \le \lambda \le \pi.$$

Clearly,

(A.3)
$$|x^*_{T_i}(\lambda)| \le T^{1/2}\|x_i\|_T.$$

Then noting that $f^\eta_r(\lambda_1, \ldots, \lambda_{r-1})$ is a bounded function and using (A.2) and (A.3), we obtain

$$\int_S |g(\tilde{\lambda})| \, d\lambda \le NT^{(n+1)/2+(q-n-1)d} \prod_{j=1}^{n+1} \|x_{i(j)}\|_T \prod_{j=q+u+1}^{r} P_{T,i(j)}(\delta)$$
$$\times \int_S \left\{ \prod_{j=n+2}^{q+u} |x^*_{T,i(j)}(\lambda_{j-1})| \right\}$$
$$\times \left| b\left(\sum_{j=1}^{r-1} \lambda_j \right) \right| \prod_{j=q+u}^{r-1} |b(-\lambda_j)| \Bigg/ \left(\prod_{j=1}^{n} |\lambda_j|^d \right) d\lambda.$$

Then similar to Lemma 3(i) of Yajima (1989), the right-hand side of the above inequality can be bounded above by

$$NT^{(1-n)/2+(q-1)d} \prod_{j=1}^{n+1} \|x_{i(j)}\|_T \left(T^{d-1/2}\|x_{i(n+2)}\|_T + T^d Q_{T,i(n+2)} \right)$$
$$\times \prod_{j=n+3}^{q+u} Q_{T,i(j)} \prod_{j=q+u+1}^{r} P_{T,i(j)}(\delta)$$

for $n < q-1$ or $u > 0$ and by

$$NT^{1-q/2+(q-1)d} \prod_{j=1}^{q} \|x_{i(j)}\|_T \prod_{j=q+1}^{r} P_{T,i(j)}(\delta)$$

for $n = q-1$ and $u = 0$, respectively. The assertion follows from (23) and (24).

(ii) We can restrict the domain of integration to

$$V = \left\{ \tilde{\lambda} \,\middle|\, |\lambda_j| > \delta, 1 \le j \le n, |\lambda_j| \le \delta, n+1 \le j \le r-1 \right\}.$$

Then similar to (i), $\int_S |g(\tilde{\lambda})| \, d\lambda$ is bounded by

$$T^{1/2}\|x_{i(1)}\|_T \left(T^{d-1/2}\|x_{i(2)}\|_T + T^d Q_{T,i(2)} \right) \prod_{j=3}^{n+1} Q_{T,i(j)} \prod_{j=n+2}^{r} P_{T,i(j)}(\delta)$$

for $n > 0$, and by

$$T^{1/2}\|x_{i(1)}\|_T \prod_{j=2}^{r} P_{T,i(j)}(\delta)$$

for $n = 0$, respectively. The assertion again follows from (23) and (24). $\quad\square$

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