Asymptotic Properties of the Solutions to Stochastic KPP Equations

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Abstract

A reduction method is used to prove existence and uniqueness of strong solutions to stochastic KPP equations, where the initial condition may be anticipating. The asymptotic behavior of the solution for large time and space is then studied under two different basic assumptions.

1 Introduction

There are numerous examples of wave phenomena in nature and in biology there seem to be particularly many. Some examples of such phenomena are insect dispersal, the progressing wave of an epidemic (e.g. the spread of the Black Death in the 14th century and the current rabies epizootic spreading across Europe), the movement of microorganisms into a food source, and the spread of killer bees in South America. Detailed discussions on these and many other examples can be found in [18].

The KPP equation

$$\frac{du}{dt} = ru(K-u) + \frac{D}{2}\Delta u, \tag{1.1}$$

where $r > 0$ is the reproduction rate, $K > 0$ the carrying capacity, and $D > 0$ the diffusion coefficient, provides a (deterministic) model for the density of a population, $u(t,x)$, living in an environment with a limited carrying capacity. We shall make the model more realistic by introducing environmental noise. More precisely, we assume the carrying capacity is stochastic and given by $K(t) = c_0 + k\dot{W}(t)$, where $c_0 > 0$ and $\dot{W}(t)$ is white noise. Substituting $K(t)$ into (1.1) gives the stochastic partial differential equation (SPDE)

$$du(t,x) = \left(u(t,x)(c_0 - u(t,x)) + \frac{D}{2}\Delta u(t,x)\right) dt + ku(t,x)\, dW_t. \tag{1.2}$$

To make the problem well posed we suppose that the spatial distribution of the population density at time $t = 0$ is known, $u(0,x) = u_0(x)$.

There are clearly other ways of introducing environmental noise in (1.1). In the spatial homogeneous case, $D = 0$, different versions are discussed and compared in [7, 13, 14, 16, 17, 20] . Some of these papers also contain discussions on whether the Itô- or Stratonovich-interpretation of the equation is most appropriate. Note that the spatial homogeneous case when the carrying capacity $K > 0$ is constant and $r = r_0 + \alpha \dot{W}(t)$ has recently been analysed in detail in [11, 15].

(1.2) is a very simple model for a population living in a stochastic environment with limited carrying capacity. It is well known that under suitable conditions the corresponding deterministic equation (1.1) develops travelling waves. In this paper we shall study how the strength of the environmental noise influences the travelling waves which are known to develop in the corresponding deterministic equation.

Generally, one would also like r, K, and D to be stochastic, time and space dependent. Apart from the practical difficulties in analysing the behavior of the solution to an equation with so many degrees of freedom, we also face the fundamental problem that (1.2) may fail to have a solution in the usual sense. It is well known that solutions of many SPDE's only exist in some generalized sense in particular in higher space dimensions. This is, for example, the case if D is assumed to be stochastic.

However, if the equation is interpreted in the Wick sense and within the context of the Kondratiev space $(\mathscr{S})_{-1}$ of stochastic distributions (see [12]), then it gets the form

$$\frac{\partial u(t,x)}{\partial t} = \frac{D}{2}\Delta u(t,x) + u(t,x) \diamond (c_0 - u(t,x)) + k(t)u(t,x) \diamond \dot{W}(t,x).$$

It has been shown recently (see [10]) that if the initial values $u(0,x)$ are specified then a unique $(\mathscr{S})_{-1}$ valued solution of this equation exists, for any space dimension.

In view of the above suppose $u(t,x)$ solves the stochastic KPP equation

$$du(t,x) = \left(\frac{D}{2}\Delta u(t,x) + u(t,x)c(u(t,x))\right) dt + k(t)u(t,x)\, dW_t, \;\; u(0,x) = u_0(x) \qquad (1.3)$$

for $t > 0$ and $x \in \mathbb{R}$, where $D > 0$ and $W = \{W_t, \mathscr{F}_t;\, t \geq 0\}$ is a Brownian motion. It is well known that if $c(u) = 1 - u$, $k \equiv 0$ and $u_0 = \chi_{(-\infty,0]}$, then (1.3) has a unique solution and it tends to a travelling wave as time and space tend to infinity, see e.g. [3, 8, 18, 19].

Assume $c \in C^1(\mathbb{R}^+)$ is strictly decreasing, $c_0 = c(0) > 0$, there is $\theta_0 > 0$ such that $c(\theta) \leq 0$ for all $\theta \geq \theta_0$, and k is not identically zero. The large time and space behavior of this stochastically perturbed KPP equation has also been studied, see [4, 5, 6]. Some computer simulations of the solution are reproduced in [5]. In [4] the authors study the asymptotic behavior of the solution to a stochastic KPP equation closely related to (1.3) using Hamilton-Jacobi theory. The authors show that the solution's behavior as time and space tend to infinity depends on the strength of the noise. If the noise is strong, the solution tends to zero. If it's moderately strong the solution may tend to a travelling wave (possibly travelling at a reduced speed) or the wave may be destroyed. The solution tends to the same travelling wave as the solution of the unperturbed deterministic problem, if the noise is weak.

We use an extension of the reduction method in [2] (see also [12]), to show that (1.3) has a unique strong solution, when u_0 is an $\mathscr{F}_T$-measurable random variable and k is deterministic. Applying Itô calculus, we use a similar argument to prove the existence of a unique strong solution to (1.3) when u_0 is $\mathscr{F}_0$-measurable and k is Itô-integrable on compact time intervals. By a strong solution of (1.3) we mean that $u(t,x)$ is almost surely twice continuously differentiable with respect to x, the integrals in (1.3) exist, and u satisfies the equation.

An implicit Feynman-Kac-like formula for the solution of (1.3) will be given. With this formula it is possible to extend the ideas in [8] for the deterministic KPP equation, to study $u(t,x)$ for large times. We characterize the asymptotic behavior of the solution in terms of k in two different cases:

(a) $u_0 = \chi_{(-\infty,f]}$, where f is an $\mathscr{F}_T$-measurable random variable for some $T \geq 0$, and k is deterministic; and

(b) $u_0 = \chi_{(-\infty,0]}$ and k is Itô integrable on compact time intervals.

As in [4, 5, 6] we find that the solution's behavior depends on the strength of the noise. In (a) we obtain the same limit behavior for a.e. ω, whereas we in (b) find that the behavior is ω dependent.

If the noise is strong, that is if

$$\liminf_{t\to\infty} \frac{1}{2t}\int_0^t k(s)^2\,ds > c(0) = c_0 \tag{1.4}$$

the solution in case (a) almost surely tends to zero as time tends to infinity. In case (b) the solution tends to zero for a.e. ω such that (1.4) holds.

We say the noise is weak if $\int_0^\infty k(s)^2\,ds < \infty$. In (a) the solution of (1.3) a.s. converges to the same travelling wave as the solution of the corresponding unperturbed deterministic KPP equation, when the noise is weak. In (b) it converges to the same wave for a.e. ω such that $\int_0^\infty k(s,\omega)^2\,ds < \infty$.

When neither of the above cases occur, we say the noise is moderately strong. The asymptotic behavior of the solution to (1.3) is in this case analysed using methods from [4]. We first compare the solution of (1.3) with the solution, w, of a stochastic partial differential equation, where ω only enters as a parameter. If in addition $2a_1 \leq k(t)^2 \leq 2a_2$ for all sufficiently large t, we are able to obtain asymptotic estimates on w. These bounds can then be used to obtain more explicit estimates on u as time tends to infinity.

In the anticipating case the limit behavior obtained below agrees with what is found for the related problem studied in [4], using different methods. When the initial condition is adapted and k is assumed Itô integrable on compact time intervals we observe a more complex behavior, in the sense that the solution's limit behavior may depend on ω.

The paper is organized as follows: In Section 2 we give two results from white noise analysis needed to understand the reduction method in Section 3, when the initial condition is assumed anticipating. In Section 3 we show how existence and uniqueness results can be obtained for (1.3). The asymptotic behavior of the solutions is studied in Section 4. In the final section we briefly sum up our results.

2 Two Results from White Noise Analysis

We refer the reader to [12] for a comprehensive introduction to white noise analysis and SPDE's. Here we only mention two results which play an important role in what follows. Let $(\mathscr{S}', \mathscr{B}, P)$ denote the white noise probability space, $W = \{W_t; 0 \leq t < \infty\}$ be the Brownian motion given by the coordinate process, and $\dot{W}_t$ denote the distributional time derivative of W_t. Then

$$\int_0^t f(s)\, dW_s = \int_0^t f(s) \diamond \dot{W}_s\, ds,$$

where the left hand side is interpreted as an Itô- or Skorohod-integral and the right hand side as a Pettis integral in the space of tempered distributions (see e.g. [12] page 45 for details). Here $\diamond$ denotes the Wick product. The definition of the Wick product will not be needed in the following, since all Wick products that occur can be expressed in terms of the ordinary product using Gjessing's translation formula (see e.g. Theorem 2.10.7 in [12]). This result says that if $\phi \in L^2(\mathbb{R}^+)$ and $X \in L^p(P)$ for some $p > 1$, then

$$(X \diamond \mathscr{E}_\infty(\phi))(\omega) = X(\omega - \phi) \cdot \mathscr{E}_\infty(\phi, \omega) \text{ a.s.},$$

where

$$\mathscr{E}_t(\phi) := \exp\left(\int_0^t \phi(s)\, dW_s - \frac{1}{2} \int_0^t \phi(s)^2\, ds \right), \quad 0 \leq t \leq \infty. \tag{2.1}$$

3 Existence and Uniqueness of a Strong Solution

Let $D > 0$, $c \in C^1(\mathbb{R}^+)$, and suppose there is $\theta_0 > 0$ such that $c(\theta) \leq 0$ for all $\theta \geq \theta_0$. We shall apply a reduction method to prove existence and uniqueness of a strong solution to

$$du(t,x) = \left(\frac{D}{2} \Delta u(t,x) + u(t,x) c(u(t,x)) \right) dt + k(t) u(t,x)\, dW_t, \;\; u(0,x) = u_0(x) \tag{3.1}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. The idea is to transform (3.1) into a deterministic equation that can be solved for each ω, separately. We do this for two different cases:

- anticipating initial data together with deterministic k, and
- $\mathscr{F}_0$-measurable initial condition together with a $k(t,\omega)$ which is Itô integrable on compact time intervals.

The former is shown using an extension of the white noise technique in [2], and the latter applying Itô calculus. We present a complete argument for the first case and sketch an argument for the second, since it is similar.

Definition 3.1 *A random field $u : [0,\infty) \times \mathbb{R} \times \mathscr{S}' \to \mathbb{R}$ is called a* (strong) solution *of (3.1) if* *(a)* $u(\cdot,\cdot,\omega) \in C^{0,2}((0,\infty)\times\mathbb{R})$ *a.s.,*

(b) $u(t,x)$, $\Delta u(t,x) \in L^2(P)$ *for all* $(t,x) \in (0,\infty)\times\mathbb{R}$, *and*

(c) u *satisfies (3.1) a.s. (in the sense below).*

Remark 3.2 *If* $(t,\omega) \mapsto u(t,x,\omega)$ *is not* $\mathscr{F}_t$*-adapted we interpret the stochastic integral in (3.1) as* $\int_0^t k(s)u(s,x) \diamond \dot{W}_s\,ds$ *and require the result to be in* $L^2(P)$*. (This is often called a generalized Skorohod interpretation of (3.1).)*

Since u_0 *may have discontinuities,* u *will satisfy the initial condition in the sense that for almost all* $x \in \mathbb{R}$,

$$\lim_{t\downarrow 0} u(t,x,\omega) = u_0(x) \text{ a.s.}$$

(in fact for all $x \in \mathbb{R}$ *at which* $u_0(\cdot,\omega)$ *is continuous.) (3.1) is satisfied in the sense that for all* $0 < t_0 < t < \infty$ *and* $x \in \mathbb{R}$,

$$u(t,x) = u(t_0,x) + \int_{t_0}^t \left(\frac{D}{2}\Delta u(s,x) + u(s,x)c(u(s,x))\right) ds + \int_{t_0}^t k(s)u(s,x)\,dW_s \text{ a.s.}$$

If $u_0(\cdot,\omega)$ *is continuous a.s., the integral formulation holds with* $t_0 = 0$.

3.1 Anticipating case

In this section we prove existence and uniqueness of a strong solution to (3.1), when $k \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$ and $u_0(x)$ is a stochastic, possibly anticipating, random variable for $x \in \mathbb{R}$.

To obtain the existence and uniqueness theorem we will extend the method in [2]. Let $\mathscr{E}_t(-k)$ be given by (2.1) for (deterministic) $k \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$ and note that

$$\frac{d}{dt}\mathscr{E}_t(-k) = -k(t)\dot{W}_t \diamond \mathscr{E}_t(-k),\ 0 < t < \infty.$$

If u solves (3.1) and $v(t,x) := u(t,x) \diamond \mathscr{E}_t(-k)$, then

$$\frac{\partial}{\partial t}v(t,x) = \left(\frac{D}{2}\Delta u(t,x) + u(t,x)c(u(t,x))\right) \diamond \mathscr{E}_t(-k).$$

The definition of v and Gjessing's formula gives

$$\frac{\partial}{\partial t}v(t,x,\omega) = \frac{D}{2}\Delta v(t,x,\omega) + v(t,x,\omega)c(u(t,x,\omega + k\chi_{[0,t]})).$$

Another application of Gjessing's formula shows that $v(t,x,\omega)$ satisfies the (deterministic) equation

$$\begin{aligned} \tfrac{\partial}{\partial t}v(t,x,\omega) &= \tfrac{D}{2}\Delta v(t,x,\omega) + v(t,x,\omega)\,c(\mathscr{E}_t(-k,\omega)^{-1}v(t,x,\omega)) \\ v(0,x,\omega) &= u_0(x,\omega) \end{aligned} \tag{3.2}$$

for almost every fixed $\omega \in \mathscr{S}'$.

If a classical solution, $v(t,x,\omega)$, of (3.2) is known for a.e. $\omega \in \mathscr{S}'$, then

$$u(t,x,\omega) = v(t,x,\omega - k\chi_{[0,t]})\mathscr{E}_t(-k,\omega - k\chi_{[0,t]})^{-1} = v(t,x,\omega - k\chi_{[0,t]})\mathscr{E}_t(k,\omega) \tag{3.3}$$

is a solution of the original problem (3.1).

This shows that to solve (3.1) it is sufficient to solve (3.2), where ω only enters as a parameter. Then (3.3) gives a solution of (3.1). Moreover, if (3.2) has a unique solution, the solution of (3.1) is unique as well.

To prove (3.2) has a unique solution for almost every $\omega \in \mathscr{S}'$ one may apply any method for deterministic nonlinear parabolic PDE's. A contraction method yields:

Proposition 3.3 *Let $D > 0$, $c \in C^1(\mathbb{R}^+)$, and let $k \in L^2_{\text{loc}}(\mathbb{R}^+)$ be deterministic. Suppose there exists $\theta_0 > 0$ such that $c(\theta) \leq 0$ for all $\theta \geq \theta_0$. Assume $u_0 \in L^\infty(\mathscr{S}'; L^\infty(\mathbb{R}))$ is such that $x \mapsto u_0(x,\omega)$ is piecewise continuous and nonnegative for a.e. $\omega \in \mathscr{S}'$. Then (3.1) has a unique strong solution.*

Proof: We begin by proving (3.2) has a unique classical solution, v, for a.e. $\omega \in \mathscr{S}'$. Let $F \subset \mathscr{S}'$ be a set of 1 measure on which $t \mapsto \mathscr{E}_t(-k,\omega)$ is continuous and $x \mapsto u_0(x,\omega)$ is piecewise continuous.

Fix any $T > 0$ and $\omega \in F$. A classical solution of (3.2) has to satisfy

$$\begin{aligned} v(t,x,\omega) &= \int_{\mathbb{R}} p(t,x;0,y)u_0(y,\omega)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}} p(t,x;s,y)v(s,y,\omega)c(v(s,y,\omega)\mathscr{E}_s(-k,\omega)^{-1})\,dyds \end{aligned} \tag{3.4}$$

for $(t,x) \in (0,T] \times \mathbb{R}$, where p denotes the Green's function associated with $\partial_t + D\Delta/2$ on $\mathbb{R}^+ \times \mathbb{R}$.

By the comparison theorem, see e.g. [9], a classical solution of (3.2) also has to satisfy the a priori bounds

$$0 \leq v(t,x,\omega) \leq \|u_0(\cdot,\omega)\|_\infty \vee \theta_0 \max_{0\leq t\leq T} \mathscr{E}_t(-k,\omega) \tag{3.5}$$

for $(t,x) \in [0,T]\times\mathbb{R}$. Based on these properties (3.4) can be used to construct a contraction, see Chapter 14 in [19] for details. Applying Banach's fixed point theorem we obtain a unique solution $v(\cdot,\cdot,\omega) \in C((0,T] \times \mathbb{R})$ of (3.4). A classical regularity result, see e.g. Chapter 1.7 in [9], ensures that $v(\cdot,\cdot,\omega) \in C^{1,2}((0,T] \times \mathbb{R})$. Since v was found using a contraction method, $v(t,x,\cdot)$ is measurable and (3.5) ensures that $v(t,x,\cdot) \in L^2(P)$ for all $(t,x) \in [0,T] \times \mathbb{R}$. To obtain a solution for all $t > 0$, note that $T > 0$ was arbitrary.

The argument preceding the proposition ensures that u is a strong solution of (3.1). The solution u is unique, since v is the unique solution of (3.2). The other statements follow easily. $\square$

The solution also enjoys the following properties:

Proposition 3.4 *Let u denote the strong solution of (3.1), then*

(a) $u(t,x) \geq 0$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ a.s.

(b) If $x \mapsto u_0(x)$ is decreasing a.s., then $x \mapsto u(t,x,\omega)$ is decreasing for each $t \in \mathbb{R}^+$ and a.e. $\omega \in \mathscr{S}'$.

Proof: (a) follows from (3.3), (3.5), and that $\mathscr{E}_t(-k) > 0$ for $t \geq 0$ a.s.

(b) Let $y \geq 0$. Suppose v and w solve (3.2) with the initial conditions $v(0,x) = u_0(x)$ and $w(0,x) = u_0(x+y)$, respectively. From the comparison theorem $v(t,x) \geq w(t,x)$. Applying (3.3) completes the proof. □

Remark 3.5 *The solution, $u(t,x)$, we obtain is the* unique strong solution *of (3.1). Note also that we have shown the existence of a unique strong solution to a nonlinear SPDE, where the nonlinear term, $uc(u)$, does not satisfy a global Lipschitz condition.*

Travelling waves do not form under arbitrary initial conditions, e.g., $u_0 \equiv \text{const} \geq 0$. Later we shall assume $u_0 = \chi_{(-\infty, f(\omega)]}$, where f is an $\mathscr{F}_T$-adapted random variable for some $T > 0$. The following remark gives a representation formula for the solution, u, for large times, when u_0 is $\mathscr{B} \otimes \mathscr{F}_T$-measurable. This formula turns out to be very useful in Section 4.

Remark 3.6 *Let u_0 be $\mathscr{B} \otimes \mathscr{F}_T$-measurable for some $T \geq 0$. Suppose $v(t,x,\omega)$ is a classical solution of (3.2) for a.e. $\omega \in \mathscr{S}'$. Then v satisfies the Feynman-Kac formula*

$$v(t,x,\omega) = \bar{E}\left[u_0(x + \sqrt{D}\bar{B}_t, \omega) \exp \int_0^t c\left(v(t-s, x+\sqrt{D}\bar{B}_s, \omega)\mathscr{E}_{t-s}(-k,\omega)^{-1}\right) ds\right]$$

for a.e. $\omega \in \mathscr{S}'$, where $\bar{B} = \{\bar{B}_t;\, t \geq 0\}$ is a Brownian motion defined on an auxiliary probability space, $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$, and $\bar{E}$ denotes the expectation with respect to $\bar{P}$. To obtain u from v, we substitute $\omega - k\chi_{[0,t]}$ for ω in v and multiply by $\mathscr{E}_t(k)$. Since $u_0(x)$ is $\mathscr{F}_T$-adapted, $u_0(x, \omega - k\chi_{[0,t]}) = u_0(x, \omega - k\chi_{[0,T]})$ for $T \leq t \leq \infty$. A straightforward calculation shows that $\mathscr{E}_s(-k, \omega - k\chi_{[0,t]})^{-1} = \mathscr{E}_s(k,\omega)$ for all $0 \leq s \leq t \leq \infty$. Therefore $v(t,x,\omega - k\chi_{[0,t]}) = v(t,x,\omega - k)$ for all $t \geq T$. Let $\tilde{v}(t,x,\omega) := v(t,x,\omega-k)$, then $u(t,x,\omega) = \tilde{v}(t,x,\omega)\mathscr{E}_t(k,\omega)$ for $t \geq T$, where $\tilde{v}$ satisfies

$$\frac{\partial}{\partial t}\tilde{v} = \frac{D}{2}\Delta\tilde{v} + \tilde{v}c(\tilde{v}\mathscr{E}_t(k)),\;\; \tilde{v}(0,x,\omega)) = u_0(x,\omega - k) \tag{3.6}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ a.e. $\omega \in \mathscr{S}'$. Since $\tilde{v}$ is a strong solution of (3.6), $\tilde{v}$ almost surely satisfies the Feynman-Kac formula

$$\tilde{v}(t,x,\omega) = \bar{E}\left[u_0(x+\sqrt{D}\bar{B}_t, \omega - k) \exp \int_0^t c\left(\tilde{v}(t-s, x+\sqrt{D}\bar{B}_s, \omega)\mathscr{E}_{t-s}(k,\omega)\right) ds\right],$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.

We sum up our results in this section the following theorem:

Theorem 3.7 *(a) Let $D > 0$, $k \in L^2_{\text{loc}}(\mathbb{R}^+)$, $c \in C^1(\mathbb{R}^+)$, and suppose there exists $\theta_0 > 0$ such that $c(\theta) \leq 0$ for all $\theta \geq \theta_0$. Assume $u_0 \in L^\infty(\mathscr{S}'; L^\infty(\mathbb{R}))$ is such that $x \mapsto u_0(x, \omega)$ is piecewise continuous and nonnegative for a.e. $\omega \in \mathscr{S}'$. Then (3.1) has a unique nonnegative strong solution.*

(b) If, in addition to the assumptions above, u_0 is $\mathscr{B} \otimes \mathscr{F}_T$-measurable for some $0 \leq T < \infty$, then

$$u(t, x, \omega) = \tilde{v}(t, x, \omega)\mathscr{E}_t(k, \omega) \text{ for } (t, x) \in [T, \infty) \times \mathbb{R},$$

where $\tilde{v}$ satisfies (3.6) for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

(c) If, in addition to the conditions in (a), $x \mapsto u_0(x, \omega)$ is decreasing a.s., then $x \mapsto u(t, x, \omega)$ is decreasing almost surely for every $t \geq 0$.

3.2 Adapted case

Suppose $u_0(x)$ is $\mathscr{F}_0$-measurable for every $x \in \mathbb{R}$, $k = k(s, \omega)$ is Itô integrable on compact time intervals, and that $u(t, x)$ is a strong solution of (3.1). Let $b = D\Delta u/2 + uc(u)$ and $\sigma = ku$, then u_t solves the diffusion equation

$$du_t = b\, dt + \sigma\, dW_t, \;\; u_t|_{t=0} = u_0,$$

for each $x \in \mathbb{R}$. Let $Y_t := (\mathscr{E}_t(k))^{-1} = \exp(-\int_0^t k_s\, dW_s + \frac{1}{2}\int_0^t k_s^2\, ds)$, then Y_t satisfies

$$dY_t = k_t^2 Y_t\, dt - k_t Y_t\, dW_t, \;\; Y_0 = 1.$$

Itô's formula shows that $\tilde{v}(t, x) = u(t, x)Y_t$ satisfies

$$\frac{\partial}{\partial t}\tilde{v}(t, x) = \frac{D}{2}\Delta\tilde{v}(t, x) + \tilde{v}(t, x)c(\mathscr{E}_t(k)\tilde{v}(t, x)), \;\; \tilde{v}(0, x) = u_0(x). \tag{3.7}$$

Again we have arrived at a PDE where ω enters as a parameter only. Moreover, it is not difficult to show that if (3.7) has a unique solution $\tilde{v}$ for almost every ω, then $u(t, x) = \tilde{v}(t, x)\mathscr{E}_t(k)$ is the unique strong solution of (3.1).

Arguing as in the previous section, one can prove the following result:

Theorem 3.8 *(a) Let $D > 0$, $c \in C^1(\mathbb{R}^+)$, and suppose there exists $\theta_0 > 0$ such that $c(\theta) \leq 0$ for all $\theta \geq \theta_0$. Assume $k(t, \omega)$ is Itô-integrable on compact time intervals and $u_0 \in L^\infty(\mathscr{S}'; L^\infty(\mathbb{R}))$ is such that $x \mapsto u_0(x, \omega)$ is piecewise continuous and nonnegative for a.e. $\omega \in \mathscr{S}'$. Suppose also that $u_0(x)$ is $\mathscr{F}_0$-measurable for $x \in \mathbb{R}$. Then (3.1) has a unique nonnegative strong solution, $u(t, x)$, given by*

$$u(t, x, \omega) = \tilde{v}(t, x, \omega)\mathscr{E}_t(k, \omega) \text{ for } (t, x) \in \mathbb{R}^+ \times \mathbb{R},$$

where $\tilde{v}$ almost surely satisfies (3.7) for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

(b) If, in addition to the assumptions above, $x \mapsto u_0(x, \omega)$ is decreasing a.s., then $x \mapsto u(t, x, \omega)$ is decreasing a.s. for every $t \geq 0$.

4 Travelling Waves for the Stochastic KPP Equation

Below we study the solution of (3.1) for large time and space. First we would like to remind the reader of the behavior in the deterministic case:

If c is strictly decreasing, $k \equiv 0$, and $u_0 = \chi_{(-\infty,0]}$, the solution of (3.1) tends to a travelling wave. With Freidlin's point of view, i.e., if we consider the solution's limit as time and space tend to infinity and ignore questions concerning the wave's shape, this can be expressed as

$$\lim_{t\to\infty} \inf_{x<t(\sqrt{2c_0D}-h)} u(t,x) = c^{-1}(0) \text{ and } \lim_{t\to\infty} \sup_{x>t(\sqrt{2c_0D}+h)} u(t,x) = 0, \tag{4.1}$$

for any $h > 0$, where $c_0 := c(0)$. We call $\alpha = \sqrt{2c_0D}$ the speed of the wave. On the right hand side of the line $x = \alpha t$ the solution tends to 0 and on the left hand side it tends to $c^{-1}(0)$, where $c^{-1}(\cdot)$ denotes the inverse of $c(\cdot)$.

In the following paragraphs we study how the solution of (3.1) behaves as time tends to infinity, when

(a) $u_0 = \chi_{(-\infty,f]}$, where f is an $\mathscr{F}_T$-measurable random variable for some $T \geq 0$, and $k \in L^2_{\text{loc}}(\mathbb{R}^+)$; and

(b) $u_0 = \chi_{(-\infty,0]}$ and $k(t,\omega)$ is Itô integrable on compact time intervals.

4.1 Strong noise

We first consider case (a). Suppose $k \in L^2_{\text{loc}}(\mathbb{R}^+)$ is deterministic and define

$$a_* := \liminf_{t\to\infty} \frac{1}{2t}\int_0^t k(s)^2\, ds.$$

Theorem 4.1 *Suppose the conditions in Theorem 3.7 (a) hold and let $u(t,x)$ denote the strong solution of (3.1). If $a_* > \max_{0\leq\theta\leq\theta_0} c(\theta)$, then for almost every $\omega \in \mathscr{S}'$,*

$$0 \leq u(t,x,\omega) \leq \|u_0\|_\infty \exp\left(t\max_\theta c(\theta) + \int_0^t k(s)\, dW_s(\omega) - \frac{1}{2}\int_0^t k(s)^2\, ds\right) \to 0,$$

as $t \to \infty$, for all $x \in \mathbb{R}$.

Proof: Let u denote the solution of (3.1). Then $u(t,x,\omega) = v(t,x,\omega - k\chi_{[0,t]})\mathscr{E}_t(k,\omega)$ from (3.3), where v satisfies

$$\begin{aligned} v(t,x,\omega) &= \bar{E}\left[u_0(x+\sqrt{D}\bar{B}_t,\omega)\exp\int_0^t c(v(t-s,x+\sqrt{D}\bar{B}_s,\omega)\mathscr{E}_{t-s}(-k,\omega)^{-1})\, ds\right] \\ &\leq \|u_0\|_\infty \exp\left(t\max_\theta c(\theta)\right), \end{aligned}$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}$. It follows that for almost every $\omega \in \mathscr{S}'$,

$$0 \le u(t,x,\omega) \le \|u_0\|_\infty \exp\left(t \max_\theta c(\theta) + \int_0^t k(s)\, dW_s(\omega) - \frac{1}{2}\int_0^t k(s)^2\, ds \right) \to 0,$$

as $t \to \infty$, for all $x \in \mathbb{R}$. □

We may argue similarly in case (b). Suppose $k(t,\omega)$ is Itô-integrable on compact time intervals and define

$$a_*(\omega) := \liminf_{t\to\infty} \frac{1}{2t}\int_0^t k(s,\omega)^2\, ds.$$

Theorem 4.2 *Suppose the conditions in Theorem 3.8 (a) are satisfied and let $u(t,x)$ denote the strong solution of (3.1). If $A = \{\omega \in \mathscr{S}';\ a_*(\omega) > \max_\theta c(\theta)\}$, then for a.e. $\omega \in A$,*

$$u(t,x,\omega) \to 0 \text{ as } t \to \infty,$$

for all $x \in \mathbb{R}$.

Thus, for a.a. ω such that $a_* > \max_\theta c(\theta)$, the solution of (3.1) tends to zero (uniformly in x) as t tends to infinity. Put differently, if the noise is sufficiently strong the wave structure for the corresponding deterministic equation is destroyed. This is not surprising considering that the solution of the SODE we obtain from (3.1) by letting $D = 0$ also vanishes if the noise is sufficiently strong. This in fact follows immediately from our results. See [1] and the references therein for alternative discussions.

4.2 Moderate noise

When the noise is moderately strong, the solution of (3.1) displays a more complex behavior than what we found in the previous section.

Theorem 4.3 *Suppose the conditions in Theorem 3.7 (a) are satisfied and let u denote the strong solution of (3.1). Assume that $c'(\theta) \le 0$ for $\theta > 0$ and $u_0(x,\omega) = \chi_{(-\infty,f(\omega)]}(x)$, where f is an $\mathscr{F}_T$-measurable random variable for some $0 \le T < \infty$. If $k \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ (deterministic) is such that the limit*

$$a := \lim_{t\to\infty} \frac{1}{2t}\int_0^t k(s)^2\, ds$$

exists and $0 \le a \le c_0 := c(0)$, then for any $h > 0$

$$\lim_{t\to\infty} \sup_{x > t(\alpha+h)} u(t,x) = 0 \text{ a.s.},$$

where $\alpha = \sqrt{2(c_0 - a)D}$.

Proof: Fix arbitrary $h > 0$ and choose $0 < \varepsilon < h(h + 2\alpha)/4D$, then for a.e. $\omega \in \mathscr{S}'$ there is $t_0 = t_0(\varepsilon, \omega) \geq T$ such that

$$e^{-(a+\varepsilon)t} \leq \mathscr{E}_t(k, \omega) \leq e^{-(a-\varepsilon)t} \text{ for } t \geq t_0. \tag{4.2}$$

Let $\omega \in \mathscr{S}'$ be such that (4.2) holds and consider

$$V_t = \frac{D}{2}\Delta V + Vc(e^{-(a+\varepsilon)t}V), \; V|_{t=t_0} = V_0.$$

By the comparison theorem, see e.g. [9],

$$V(t, x) \geq \tilde{v}(t, x) \text{ for } (t, x) \in [t_0, \infty) \times \mathbb{R}, \tag{4.3}$$

if $V_0(x) \geq \tilde{v}(t_0, x)$ for $x \in \mathbb{R}$, where $\tilde{v}$ denotes the solution of (3.6). We define

$$V_0(x) := \frac{e^{c_0 t_0}}{\sqrt{2\pi t_0}} \int_{-\infty}^{(f(\omega-k)-x)/\sqrt{D}} e^{-z^2/2t_0}\, dz \geq \tilde{v}(t_0, x), \; x \in \mathbb{R}.$$

Let

$$w(t, x) = V(t + t_0, x) \exp(-(a + \varepsilon)(t + t_0)), \tag{4.4}$$

then w satisfies

$$w_t = \frac{D}{2}\Delta w + w(c(w) - a - \varepsilon), \; w|_{t=0} = e^{-(a+\varepsilon)t_0} V_0.$$

The implicit Feynman-Kac formula for w shows that

$$w(t, x) \leq e^{c_0 t - (a+\varepsilon)(t_0+t)} \bar{E}[V_0(x + \sqrt{D}\bar{B}_t)] \text{ for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{4.5}$$

From Theorem 3.7 (b), (4.3), (4.4), and (4.5) we obtain

$$u(t + t_0, x) = \tilde{v}(t + t_0, x)\mathscr{E}_{t+t_0}(k) \leq e^{c_0 t}\bar{E}[V_0(x + \sqrt{D}\bar{B}_t)]\mathscr{E}_{t+t_0}(k) \tag{4.6}$$

for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$. It is not difficult to show that

$$e^{c_0 t}\bar{E}[V_0(x + \sqrt{D}\bar{B}_t)] = e^{c_0(t_0+t)}\bar{P}\left(\bar{B}_{t+t_0} < \frac{f(\omega - k) - x}{\sqrt{D}}\right).$$

Large deviation bounds for Brownian motion ensures that for any $\beta > 0$ there is a $t_1 = t_1(\varepsilon, \beta, \omega) \geq 0$ such that

$$\bar{P}\left(\bar{B}_{t+t_0} < \frac{f(\omega - k) - \beta(t + t_0)}{\sqrt{D}}\right) \leq \exp\left\{\left(-\frac{\beta^2}{2D} + \varepsilon\right)(t + t_0)\right\},$$

for $t > t_1$. Combining this with (4.2) and (4.6) gives

$$0 \leq u(t + t_0, \beta(t + t_0), \omega) \leq K \exp((c_0 - a + 2\varepsilon - \frac{\beta^2}{2D})t)$$

for $t > t_1$. Our choice of ε ensures that

$$u(t, (\alpha + h)t, \omega) \to 0 \text{ as } t \to \infty.$$

The theorem now follows from Theorem 3.7 (c) and observing that $h > 0$ and ω were arbitrary. $\square$

An argument similar to the one above gives the corresponding result in case (b), when $u_0 = \chi_{(-\infty,0]}$ and $k(t,\omega)$ is Itô integrable on compact time intervals:

Theorem 4.4 *Suppose the conditions in Theorem 3.8 (a) are satisfied and let u denote the strong solution of (3.1). Assume c is decreasing and $u_0 = \chi_{(-\infty,0]}$. Let*

$$a(\omega) := \lim_{t\to\infty} \frac{1}{2t} \int_0^t k(s,\omega)^2\, ds,$$

for those $\omega \in \mathscr{S}'$ for which the limit exists and leave $a(\omega)$ undefined otherwise. Then for a.e. $\omega \in \mathscr{S}'$ such that $0 \leq a(\omega) \leq c_0 := c(0)$,

$$\lim_{t\to\infty} \sup_{x > (\alpha(\omega)+h)t} u(t,x,\omega) = 0$$

for any $h > 0$, where $\alpha(\omega) = \sqrt{2(c_0 - a(\omega))D}$.

It is more complicated to obtain bounds on the solution when $x < t\sqrt{2(c_0 - a)D}$. We begin by comparing u with w, the solution of another partial differential equation. The proof is essentially the same as the proof of Lemma 3.1 in [6] and Lemma 1.6 in [4]. Note, however, that for the problems studied here, w satisfies a *random* partial differential equation. But since ω only enters as a parameter in the equation for w, it is easier to study the asymptotic properties of w than it is to study the asymptotic properties of u directly.

Theorem 4.5 *Assume the conditions in Theorem 3.7 (a) and (b) are satisfied and let u be the strong solution of (3.1). Suppose that for a.e. $\omega \in \mathscr{S}'$, w is a classical solution of*

$$\frac{\partial w}{\partial t} = \frac{D}{2}\Delta w + w(c(w) - \frac{1}{2}k^2),\ w(0,\cdot,\omega) = u_0(\cdot, \omega - k);\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{4.7}$$

If c is decreasing, then

$$w(t,x) \exp \inf_{0\leq\sigma\leq t} \int_\sigma^t k_s\, dW_s \leq u(t,x) \leq w(t,x) \exp \sup_{0\leq\sigma\leq t} \int_\sigma^t k_s\, dW_s,\ t \geq T,\ x \in \mathbb{R}; \tag{4.8}$$

for a.e. $\omega \in \mathscr{S}'$.

Proof: Suppose, to obtain a contradiction, that there is $(t', x') \in [T, \infty) \times \mathbb{R}$ such that

$$u(t', x') > w(t', x') \exp \sup_{0\leq\sigma\leq t'} \int_\sigma^{t'} k_s\, dW_s, \tag{4.9}$$

then $u(t',x') > w(t',x')$. To simplify notation let $\bar{X}_s^{t',x'} = (t'-s, x'+\sqrt{D}\bar{B}_s)$ for $s \geq 0$ and let $\tilde{u}(t,x) := \tilde{v}(t,x)\mathscr{E}_t(k)$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, where $\tilde{v}$ solves (3.6). Recall from Theorem 3.7 (b) that $\tilde{u}(t,x) = u(t,x)$ when $t \geq T$. Define the stopping time

$$\bar{\tau} := \inf\{s > 0;\ \tilde{u}(\bar{X}_s^{t',x'})) = w(\bar{X}_s^{t',x'})\},$$

for each $\omega \in \mathscr{S}'$. Using the strong Markov property we obtain that

$$\begin{aligned}
u(t',x') &= \bar{E}\left[\tilde{u}(\bar{X}_{\bar{\tau}}^{t',x'}) \exp\left(\int_0^{\bar{\tau}} c(\tilde{u}(\bar{X}_s^{t',x'}))\,ds + \int_{t'-\bar{\tau}}^{t'} k_s\,dW_s - \frac{1}{2}\int_{t'-\bar{\tau}}^{t'} k_s^2\,ds\right)\right] \\
&< \bar{E}\left[w(\bar{X}_{\bar{\tau}}^{t',x'}) \exp\left(\int_0^{\bar{\tau}} c(w(\bar{X}_s^{t',x'}))\,ds - \frac{1}{2}\int_{t'-\bar{\tau}}^{t'} k_s^2\,ds + \int_{t'-\bar{\tau}}^{t'} k_s\,dW_s\right)\right] \\
&\leq \bar{E}\left[w(\bar{X}_{\bar{\tau}}^{t',x'}) \exp\int_0^{\bar{\tau}} [c(w(\bar{X}_s^{t',x'})) - \frac{1}{2}k_{t'-s}^2]\,ds\right] \exp \sup_{0\leq\sigma\leq t'} \int_{t'-\sigma}^{t'} k_s\,dW_s \\
&= w(t',x') \exp \sup_{0\leq\sigma\leq t'} \int_\sigma^{t'} k_s\,dW_s,
\end{aligned}$$

which contradicts (4.9) and proves the upper bound in (4.8). The lower bound is shown similarly. □

Arguing as above gives the corresponding result when u_0 is $\mathscr{F}_0$-measurable and $k(t,\omega)$ is Itô integrable on compact time intervals:

Theorem 4.6 *Suppose the conditions in Theorem 3.8 (a) are satisfied and let u denote the strong solution of (3.1). If c is decreasing and w is the classical solution of*

$$\frac{\partial w}{\partial t} = \frac{D}{2}\Delta w + w(c(w) - \frac{1}{2}k^2),\ \ w|_{t=0} = u_0, \tag{4.10}$$

for almost every $\omega \in \mathscr{S}'$, then

$$w(t,x) \exp \inf_{0\leq\sigma\leq t} \int_\sigma^t k_s\,dW_s \leq u(t,x) \leq w(t,x) \exp \sup_{0\leq\sigma\leq t} \int_\sigma^t k_s\,dW_s;\ \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

To obtain more explicit bounds on the solution of (3.1), we shall study the asymptotic behavior of the solutions to (4.7) and (4.10). The following standard result for deterministic PDE's, presented without proof, will play an important role:

Lemma 4.7 *Let $D > 0$, $k \in C(\mathbb{R}^+)$, and let $x \mapsto w_0(x) \geq 0$ be bounded and piecewise continuous. Suppose $c \in C^1(\mathbb{R}^+)$ and that there is $\theta_0 > 0$ such that $c(\theta) \leq 0$ for all $\theta \geq \theta_0$. Let w be the unique classical solution of (the deterministic equation)*

$$\frac{\partial w}{\partial t} = \frac{D}{2}\Delta w + w(c(w) - \frac{1}{2}k(t)^2),\ \ w|_{t=0} = w_0,\ \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

If there is $a \geq 0$ such that $a \leq \frac{1}{2}k(t)^2$ (resp. $\frac{1}{2}k(t)^2 \leq a$) for all $t \geq t_0 \geq 0$, then

$$0 \leq w(t,x) \leq q(t,x) \text{ (resp. } 0 \leq q(t,x) \leq w(t,x)) \text{ for } (t,x) \in [t_0,\infty) \times \mathbb{R},$$

where

$$\frac{\partial q}{\partial t} = \frac{D}{2}\Delta q + q(c(q) - a), \;\; q|_{t=t_0}(\cdot) = w(t_0,\cdot); \;\; (t,x) \in [t_0,\infty) \times \mathbb{R}.$$

Moreover, if $a > \max_\theta c(\theta)$, then $q(t,x) \downarrow 0$ (uniformly in x) as $t \to \infty$. If c is strictly decreasing and $0 \leq a < c_0 := c(0)$, then for any $h > 0$

$$\lim_{t\to\infty} \inf_{x<t(\alpha-h)} q(t,x) = c^{-1}(a) \text{ and } \lim_{t\to\infty} \sup_{x>t(\alpha+h)} q(t,x) = 0,$$

where $\alpha = \sqrt{2(c_0 - a)D}$.

With this lemma we can study the asymptotic behavior of the solutions to (4.7) and (4.10). The following argument applies to w satisfying (4.7) when $u_0 = \chi_{(-\infty,f]}$, for an $\mathscr{F}_T$-measurable random variable f, as well as to w satisfying (4.10) when $u_0 = \chi_{(-\infty,0]}$. Assume in addition that $t \mapsto k(t)$ is continuous (resp. continuous almost surely). Since ω only enters as a parameter in the SPDE's for w, we fix $\omega \in \mathscr{S}'$. If

$$0 \leq a_1 \leq \frac{1}{2}k(t)^2 \leq a_2, \tag{4.11}$$

for $t \geq t_0 \geq 0$, then Lemma 4.7 ensures

$$w_2(t,x) \leq w(t,x) \leq w_1(t,x) \text{ for } (t,x) \in [t_0,\infty) \times \mathbb{R},$$

where

$$\frac{\partial}{\partial t} w_i = \frac{D}{2}\Delta w_i + w_i(c(w_i) - a_i), \;\; w_i|_{t=t_0} = w|_{t=t_0};$$

for $t \geq t_0$, $x \in \mathbb{R}$, and $i = 1, 2$.

We can now apply the last part of Lemma 4.7 to obtain explicit bounds on w_1 and w_2 as time tends to infinity. We thereby also obtain explicit bounds on u. We consider the situation in Theorem 4.5 in detail. If $c \in C^1(\mathbb{R}^+)$ is strictly decreasing and $k \in C(\mathbb{R}^+)$ satisfies (4.11) with $0 \leq a_1 \leq a_2 \leq c_0$, we observe three different types of behavior: For any ε, $h > 0$ and almost every $\omega \in \mathscr{S}'$ there is $t_1 = t_1(\omega,\varepsilon,h) > 0$ such that

(i) if $x < (\sqrt{2(c_0 - a_2)D} - h)t$ and $t \geq t_1$,

$$(c^{-1}(a_2) - \varepsilon) \exp \inf_{0\leq\sigma\leq t} \int_\sigma^t k_s\, dW_s \leq u(t,x) \leq (c^{-1}(a_1) + \varepsilon) \exp \sup_{0\leq\sigma\leq t} \int_\sigma^t k_s\, dW_s;$$

(ii) if $(\sqrt{2(c_0 - a_2)D} - h)t \le x < (\sqrt{2(c_0 - a_1)D} - h)t$ and $t \ge t_1$,

$$0 \le u(t,x) \le (c^{-1}(a_1) + \varepsilon) \exp \sup_{0 \le \sigma \le t} \int_\sigma^t k_s \, dW_s; \text{ and}$$

(iii) if $(\sqrt{2(c_0 - a_1)D} + h)t < x$ and $t \ge t_1$, then

$$0 \le u(t,x) \le \varepsilon.$$

If in addition to (4.11), the limit $a = \lim_{t\to\infty} \int_0^t k(s)^2 \, ds/2t \in [a_1, a_2]$ exists, we may apply Theorem 4.3 to improve the two last estimates above as follows:

(ii') if $(\sqrt{2(c_0 - a_2)D} - h)t \le x < (\sqrt{2(c_0 - a)D} - h)t$ and $t \ge t_1$,

$$0 \le u(t,x) \le (c^{-1}(a) + \varepsilon) \exp \sup_{0 \le \sigma \le t} \int_\sigma^t k_s \, dW_s; \text{ and}$$

(iii') if $(\sqrt{2(c_0 - a)D} + h)t < x$ and $t \ge t_1$, then

$$0 \le u(t,x) \le \varepsilon.$$

Several remarks are in order. If $a_2 > c_0$, (i) no longer applies and the estimate in (ii) (or (ii')) holds for all $x < (\sqrt{2(c_0 - a_1)D} - h)t$ (or $x < (\sqrt{2(c_0 - a)D} - h)t$, respectively). If $a_1 > c_0$ also, then u converges uniformly to 0 by the results in Section 4.1.

Observe that if $k(t)$ doesn't vary much for large times, i.e., if $a_1 - a_2$ is close to zero, the region (ii) (and (ii') if a exists) is small and we obtain more accurate estimates on u. In particular, if $k_\infty = \lim_{t\to\infty} k(t)$ exists, the KPP equation for w tends to a travelling wave as time tends to infinity. It follows that for any $\varepsilon > 0$ and $h > 0$

$$(c^{-1}(k_\infty^2/2) - \varepsilon) \exp \inf_{0 \le \sigma \le t} \int_\sigma^t k_s \, dW_s \le u(t,x) \le (c^{-1}(k_\infty^2/2) + \varepsilon) \exp \sup_{0 \le \sigma \le t} \int_\sigma^t k_s \, dW_s$$

when $x < (\alpha - h)t$ and t is sufficiently large, where $\alpha = \sqrt{2(c_0 - k_\infty^2/2)D}$. Moreover,

$$\lim_{t\to\infty} \sup_{x > (\alpha + h)t} u(t,x) = 0.$$

Similar results are easily obtained if the assumptions in Theorem 4.6 are satisfied, $c \in C^1(\mathbb{R}^+)$ is strictly decreasing, $t \mapsto k(t,\omega)$ is continuous almost surely, and $u_0 = \chi_{(-\infty,0]}$. Note that in this case we find that (i)-(iii) (respectively, (i), and (ii')–(iii')) hold for a.e. $\omega \in \mathscr{S}'$ for which (4.11) hold with $0 \le a_1(\omega) \le a_2(\omega) \le c_0$.

4.3 Weak noise

If the noise is weak, a concept which is made precise in the theorems below, the solution of (3.1) tends to the solution of the corresponding deterministic equation ($k = 0$ and $u_0 = \chi_{(-\infty,0]}$) as time tends to infinity.

Theorem 4.8 *Suppose the conditions in Theorem 3.7 (a) are satisfied and let u be the strong solution of (3.1). Assume $c \in C^1(\mathbb{R}^+)$ is strictly decreasing and $u_0 = \chi_{(-\infty,f]}$, where f is an $\mathscr{F}_T$-measurable random variable for some $T \geq 0$. If $k \in L^2(\mathbb{R}^+)$ is deterministic, then for any $h > 0$*

$$\lim_{t\to\infty} \sup_{x<t(\alpha-h)} u(t,x) = c^{-1}(0) \text{ and } \lim_{t\to\infty} \sup_{x>t(\alpha+h)} u(t,x) = 0 \text{ a.s.},$$

where $\alpha = \sqrt{2c_0 D}$.

Proof: From Theorem 4.3 it is sufficient to show that $u(t,x)$ tends to $c^{-1}(0)$ on the left hand side of $x = t\sqrt{2c_0D}$. If $k \in L^2(\mathbb{R}^+)$, there exists $g \in L^1(\mathscr{S}')$ such that

$$\int_0^t k_s\, dW_s - \frac{1}{2}\int_0^t k_s^2\, ds \to g \text{ as } t \to \infty \tag{4.12}$$

for almost every $\omega \in \mathscr{S}'$. Let $\omega \in \mathscr{S}'$ be such that (4.12) holds and $h > 0$, then for any $\varepsilon > 0$ there is $t_0 = t_0(\omega, \varepsilon) \geq T$ such that

$$g - \varepsilon \leq \int_0^t k_s\, dW_s - \frac{1}{2}\int_0^t k_s^2\, ds \leq g + \varepsilon,\ t \geq t_0.$$

From the comparison theorem and (4.1) we see that

$$e^{-2\varepsilon}c^{-1}(0) \leq u(t,x,\omega) = \tilde{v}(t,x,\omega)\mathscr{E}_t(k,\omega) \leq e^{2\varepsilon}c^{-1}(0),$$

when $x < t(\sqrt{2c_0D} - h)$ and t is sufficiently large. Since h and ε were arbitrary the result follows. □

The following theorem is shown similarly.

Theorem 4.9 *Suppose the conditions in Theorem 3.8 (a) are satisfied and let u be the strong solution of (3.1). Let $c \in C^1(\mathbb{R}^+)$ be strictly decreasing, $u_0 = \chi_{(-\infty,0]}$, and let $k(t,\omega)$ be path continuous and Itô integrable on compact time intervals. Then for a.e. $\omega \in \mathscr{S}'$ such that $\int_0^\infty k(s,\omega)^2\, ds$ is finite,*

$$\lim_{t\to\infty} \sup_{x<t(\alpha-h)} u(t,x,\omega) = c^{-1}(0) \text{ and } \lim_{t\to\infty} \sup_{x>t(\alpha+h)} u(t,x,\omega) = 0,$$

where $h > 0$ is arbitrary and $\alpha = \sqrt{2c_0D}$.

Note that the wave speed found above, $\alpha = \sqrt{2c_0D}$, coincides with the wave speed in the deterministic case. Moreover, we have shown that the solution tends to $c^{-1}(0)$ on the left hand side and 0 on the right hand side of the wave. It follows that if the noise is weak, the stochastically perturbed equation has the same limit behavior as the corresponding deterministic equation ($k \equiv 0$).

5 Concluding Remarks

We have considered the asymptotic behavior of the solution to (3.1) in two different cases:

(a) for a family of anticipating initial conditions together with deterministic k's, and

(b) for deterministic initial conditions together with adapted k's.

In both cases the solutions' behavior in the limit depends on the strength of the noise, i.e., the asymptotic properties of $\int_0^t k(s)^2\,ds$. The main difference between the results in the two cases is that in (a) the solution behaves the same way for a.e. $\omega \in \mathscr{S}'$, whereas it in (b) may behave differently for different $\omega \in \mathscr{S}'$ depending on the asymptotic properties of $\int_0^t k(s,\omega)^2\,ds$.

We have shown that if the noise is strong, i.e., if

$$\liminf_{t\to\infty} \frac{1}{2t}\int_0^t k(s)^2\,ds > \max_\theta c(\theta),$$

the solution of (3.1) tends to zero (uniformly in x) as t tends to infinity. This should not come as a surprise since the SODE that results by putting $D = 0$ in (3.1), behaves similarly (see also [1] and the references therein).

When the noise is weak, i.e., if $k \in L^2(\mathbb{R}^+)$ or for almost evert $\omega \in \mathscr{S}'$ such that $\int_0^\infty k(s,\omega)^2\,ds < \infty$, the solutions of the two equations we have considered tend to the solution of the corresponding unperturbed deterministic equation.

If the noise is moderately strong, the solution of (3.1) displays a more complex behavior than it does in the corresponding deterministic case. Note that our estimates on the solution are not as accurate as the ones in the deterministic case, cf. (4.1). This is not surprising considering that $t \mapsto u(t,x)$ for $x \ll 0$ behaves essentially as the SODE one obtains from (3.1) by letting $D = 0$ and $u(0) = u_0 > 0$. It has been shown that the solution of this equation with $c(u) = r(1-\gamma u)$, $k = \sigma$, and $u_0 > 0$, where $r > \sigma^2/2 > 0$, has a χ^2 stationary distribution with parameter $\eta = 2r/\sigma^2 - 1$ (see [1] and the references therein). We can therefore not expect the solution of (3.1) in the stochastic case to converge to specific values as the solution of the corresponding deterministic problem does.

Suppose there is a constant a_1 such that $0 \le 2a_1 \le k^2(t)$, for all t large enough, then there are constants d_1 and d_2 with $d_1 < 0 < d_2$ such that for any $h > 0$

$$\frac{1}{t}\log u(t,x) < d_1 \text{ if } x > (\sqrt{2(c_0 - a_1)D} + h)t,$$

and

$$\frac{\log u(t,x)}{\sqrt{2t\log\log t}} \le d_2 \text{ if } x < (\sqrt{2(c_0 - a_1)D} - h)t;$$

for all sufficiently large t. If there, in addition, is a constant a_2 such that $k^2(t) \leq 2a_2 < 2c_0$ for all t large enough, we can find $d_3 < 0$ such that

$$d_3 \leq \frac{\log u(t,x)}{\sqrt{2t \log\log t}} \leq d_2 \text{ if } x < (\sqrt{2(c_0 - a_2)D} - h)t,$$

for all sufficiently large t.

Observe the similarities with the deterministic case. We do not obtain a convergence as in the deterministic case, but we still observe two distinct types of behavior separated by a cone. The width of the cone depends on how much k varies for large t and tends to zero if $a_2 - a_1$ tends to zero.

Note also that as a_1 is increased from 0 to c_0 the region, where the solution converges to zero exponentially, grows. For $a_1 = 0$, the region coincides with the one found in the deterministic case and as a_1 approaches c_0, the region approaches the first quadrant in the plane. One may interpret this as the speed of the wave is reduced as a_1 is increased. If $a_1 > c_0$, the noise is strong and the solution converges exponentially to 0 (uniformly in x) as t tends to infinity.

Our results also agree with the ones in [4], where a related problem for deterministic k and smooth bell-shaped initial conditions was studied using Hamilton-Jacobi theory.

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