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# Asymptotic results for Fourier-PARMA time series

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**Periodically stationary times series are useful to model physical systems whose mean behavior and covariance structure varies with the season. The Periodic Auto-Regressive Moving Average (PARMA) process provides a powerful tool for modelling periodically stationary series. Since the process is non-stationary, the innovations algorithm is useful to obtain parameter estimates. Fitting a PARMA model to high-resolution data, such as weekly or daily time series, is problematic because of the large number of parameters. To obtain a more parsimonious model, the discrete Fourier transform (DFT) can be used to represent the model parameters. This article proves asymptotic results for the DFT coefficients, which allow identification of the statistically significant frequencies to be included in the PARMA model.**

**Keywords:** Discrete Fourier transform, periodic auto-regressive moving average, parameter estimation, innovations algorithm, asymptotic distribution.

## 1. INTRODUCTION

A stochastic process $X_t$ is called periodically stationary (in the wide sense) if $\mu_t = EX_t$ and $\gamma_t(h) = \text{Cov}(X_t, X_{t+h})$ for $h = 0, \pm1, \pm2,\ldots$ are all periodic functions of time $t$ with the same period $\nu \geq 1$. If $\nu = 1$, then the process is stationary. Periodically stationary processes manifest themselves in such fields as economics, hydrology and geophysics, where the observed time series are characterized by seasonal variations in both the mean and covariance structure. An important class of stochastic models for describing such time series are the periodic Auto-Regressive Moving Average (ARMA) models, which allows the model parameters in the classical ARMA model to vary with the season. A periodic ARMA process $\{\tilde{X}_t\}$ with period $\nu$ [denoted by PARMA$_\nu(p,q)$] has representation

$$X_t - \sum_{j=1}^{p} \phi_t(j)X_{t-j} = \varepsilon_t - \sum_{j=1}^{q} \theta_t(j)\varepsilon_{t-j}, \tag{1}$$

where $X_t = \tilde{X}_t - \mu_t$ and $\{\varepsilon_t\}$ is a sequence of random variables with mean zero and scale $\sigma_t$ such that $\{\delta_t = \sigma_t^{-1}\varepsilon_t\}$ is i.i.d. The notation in eqn (1) is consistent with that of Box and Jenkins (1976). The autoregressive parameters $\phi_t(j)$, the moving average parameters $\theta_t(j)$ and the residual standard deviations $\sigma_t$ are all periodic functions of $t$ with the same period $\nu \geq 1$. Periodic time series models and their practical applications are discussed in Adams and Goodwin (1995), Anderson and Vecchia (1993), Anderson and Meerschaert (1997, 1998), Anderson *et al.* (1999), Basawa *et al.* (2004), Boshnakov (1996), Gautier (2006), Jones and Brelsford (1967), Lund and Basawa (1999, 2000), Lund (2006), Nowicka-Zagrajek and Wyłomańska (2006), Pagano (1978), Roy and Saidi (2008), Salas *et al.* (1982, 1985), Shao and Lund (2004), Tesfaye *et al.* (2005), Tjøstheim and Paulsen (1982), Troutman (1979), Vecchia (1985a, 1985b), Vecchia and Ballerini (1991), Ula (1990, 1993), Ula and Smadi (1997, 2003) and Wyłomańska (2008). See also the recent book of Franses and Paap (2004) as well as Hipel and McLeod (1994).

In this article, we will assume:

(i) Finite variance: $E\varepsilon_t^2 < \infty$.

(ii) Either $E\varepsilon_t^4 < \infty$ (*Finite Fourth Moment Case*); or the i.i.d. sequence $\delta_t = \sigma_t^{-1}\varepsilon_t$ is RV($\alpha$) for some $2 < \alpha < 4$ (*Infinite Fourth Moment Case*), meaning that $P[|\delta_t| > x]$ varies regularly with index $-\alpha$ and $P[\delta_t > x]/P[|\delta_t| > x] \to p$ for some $p \in [0,1]$.

(iii) The model admits a causal representation

$$X_t = \sum_{j=0}^{\infty} \psi_t(j)\varepsilon_{t-j}, \tag{2}$$

where $\psi_t(0) = 1$ and $\sum_{j=0}^{\infty} |\psi_t(j)| < \infty$ for all $t$. Note that $\psi_t(j) = \psi_{t+k\nu}(j)$ for all $j$.

(iv) The model also satisfies an invertibility condition

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$$\varepsilon_t = \sum_{j=0}^{\infty} \pi_t(j) X_{t-j}, \tag{3}$$

where $\pi_t(0) = 1$ and $\sum_{j=0}^{\infty} |\pi_t(j)| < \infty$ for all $t$. Again, $\pi_t(j) = \pi_{t+k\nu}(j)$ for all $j$.

In the infinite fourth moment case, the RV($\alpha$) assumption implies that $E|\delta_t|^p < \infty$ if $0 < p < \alpha$, and in particular, the variance of $\varepsilon_t$ exists, while $E|\delta_t|^p = \infty$ for $p > \alpha$, so that $E\varepsilon_t^4 = \infty$. Anderson and Meerschaert (1997) show that, in this case, the sample autocovariance is a consistent estimator of the autocovariance, and asymptotically stable with tail index $\alpha/2$. Stable laws and processes, and the theory of regular variation, are comprehensively treated in Feller (1971) and Meerschaert and Scheffler (2001), see also Samorodnitsky and Taqqu (1994). Time series with infinite fourth moments are often seen in natural river flows, see, for example, Anderson and Meerschaert (1998).

The main results of this article, and their relation to previous published results, are as follows. The innovations algorithm can be used to estimate parameters of a non-stationary time series model, based on the infinite order moving average representation (2). Anderson *et al.* (1999) proved consistency of the innovations algorithm estimates for the infinite order moving average parameters $\psi_t(j)$. In the finite fourth moment case, Anderson and Meerschaert (2005) developed the asymptotics necessary to determine which of these parameter estimates are statistically different from zero. Anderson *et al.* (2008) extended those results to the infinite fourth moment case. Theorem 1 in this article reformulates those results in a manner suitable for the application to discrete fourier transform (DFT) asymptotics. To obtain estimates of the auto-regressive parameters $\phi_t(j)$ and moving average parameters $\theta_t(j)$ of the PARMA process in eqn (1), it is typically necessary to solve a system of difference equations that relate these PARMA parameters back to the infinite order moving average representation. Section 3 discusses the general form of those difference equations, and develops some useful examples. The PARMA$_\nu$(1,1) model (eqn 1) with $p = q = 1$ is an important example, relatively simple to analyse, yet sufficiently flexible to handle many practical applications (e.g. see Anderson and Meerschaert, 1998; Anderson *et al.*, 2007). Theorem 2 provides asymptotics for the Yule–Walker estimates, and Theorem 3 gives the asymptotics of the innovations estimates, for the PARMA$_\nu$(1,1) model. These asymptotic results can be used for model identification, to determine which seasons have non-zero PARMA coefficients in the model. For high-resolution data (e.g. weekly or daily data), even a first-order PARMA model can involve numerous parameters, which can lead to over-fitting. This can be overcome using DFT. Theorem 5 gives DFT asymptotics for the infinite order moving average parameters $\psi_t(j)$. For the PARMA$_\nu$(1,1) model, Theorems 7 and 8 provide DFT asymptotics for the autoregressive parameters, and the moving average parameters respectively. These results can be used to determine which Fourier frequencies need to be retained in a DFT model for the PARMA$_\nu$(1,1) parameters. Section 6 briefly reviews results from Anderson *et al.* (2007), where results of this article were used to work out several practical applications. Theorems 1, 7 and 8 were stated in Anderson *et al.* (2007) without proof.

## 2. THE INNOVATIONS ALGORITHM

The innovations algorithm (Brockwell and Davis, 1991, Propn 5.2.2) was adapted by Anderson *et al.* (1999) to yield parameter estimates for PARMA models. Since the parameters are seasonally dependent, there is a notational difference between the innovations algorithm for PARMA processes and that for ARMA processes (compare Brockwell and Davis, 1991). We introduce this difference through the 'season', $i$. For monthly data, we have $\nu = 12$ seasons and our convention is to let $i = 0$ represents the first month, $i = 1$ represents the second, …, and $i = \nu - 1 = 11$ represents the last.

Let $\hat{X}_{i+k}^{(i)} = P_{\mathcal{H}_{k,i}} X_{i+k}$ denotes the one-step predictors, where $\mathcal{H}_{k,i} = \overline{\text{sp}}\{X_i, \ldots, X_{i+k-1}\}$ is the data vector starting at season $0 \le i \le \nu - 1$, $k \ge 1$ and $P_{\mathcal{H}_{k,i}}$ is the orthogonal projection onto this space, which minimizes the mean-squared error

$$v_{k,i} = \|X_{i+k} - \hat{X}_{i+k}^{(i)}\|^2 = E(X_{i+k} - \hat{X}_{i+k}^{(i)})^2.$$

Then,

$$\hat{X}_{i+k}^{(i)} = \phi_{k,1}^{(i)} X_{i+k-1} + \cdots + \phi_{k,k}^{(i)} X_i, \; k \ge 1, \tag{4}$$

where the vector of coefficients $\phi_k^{(i)} = (\phi_{k,1}^{(i)}, \ldots, \phi_{k,k}^{(i)})'$ solves the prediction equations

$$\Gamma_{k,i} \phi_k^{(i)} = \gamma_k^{(i)} \tag{5}$$

with $\gamma_k^{(i)} = (\gamma_{i+k-1}(1), \gamma_{i+k-2}(2), \ldots, \gamma_i(k))'$ and

$$\Gamma_{k,i} = [\gamma_{i+k-\ell}(\ell - m)]_{\ell,m=1,\ldots,k} \tag{6}$$

is the covariance matrix of $(X_{i+k-1},\ldots, X_i)'$ for each $i = 0,\ldots,\nu - 1$. Let

$$\hat{\gamma}_i(\ell) = N^{-1} \sum_{j=0}^{N-1} X_{j\nu+i} X_{j\nu+i+\ell} \tag{7}$$

denotes the (uncentered) sample autocovariance, where $X_t = \tilde{X}_t - \mu_t$. If we replace the autocovariances in the prediction eqn (5) with their corresponding sample autocovariances, we obtain the estimator $\hat{\phi}_{k,j}^{(i)}$ of $\phi_{k,j}^{(i)}$.

As the scalar-valued process $X_t$ is non-stationary, the Durbin–Levinson algorithm (Brockwell and Davis, 1991, Propn 5.2.1) for computing $\hat{\phi}_{k,j}^{(i)}$ does not apply. However, the innovations algorithm still applies to a non-stationary process. Writing

$$\hat{X}_{i+k}^{(i)} = \sum_{j=1}^{k} \theta_{k,j}^{(i)} (X_{i+k-j} - \hat{X}_{i+k-j}^{(i)}), \tag{8}$$

yields the one-step predictors in terms of the innovations $X_{i+k-j} - \hat{X}_{i+k-j}^{(i)}$. Lund and Basawa (1999, Propn 4) shows that if $\sigma_i^2 > 0$ for $i = 0, \ldots, \nu - 1$, then for a causal PARMA$_\nu(p,q)$ process, the covariance matrix $\Gamma_{k,i}$ is non-singular for every $k \geq 1$ and each $i$. Anderson *et al.* (1999) show that if $EX_t = 0$ and $\Gamma_{k,i}$ is non-singular for each $k \geq 1$, then the one-step predictors $\hat{X}_{i+k}^{(i)}$ starting at season $i$ and their mean-square errors $v_{k,i}$ are given by

$$\begin{aligned} v_{0,i} &= \gamma_i(0), \\ \theta_{k,k-\ell}^{(i)} &= (v_{\ell,i})^{-1}\left[\gamma_{i+\ell}(k-\ell) - \sum_{j=0}^{\ell-1} \theta_{\ell,\ell-j}^{(i)} \theta_{k,k-j}^{(i)} v_{j,i}\right], \\ v_{k,i} &= \gamma_{i+k}(0) - \sum_{j=0}^{k-1} (\theta_{k,k-j}^{(i)})^2 v_{j,i}, \end{aligned} \tag{9}$$

where eqn (7) is solved in the order $v_{0,i}$, $\theta_{1,1}^{(i)}$, $v_{1,i}$, $\theta_{2,2}^{(i)}$, $\theta_{2,1}^{(i)}$, $v_{2,i}$, $\theta_{3,3}^{(i)}$, $\theta_{3,2}^{(i)}$, $\theta_{3,1}^{(i)}$, $v_{3,i}$,… and so forth. The results in Anderson *et al.* (1999) show that

$$\begin{aligned} \theta_{k,j}^{(\langle i-k \rangle)} &\to \psi_i(j), \\ v_{k,\langle i-k \rangle} &\to \sigma_i^2, \\ \phi_{k,j}^{(\langle i-k \rangle)} &\to -\pi_i(j), \end{aligned} \tag{10}$$

for all $i,j$, where $\langle t \rangle$ is the season corresponding to index $t$, so that $\langle j\nu + i \rangle = i$.

If we replace the autocovariances in eqn (9) with the corresponding sample autocovariances in eqn (7), we obtain the innovations estimates $\hat{\theta}_{k,\ell}^{(i)}$ and $\hat{v}_{k,i}$. Similarly, replacing the autocovariances in eqn (5) with the corresponding sample autocovariances yields the Yule–Walker estimators $\hat{\phi}_{k,\ell}^{(i)}$. The consistency of these estimators was also established in Anderson *et al.* (1999).

Suppose that the PARMA process given by eqn (1) satisfies assumptions (i) through (iv) and that:

(v) The spectral density matrix $f(\lambda)$ of the equivalent vector ARMA process (Anderson and Meerschaert, 1997, p. 778) is such that for some $0 < m \leq M < \infty$, we have

$$mz'z \leq z'f(\lambda)z \leq Mz'z, \quad -\pi \leq \lambda \leq \pi,$$

for all $z$ in $\mathrm{R}^\nu$;

(vi) In the finite fourth moment case $E\varepsilon_t^4 < \infty$, we choose $k$ as a function of the sample size $N$ so that $k^2/N \to 0$ as $N \to \infty$ and $k \to \infty$. In the infinite fourth moment case, where the i.i.d. noise sequence $\delta_t = \sigma_t^{-1}\varepsilon_t$ is RV($\alpha$) for some $2 < \alpha < 4$, define

$$a_N = \inf\{x : P(|\delta_t| > x) < 1/N\} \tag{11}$$

a regularly varying sequence with index $1/\alpha$, (see, e.g., Propn 6.1.37 in Meerschaert and Scheffler, 2001). Here, we choose $k$ as a function of the sample size $N$ so that $k^{5/2}a_N^2/N \to 0$ as $N \to \infty$ and $k \to \infty$.

Then, the results in Anderson *et al.* (1999) show that for all $i,j$, where "$\xrightarrow{P}$" denotes

$$\begin{aligned} \hat{\theta}_{k,j}^{(\langle i-k \rangle)} &\xrightarrow{P} \psi_i(j), \\ \hat{v}_{k,\langle i-k \rangle} &\xrightarrow{P} \sigma_i^2, \\ \hat{\phi}_{k,j}^{(\langle i-k \rangle)} &\xrightarrow{P} -\pi_i(j), \end{aligned} \tag{12}$$

convergence in probability.

Suppose that the PARMA process given by eqn (1) satisfies assumptions (i) through (v) and that:

(vii) In the finite fourth moment case, we suppose that $k = k(N) \to \infty$ as $N \to \infty$ with $k^3/N \to 0$. In the infinite fourth moment case, we suppose $k^3 a_N^2/N \to 0$ where $a_N$ is defined by eqn (11).

Then, results in Anderson and Meerschaert (2005) and Anderson *et al.* (2008) show that for any fixed positive integer $D$, we have

$$N^{1/2}(\hat{\theta}_{k,u}^{(\langle i-k \rangle)} - \psi_i(u) : u = 1, \ldots, D, i = 0, \ldots, \nu - 1) \Rightarrow \mathcal{N}(0, W), \tag{13}$$

where

$$W = A \,\mathrm{diag}(\sigma_0^2 D^{(0)}, \ldots, \sigma_{\nu-1}^2 D^{(\nu-1)})A', \tag{14}$$

$$A = \sum_{n=0}^{D-1} E_n \Pi^{[D\nu - n(D+1)]},$$

$$D^{(i)} = \mathrm{diag}(\sigma_{i-1}^{-2}, \sigma_{i-2}^{-2}, \ldots, \sigma_{i-D}^{-2}),$$

$$E_n = \mathrm{diag}\left\{ \underbrace{0, \ldots, 0}_{n}, \underbrace{\psi_0(n), \ldots, \psi_0(n)}_{D-n}, \ldots, \underbrace{0, \ldots, 0}_{n}, \underbrace{\psi_{\nu-1}(n), \ldots, \psi_{\nu-1}(n)}_{D-n} \right\}$$

and $\Pi$ an orthogonal $D\nu \times D\nu$ cyclic permutation matrix defined as

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{15}$$

Note that $\Pi^{-q} = \Pi^{\nu-q}$, $\Pi^0 = \Pi^\nu = I$, and $\Pi' = \Pi^{\nu-1} = \Pi^{-1}$.

For the purposes of this article, it is useful to express the joint asymptotics of the innovations estimates as follows.

Theorem 1. *Let $X_t = \tilde{X}_t - \mu_t$, where $X_t$ is the periodic moving average process (eqn 2) and $\mu_t$ is a periodic mean function with period $\nu$. Suppose that (i) through (v) and (vii) hold. Letting $\hat{\theta}_{k,\ell}^{(\langle i-k \rangle)} = \hat{\psi}_i(\ell)$, for any non-negative integers j and h with $j \neq h$ we have*

$$N^{1/2} \begin{pmatrix} \hat{\psi}(j) - \psi(j) \\ \hat{\psi}(h) - \psi(h) \end{pmatrix} \Rightarrow \mathcal{N}\left(0, \begin{pmatrix} V_{jj} & V_{jh} \\ V_{hj} & V_{hh} \end{pmatrix}\right), \tag{16}$$

*where* $\hat{\psi}(\ell) = [\hat{\psi}_0(\ell), \hat{\psi}_1(\ell), \ldots, \hat{\psi}_{\nu-1}(\ell)]'$, $\psi(\ell) = [\psi_0(\ell), \psi_1(\ell), \cdots, \psi_{\nu-1}(\ell)]'$,

$$V_{jh} = \sum_{n=1}^{x} \{F_{j-n} \Pi^{-(j-n)} B_n (F_{h-n} \Pi^{-(h-n)})'\}, \tag{17}$$

*with* $x = \min(h, j)$, *and*

$$\begin{aligned} F_n &= \mathrm{diag}\{\psi_0(n), \psi_1(n), \ldots, \psi_{\nu-1}(n)\}, \\ B_n &= \mathrm{diag}\{\sigma_0^2 \sigma_{0-n}^{-2}, \sigma_1^2 \sigma_{1-n}^{-2}, \ldots, \sigma_{\nu-1}^2 \sigma_{\nu-1-n}^{-2}\}, \end{aligned} \tag{18}$$

*where $\Pi$ is the orthogonal $\nu \times \nu$ cyclic permutation matrix (eqn 15).*

Proof. For a $p \times q$ matrix $M$, we will write $M_{ij}$ for its $ij$ entry, and we will write $M_i$ for the $ii$ entry of a diagonal matrix. We will also use modulo arithmetic to compute subscripts, so that $M_{i+p,j} = M_{i,j+q} = M_{ij}$. Since $\Pi_{ij} = 1_{\{i=j-1\}}$, we have for any matrix $M$ that

$$[M\Pi]_{ij} = \sum_k M_{ik} \Pi_{kj} = \sum_k M_{ik} 1_{\{k=j-1\}} = M_{i,j-1}$$

and hence we also have $[M\Pi^s]_{ij} = M_{i,j-s}$. Since $[\Pi^{-1}]_{ij} = 1_{\{j=i-1\}}$, we also have

$$[\Pi^{-1} M \Pi]_{ij} = \sum_k 1_{\{k=i-1\}} M_{k,j-1} = M_{i-1,j-1},$$

so that $[\Pi^{-t} M \Pi^t]_{ij} = M_{i-t,j-t}$.

Equation (13) shows that the $D\nu$ dimensional column vector $\hat{\psi}_S$ with entries grouped by season, $[\hat{\psi}_S]_{iD+\ell} = \hat{\psi}_i(\ell)$, has asymptotic covariance matrix $W = AD_0A'$, where $[D_0]_{iD+\ell} = \sigma_i^2 \sigma_{i-\ell}^{-2}$ for $0 \le i \le \nu - 1, 1 \le \ell \le D$, and

$$A = \sum_{m=0}^{D-1} E_m \Pi^{-m(D+1)},$$

where $[E_m]_{iD+\ell} = \psi_i(m) 1_{\{\ell > m\}}$. Define $D_m = \Pi^{-m(D+1)} D_0 \Pi^{m(D+1)}$ and $E_{\ell,t} = \Pi^{-t(D+1)} E_\ell \Pi^{t(D+1)}$. Then, $D_m$ and $E_{\ell,t}$ are diagonal matrices formed by a permutation of coordinates, and $D_m \Pi^{-m(D+1)} = \Pi^{-m(D+1)} D_0$, $E_{\ell,t} \Pi^{-t(D+1)} = \Pi^{-t(D+1)} E_\ell$. Since $(\Pi^t)' = \Pi^{-t}$, we can write

$$\begin{aligned} W &= AD_0A' \\ &= \sum_{m=0}^{D-1} E_m \Pi^{-m(D+1)} D_0 \sum_{\ell=0}^{D-1} \Pi^{\ell(D+1)} E_\ell \\ &= \sum_{m=0}^{D-1}\sum_{\ell=0}^{D-1} E_m D_m \Pi^{-m(D+1)} \Pi^{\ell(D+1)} E_\ell \\ &= \sum_{m=0}^{D-1}\sum_{\ell=0}^{D-1} E_m D_m \Pi^{-(m-\ell)(D+1)} E_\ell \\ &= \sum_{m=0}^{D-1}\sum_{\ell=0}^{D-1} E_m D_m E_{\ell,m-\ell} \Pi^{-(m-\ell)(D+1)}. \end{aligned} \tag{19}$$

Substitute $p = m - \ell$ to obtain

$$W = \sum_{p=1-D}^{D-1} G_p \Pi^{-p(D+1)}, \quad \text{where } G_p = \sum_m E_m D_m E_{m-p,p} \tag{20}$$

is a diagonal matrix and $m$ ranges over the set of $0 \le m \le D-1$ such that $0 \le m - p \le D - 1$.

Note that $[G_p\Pi^{-p(D+1)}]_{rs} = [G_p]_{r,s+p(D+1)}$ in eqn (20). Since $G_p$ is a diagonal matrix, it follows that $[G_p\Pi^{-p(D+1)}]_{rs}$ can be non-zero only if $s = r - p(D + 1)$. Hence, the only non-zero entries of the matrix $W$ are

$$W_{iD+j,iD+j-p(D+1)} = [G_p]_{iD+j}, \tag{21}$$

for integers $1 - D \le p \le D - 1$. Then, we can compute

$$\begin{aligned} [E_m]_{iD+j} &= \psi_i(m)1_{\{j>m\}}, \\ [D_m]_{iD+j} &= [\Pi^{-m(D+1)} D_0 \Pi^{m(D+1)}]_{iD+j} \\ &= [D_0]_{iD+j-m(D+1)} = [D_0]_{(i-m)D+(j-m)} = \sigma_{i-m}^2\sigma_{i-j}^{-2}, \\ [E_{m-p,p}]_{iD+j} &= [\Pi^{-p(D+1)} E_{m-p} \Pi^{p(D+1)}]_{iD+j} \\ &= [E_{m-p}]_{iD+j-p(D+1)} = \psi_{i-p}(m-p)1_{\{j-p>m-p\}} = \psi_{i-p}(m-p)1_{\{j>m\}}, \end{aligned} \tag{22}$$

so that the diagonal matrix $G_p$ has entries

$$[G_p]_{iD+j} = \sum_m [E_m D_m E_{m-p,p}]_{iD+j} = \sum_m \sigma_{i-m}^2\sigma_{i-j}^{-2}\psi_i(m)\psi_{i-p}(m-p)1_{\{j>m\}}, \tag{23}$$

where $m$ ranges over the set of $0 \le m \le D-1$ such that $0 \le m - p \le D - 1$.

Since $j \le D$, the condition $m < j$, equivalent to $m \le j - 1$, is stronger than $m \le D - 1$. Hence, $m$ ranges over the set of $0 \le m - p \le D - 1$ such that $0 \le m \le j - 1$.

Since $W$ is symmetric, it suffices to consider $0 \le j \le h$. Substitute $p = j - h$ to see that

$$[G_{j-h}]_{iD+j} = \sum_m \sigma_{i-m}^2\sigma_{i-j}^{-2}\psi_i(m)\psi_{i-j+h}(m-j+h),$$

where $m$ ranges over the set of $0 \le m \le j - 1$ such that $0 \le m - j + h \le D - 1$. Since $D$ is arbitrary, we may take $D = h$. Then, the condition $0 \le m - j + h \le D - 1$, equivalent to $j - D \le m \le D - 1 + j - D$, reduces to $j - D \le m \le j - 1$. Since $j - D \le 0$, this together with the remaining condition $0 \le m \le j - 1$ shows that the only non-zero entries $W_{k\ell}$ with $\ell \ge k$ are of the form

$$[G_{j-h}]_{iD+j} = \sum_{m=0}^{j-1} \sigma_{i-m}^2\sigma_{i-j}^{-2}\psi_i(m)\psi_{i-j+h}(m-j+h).$$

Substitute $n = j - m$ and use eqn (21) to arrive at

$$W_{iD+j,(i-j+h)D+h} = [G_{j-h}]_{iD+j} = \sum_{n=1}^{j} \sigma_{i-j+n}^2\sigma_{i-j}^{-2}\psi_i(j-n)\psi_{i-j+h}(h-n), \tag{24}$$

for $0 \le j \le h$.

Now define a $D\nu$-dimensional column vector $\hat{\psi}_L$ with entries grouped by lag: $[\hat{\psi}_L]_{(\ell-1)\nu+i+1} = \hat{\psi}_i(\ell)$. We want to characterize the variance–covariance matrix $V = CWC'$ of $\hat{\psi}_L$, where $C$ is the transition matrix such that $\hat{\psi}_L = C\hat{\psi}_S$. Then, $C_{r,iD+\ell} = 1_{\{r=(\ell-1)\nu+i+1\}}$. Write $C$ as the block matrix

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1\nu} \\ C_{21} & C_{22} & \dots & C_{2\nu} \\ \vdots & \vdots & & \vdots \\ C_{D1} & C_{D2} & \dots & C_{D\nu} \end{pmatrix},$$

where the submatrix $C_{jr}$ is $\nu \times D$. Since $[C_{jr}]_{hk} = [C]_{(j-1)\nu+h,(r-1)D+k}$, we see that $[C_{j,i+1}]_{h\ell} = [C]_{(j-1)\nu+h,iD+\ell}$ equals zero unless $j = \ell$ and $h = i + 1$. This shows that $C_{jr} = J_{rj}$, the $\nu \times D$ indicator matrix with $[J_{rj}]_{st} = 1_{\{s=r,t=j\}}$. Write

$$W = \begin{pmatrix} W_{11} & W_{12} & \dots & W_{1\nu} \\ W_{21} & W_{22} & \dots & W_{2\nu} \\ \vdots & \vdots & & \vdots \\ W_{\nu 1} & W_{\nu 2} & \dots & W_{\nu\nu} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1D} \\ V_{21} & V_{22} & \dots & V_{2D} \\ \vdots & \vdots & & \vdots \\ V_{D1} & V_{D2} & \dots & V_{DD} \end{pmatrix},$$

where $W_{rs}$ is a $D \times D$ matrix and $V_{rs}$ is a $\nu \times \nu$ matrix. Block-matrix multiplication yields

$$V_{jh} = (CWC')_{jh} = \sum_{s=1}^{\nu} \sum_{r=1}^{\nu} C_{jr} W_{rs} C'_{hs}.$$

It is easy to check that indicator matrices have the property $J_{ij} A J_{rs} = a_{jr} J_{is}$ for any matrix $A$ with $[A]_{ij} = a_{ij}$. Then, $C_{jr} W_{rs} C'_{hs} = J_{rj} W_{rs} J_{hs} = [W_{rs}]_{jh} J_{rs}$. In other words, the $rs$ entry of $V_{jh}$ equals the $jh$ entry of $W_{rs}$. Then, from eqn (24) we obtain

$$[V_{jh}]_{i+1,i-j+h+1} = [W_{i+1,i-j+h+1}]_{jh} = \sum_{n=1}^{j} \sigma^2_{i-j+n} \sigma^{-2}_{i-j} \psi_i(j-n) \psi_{i-j+h}(h-n), \tag{25}$$

for $0 \le j \le h$. Furthermore, $[V_{jh}]_{i+1,s+1} = [W_{i+1,s+1}]_{jh} = W_{iD+j,sD+h} = 0$ for all $s \ne i - j + h$, i.e. there is only one non-zero entry in each row of $V_{jh}$. Since $V$ is symmetric, this determines every entry of $V$.

Finally, we want to establish eqn (17). The diagonal matrices in eqn (18) are such that $[F_n]_{i+1} = \psi_i(n)$ and $[B_n]_{i+1} = \sigma_i^2 \sigma_{i-n}^{-2}$. Then,

$$\begin{aligned} [F_{j-n} \Pi^{-(j-n)}]_{i+1,s} &= [F_{j-n}]_{i+1,s-j+n} = \psi_i(j-n) 1_{\{i+1=s+j-n\}}, \\ [B_n]_{st} &= \sigma^2_{s-1} \sigma^{-2}_{s-1-n} 1_{\{t=s\}}, \\ [(F_{h-n} \Pi^{-(h-n)})']_{tv} &= [F_{h-n} \Pi^{-(h-n)}]_{vt} = \psi_{v-1}(h-n) 1_{\{v=t+h-n\}}, \end{aligned}$$

so that

$$\begin{aligned} [F_{j-n} \Pi^{-(j-n)} B_n]_{i+1,t} &= \sum_{s=1}^{\nu} \psi_i(j-n) 1_{\{i+1=s+j-n\}} \sigma^2_{s-1} \sigma^{-2}_{s-1-n} 1_{\{t=s\}} \\ &= \sigma^2_{t-1} \sigma^{-2}_{t-1-n} \psi_i(j-n) 1_{\{i+1=t+j-n\}} \end{aligned}$$

and

$$\begin{aligned} [F_{j-n} \Pi^{-(j-n)} B_n (F_{h-n} \Pi^{-(h-n)})']_{i+1,v} &= \sum_{t=1}^{\nu} \sigma^2_{t-1} \sigma^{-2}_{t-1-n} \psi_i(j-n) 1_{\{i+1=t+j-n\}} \times \psi_{v-1}(h-n) 1_{\{v=t+h-n\}} \\ &= \sigma^2_{i-j+n} \sigma^{-2}_{i-j} \psi_i(j-n) \psi_{i-j+h}(h-n) 1_{\{v=i-j+h+1\}}. \end{aligned}$$

Then, a comparison with eqn (25) shows that eqn (17) holds. This completes the proof. □

Corollary 1. *Under the assumptions of Theorem* 1, *we have*

$$N^{1/2} \begin{pmatrix} \hat{\psi}_i(j) - \psi_i(j) \\ \hat{\psi}_k(\ell) - \psi_k(\ell) \end{pmatrix} \Rightarrow \mathcal{N}\left(0, \begin{pmatrix} v_{ijij} & v_{ijk\ell} \\ v_{ijk\ell} & v_{k\ell k\ell} \end{pmatrix}\right), \tag{26}$$

*for all* $0 \le i \le \nu - 1$ *and* $0 \le j \le \ell$, *where*

$$v_{ijk\ell} = \sum_{n=1}^{j} \sigma^2_{i-j+n} \sigma^{-2}_{i-j} \psi_i(j-n) \psi_{i-j+\ell}(\ell-n),$$

*if* $k = i + \ell - j \bmod \nu$, *and* $v_{ijk\ell} = 0$ *otherwise.*

For a second-order stationary process, where the period $\nu = 1$, we have $\sigma_i^2 = \sigma^2$, and then substituting $m = j - n$ in Corollary 1 yields

  

$$N^{1/2}(\hat{\psi}(u) - \psi(u)) \Rightarrow \mathcal{N}\left(0, \sum_{m=0}^{j-1} \psi(m)^2\right), \tag{27}$$

which agrees with Thm 2.1 in Brockwell and Davis (1988).

## 3. DIFFERENCE EQUATIONS

For a PARMA$_\nu$($p$,$q$) model given by eqn (1), we develop a vector difference equation for the $\psi$-weights of the PARMA process so that we can determine feasible values of $p$ and $q$. For fixed $p$ and $q$ with $p + q = m$, assuming $m$ statistically significant values of $\psi_t(j)$, to determine the parameters $\phi_t(\ell)$ and $\theta_t(\ell)$, eqn (2) can be substituted in to eqn (1) to obtain

$$\sum_{j=0}^{\infty} \psi_t(j)\varepsilon_{t-j} - \sum_{\ell=1}^{p} \phi_t(\ell) \sum_{j=0}^{\infty} \psi_{t-\ell}(j)\varepsilon_{t-\ell-j} = \varepsilon_t - \sum_{\ell=1}^{q} \theta_t(\ell)\varepsilon_{t-\ell} \tag{28}$$

and then the coefficients on both sides can be equated so as to calculate $\phi_t(\ell)$ and $\theta_t(\ell)$:

$$\begin{aligned} \psi_t(0) &= 1, \\ \psi_t(1) - \phi_t(1)\psi_{t-1}(0) &= -\theta_t(1), \\ \psi_t(2) - \phi_t(1)\psi_{t-1}(1) - \phi_t(2)\psi_{t-2}(0) &= -\theta_t(2), \\ \psi_t(3) - \phi_t(1)\psi_{t-1}(2) - \phi_t(2)\psi_{t-2}(1) - \phi_t(3)\psi_{t-3}(0) &= -\theta_t(3), \\ &\vdots \end{aligned} \tag{29}$$

where we take $\phi_t(\ell) = 0$ for $\ell > p$, and $\theta_t(\ell) = 0$ for $\ell > q$. The $\psi$-weights satisfy the homogeneous difference equations

$$\begin{cases} \psi_t(j) - \sum_{k=1}^{p} \phi_t(k)\psi_{t-k}(j-k) = 0 & j \geq \max(p, q+1) \\ \psi_t(j) - \sum_{k=1}^{j} \phi_t(k)\psi_{t-k}(j-k) = -\theta_t(j) & 0 \leq j \leq \max(p, q+1), \end{cases} \tag{30}$$

for $0 \leq t \leq \nu - 1$. Defining

$$\begin{aligned} A_\ell &= \operatorname{diag}\{\phi_0(\ell), \phi_1(\ell), \ldots, \phi_{\nu-1}(\ell)\}, \\ \psi(j) &= (\psi_0(j), \psi_1(j), \ldots, \psi_{\nu-1}(j))', \\ \theta(j) &= (\theta_0(j), \theta_1(j), \ldots, \theta_{\nu-1}(j))', \\ \psi_{jk}(j-k) &= (\psi_{-k}(j-k), \psi_{-k+1}(j-k), \ldots, \psi_{-k+\nu-1}(j-k))', \end{aligned}$$

then eqn (30) leads to the vector difference equations

$$\begin{cases} \psi(j) - \sum_{k=1}^{p} A_k\psi_{jk}(j-k) = 0 & j \geq \max(p, q+1) \\ \psi(j) - \sum_{k=1}^{j} A_k\psi_{jk}(j-k) = -\theta(j) & 0 \leq j \leq \max(p, q+1), \end{cases} \tag{31}$$

where $\psi_t(0) = 1$.

Since $\psi_{jk}(j-k) = \Pi^{-k}\psi(j-k)$ where $\Pi$ is the orthogonal $\nu \times \nu$ cyclic permutation matrix given by eqn (15), it follows from eqn (31) that

$$\begin{cases} \psi(j) - \sum_{k=1}^{p} A_k\Pi^{-k}\psi(j-k) = 0 & j \geq \max(p, q+1) \\ \psi(j) - \sum_{k=1}^{j} A_k\Pi^{-k}\psi(j-k) = -\theta(j) & 0 \leq j \leq \max(p, q+1). \end{cases} \tag{32}$$

The vector difference eqn (32) can be helpful for the analysis of higher-order PARMA models using matrix algebra. The following are special cases of eqn (32).

### 3.1. Periodic moving average

The periodic moving average process, denoted by $\mathrm{PMA}_\nu(q)$, is obtained by setting $p = 0$ in eqn (1). The vector difference equation for this process, from (32), is

$$\begin{cases} \psi(j) = -\theta(j) & 0 \le j \le q \\ \psi(j) = 0 & j > q. \end{cases} \tag{33}$$

Then, Theorem 1 can be directly applied to identify the order of the PMA process via

$$N^{1/2}(-\hat{\theta}(j) + \theta(j)) \Rightarrow \mathcal{N}(0, V_{jj}), \tag{34}$$

where $V_{jj}$ is obtained from eqn (17).

### 3.2. Periodic autoregressive processes

The periodic moving average process, denoted by $\mathrm{PAR}_\nu(p)$, is obtained by setting $q = 0$ in eqn (1). The vector difference equation for this process given by eqn (32) is

$$\psi(j) = \sum_{k=1}^{p} A_k \Pi^{-k} \psi(j-k) \quad j \ge p. \tag{35}$$

For the $\mathrm{PAR}_\nu(1)$ with $X_t = \phi_t X_{t-1} + \varepsilon_t$, we have $\psi(1) = A_1 \Pi^{-1} \psi(0) = \phi = \{\phi_0, \phi_1, \ldots, \phi_{\nu-1}\}'$.

### 3.3. First order PARMA process

For higher-order PAR or PARMA models, it is difficult to obtain explicit solutions for $\phi(\ell)$ and $\theta(\ell)$, hence model identification is a complicated problem. However, for the $\mathrm{PARMA}_\nu(1,1)$ model

$$X_t = \phi_t X_{t-1} + \varepsilon_t - \theta_t \varepsilon_{t-1}, \tag{36}$$

it is possible to solve directly in eqn (29) to obtain $\psi_t(0) = 1$ and

$$\theta_t = \phi_t - \psi_t(1), \tag{37}$$

$$\psi_t(2) = \phi_t \psi_{t-1}(1). \tag{38}$$

## 4. ASYMPTOTICS FOR PARMA PARAMETER ESTIMATES

Here, we will apply Theorem 1 to derive the asymptotic distribution of the autoregressive and moving average parameters in the $\mathrm{PARMA}_\nu(1,1)$ model (eqn 36).

Theorem 2. *Under the assumption of Theorem* 1, *we have*

$$N^{1/2}(\hat{\phi} - \phi) \Rightarrow \mathcal{N}(0, Q), \tag{39}$$

*where* $\hat{\phi} = [\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_{\nu-1}]'$, $\phi = [\phi_0, \phi_1, \ldots, \phi_{\nu-1}]'$ *and the* $\nu \times \nu$ *matrix* $Q$ *is defined by*

$$Q = \sum_{k,\ell=1}^{2} H_\ell V_{\ell k} H_k', \tag{40}$$

*where* $V_{\ell k}$ *is given by eqn (17),* $H_1 = -F_2 \Pi^{-1} F_1^{-2}$ *and* $H_2 = \Pi^{-1} F_1^{-1} \Pi$ *with* $\Pi$ *the* $\nu \times \nu$ *permutation matrix eqn (15) and* $F_n$ *is from eqn (18).*

Proof. We will use a continuous mapping argument (Brockwell and Davis, 1991, Propn 6.4.3): we say that a sequence of random vectors $X_n$ is $\mathrm{AN}(\mu_n, c_n^2 \Sigma)$ if $c_n(X_n - \mu_n) \Rightarrow \mathcal{N}(0, \Sigma)$, where $\Sigma$ is a symmetric non-negative definite matrix and $c_n \to 0$ as $n \to \infty$. If $X_n$ is $\mathrm{AN}(\mu, c_n^2 \Sigma)$ and $g(x) = (g_1(x), \ldots, g_m(x))'$ is a mapping from $\mathbb{R}^k$ into $\mathbb{R}^m$ such that each $g_i(.)$ is continuously differentiable in a neighborhood of $\mu$, and if $D \Sigma D'$ has all of its diagonal elements non-zero, where $D$ is the $m \times k$ matrix $[(\partial g_i / \partial x_j)(\mu)]$, then $g(X_n)$ is $\mathrm{AN}(g(\mu), c_n^2 D \Sigma D')$.

Theorem 1 with $j = 1$ and $h = 2$ yields that $X_n = (\hat{\psi}(1), \hat{\psi}(2))'$ is AN($\mu$,$N^{-1}V$) with $\mu = (\psi(1),\psi(2))'$ and

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where $V_{\ell k}$ is given by eqn (17). Apply the continuous mapping $g(\mu) = \phi$ to see that eqn (39) holds with $Q = HVH'$, where $H$ is the $v \times 2v$ matrix of partial derivatives

$$H = (H_1, H_2) = \left( \frac{\partial \phi_{\ell-1}}{\partial \psi_{m-1}(1)}, \frac{\partial \phi_{\ell-1}}{\partial \psi_{m-1}(2)} \right)_{\ell,m=1,\ldots,v}, \tag{41}$$

which we now compute. Use $\phi_\ell = \psi_\ell(2)/\psi_{\ell-1}(1)$ from eqn (38) to compute

$$[H_1]_{\ell m} = \frac{\partial}{\partial \psi_{m-1}(1)} \left( \frac{\psi_{\ell-1}(2)}{\psi_{\ell-2}(1)} \right) = -\psi_m(2)\psi_{m-1}(1)^{-2} 1_{\{m=\ell-1\}}$$

and recall from eqn (18) that $[F_n]_{ij} = \psi_{i-1}(n)1_{\{j=i\}}$. Since $[\Pi^{-1}]_{ij} = 1_{\{j=i-1\}}$, we have

$$[\Pi^{-1}F_1^{-2}]_{im} = \sum_{k=1}^{v} 1_{\{k=i-1\}}\psi_{k-1}(1)^{-2}1_{\{k=m\}} = \psi_{m-1}(1)^{-2}1_{\{m=i-1\}}$$

and then

$$[-F_2\Pi^{-1}F_1^{-2}]_{\ell m} = -\sum_{i=1}^{v} \psi_{\ell-1}(2)1_{\{\ell=i\}}\psi_{m-1}(1)^{-2}1_{\{m=i-1\}} = -\psi_m(2)\psi_{m-1}(1)^{-2}1_{\{m=\ell-1\}},$$

which shows that $H_1 = -F_2\Pi^{-1}F_1^{-2}$. Since

$$[H_2]_{\ell m} = \frac{\partial}{\partial \psi_{m-1}(2)} \left( \frac{\psi_{\ell-1}(2)}{\psi_{\ell-2}(1)} \right) = \psi_{m-2}(1)^{-1}1_{\{m=\ell\}}$$

and recalling that $[\Pi^{-1}M\Pi]_{ij} = M_{i-1,\, j-1}$, we also have $H_2 = \Pi^{-1}F_1^{-1}\Pi$. □

Corollary 2. *Regarding Theorem 2, in particular, we have that*

$$N^{1/2}(\hat{\phi}_i - \phi_i) \Rightarrow \mathcal{N}(0, w_{\phi i}^2), \tag{42}$$

for $0 \le i \le v - 1$, where

$$w_{\phi i}^2 = \psi_{i-1}^{-4}(1)\left\{ \psi_i^2(2)\sigma_{i-2}^{-2}\sigma_{i-1}^2 \left( 1 - \frac{2\psi_i(1)\psi_{i-1}(1)}{\psi_i(2)} \right) + \psi_{i-1}^2(1)\sigma_{i-2}^{-2} \sum_{n=0}^{1} \sigma_{i-n}^2 \psi_i^2(n) \right\}. \tag{43}$$

Proof. We need to compute the matrix $Q$ in eqn (40). From eqns (17) and (18), we obtain $V_{11} = B_1$ and $[B_n]_{mj} = \sigma_{m-1}^2\sigma_{m-1-n}^{-2}1_{\{j=m\}}$. Then,

$$[H_1V_{11}]_{\ell j} = -\sum_{m=1}^{v} \frac{\psi_m(2)}{\psi_{m-1}(1)^2} 1_{\{m=\ell-1\}} \frac{\sigma_{m-1}^2}{\sigma_{m-2}^2} 1_{\{j=m\}} = \frac{\psi_j(2)\sigma_{j-1}^2}{\psi_{j-1}(1)^2\sigma_{j-2}^2} 1_{\{j=\ell-1\}}$$

and

$$[H_1V_{11}H_1']_{\ell k} = -\sum_{j=1}^{v} \frac{\psi_j(2)\sigma_{j-1}^2}{\psi_{j-1}(1)^2\sigma_{j-2}^2} 1_{\{j=\ell-1\}} \frac{\psi_j(2)}{\psi_{j-1}(1)^2} 1_{\{j=k-1\}} = \frac{\psi_{k-1}(2)^2\sigma_{k-2}^2}{\psi_{k-2}(1)^4\sigma_{k-3}^2} 1_{\{k=\ell\}},$$

so that $H_1V_{11}H_1'$ is a diagonal matrix. Next, note that $[B_1\Pi]_{ij} = [B_1]_{i,j-1} = \sigma_{i-1}^2\sigma_{i-2}^{-2}1_{\{j-1=i\}}$ so that in view of eqn (17), we have

$$[V_{12}]_{ij} = [B_1\Pi F_1]_{ij} = \sum_{k=1}^{v} \frac{\sigma_{i-1}^2}{\sigma_{i-2}^2} 1_{\{k-1=i\}}\psi_{k-1}(1)1_{\{k=j\}} = \frac{\sigma_{i-1}^2\psi_{j-1}(1)}{\sigma_{i-2}^2} 1_{\{j-1=i\}},$$

so that

$$[H_1V_{12}]_{\ell j} = -\sum_{i=1}^{v} \frac{\psi_i(2)}{\psi_{i-1}(1)^2} 1_{\{i=\ell-1\}} \frac{\sigma_{i-1}^2\psi_{j-1}(1)}{\sigma_{i-2}^2} 1_{\{j-1=i\}} = -\frac{\psi_{\ell-1}(2)\psi_{\ell-1}(1)\sigma_{\ell-2}^2}{\psi_{\ell-2}(1)^2\sigma_{\ell-3}^2} 1_{\{j=\ell\}}$$

and finally,

$$[H_1V_{12}H_2']_{\ell k} = -\sum_{j=1}^{\nu}\frac{\psi_{\ell-1}(2)\psi_{\ell-1}(1)\sigma_{\ell-2}^2}{\psi_{\ell-2}(1)^2\sigma_{\ell-3}^2}1_{\{j=\ell\}}\psi_{j-2}(1)^{-1}1_{\{k=j\}} = -\frac{\psi_{\ell-1}(2)\psi_{\ell-1}(1)\sigma_{\ell-2}^2}{\psi_{\ell-2}(1)^3\sigma_{\ell-3}^2}1_{\{k=\ell\}},$$

so that $H_1V_{12}H_2'$ is another diagonal matrix. Then, $H_2V_{21}H_1' = (H_1V_{12}H_2')' = H_1V_{12}H_2'$ is the same matrix. Lastly, note that since $[\Pi^{-1}B_1\Pi]_{ij} = [B_1]_{i-1,\,j-1} = \sigma_{i-2}^2\sigma_{i-3}^{-2}1_{\{j=i\}}$, $B_2$, $F_1$ and $H_2$ are all diagonal matrices, it follows from the definition (17) that

$$[V_{22}]_{\ell k} = [B_2 + F_1\Pi^{-1}B_1\Pi F_1]_{\ell k} = \left(\frac{\sigma_{\ell-1}^2}{\sigma_{\ell-3}^2} + \frac{\sigma_{\ell-2}^2\psi_{\ell-1}(1)^2}{\sigma_{\ell-3}^2}\right)1_{\{k=\ell\}}$$

and

$$[H_2V_{22}H_2']_{\ell k} = \frac{1}{\psi_{\ell-2}(1)^2}\left(\frac{\sigma_{\ell-1}^2}{\sigma_{\ell-3}^2} + \frac{\sigma_{\ell-2}^2\psi_{\ell-1}(1)^2}{\sigma_{\ell-3}^2}\right)1_{\{k=\ell\}}$$

is also diagonal. Then, it follows from eqn (40) that $Q$ is a diagonal matrix with entries $[Q]_{i+1} = w_{\phi i}^2$ given by eqn (43). □

Theorem 3. *Under the assumption of Theorem* 1, *we have*

$$N^{1/2}(\hat{\theta} - \theta) \Rightarrow \mathcal{N}(0, S), \tag{44}$$

*where* $\hat{\theta} = [\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_{\nu-1}]'$, $\theta = [\theta_0,\theta_1,\ldots,\theta_{\nu-1}]'$, *and the* $\nu \times \nu$ *matrix S is defined by*

$$S = \sum_{k,\ell=1}^{2} M_\ell V_{\ell k} M_k', \tag{45}$$

*where* $V_{\ell k}$ *is given in eqn (17),* $M_1 = -I - F_2\Pi^{-1}F_1^{-2}$ and $M_2 = \Pi^{-1}F_1^{-1}\Pi$. *Here, I is the* $\nu \times \nu$ *identity matrix,* $\Pi$ *is the* $\nu \times \nu$ *permutation matrix (eqn 15), and* $F_n$ *is from eqn (18).*

Proof. Theorem 1 with $j = 1$ and $h = 2$ yields that $X_n = (\hat{\psi}(1), \hat{\psi}(2))'$ is AN($\mu, N^{-1}V$) with $\mu, V$ as in the proof of Theorem 2. Recall from eqn (37) that $\theta_t = \phi_t - \psi_t(1)$, and apply the continuous mapping $g(\mu) = \theta$ to see that eqn (44) holds with

$$S = MVM',$$

where $M$ is a $\nu \times 2\nu$ matrix of partial derivatives

$$M = (M_1, M_2) = \left(\frac{\partial\theta_{\ell-1}}{\partial\psi_{m-1}(1)}, \frac{\partial\theta_{\ell-1}}{\partial\psi_{m-1}(2)}\right)_{\ell,m=1,\ldots,\nu}, \tag{46}$$

and then it follows immediately from eqns (37) and (41) that $M_1 = H_1 - I$ and $M_2 = H_2$. □

Corollary 3. *Regarding Theorem* 3, *in particular, we have that*

$$N^{1/2}(\hat{\theta}_i - \theta_i) \Rightarrow \mathcal{N}(0, w_{\theta i}^2), \tag{47}$$

*for* $0 \le i \le \nu - 1$, *where*

$$w_{\theta i}^2 = \psi_{i-1}^{-4}(1)\left\{\psi_i^2(2)\sigma_{i-2}^{-2}\sigma_{i-1}^2\left(1 - \frac{2\psi_i(1)\psi_{i-1}(1)}{\psi_i(2)}\right) + \sum_{j=1}^{2}\psi_{i-1}^{4/j}(1)\sigma_{i-j}^{-2}\sum_{n=0}^{j-1}\sigma_{i-n}^2\psi_i^2(n)\right\}. \tag{48}$$

□

Proof. Since $M_1 = H_1 - I$ and $M_2 = H_2$, it follows from block matrix multiplication that $S = MVM' = HVH' + \bar{S} = Q + \bar{S}$ where

$$\bar{S} = V_{11} - H_1V_{11} - V_{11}H_1' - H_2V_{21} - V_{12}H_2', \tag{49}$$

with $[V_{11}]_{\ell\ell} = \sigma_{\ell-1}^2\sigma_{\ell-2}^{-2}$ and the remaining matrix terms on the right-hand side of eqn (49) have zero entries along the diagonal. Then, $[S]_{i+1,i+1} = \sigma_i^2\sigma_{i-1}^{-2} + [Q]_{i+1,i+1}$ which reduces to $w_{\theta i}^2$ in eqn (48). □

Using Corollaries 2 and 3, we can write the $(1 - \alpha)100\%$ confidence intervals for $\phi_i$ and $\theta_i$ as

$$(\hat{\phi}_i - z_{\alpha/2}N^{-1/2}w_{\phi i}, \hat{\phi}_i + z_{\alpha/2}N^{-1/2}w_{\phi i}),$$
$$(\hat{\theta}_i - z_{\alpha/2}N^{-1/2}w_{\theta i}, \hat{\theta}_i + z_{\alpha/2}N^{-1/2}w_{\theta i}),$$

where $P(Z > z_\alpha) = \alpha$ for $Z \sim \mathcal{N}(0, 1)$.

## 5. ASYMPTOTICS FOR DISCRETE FOURIER TRANSFORMS

The PARMA$_\nu$($p$,$q$) model (eqn 1) has ($p$ + $q$ + 1)$\nu$ total parameters. For example, for a monthly series ($\nu = 12$) with $p = q = 1$, there are 36 parameters. For a weekly series with $p = q = 1$, there are 156 parameters, representing three periodic functions with period $\nu = 52$. When the period $\nu$ is large, the authors have found that the model parameters often vary smoothly with time, and can therefore be explained by just a few non-zero discrete Fourier coefficients. In fact, increasing $\nu$ often makes the periodically varying parameter functions smoother (see, e.g, Anderson *et al.*, 2007). The statistical basis for selecting the significant harmonics in the DFT of the periodically varying model parameters in eqns (1) and (2) depends on the asymptotic distribution of the DFT coefficients. The PARMA model parameters in eqn (1) can be expressed in terms of the infinite order moving average parameters in eqn (2), as shown in Section 3. Hence, we begin by computing DFT asymptotics for parameter estimates obtained from the innovations algorithm.

### 5.1. Moving averages

Write the moving average parameters in eqn (2) at lag $j$ in the form

$$\psi_t(j) = c_0(j) + \sum_{r=1}^{k}\left\{c_r(j)\cos\left(\frac{2\pi rt}{\nu}\right) + s_r(j)\sin\left(\frac{2\pi rt}{\nu}\right)\right\}, \tag{50}$$

where $c_r(j)$ and $s_r(j)$ are the Fourier coefficients, $r$ is the harmonic and $k$ is the total number of harmonics, which is equal to $\nu/2$ or $(\nu - 1)/2$ depending on whether $\nu$ is even or odd respectively. Write the vector of Fourier coefficients at lag $j$ in the form

$$f(j) = \begin{cases} \left[c_0(j), c_1(j), s_1(j), \ldots, c_{(\nu-1)/2}(j), s_{(\nu-1)/2}(j)\right]' & (\nu \text{ odd}) \\ \left[c_0(j), c_1(j), s_1(j), \ldots, s_{(\nu/2-1)}(j), c_{(\nu/2)}(j)\right]' & (\nu \text{ even}). \end{cases} \tag{51}$$

Similarly, define $\hat{f}_j$ to be the vector of Fourier coefficients for the innovations estimates $\hat{\psi}_t(j)$, defined by replacing $\psi_t(j)$ by $\hat{\psi}_t(j)$, $c_r(j)$ by $\hat{c}_r(j)$ and $s_r(j)$ by $\hat{s}_r(j)$ in eqns (50) and (51). We wish to describe the asymptotic distributional properties of these Fourier coefficients to determine those that are statistically significantly different from zero. These are the coefficients that will be included in our model.

To compute the asymptotic distribution of the Fourier coefficients, it is convenient to work with the complex DFT and its inverse

$$\begin{aligned} \psi_r^*(j) &= \nu^{-1/2}\sum_{t=0}^{\nu-1} exp\left(\frac{-2i\pi rt}{\nu}\right)\psi_t(j), \\ \psi_t(j) &= \nu^{-1/2}\sum_{r=0}^{\nu-1} exp\left(\frac{2i\pi rt}{\nu}\right)\psi_r^*(j), \end{aligned} \tag{52}$$

and similarly $\hat{\psi}_r^*(j)$ is the complex DFT of $\hat{\psi}_m(j)$. The complex DFT can also be written in matrix form. Recall from Theorem 1 the definitions $\hat{\psi}(\ell) = [\hat{\psi}_0(\ell), \hat{\psi}_1(\ell), \ldots, \hat{\psi}_{\nu-1}(\ell)]'$ and $\psi(\ell) = [\psi_0(\ell), \psi_1(\ell), \ldots, \psi_{\nu-1}(\ell)]'$ and similarly define

$$\begin{aligned} \hat{\psi}^*(j) &= [\hat{\psi}_0^*(j), \hat{\psi}_1^*(j), \ldots, \hat{\psi}_{\nu-1}^*(j)]', \\ \psi^*(j) &= [\psi_0^*(j), \psi_1^*(j), \cdots, \psi_{\nu-1}^*(\ell)]', \end{aligned} \tag{53}$$

noting that these are all $\nu$-dimensional vectors. Define a $\nu \times \nu$ matrix $U$ with complex entries

$$U = \nu^{-1/2}(e^{\frac{-i2\pi rt}{\nu}})_{r,t=0,1,\ldots,\nu-1}, \tag{54}$$

so that $\psi^*(j) = U\psi(j)$ and $\hat{\psi}^*(j) = U\hat{\psi}(j)$. This matrix form is useful because it is easy to invert. Obviously $\psi^*(j) = U\psi(j)$ is equivalent to $\psi(j) = U^{-1}\psi^*(j)$ since the matrix $U$ is invertible. This is what guarantees that there exists a unique vector of complex DFT coefficients $\psi^*(j)$ corresponding to any vector $\psi(j)$ of moving average parameters. However, in this case, the matrix $U$ is also unitary (i.e. $U\tilde{U}' = I$) which means that $U^{-1} = \tilde{U}'$, and the latter is easy to compute. Here, $\tilde{U}$ denotes the matrix whose entries are the complex conjugates of the respective entries in $U$. Then, we also have $\psi(j) = \tilde{U}'\psi^*(j)$ which is the matrix form of the second relation in eqn (52).

Next, we convert from complex to real DFT, and it is advantageous to do this in a way that also involves a unitary matrix. Define

$$\begin{aligned} a_r(j) &= 2^{-1/2}\{\psi_r^*(j) + \psi_{\nu-r}^*(j)\} \quad (r = 1, 2, \ldots, [(\nu-1)/2]) \\ a_r(j) &= \psi_r^*(j) \quad (r = 0 \text{ or } \nu/2) \\ b_r(j) &= i2^{-1/2}\{\psi_r^*(j) - \psi_{\nu-r}^*(j)\} \quad (r = 1, 2, \ldots, [(\nu-1)/2]) \end{aligned} \tag{55}$$

and let

$$e(j) = \begin{cases} \left[a_0(j), a_1(j), b_1(j), \ldots, a_{(\nu-1)/2}(j), b_{(\nu-1)/2}(j)\right]' & (\nu \text{ odd}) \\ \left[a_0(j), a_1(j), b_1(j), \ldots, b_{(\nu/2-1)}(j), a_{(\nu/2)}(j)\right]' & (\nu \text{ even}) \end{cases} \tag{56}$$

and likewise for the coefficients of $\hat{\psi}_t(j)$. These relations (eqn 55) define another $\nu \times \nu$ matrix $P$ with complex entries such that

$$e(j) = PU\psi(j) = P\psi^*(j), \tag{57}$$

and it is not hard to check that $P$ is also unitary, so that $\psi^*(j) = P^{-1}e(j) = \tilde{P}'e(j)$, the latter form being most useful for computations. The DFT coefficients $a_r(j)$ and $b_r(j)$ are not the same as the coefficients $c_r(j)$ and $s_r(j)$ in eqn (50) but they are closely related. Substitute the first line of eqn (52) into eqn (55) and simplify to obtain

$$\begin{aligned} a_r(j) &= \nu^{-1/2}\sum_{t=0}^{\nu-1}\cos\left(\frac{2\pi rt}{\nu}\right)\psi_t(j) \quad (r = 0 \text{ or } \nu/2),\\ a_r(j) &= \sqrt{\frac{2}{\nu}}\sum_{t=0}^{\nu-1}\cos\left(\frac{2\pi rt}{\nu}\right)\psi_t(j) \quad (r = 1,2,\ldots,[(\nu-1)/2]),\\ b_r(j) &= \sqrt{\frac{2}{\nu}}\sum_{tm=0}^{\nu-1}\sin\left(\frac{2\pi rt}{\nu}\right)\psi_t(j) \quad (r = 1,2,\ldots,[(\nu-1)/2]). \end{aligned} \tag{58}$$

Inverting the relations (eqn 55) or, equivalently, using the matrix equation $\psi^*(j) = \tilde{P}'e(j)$, we obtain $\psi_r^*(j) = a_r(j)$ for $r = 0$ or $\nu/2$ and

$$\psi_r^*(j) = 2^{-1/2}\{a_r(j) - ib_r(j)\} \quad \text{and} \quad \psi_{\nu-r}^*(j) = \tilde{\psi}_r^*(j) = 2^{-1/2}\{a_r(j) + ib_r(j)\},$$

for $r = 1,\ldots, k = [(\nu - 1)/2]$. Substitute these relations into the second expression in eqn (52) and simplify to obtain

$$\psi_t(j) = \nu^{-1/2}a_0(j) + \sqrt{\frac{2}{\nu}}\sum_{r=1}^{k}\left\{a_r(j)\cos\left(\frac{2\pi rt}{\nu}\right) + b_r(j)\sin\left(\frac{2\pi rt}{\nu}\right)\right\},$$

for $\nu$ odd and

$$\psi_t(j) = \nu^{-1/2}(a_0(j) + a_k(j)) + \sqrt{\frac{2}{\nu}}\sum_{r=1}^{k-1}\left\{a_r(j)\cos\left(\frac{2\pi rt}{\nu}\right) + b_r(j)\sin\left(\frac{2\pi rt}{\nu}\right)\right\},$$

for $\nu$ even, where $k$ is the total number of harmonics, which is equal to $\nu/2$ or $(\nu - 1)/2$ depending on whether $\nu$ is even or odd respectively. Comparison with eqn (50) reveals that

$$\begin{aligned} c_r &= \sqrt{\frac{2}{\nu}}\,a_r \quad (r = 1,2,\ldots,[(\nu-1)/2]),\\ c_r &= \nu^{-1/2}a_r \quad (r = 0 \text{ or } \nu/2),\\ s_r &= \sqrt{\frac{2}{\nu}}\,b_r \quad (r = 1,2,\ldots,[(\nu-1)/2]). \end{aligned} \tag{59}$$

Substituting into eqn (58) yields

$$\begin{aligned} c_r(j) &= \nu^{-1}\sum_{m=0}^{\nu-1}\cos\left(\frac{2\pi rm}{\nu}\right)\psi_m(j) \quad (r = 0 \text{ or } \nu/2),\\ c_r(j) &= 2\nu^{-1}\sum_{m=0}^{\nu-1}\cos\left(\frac{2\pi rm}{\nu}\right)\psi_m(j) \quad (r = 1,2,\ldots,[(\nu-1)/2]),\\ s_r(j) &= 2\nu^{-1}\sum_{m=0}^{\nu-1}\sin\left(\frac{2\pi rm}{\nu}\right)\psi_m(j) \quad (r = 1,2,\ldots,[(\nu-1)/2]), \end{aligned} \tag{60}$$

and likewise for the Fourier coefficients of $\hat{\psi}_m(j)$. Define the $\nu \times \nu$ diagonal matrix

$$L = \begin{cases} \operatorname{diag}(\nu^{-1/2}, \sqrt{2/\nu}, \ldots, \sqrt{2/\nu}) & (\nu \text{ odd})\\ \operatorname{diag}(\nu^{-1/2}, \sqrt{2/\nu}, \ldots, \sqrt{2/\nu}, \nu^{-1/2}) & (\nu \text{ even})' \end{cases} \tag{61}$$

so that in view of eqn (59), we have $f(j) = Le(j)$ and $\hat{f}(j) = L\hat{e}(j)$. Substituting into eqn (57) we obtain

$$f(j) = LPU\psi(j) \quad \text{and} \quad \hat{f}(j) = LPU\hat{\psi}(j). \tag{62}$$

Theorem 4. *For any positive integer j*

$$N^{1/2}[\hat{f}(j) - f(j)] \Rightarrow \mathcal{N}(0, R_V), \tag{63}$$

*where*

$$R_V = LPUV_{jj}\tilde{U}'\tilde{P}'L'. \tag{64}$$

PROOF. From Theorem 1, we have

$$N^{1/2}[\hat{\psi}(j) - \psi(j)] \Rightarrow \mathcal{N}(0, V_{jj}), \tag{65}$$

where $V_{jj}$ is given by eqn (17). Define $B = LPU$ so that $f(j) = B\psi(j)$ and $\hat{f}(j) = B\hat{\psi}(j)$ using eqn (62). Apply continuous mapping to obtain $N^{1/2}[B\hat{\psi}(j) - B\psi(j)] \Rightarrow \mathcal{N}(0, BV_{jj}B')$ or in other words $N^{1/2}[\hat{f}(j) - Bf(j)] \Rightarrow \mathcal{N}(0, BV_{jj}B')$. Although $P$ and $U$ are complex matrices, the product $B = LPU$ is a real matrix, and therefore $B' = \tilde{B}' = \tilde{U}'\tilde{P}'L'$. Then eqns (63) and (64) follow, which finishes the proof. □

THEOREM 5. *Let $X_t$ be the periodically stationary infinite order moving average process* (*eqn* 2). *Then, under the null hypothesis that the process is stationary with $\psi_t(h) = \psi(h)$ and $\sigma_t = \sigma$, the elements of eqn* (51) *are asymptotically independent with*

$$\begin{aligned} N^{1/2}\{\hat{c}_m(h) - \mu_m(h)\} &\Rightarrow \mathcal{N}(0, \nu^{-1}\eta_V(h)) \quad (m = 0 \text{ or } \nu/2), \\ N^{1/2}\{\hat{c}_m(h) - \mu_m(h)\} &\Rightarrow \mathcal{N}(0, 2\nu^{-1}\eta_V(h)) \quad (m = 1, 2, \ldots, [(\nu-1)/2]), \\ N^{1/2}\{\hat{s}_m(h) - \mu_m(h)\} &\Rightarrow \mathcal{N}(0, 2\nu^{-1}\eta_V(h)) \quad (m = 1, 2, \ldots, [(\nu-1)/2]), \end{aligned} \tag{66}$$

*for all $h \geq 1$, where*

$$\mu_m(h) = \begin{cases} \psi(h) & (m = 0) \\ 0 & (m > 0) \end{cases}. \tag{67}$$

$$\eta_V(h) = \sum_{n=0}^{h-1} \psi^2(n). \tag{68}$$

PROOF. Under the null hypothesis, $\psi_t(h) = \psi(h)$ and $\sigma_t = \sigma$, is constant in $t$ for each $h$ and hence the $F_n$ and $B_n$ matrices in eqn (18) become respectively, a scalar multiple of the identity matrix: $F_n = \psi(n)I$, and an identity matrix: $B_n = I$. Then, from eqn (17), using $(\Pi^t)' = \Pi^{-t}$, we have

$$V_{hh} = \sum_{n=1}^{h} \psi(h-n)\Pi^{-(h-n)}\psi(h-n)\Pi^{h-n} = \sum_{m=0}^{h-1} \psi^2(m)I = \eta_V(h)I$$

is also a scalar multiple of the identity matrix. Hence, since scalar multiples of the identity matrix commute in multiplication with any other matrix, we have from eqn (64) that $PUV_{hh}\tilde{U}'\tilde{P}' = V_{hh}PU\tilde{U}'\tilde{P}' = V_{hh}$ since $P$ and $U$ are unitary matrices (i.e. $\tilde{P}'P = I$ and $U\tilde{U}' = I$). Then, in Theorem 4, we have

$$N^{1/2}[\hat{f}(h) - f(h)] \Rightarrow \mathcal{N}(0, R_V), \tag{69}$$

where $R_V = LPUV_{hh}\tilde{U}'\tilde{P}'L' = V_{hh}LL'$, so that

$$R_V = \begin{cases} \eta_V(h) \text{ diag}(\nu^{-1}, 2\nu^{-1}, \ldots, 2\nu^{-1}, 2\nu^{-1}) & (\nu \text{ odd}) \\ \eta_V(h) \text{ diag}(\nu^{-1}, 2\nu^{-1}, \ldots, 2\nu^{-1}, \nu^{-1}) & (\nu \text{ even}). \end{cases}$$

Under the null hypothesis, $f(h) = [\psi(h), 0, \ldots, 0]'$ and then the theorem follows by considering the individual elements of the vector convergence (eqn 69).

Theorem 5 can be used to test whether the coefficients in the infinite order moving average model (2) vary with the season. Suppose, for example, that $\nu$ is odd. Then, under the null hypothesis that $c_m(h)$ and $s_m(h)$ are zero for all $m \geq 1$ and $h \neq 0$, $\{\hat{c}_1(h), \hat{s}_1(h), \ldots, \hat{c}_{(\nu-1)/2}(h), \hat{s}_{(\nu-1)/2}(h)\}$ form $\nu - 1$ independent and normally distributed random variables with mean zero and standard error $(2\nu^{-1}\hat{\eta}_V(h)/N)^{1/2}$. The Bonferroni $\alpha$-level test rejects the null hypothesis that $c_m(h)$ and $s_m(h)$ are all zero if $|Z_c(m)| > z_{\alpha'/2}$ and $|Z_s(m)| > z_{\alpha'/2}$ for $m = 1, \ldots, (\nu - 1)/2$, where

$$Z_c(m) = \frac{\hat{c}_m(h)}{(2\nu^{-1}\hat{\eta}_V(h)/N)^{1/2}}, \quad Z_s(m) = \frac{\hat{s}_m(h)}{(2\nu^{-1}\hat{\eta}_V(h)/N)^{1/2}}, \tag{70}$$

$\hat{\eta}_V(h)$ is given by eqn (68) with $\psi(n)$ replaced by $\hat{\psi}(n)$, $\alpha' = \alpha/(\nu - 1)$ and $P(Z > z_\alpha) = \alpha$ for $Z \sim \mathcal{N}(0, 1)$. A similar formula holds when $\nu$ is even, except that $2\nu^{-1}$ is replaced by $\nu^{-1}$ when $m = 0$ or $\nu/2$ in view of eqn (66). When $\alpha = 5\%$ and $\nu = 12$, $\alpha' = 0.05/11 = 0.0045$, $z_{\alpha'/2} = z_{0.0023} = 2.84$, and the null hypothesis is rejected when any $|Z_{c,s}(m)| > 2.84$, indicating that at least one of the corresponding Fourier coefficients are statistically significantly different from zero. In that case, that the moving average parameter at this lag should be represented by a periodic function in the infinite order moving average model (2).

### 5.2. PARMA model parameters

For large $\nu$, it is often the case that the PARMA model parameters $\phi_t(\ell)$, $\theta_t(\ell)$ and $\sigma_t$, will vary smoothly w.r.t. $t$, and can therefore be explained by a few of their non-zero Fourier coefficients. For a PARMA$_\nu$($p$,$q$) model, the DFT of $\phi_t(\ell)$, $\theta_t(\ell)$ and $\sigma_t$, can be written as

$$\begin{aligned}
\theta_t(\ell) &= c_{a0}(\ell) + \sum_{r=1}^{k}\left\{c_{ar}(\ell)\cos\left(\frac{2\pi rt}{\nu}\right) + s_{ar}(\ell)\sin\left(\frac{2\pi rt}{\nu}\right)\right\},\\
\phi_t(\ell) &= c_{b0}(\ell) + \sum_{r=1}^{k}\left\{c_{br}(\ell)\cos\left(\frac{2\pi rt}{\nu}\right) + s_{br}(\ell)\sin\left(\frac{2\pi rt}{\nu}\right)\right\},\\
\sigma_t &= c_{d0} + \sum_{r=1}^{k}\left\{c_{dr}\cos\left(\frac{2\pi rt}{\nu}\right) + s_{dr}\sin\left(\frac{2\pi rt}{\nu}\right)\right\}.
\end{aligned} \tag{71}$$

$c_{ar,br,dr}$ and $s_{ar,br,dr}$ are the Fourier coefficients, $r$ is the harmonic and $k$ is the total number of harmonics as in eqn (50). For instance, for monthly series where $\nu = 12$, we have $k = 6$; for weekly series with $\nu = 52$, $k = 26$ and for daily series with $\nu = 365$, $k = 182$. In practice, a small number of harmonics $k^* < k$ is used.

Fourier analysis of PARMA$_\nu$($p$,$q$) models can be accomplished using the vector difference equations (eqn 32) to write the Fourier coefficients of the model parameters in terms of the DFT of $\psi_t(j)$. This procedure is complicated, in general, by the need to solve the nonlinear system (eqn 32). Here we illustrate the general procedure by developing asymptotics of the DFT coefficients for a PARMA$_\nu$(1,1) model, using the relationships (37) and (38). Since first order PARMA models are often sufficient to capture periodically varying behavior (see, e.g. Anderson and Meerschaert, 1998; Anderson *et al.*, 2007), these results are also useful in their own right.

Consider again the PARMA$_\nu$(1,1) model given in eqn (36). To simplify notation, we will express the model parameters, along with their Fourier coefficients, in terms of vector notation. Let $\theta = [\theta_0, \theta_1, \cdots, \theta_{\nu-1}]'$, $\phi = [\phi_0, \phi_1, \cdots, \phi_{\nu-1}]'$ and $\sigma = [\sigma_0, \sigma_1, \cdots, \sigma_{\nu-1}]'$ be the vector of PARMA$_\nu$(1,1) model parameters. These model parameters may be defined in terms of their complex DFT coefficients $\theta_t^*$, $\phi_t^*$ and $\sigma^*$ as follows:

$$\begin{aligned}
\theta^*(\ell) = U\theta(\ell) \quad &\text{and} \quad \theta(\ell) = \tilde{U}'\theta^*(\ell),\\
\phi^*(\ell) = U\phi(\ell) \quad &\text{and} \quad \phi(\ell) = \tilde{U}'\phi^*(\ell),\\
\sigma^* = U\sigma \quad &\text{and} \quad \sigma = \tilde{U}'\sigma^*,
\end{aligned} \tag{72}$$

where $U$ is the $\nu \times \nu$ Fourier transform matrix defined in eqn (54) and

$$\begin{aligned}
\theta^* &= \left[\theta_0^*, \theta_1^*, \cdots, \theta_{\nu-1}^*\right]',\\
\phi^* &= \left[\phi_0^*, \phi_1^*, \cdots, \phi_{\nu-1}^*\right]',\\
\sigma^* &= \left[\sigma_0^*, \sigma_1^*, \cdots, \sigma_{\nu-1}^*\right]'.
\end{aligned}$$

As in Theorem 4 let the vector form for transformed $\theta$ and $\phi$ be given by

$$\begin{aligned}
f_\theta &= LP\theta^* = LPU\theta,\\
f_\phi &= LP\phi^* = LPU\phi,
\end{aligned} \tag{73}$$

where

$$f_\theta = \begin{cases} \left[c_{a0}, c_{a1}, s_{a1}, \ldots, c_{a(\nu-1)/2}, s_{a(\nu-1)/2}\right]' & (\nu \text{ odd})\\ \left[c_{a0}, c_{a1}, s_{a1}, \ldots, s_{a(\nu/2-1)}, c_{a(\nu/2)}\right]' & (\nu \text{ even}), \end{cases} \tag{74}$$

$$f_\phi = \begin{cases} \left[c_{b0}, c_{b1}, s_{b1}, \ldots, c_{b(\nu-1)/2}, s_{b(\nu-1)/2}\right]' & (\nu \text{ odd})\\ \left[c_{b0}, c_{b1}, s_{b1}, \ldots, s_{b(\nu/2-1)}, c_{b(\nu/2)}\right]' & (\nu \text{ even}), \end{cases} \tag{75}$$

$$\begin{aligned}
c_{ar} &= \nu^{-1}\sum_{m=0}^{\nu-1}\cos\left(\frac{2\pi rm}{\nu}\right)\theta_m \quad (r = 0 \text{ or } \nu/2),\\
c_{ar} &= 2\nu^{-1}\sum_{m=0}^{\nu-1}\cos\left(\frac{2\pi rm}{\nu}\right)\theta_m \quad (r = 1, 2, \ldots, [(\nu-1)/2]),\\
s_{ar} &= 2\nu^{-1}\sum_{m=0}^{\nu-1}\sin\left(\frac{2\pi rm}{\nu}\right)\theta_m \quad (r = 1, 2, \ldots, [(\nu-1)/2])
\end{aligned} \tag{76}$$

and

$$
\begin{aligned}
c_{br} &= v^{-1}\sum_{m=0}^{v-1}\cos\left(\frac{2\pi rm}{v}\right)\phi_m \quad (r = 0 \text{ or } v/2),\\
c_{br} &= 2v^{-1}\sum_{m=0}^{v-1}\cos\left(\frac{2\pi rm}{v}\right)\phi_m \quad (r = 1, 2, \ldots, [(v-1)/2]),\\
s_{br} &= 2v^{-1}\sum_{m=0}^{v-1}\sin\left(\frac{2\pi rm}{v}\right)\phi_m \quad (r = 1, 2, \ldots, [(v-1)/2])
\end{aligned} \tag{77}
$$

and likewise for the Fourier coefficients of $\hat{\theta}_m$ and $\hat{\phi}_m$. We wish to describe the asymptotic distributional properties of the elements of eqns (74) and (75).

Theorem 6. *Regarding the DFT of the PARMA$_v$(1,1) model coefficients in eqns* (74) *and* (75), *under the assumption of Theorem* 1, *we have*

$$
\begin{aligned}
N^{1/2}[\hat{f}_\theta - f_\theta] &\Rightarrow \mathcal{N}(0, R_S),\\
N^{1/2}[\hat{f}_\phi - f_\phi] &\Rightarrow \mathcal{N}(0, R_Q),
\end{aligned} \tag{78}
$$

*where*

$$
\begin{aligned}
f_\theta &= LP\theta^* = LPU\theta,\\
R_S &= LPUS\tilde{U}'\tilde{P}'L',\\
f_\phi &= LP\phi^* = LPU\phi,\\
R_Q &= LPUQ\tilde{U}'\tilde{P}'L',
\end{aligned} \tag{79}
$$

*with Q given by eqn (40) and S given by eqn (45).*

Proof. The proof is similar to Theorem 4. Apply continuous mapping along with Theorems 2 and 3. □

Theorem 7. *Let $X_t$ be the mean-standardized PARMA$_v$(1,1) process (eqn 36), and suppose that the assumptions of Theorem* 1 *hold. Then, under the null hypothesis that the $X_t$ is stationary with $\phi_t = \phi$, $\theta_t = \theta$ and $\sigma_t = \sigma$, the elements of $\hat{f}_\phi$, defined by eqn* (75) *with $c_{br}$ replaced by $\hat{c}_{br}$ and $s_{br}$ replaced by $\hat{s}_{br}$, are asymptotically independent with*

$$
\begin{aligned}
N^{1/2}\{\hat{c}_{bm} - \mu_{bm}\} &\Rightarrow \mathcal{N}(0, v^{-1}\eta_Q) \quad (m = 0 \text{ or } v/2),\\
N^{1/2}\{\hat{c}_{bm} - \mu_{bm}\} &\Rightarrow \mathcal{N}(0, 2v^{-1}\eta_Q) \quad (m = 1, 2, \ldots, [(v-1)/2]),\\
N^{1/2}\{\hat{s}_{bm} - \mu_{bm}\} &\Rightarrow \mathcal{N}(0, 2v^{-1}\eta_Q) \quad (m = 1, 2, \ldots, [(v-1)/2]),
\end{aligned} \tag{80}
$$

*where*

$$
\mu_{bm} = \begin{cases} \phi & (m = 0) \\ 0 & (m > 0)' \end{cases} \tag{81}
$$

$$
\eta_Q = \psi^{-4}(1)\left\{\psi^2(2)\left(1 - \frac{2\psi^2(1)}{\psi(2)}\right) + \psi^2(1)\sum_{n=0}^{1}\psi^2(n)\right\} \tag{82}
$$

*and* $\psi(1) = \phi - \theta$, $\psi(2) = \phi\ \psi(1)$.

Proof. The proof follows along the same lines as Theorem 2 and hence we adopt the same notation. As in proof of Theorem 5, we have $B_n = I$ and $F_n = \psi(n)I$ in eqn (18) and so eqn (17) implies ($x = \min(h, j)$):

$$
V_{jh} = \sum_{n=1}^{x}\psi(j-n)\Pi^{-(j-n)}\Pi^{h-n}\psi(h-n) = \sum_{n=1}^{x}\psi^2(j-n)I \quad \text{if } j = h,
$$

so

$$
\begin{aligned}
V_{11} &= \psi^2(0)I = I,\\
V_{22} &= [\psi^2(1) + \psi^2(0)]I = [\psi^2(1) + 1]I,\\
V_{12} &= \psi(0)\Pi^0\Pi^1\psi(1) = \psi(1)\Pi,\\
V_{21} &= \psi(1)\Pi^{-1}\psi(0) = \psi(1)\Pi^{-1} = \psi(1)\Pi'
\end{aligned}
$$

and

$$Q = \begin{pmatrix} H_1 & H_2 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} H_1' \\ H_2' \end{pmatrix},$$

where $V_{21} = V_{12}'$ so that $Q$ is symmetric. Since $X_t$ is stationary, every $\psi_\ell(t) = \psi(t)$ in eqn (1) and so

$$H_1 = -\frac{\psi(2)}{\psi^2(1)}\Pi^{-1}$$

and $H_2 = \psi(1)^{-1}I$ so that

$$\begin{aligned} Q &= H_1V_{11}H_1' + H_2V_{21}H_1' + H_1V_{12}H_2' + H_2V_{22}H_2' \\ &= \frac{-\psi(2)}{\psi^2(1)}\Pi^{-1}I\left(\frac{-\psi(2)}{\psi^2(1)}\Pi\right) + \frac{-\psi(2)}{\psi^2(1)}\Pi^{-1}\psi(1)\Pi\frac{1}{\psi(1)} + \frac{1}{\psi(1)}I\psi(1)\Pi^{-1}\left(\frac{-\psi(2)}{\psi^2(1)}\Pi\right) + \frac{1}{\psi(1)}I[\psi^2(1)+1]I\frac{1}{\psi(1)}I \\ &= \left\{\frac{\psi^2(2) - 2\psi^2(1)\psi(2) + [\psi^2(1)+1]\psi^2(1)}{\psi^4(1)}\right\}I. \end{aligned}$$

So $Q = \eta_Q I$ is actually a scalar multiple of the identity matrix $I$. Then, $PUQ\tilde{U}'\tilde{P}' = QPU\tilde{U}'\tilde{P}' = Q$ and hence $R_Q = LQL' = QLL'$ or in other words

$$R_Q = \begin{cases} \eta_Q \operatorname{diag}(v^{-1}, 2v^{-1}, \ldots, 2v^{-1}, 2v^{-1}) & (v \text{ odd}) \\ \eta_Q \operatorname{diag}(v^{-1}, 2v^{-1}, \ldots, 2v^{-1}, v^{-1}) & (v \text{ even}). \end{cases}$$

Under the null hypothesis, $f_\phi = [\phi, 0,\ldots, 0]'$ and then the theorem follows by considering the individual elements of the vector convergence from the second line of eqn (78). □

THEOREM 8. *Under the assumption of Theorem 7, the elements of $\hat{f}_\theta$, defined by eqn (74) with $c_{ar}$ replaced by $\hat{c}_{ar}$ and $s_{ar}$ replaced by $\hat{s}_{ar}$, are asymptotically independent with*

$$\begin{aligned} N^{1/2}\{\hat{c}_{am} - \mu_{am}\} &\Rightarrow \mathcal{N}(0, v^{-1}\eta_S) \quad (m = 0 \text{ or } v/2), \\ N^{1/2}\{\hat{c}_{am} - \mu_{am}\} &\Rightarrow \mathcal{N}(0, 2v^{-1}\eta_S) \quad (m = 1, 2, \ldots, [(v-1)/2]), \\ N^{1/2}\{\hat{s}_{am} - \mu_{am}\} &\Rightarrow \mathcal{N}(0, 2v^{-1}\eta_S) \quad (m = 1, 2, \ldots, [(v-1)/2]), \end{aligned} \tag{83}$$

*where*

$$\mu_{am} = \begin{cases} \theta & (m = 0) \\ 0 & (m > 0) \end{cases}, \tag{84}$$

$$\eta_S = \psi^{-4}(1)\left\{\psi^2(2)\left(1 - \frac{2\psi^2(1)}{\psi(2)}\right) + \sum_{j=1}^{2}\psi^{4/j}(1)\sum_{n=0}^{j-1}\psi^2(n)\right\}, \tag{85}$$

and $\psi(1) = \phi - \theta$, $\psi(2) = \phi\psi(1)$.

PROOF. The proof follows along the same lines as Theorem 3 and hence we adopt the same notation. Note that $S = Q + \bar{S}$, where

$$\bar{S} = V_{11} - H_1V_{11} - V_{11}H_1' - H_2V_{21} - V_{12}H_2' = I,$$

so as in the proof of Theorem 7 we obtain $R_S = LPUS\tilde{U}'\tilde{P}'L' = SLPU\tilde{U}'\tilde{P}'L' = SLL'$, where

$$S = R_Q - I = \left\{\frac{\psi^2(2) - 2\psi^2(1)\psi(2) + [\psi^2(1)+1]\psi^2(1) + \psi^4(1)}{\psi^4(1)}\right\}I,$$

so that

$$R_S = \begin{cases} \eta_S \operatorname{diag}(v^{-1}, 2v^{-1}, \ldots, 2v^{-1}, 2v^{-1}) & (v \text{ odd}) \\ \eta_S \operatorname{diag}(v^{-1}, 2v^{-1}, \ldots, 2v^{-1}, v^{-1}) & (v \text{ even}). \end{cases}$$

Under the null hypothesis, $f_\theta = [\theta, 0,\ldots, 0]'$ and then the theorem follows by considering the individual elements of the vector convergence from the first line of eqn (78). □

Theorems 7 and 8 can be used to test whether the coefficients in the first-order PARMA model (eqn 36) vary with the season. Under the null hypothesis that $\phi_t \equiv \phi$, the Bonferroni $\alpha$-level test statistic rejects if

$$\left|\frac{\hat{c}_{bm}(h)}{(\lambda\hat{\eta}_Q/N)^{1/2}}\right| > z_{\alpha'/2} \quad \text{and} \quad \left|\frac{\hat{s}_{bm}(h)}{(\lambda\hat{\eta}_Q/N)^{1/2}}\right| > z_{\alpha'/2}, \tag{86}$$

for all $m > 0$, where $\alpha' = \alpha/(\nu - 1)$, $\lambda = \nu^{-1}$ for $m = \nu/2$, $\lambda = 2\nu^{-1}$ for $m = 1,2,\ldots,[(\nu - 1)/2]$, and

$$\hat{\eta}_Q = \hat{\psi}^{-4}(1)\left\{\hat{\psi}^2(2)\left(1 - \frac{2\hat{\psi}^2(1)}{\hat{\psi}(2)}\right) + \hat{\psi}^2(1)\sum_{n=0}^{1}\hat{\psi}^2(n)\right\}.$$

Similarly, the Bonferroni $\alpha$-level test statistic rejects the null hypothesis $\theta_t \equiv \theta$ if

$$\left|\frac{\hat{c}_{am}(h)}{(\lambda\hat{\eta}_S/N)^{1/2}}\right| > z_{\alpha'/2} \quad \text{and} \quad \left|\frac{\hat{s}_{am}(h)}{(\lambda\hat{\eta}_S/N)^{1/2}}\right| > z_{\alpha'/2}, \tag{87}$$

where

$$\hat{\eta}_S = \hat{\psi}^{-4}(1)\left\{\hat{\psi}^2(2)\left(1 - \frac{2\hat{\psi}^2(1)}{\hat{\psi}(2)}\right) + \sum_{j=1}^{2}\hat{\psi}^{4/j}(1)\sum_{n=0}^{j-1}\hat{\psi}^2(n)\right\}.$$

## 6. DISCUSSION

The results of this article were applied in Anderson *et al.* (2007) to a time series of average discharge for the Fraser river near Hope, British Columbia in Canada. The time series of monthly average flows was adequately fit by a $\text{PARMA}_{12}(1,1)$ model with 12 autoregressive and 12 moving average parameters. The DFT methods of this article were then used to identify eight statistically significant Fourier coefficients. Model diagnostics indicated a pleasing fit, and a reduction from 24 to 8 parameters. In fact, there were some indications that the original $\text{PARMA}_{12}(1,1)$ model suffered from 'overfitting' because of the large number of parameters. The weekly flow series at the same site was also considered. In that case, the DFT methods of this article reduced the number of autoregressive and moving average parameters in the $\text{PARMA}_{52}(1,1)$ from 104 to 4. The parameters in the weekly model varied more smoothly, leading to fewer significant frequencies that for the monthly data. In fact, the weekly autoregressive parameters collapsed to a constant, as only the zero frequency was significant. In summary, the results in this article render the PARMA model a useful and practical method for modeling high-frequency time series data with significant seasonal variations in the underlying correlation structure.

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