Asymptotic Ruin Probabilities for a Bivariate Lévy-driven Risk Model with Heavy-tailed Claims and Risky Investments

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Abstract

Consider a general bivariate Lévy-driven risk model. The surplus process Y, starting with $Y_0 = x > 0$, evolves according to $\mathrm{d}Y_t = Y_{t-}\mathrm{d}R_t - \mathrm{d}P_t$ for $t > 0$, where P and R are two independent Lévy processes representing, respectively, a loss process in a world without economic factors and a process describing return on investments in real terms. Motivated by a conjecture of Paulsen, we study the finite-time and infinite-time ruin probabilities for the case in which the loss process P has a Lévy measure of extended regular variation and the stochastic exponential of R fulfills a moment condition. We obtain a simple and unified asymptotic formula as $x \to \infty$, which confirms Paulsen's conjecture.

Keywords: Asymptotics; (Extended) regular variation; Finite-time and infinite-time ruin probabilities; Lévy process; Stochastic difference equation; Tail probabilities

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1 Introduction

Consider a bivariate Lévy-driven risk model in which the surplus process, Y, of an insurance company is modeled by

$$Y_t = x - P_t + \int_0^t Y_{s-}\mathrm{d}R_s, \qquad t > 0, \tag{1.1}$$

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with $Y_0 = x > 0$ the initial surplus level, and P and R two independent Lévy processes representing, respectively, a loss process in a world without economic factors and a process describing return on investments in real terms. See Paulsen (1998a, 1998b, 2002, 2008) for detailed explanations.

This model does not mean that the surplus must be completely invested in a risky asset. An understanding on the stochastic process R is as follows. Consider a financial market consisting of a risk-free bond with price $S_{0,t}$ and d risky stocks with prices $S_{k,t}$ for $k = 1, \ldots, d$. Denote by π_0 the proportion of the surplus invested in the bond and by π_k the proportion invested in stock k. Thus, $\pi = (\pi_0, \pi_1, \ldots, \pi_d)$, called a relative investment portfolio and assumed to be time invariant, satisfies $\pi_0 + \pi_1 + \cdots + \pi_d = 1$. Then the differential in (1.1) is

$$\mathrm{d}R_t = \sum_{k=0}^{d} \pi_k \frac{\mathrm{d}S_{k,t}}{S_{k,t-}}, \qquad t > 0.$$

In particular, consider the Black–Scholes market with

$$\begin{cases} \mathrm{d}S_{0,t} = rS_{0,t}\mathrm{d}t, & t > 0, \\ \mathrm{d}S_{k,t} = S_{k,t}\left(\mu_k \mathrm{d}t + \sigma_k \mathrm{d}W_{k,t}\right), & t > 0, \end{cases}$$

where $\mathbf{W}_t = (W_{1,t}, \ldots W_{d,t})$ is a $d-$dimensional Wiener process, $r \geq 0$, $-\infty < \mu_k < \infty$ and $\sigma_k > 0$ for $k = 1, \ldots, d$. Then

$$\mathrm{d}R_t = \pi_0 r \mathrm{d}t + \sum_{k=1}^{d} \pi_k \left(\mu_k \mathrm{d}t + \sigma_k \mathrm{d}W_{k,t}\right), \qquad t > 0.$$

Therefore, the assumption that R is a Lévy process is fulfilled in this particular case.

The solution of (1.1) is given by

$$Y_t = \mathrm{e}^{\tilde{R}_t}\left(x - \int_0^t \mathrm{e}^{-\tilde{R}_s} \mathrm{d}P_s\right), \tag{1.2}$$

where $\tilde{R}$, also a Lévy process, is the logarithm of the stochastic exponential (also called the Doléans-Dade exponential) of R; see, e.g. Protter (2005) for details. For simplicity, we write

$$Z_t = \int_0^t \mathrm{e}^{-\tilde{R}_s} \mathrm{d}P_s,$$

so that $Y_t = \mathrm{e}^{\tilde{R}_t}(x - Z_t)$. The stochastic process Z is usually called the discounted net loss process. We shall start with (1.2) instead of (1.1), as has been done by many researchers including Kalashnikov and Norberg (2002) and Klüppelberg and Kostadinova (2008).

We are interested in the asymptotic behavior of the finite-time and infinite-time ruin probabilities of this bivariate Lévy-driven risk model. As usual, the finite-time ruin probability is defined as

$$\psi(x, T) = \mathbb{P}\left(\inf_{0 \leq t \leq T} Y_t < 0 \,\middle|\, Y_0 = x\right), \qquad T \geq 0,$$

and the infinite-time ruin probability as

$$\psi(x,\infty)=\lim_{T\to\infty}\psi(x,T)=\mathbb{P}\left(\inf_{0\le t<\infty}Y_t<0\middle|Y_0=x\right).$$

Introduce

$$Z_T^*=\sup_{0\le t\le T}Z_t=\sup_{0\le t\le T}\int_0^t \mathrm{e}^{-\tilde{R}_s}\mathrm{d}P_s,\qquad 0\le T\le\infty,$$

where the supremum is taken over $0\le t<\infty$ if $T=\infty$. Noticing that $Y_t<0$ and $Z_t>x$ are equivalent, we have

$$\psi(x,T)=\mathbb{P}\left(Z_T^*>x\right),\qquad 0<T\le\infty. \tag{1.3}$$

In this paper, we aim at a simple and unified asymptotic expression for $\psi(x,T)$ with $0<T\le\infty$ as the initial surplus level x becomes large.

2 Main Results

Hereafter, all limit relationships are for $x\to\infty$ unless otherwise stated. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x)\sim b(x)$ if $\lim a(x)/b(x)=1$, write $a(x)\lesssim b(x)$ or $b(x)\gtrsim a(x)$ if $\limsup a(x)/b(x)\le 1$ and write $a(x)\asymp b(x)$ if $0<\liminf a(x)/b(x)\le\limsup a(x)/b(x)<\infty$. For a real number x, we write $x^+=x\vee 0$ and $x^-=(-x)\vee 0$ as the positive and negative parts of x, respectively.

For a general Lévy process L, its characteristic exponent, $\Psi_L(u)=-\log\left(\mathbb{E}\mathrm{e}^{\mathrm{i}uL_1}\right)$, has the following Lévy-Khintchine representation:

$$\Psi_L(u)=\mathrm{i}lu+\frac{1}{2}\sigma^2u^2+\int_{-\infty}^{\infty}\left(1-\mathrm{e}^{\mathrm{i}ux}+\mathrm{i}ux1_{(|x|<1)}\right)\nu(\mathrm{d}x) \tag{2.1}$$

with $l\in\mathbb{R}$, $\sigma\ge 0$, and Lévy measure ν on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{-\infty}^{\infty}(x^2\wedge 1)\,\nu(\mathrm{d}x)<\infty$. We further denote by

$$\varphi_L(u)=-\Psi_L(\mathrm{i}u)=\log\mathbb{E}\mathrm{e}^{-uL_1}$$

the Laplace exponent of L. Clearly, $\log\mathbb{E}\mathrm{e}^{-uL_t}=t\varphi_L(u)$ for every $t\ge 0$.

Now we turn to the loss process P, which we assume to be a Lévy process. When its Lévy measure satisfies $\overline{\nu}_P(1)=\nu_P((1,\infty))>0$, introduce $\Pi_P(\cdot)=\nu_P(\cdot)1_{(1,\infty)}/\overline{\nu}_P(1)$, which is a proper probability measure on $(1,\infty)$. We assume that Π_P is of extended regular variation (ERV). Formally, a distribution F belongs to the class $\mathrm{ERV}(-\alpha,-\beta)$ for some $0\le\alpha\le\beta<\infty$ if $\overline{F}(x)=1-F(x)>0$ holds for all x and the relations

$$v^{-\beta}\le\liminf\frac{\overline{F}(vx)}{\overline{F}(x)}\le\limsup\frac{\overline{F}(vx)}{\overline{F}(x)}\le v^{-\alpha} \tag{2.2}$$

hold for all $v\ge 1$. Note that relations (2.2) with $\alpha=\beta$ define the class $\mathcal{R}_{-\alpha}$ of distributions of regular variation.

Here comes our main result:

Theorem 2.1 *Consider the bivariate Lévy-driven risk process Y given by (1.2), where P and $\tilde{R}$ are two independent Lévy processes. Assume $\Pi_P \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$.*

(1) If $\mathbb{E}\left(\mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_1}\right) < 1$ for some $\varepsilon > 0$, then it holds for every $T \in (0, \infty)$ that

$$\psi(x, T) \sim \lambda \int_0^T \mathbb{P}\left(X\mathrm{e}^{-\tilde{R}_t} > x\right) \mathrm{d}t, \tag{2.3}$$

where $\lambda = \overline{\nu}_P(1)$ and X is distributed by Π_P and independent of P and $\tilde{R}$.

(2) Furthermore, if $\mathbb{E}\left(\mathrm{e}^{-(\alpha-\varepsilon)\tilde{R}_1} \vee \mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_1}\right) < 1$ for some $\varepsilon \in (0, \alpha)$, then relation (2.3) also holds for $T = \infty$.

The condition $\Pi_P \in \mathrm{ERV}(-\alpha, -\beta)$ means that insurance claims are heavy tailed. Roughly speaking, the exponential moment condition in each case means that insurance claims control the uncertainty of negative returns of investments. When $\alpha = \beta$, it is easy to see that the exponential moment conditions in both cases are equivalent. For this case, the statement of Theorem 2.1 is simplified to the following:

Corollary 2.1 *Let $\alpha = \beta$ in Theorem 2.1. Then it holds for every $T \in (0, \infty]$ that*

$$\psi(x, T) \sim \frac{1 - \mathrm{e}^{\varphi_{\tilde{R}}(\alpha)T}}{-\varphi_{\tilde{R}}(\alpha)} \overline{\nu}_P(x). \tag{2.4}$$

Relation (2.4) with $T = \infty$ was ever conjectured by Paulsen (2002, Theorem 3.2(b) and Remark 3.2(b)). Therefore, our work shows that Paulsen's conjecture is indeed true.

Admittedly, the class ERV is marginally larger than the class $\mathcal{R}$, but it incurs a lot more technicalities to the study. A self-contained proof targeting Corollary 2.1 only can be much simpler than the proof of Theorem 2.1 given below. However, we have reasons to believe that the ruin probabilities for the more general subexponential case will asymptotically behave in the form of (2.3), rather than (2.4). Therefore, we carry out our research within the class ERV, hoping that it will offer insights in the subexponential case.

The asymptotic behavior of the ruin probabilities in presence of investments has been extensively investigated in the literature. Early works focused on a special case in which P is a compound Poisson process with subexponential jumps and R is a deterministic linear function (corresponding to a constant rate of compound interest); see Klüppelberg and Stadtmüller (1998), Asmussen (1998), Kalashnikov and Konstantinides (2000) and Konstantinides *et al.* (2002). Later on, for the subexponential case Tang (2005) established a formula for the finite-time ruin probability, which is in line with the formula for the infinite-time ruin probability. The formulas obtained in these papers are essentially the same as (2.3) with $\tilde{R}$ replaced by a deterministic linear function.

It is more interesting to consider that the insurance company makes both risk-free and risky investments. For a very general case, Kalashnikov and Norberg (2002) proved that the infinite-time ruin probability necessarily decays to 0 as a power function as the initial surplus level increases no matter what tail behavior the jumps of P have. Frolova *et al.* (2002) obtained an explicit asymptotic formula for the infinite-time ruin probability for the case in which P is a compound Poisson process with exponential (hence light-tailed) jumps and R is a Wiener process with drift.

Most of recent research has been done for the case in which P has regularly varying jumps. For example, Paulsen (2002) showed relation (2.4) with $T = \infty$ for a special case in which P is a compound Poisson process with regularly varying jumps and R (or, equivalently, $\tilde{R}$) is a Wiener process with drift. Klüppelberg and Kostadinova (2008) extended this result to a general Lévy process $\tilde{R}$. Heyde and Wang (2009) obtained similar results to our relations (2.3) and (2.4) with $T < \infty$ for the case in which P is a compound Poisson process with heavy-tailed jumps and $\tilde{R}$ is a general Lévy process. Tang *et al.* (2010) established a result similar to our relation (2.3) for both $T < \infty$ and $T = \infty$ for the case in which P is a compound renewal process with regularly varying jumps and $\tilde{R}$ is a general Lévy process. Albrecher *et al.* (2012) also investigated the asymptotic behavior of the infinite-time ruin probability and related quantities for the case in which P is a compound renewal process having light-tailed or heavy-tailed jumps and $\tilde{R}$ is a Brownian motion with drift. Hult and Lindskog (2011) considered a more general case in which P is a Lévy process with regularly varying jumps and R is a semimartingale and they established an asymptotic formula for the finite-time ruin probability, which holds uniformly for all R with stochastic exponential $\mathrm{e}^{\tilde{R}_t}$ fulfilling a certain moment condition. See also Asmussen and Albrecher (2010, Sections VIII.5-6) for a brief review of ruin theory in presence of investments.

We are going to prepare some lemmas in Section 3 and then prove Theorem 2.1 and Corollary 2.1 in Section 4.

3 Lemmas

In this section we prepare some lemmas for the main result. The first lemma below describes some well-known properties of distributions of extended regular variation; see Bingham *et al.* (1987, Proposition 2.2.3) and Tang and Tsitsiashvili (2003, Lemma 3.5):

Lemma 3.1 *Suppose $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \le \beta < \infty$.*

(1) For every $\varepsilon \in (0, \alpha)$ and every $b > 1$, there is some $x_0 > 0$ such that the inequalities

$$\frac{1}{b}\left(y^{-(\alpha-\varepsilon)} \wedge y^{-(\beta+\varepsilon)}\right) \le \frac{\overline{F}(xy)}{\overline{F}(x)} \le b\left(y^{-(\alpha-\varepsilon)} \vee y^{-(\beta+\varepsilon)}\right)$$

hold whenever $x > x_0$ and $xy > x_0$.

(2) It holds for every $\varepsilon > 0$ that $\overline{F}(x) = o\left(x^{-(\alpha-\varepsilon)}\right)$ and $x^{-(\beta+\varepsilon)} = o\left(\overline{F}(x)\right)$.

A feature of the following lemma is that inequality (3.1) holds uniformly for all nonnegative random variables Y independent of X:

Lemma 3.2 *Let X be a real-valued random variable whose distribution belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$. Then for every $\varepsilon \in (0, \alpha)$ and every $b > 1$, there is some $x_0 = x_0(\varepsilon, b) > 0$ such that, for all $x > x_0$ and all nonnegative random variables Y independent of X,*

$$\frac{\mathbb{P}(XY > x)}{\mathbb{P}(X > x)} \leq b\mathbb{E}\left(Y^{\alpha-\varepsilon} \vee Y^{\beta+\varepsilon}\right). \tag{3.1}$$

Proof. For arbitrarily chosen $b' \in (1, b)$, by Lemma 3.1(1) there is some $x_0' > 0$ such that, for all $x > x_0'$,

$$\begin{aligned}\mathbb{P}(XY > x) &\leq \mathbb{P}(XY > x, Y \leq x/x_0') + \mathbb{P}(Y > x/x_0') \\ &\leq b'\mathbb{P}(X > x)\mathbb{E}\left(Y^{\alpha-\varepsilon} \vee Y^{\beta+\varepsilon}\right) + (x/x_0')^{-(\beta+\varepsilon)}\mathbb{E}Y^{\beta+\varepsilon}.\end{aligned}$$

Then inequality (3.1) follows since $x^{-(\beta+\varepsilon)} = o(\mathbb{P}(X > x))$ by Lemma 3.1(2). ∎

By going along the same lines of the proof of Lemma 4.4.2 of Samorodnitsky and Taqqu (1994), we obtain the following:

Lemma 3.3 *Let (X, Y) be jointly distributed random variables. If the distribution of X belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$ for some $0 \leq \alpha \leq \beta < \infty$ and $\mathbb{P}(|Y| > x) = o(\mathbb{P}(X > x))$, then $\mathbb{P}(X + Y > x) \sim \mathbb{P}(X > x)$.*

The tail behavior of randomly weighted sums of heavy-tailed random variables has become a hot topic in applied probability since Resnick and Willekens (1991). A very recent work on the topic is Olvera-Cravioto (2012). The next lemma summarizes several known results:

Lemma 3.4 *Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution F and let $\{\omega_k, k \in \mathbb{N}\}$ be another sequence of nonnegative random variables, non-degenerate at zero and independent of $\{X_k, k \in \mathbb{N}\}$. Assume $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$ and one of the following two conditions holds:*

(1) in case $\beta \in (0, 1)$, there is some $\varepsilon \in (0, \alpha)$ such that

$$\sum_{k=1}^{\infty} \mathbb{E}\left(\omega_k^{\alpha-\varepsilon} \vee \omega_k^{\beta+\varepsilon}\right) < \infty;$$

(2) in case $\beta \in [1, \infty)$, there is some $\varepsilon \in (0, \alpha)$ such that

$$\sum_{k=1}^{\infty} \left(\mathbb{E} \left(\omega_k^{\alpha-\varepsilon} \vee \omega_k^{\beta+\varepsilon} \right) \right)^{\frac{1}{\beta+\varepsilon}} < \infty.$$

Then

$$\mathbb{P} \left(\max_{1 \le n < \infty} \sum_{k=1}^{n} \omega_k X_k > x \right) \sim \mathbb{P} \left(\sum_{k=1}^{\infty} \omega_k X_k^+ > x \right) \sim \sum_{k=1}^{\infty} \mathbb{P} \left(\omega_k X_k > x \right). \tag{3.2}$$

Furthermore, the distributions of $\max_{1 \le n < \infty} \sum_{k=1}^{n} \omega_k X_k$ *and* $\sum_{k=1}^{\infty} \omega_k X_k^+$ *both belong to the class* $\mathrm{ERV}(-\alpha, -\beta)$.

Proof. The second relation in (3.2) is proved by Theorem 3.1(b) of Zhang *et al.* (2009).

Let us check the extended regular variation of the distribution, denoted by F_ω^+, of $\sum_{k=1}^{\infty} \omega_k X_k^+$. It is easy to prove that, for each $k \in \mathbb{N}$, the relation $\mathbb{P}(\omega_k X_k > x) \asymp \overline{F}(x)$ holds and the distribution of $\omega_k X_k$ belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$; see Cline and Samorodnitsky (1994, Theorem 3.5) for these facts. By Lemma 3.2, for every $b > 1$ there is some $x_0 > 0$ such that the inequality

$$\mathbb{P}(\omega_k X_k > x) \le b\overline{F}(x)\mathbb{E}\left(\omega_k^{\alpha-\varepsilon} \vee \omega_k^{\beta+\varepsilon}\right)$$

holds for all $k \in \mathbb{N}$ and all $x > x_0$. Hence, for arbitrarily given $\delta > 0$, all $x > x_0$ and all large n,

$$\sum_{k=n}^{\infty} \mathbb{P}(\omega_k X_k > x) \le b\overline{F}(x) \sum_{k=n}^{\infty} \mathbb{E}\left(\omega_k^{\alpha-\varepsilon} \vee \omega_k^{\beta+\varepsilon}\right) \le \delta \mathbb{P}(\omega_1 X_1 > x). \tag{3.3}$$

By the second relation in (3.2), inequalities (3.3) and the fact that each $\omega_k X_k$ follows a distribution in $\mathrm{ERV}(-\alpha, -\beta)$, it holds for arbitrarily fixed $v > 1$ and some large n_0 that

$$\frac{\bar{F}_\omega^+(vx)}{\bar{F}_\omega^+(x)} \sim \frac{\sum_{k=1}^{\infty} \mathbb{P}(\omega_k X_k > vx)}{\sum_{k=1}^{\infty} \mathbb{P}(\omega_k X_k > x)} \lesssim (1+\delta) \frac{\sum_{k=1}^{n_0} \mathbb{P}(\omega_k X_k > vx)}{\sum_{k=1}^{n_0} \mathbb{P}(\omega_k X_k > x)} \lesssim (1+\delta) v^{-\alpha}.$$

Symmetrically,

$$\frac{\bar{F}_\omega^+(vx)}{\bar{F}_\omega^+(x)} \gtrsim \frac{1}{1+\delta} v^{-\beta}.$$

The arbitrariness of δ implies that $F_\omega^+ \in \mathrm{ERV}(-\alpha, -\beta)$.

The extended regular variation of the distribution of $\max_{1 \le n < \infty} \sum_{k=1}^{n} \omega_k X_k$ easily follows from the extended regular variation of F_ω^+ and the first relation in (3.2).

It remains to verify the first relation in (3.2). Since the inequality $\sum_{k=1}^{\infty} \omega_k X_k^+ \ge \max_{1 \le n < \infty} \sum_{k=1}^{n} \omega_k X_k$ is obvious, we only need to prove that

$$\mathbb{P} \left(\max_{1 \le n < \infty} \sum_{k=1}^{n} \omega_k X_k > x \right) \gtrsim \sum_{k=1}^{\infty} \mathbb{P}(\omega_k X_k > x). \tag{3.4}$$

Actually, by Theorem 3.2 of Chen and Yuen (2009), it holds for all $n \in \mathbb{N}$ that

$$\mathbb{P}\left(\sum_{k=1}^{n} \omega_k X_k > x\right) \sim \sum_{k=1}^{n} \mathbb{P}\left(\omega_k X_k > x\right). \tag{3.5}$$

By (3.5) and (3.3), it holds for arbitrarily given $\delta > 0$ and some large n_0 that

$$\mathbb{P}\left(\max_{1\le n<\infty} \sum_{k=1}^{n} \omega_k X_k > x\right) \gtrsim \sum_{k=1}^{n_0} \mathbb{P}\left(\omega_k X_k > x\right) \gtrsim \frac{1}{1+\delta} \sum_{k=1}^{\infty} \mathbb{P}\left(\omega_k X_k > x\right).$$

Then the arbitrariness of δ implies (3.4). ■

Consider the compound Poisson process

$$C_t = \sum_{k=1}^{N_t} X_k, \qquad t \ge 0, \tag{3.6}$$

where $\{X_k, k \in \mathbb{N}\}$ is a sequence of i.i.d. random variables with generic random variable X and common distribution F, while $\{N_t, t \ge 0\}$ is a Poisson process, independent of $\{X_k, k \in \mathbb{N}\}$, with intensity $\lambda > 0$. The following lemma plays a crucial role in proving Theorem 2.1:

Lemma 3.5 *Let C be a compound Poisson process given by (3.6) and let L be a Lévy process independent of C. If $F \in \mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \le \beta < \infty$ and $\varphi_L(\beta + \varepsilon) < 0$ for some $\varepsilon \in (0, \alpha)$, then it holds for every $T \in (0, \infty]$ that*

$$\mathbb{P}\left(\sup_{0\le t\le T} \int_0^t \mathrm{e}^{-L_s} \mathrm{d}C_s > x\right) \sim \lambda \int_0^T \mathbb{P}\left(X \mathrm{e}^{-L_t} > x\right) \mathrm{d}t.$$

Furthermore, $\sup_{0\le t\le T} \int_0^t \mathrm{e}^{-L_s} \mathrm{d}C_s$ follows a distribution in $\mathrm{ERV}(-\alpha, -\beta)$.

Proof. Let $\tau_k, k \in \mathbb{N}$, be the arrival times of the Poisson process $\{N_t, t \ge 0\}$. Then

$$\sup_{0\le t\le T} \int_0^t \mathrm{e}^{-L_s} \mathrm{d}C_s = \sup_{0\le t\le T} \sum_{k=1}^{N_t} X_k \mathrm{e}^{-L_{\tau_k}} = \max_{0\le n<\infty} \sum_{k=1}^{n} X_k \mathrm{e}^{-L_{\tau_k}} 1_{(\tau_k \le T)},$$

where, by convention, the summation over an empty set of indices produces a value 0. This enables us to apply Lemma 3.4. By its convexity, $\varphi_L(u) < 0$ for all $u \in (0, \beta + \varepsilon]$. Then it is straightforward to verify the corresponding conditions in order to apply Lemma 3.4. Therefore,

$$\mathbb{P}\left(\sup_{0\le t\le T} \int_0^t \mathrm{e}^{-L_s} \mathrm{d}C_s > x\right) \sim \sum_{k=1}^{\infty} \mathbb{P}\left(X_k \mathrm{e}^{-L_{\tau_k}} 1_{(\tau_k \le T)} > x\right) = \lambda \int_0^T \mathbb{P}\left(X \mathrm{e}^{-L_t} > x\right) \mathrm{d}t,$$

where in the last step we used the fact that $\sum_{k=1}^{\infty} \mathbb{P}\left(\tau_k \in \mathrm{d}t\right) = \lambda \mathrm{d}t$. ■

The following lemma, which is a natural generalization of Lemma 2 of Grey (1994), does not require any information on the dependence structure of (A, B) or on the left tail of A:

Lemma 3.6 *Let A, B and Q be three random variables, with Q independent of (A, B). If each of A and Q follows a distribution in the class* $\mathrm{ERV}(-\alpha, -\beta)$ *for some* $0 < \alpha \leq \beta < \infty$, *and B is nonnegative satisfying $\mathbb{E}B^{\beta+\varepsilon} < \infty$ for some $\varepsilon \in (0, \alpha)$, then the distribution of $A + QB$ belongs to the class* $\mathrm{ERV}(-\alpha, -\beta)$ *and*

$$\mathbb{P}(A + QB > x) \sim \mathbb{P}(A > x) + \mathbb{P}(QB > x). \tag{3.7}$$

Proof. We only focus on the proof of relation (3.7) since the other conclusion follows immediately from relation (3.7) and the fact that the distribution of QB belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$.

Arbitrarily choose an increasing function $l(\cdot) : (0, \infty) \to (0, \infty)$ satisfying $l(x) < x/2$, $l(x) = o(x)$ and $l(x) \to \infty$. Moreover, arbitrarily choose constants $\delta \in (0, 1/2)$ and $p \in (\beta/(\beta + \varepsilon), 1)$. We split the left-hand side of (3.7) as

$$\begin{aligned}
\mathbb{P}(A + QB > x) = \mathbb{P}&(A > (1 + \delta)x) \\
&- \mathbb{P}(A > (1 + \delta)x, A + QB \leq x) \\
&+ \mathbb{P}((1 - \delta)x < A \leq (1 + \delta)x, A + QB > x) \\
&+ \mathbb{P}(A < -l(x) \text{ or } l(x) < A \leq (1 - \delta)x, A + QB > x) \\
&+ \mathbb{P}(|A| \leq l(x), A + QB > x) \\
= I_1&(x) - I_2(x) + I_3(x) + I_4(x) + I_5(x).
\end{aligned} \tag{3.8}$$

We estimate the five terms on the right-hand side of (3.8), in turn. Clearly,

$$(1 + \delta)^{-\beta}\, \mathbb{P}(A > x) \lesssim I_1(x) \lesssim (1 + \delta)^{-\alpha}\, \mathbb{P}(A > x).$$

Noticing that, by Lemma 3.1(2), $x^{-p(\beta+\varepsilon)} = o(\mathbb{P}(A > x))$, we have

$$\begin{aligned}
I_2(x) &= \mathbb{P}(A > (1 + \delta)x, A + QB \leq x, (B \leq x^p) \cup (B > x^p)) \\
&\leq \mathbb{P}(A > (1 + \delta)x)\, \mathbb{P}\left(Q < -\delta x^{1-p}\right) + \mathbb{P}(B > x^p) \\
&= o(\mathbb{P}(A > (1 + \delta)x)) + O\left(x^{-p(\beta+\varepsilon)}\right) \\
&= o(\mathbb{P}(A > x)).
\end{aligned}$$

By the definition of the class $\mathrm{ERV}(-\alpha, -\beta)$,

$$I_3(x) \leq \mathbb{P}((1 - \delta)x < A \leq (1 + \delta)x) \lesssim \left((1 - \delta)^{-\beta} - (1 + \delta)^{-\beta}\right) \mathbb{P}(A > x).$$

By conditioning on A and applying Lemma 3.2, it holds for every $b > 1$ and all large x that

$$\begin{aligned}
I_4(x) &\leq \mathbb{P}(|A| > l(x), QB > \delta x) \\
&\leq b\, \mathbb{P}(Q > \delta x)\, \mathbb{E}\left(B^{\alpha-\varepsilon} \vee B^{\beta+\varepsilon}\right) 1_{(|A|>l(x))} \\
&= o(\mathbb{P}(Q > x)).
\end{aligned}$$

As in dealing with $I_2(x)$ above, we further split $I_5(x)$ into two parts according to $(B \leq x^p)$ and $(B > x^p)$. For the first part, we condition it on (A, B) and apply the uniformity over $|y| \leq l(x)$ of the asymptotic relation $\mathbb{P}(Q > x + y) \sim \mathbb{P}(Q > x)$. We have

$$\begin{aligned}
I_5(x) &= \mathbb{P}(|A| \leq l(x), A + QB > x, (B \leq x^p) \cup (B > x^p)) \\
&= (1 + o(1))\mathbb{P}(|A| \leq l(x), QB > x, B \leq x^p) + O(1)\mathbb{P}(B > x^p) \\
&= (1 + o(1))\mathbb{P}(|A| \leq l(x), QB > x) + o(1)\mathbb{P}(Q > x) \\
&= (1 + o(1))(\mathbb{P}(QB > x) - \mathbb{P}(|A| > l(x), QB > x)) + o(1)\mathbb{P}(Q > x) \\
&= (1 + o(1))\mathbb{P}(QB > x) + o(1)\mathbb{P}(Q > x) \\
&\sim \mathbb{P}(QB > x),
\end{aligned}$$

where in the last but one step we used $\mathbb{P}(|A| > l(x), QB > x) = o(1)\mathbb{P}(Q > x)$, as in the treatment of $I_4(x)$ above, and in the last step we used $\mathbb{P}(QB > x) \asymp \mathbb{P}(Q > x)$. Simply plugging these estimates for $I_1(x)$, ..., $I_5(x)$ into (3.8) and noticing the arbitrariness of δ, we obtain (3.7), as desired. ■

The following lemma partially extends Theorem 1 of Grey (1994):

Lemma 3.7 *Let (A, B) be a random pair satisfying $\mathbb{E}\log(|A| \vee 1) < \infty$, $\mathbb{P}(B \geq 0) = 1$ and $-\infty \leq \mathbb{E}\log B < 0$. Let Q be a random variable independent of (A, B).*

(1) Then there is exactly one distribution for Q satisfying the stochastic difference equation

$$Q \stackrel{\mathrm{D}}{=} A + QB, \tag{3.9}$$

where $\stackrel{\mathrm{D}}{=}$ denotes equality in distribution.

(2) Furthermore, if the distribution of A belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$ for some $0 < \alpha \leq \beta < \infty$ and $\mathbb{E}\left(B^{\alpha-\varepsilon} \vee B^{\beta+\varepsilon}\right) < 1$ for some $\varepsilon \in (0, \alpha)$, then

$$\mathbb{P}(Q > x) \sim \sum_{i=1}^{\infty} \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right), \tag{3.10}$$

where $\{(A_k, B_k), k \in \mathbb{N}\}$ is a sequence of i.i.d. copies of (A, B) and, by convention, the multiplication over an empty set of indices produces a value 1.

Proof. (1) The existence and uniqueness of the weak solution of (3.9) are justified by Theorem 1.6(b, c) and Theorem 1.5(i) of Vervaat (1979).

(2) Let $\{Q_k, k \in \mathbb{N}\}$ be a sequence of random variables defined recursively by

$$Q_k = A_k + Q_{k-1}B_k, \qquad k \in \mathbb{N}, \tag{3.11}$$

where Q_0 is an arbitrary starting random variable independent of $\{(A_k, B_k), k \in \mathbb{N}\}$. Then by Theorem 1.5(i) of Vervaat (1979), the sequence $\{Q_k, k \in \mathbb{N}\}$ weakly converges with a limit distribution which does not depend on Q_0 and coincides with the distribution of Q in (3.9). See also Goldie (1991) for these statements. To prove relation (3.10), we apply the method developed by Grey (1994) to establish the following two relations:

$$\mathbb{P}(Q > x) \lesssim \sum_{i=1}^{\infty} \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right), \qquad \mathbb{P}(Q > x) \gtrsim \sum_{i=1}^{\infty} \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right). \tag{3.12}$$

Let us prove the first relation in (3.12) now. Introduce a nonnegative random variable Q_0' independent of (A, B) and satisfying $\mathbb{P}(Q_0' > x) \sim c\,\mathbb{P}(A > x)$ for some constant $c > \left(1 - \mathbb{E}\left(B^{\alpha-\varepsilon} \vee B^{\beta+\varepsilon}\right)\right)^{-1}$. By Lemmas 3.6 and 3.2,

$$\mathbb{P}\left(A + Q_0'B > x\right) \lesssim \left(1 + c\,\mathbb{E}\left(B^{\alpha-\varepsilon} \vee B^{\beta+\varepsilon}\right)\right)\mathbb{P}(A > x).$$

Since $1 + c\,\mathbb{E}\left(B^{\alpha-\varepsilon} \vee B^{\beta+\varepsilon}\right) < c$, there is some $x_0 > 0$ such that, for all $x > x_0$,

$$\mathbb{P}(A + Q_0'B > x) \leq \mathbb{P}(Q_0' > x).$$

Construct a starting random variable Q_0 which is independent of (A, B) and follows the distribution of Q_0' conditional on $Q_0' > x_0$, which belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$.

Substituting Q_0 to (3.11) and applying Lemma 3.6, the distribution of Q_1 belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$ and

$$\mathbb{P}\left(Q_1 > x\right) = \mathbb{P}\left(A_1 + Q_0 B_1 > x\right) \sim \mathbb{P}\left(A_1 > x\right) + \mathbb{P}\left(Q_0 B_1 > x\right).$$

We claim that Q_1 is stochastically not greater than Q_0, written as $Q_1 \leq_{\mathrm{st}} Q_0$. Actually, for $x > x_0$,

$$\mathbb{P}\left(Q_1 > x\right) = \mathbb{P}(A + Q_0'B > x \,|Q_0' > x_0) \leq \frac{\mathbb{P}(A + Q_0'B > x)}{\mathbb{P}(Q_0' > x_0)} \leq \frac{\mathbb{P}(Q_0' > x)}{\mathbb{P}(Q_0' > x_0)} = \mathbb{P}(Q_0 > x),$$

while for $x \leq x_0$, $\mathbb{P}\left(Q_1 > x\right) \leq 1 = \mathbb{P}(Q_0 > x)$.

Using (3.11) with $k = 2$ and $Q_1 \leq_{\mathrm{st}} Q_0$, one easily infers that $Q_2 \leq_{\mathrm{st}} Q_1$. Furthermore, by using Lemma 3.6 twice, the distribution of Q_2 belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$ and

$$\begin{aligned}
\mathbb{P}\left(Q_2 > x\right) &\sim \mathbb{P}\left(A_2 > x\right) + \mathbb{P}\left(Q_1 B_2 > x\right) \\
&= \mathbb{P}\left(A_2 > x\right) + \mathbb{P}\left(A_1 B_2 + Q_0 B_1 B_2 > x\right) \\
&\sim \mathbb{P}\left(A_2 > x\right) + \mathbb{P}\left(A_1 B_2 > x\right) + \mathbb{P}\left(Q_0 B_1 B_2 > x\right).
\end{aligned}$$

Repeating this procedure, we can prove that the sequence $\{Q_k, k \in \mathbb{N}\}$, starting with Q_0, is stochastically non-increasing and that, for each $k \in \mathbb{N}$, the distribution of Q_k belongs to the class ERV$(-\alpha,-\beta)$ with tail satisfying

$$\begin{aligned}\mathbb{P}\left(Q_k > x\right) &\sim \sum_{i=1}^{k} \mathbb{P}\left(A_i \prod_{j=i+1}^{k} B_j > x\right) + \mathbb{P}\left(Q_0 \prod_{j=1}^{k} B_j > x\right) \\ &= \sum_{i=1}^{k} \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right) + \mathbb{P}\left(Q_0 \prod_{j=1}^{k} B_j > x\right).\end{aligned}$$

It follows that

$$\mathbb{P}\left(Q > x\right) \lesssim \sum_{i=1}^{k} \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right) + \mathbb{P}\left(Q_0 \prod_{j=1}^{k} B_j > x\right).$$

The last tail probability is negligible as k becomes large because, by Lemma 3.2,

$$\mathbb{P}\left(Q_0 \prod_{j=1}^{k} B_j > x\right) \lesssim c\left(\mathbb{E}\left(B^{\alpha-\varepsilon} \vee B^{\beta+\varepsilon}\right)\right)^k \mathbb{P}\left(A > x\right)$$

and $\mathbb{E}\left(B^{\alpha-\varepsilon} \vee B^{\beta+\varepsilon}\right) < 1$. This proves the first relation in (3.12).

We turn to the second relation in (3.12). The fact $\mathbb{P}(Q > 0) > 0$ is explained in the proof of Theorem 1 of Grey (1994). Construct another starting random variable Q_0 independent of (A, B) with tail

$$\mathbb{P}(Q_0 > x) = \mathbb{P}(Q > 0)\mathbb{P}(A > x)1_{(x\geq 0)} + \mathbb{P}(Q > x)1_{(x<0)}.$$

Clearly, the distribution of Q_0 belongs to the class ERV$(-\alpha,-\beta)$ and $Q_0 \leq_{\mathrm{st}} Q$. It is easy to see that the sequence $\{Q_k, k \in \mathbb{N}\}$, starting with this Q_0, is stochastically bounded from above by Q. Similarly as before, applying Lemma 3.6 and relation (3.11) recursively, we obtain that, for each $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}\left(Q > x\right) &\geq \mathbb{P}(Q_k > x) \\ &\sim \sum_{i=1}^{k} \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right) + \mathbb{P}\left(Q_0 \prod_{j=1}^{k} B_j > x\right) \\ &\geq \sum_{i=1}^{\infty} \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right) - \sum_{i=k+1}^{\infty} \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right).\end{aligned}$$

By Lemma 3.2, the last sum above is negligible as k becomes large. This proves the second relation in (3.12). ■

Finally, we list several useful martingale inequalities. The proof of the following lemma is an excise of Doob's inequality; see also the proof of Lemma 3.2 of Paulsen (2002):

Lemma 3.8 *Let L be a Lévy process with Laplace exponent $\varphi_L(\cdot)$. If $\varphi_L(u) < \infty$ for some $u > 0$, then $\mathbb{E}\left(\sup_{0\leq t\leq T} \mathrm{e}^{-uL_t}\right) < \infty$ for every fixed $T \in (0, \infty)$.*

For a stochastic process M, denote by $[M,M]$ and $\langle M,M\rangle$ its quadratic variation and predictable quadratic variation, respectively. The following lemma recalls some well-known martingale inequalities of which the first one is the Burkholder-Gundy inequality. Their proofs can be seen, for example, in Liptser and Shiryayev (1989).

Lemma 3.9 *For a local martingale M, write $M_T^* = \sup_{0\le t\le T}|M_t|$ for $0 \le T \le \infty$.*

(1) For every $q \in (1,\infty)$, there are positive constants c_q and c_q' such that, uniformly for all local martingales M with $M_0 = 0$ and all $0 \le T \le \infty$,

$$c_q'\mathbb{E}[M,M]_T^{q/2} \le \mathbb{E}\left(M_T^*\right)^q \le c_q\mathbb{E}[M,M]_T^{q/2}.$$

Moreover, if M_t is continuous, then the inequalities above hold for all $0 < q < \infty$.

(2) If M is a local square integrable martingale with $M_0 = 0$, then it holds for every $q \in (0,2)$ that

$$\mathbb{E}\left(M_T^*\right)^q \le \frac{4-q}{2-q}\mathbb{E}\left\langle M,M\right\rangle_T^{q/2}.$$

4 Proofs of the Main Results

For the process P, the Lévy-Khintchine representation (2.1) for its characteristic exponent becomes

$$\Psi_P(u) = \mathrm{i}pu + \frac{1}{2}\sigma_P^2u^2 + \int_{|x|<1}\left(1-\mathrm{e}^{\mathrm{i}ux}+\mathrm{i}ux\right)\nu_P(\mathrm{d}x) + \int_{|x|\ge 1}\left(1-\mathrm{e}^{\mathrm{i}ux}\right)\nu_P(\mathrm{d}x).$$

Consequently, its Lévy-Itô decomposition is given by

$$P_t = pt + \sigma_P W_t + M_t + C_t, \tag{4.1}$$

where W is a standard Wiener process, M is a square integrable martingale with almost surely countably many jumps of magnitude less than 1, and C is a compound Poisson process in the form of (3.6) in which the Poisson intensity is $\lambda^* = \nu_P(\mathbb{R}\setminus(-1,1))$ and F is given by $\nu_P(\cdot)1_{\mathbb{R}\setminus(-1,1)}/\lambda^*$. In particular, W, M and C are three independent Lévy processes. See, e.g. Kyprianou (2006) for more details.

4.1 Proof of Theorem 2.1(1)

Recall relation (1.3) with $T<\infty$. The basic idea of our proof is that, when considering the tail behavior of Z_T^*, the Wiener process and small jumps of the process P are negligible.

Clearly, by (4.1) it holds that

$$\sum_{j=1}^{3}\inf_{0\le t\le T} I_{j,t} + \sup_{0\le t\le T} I_{4,t} \le Z_T^* \le \sum_{j=1}^{4}\sup_{0\le t\le T} I_{j,t}, \tag{4.2}$$

where $I_{1,t} = p\int_0^t \mathrm{e}^{-\tilde{R}_s}\mathrm{d}s$, $I_{2,t} = \sigma_P \int_0^t \mathrm{e}^{-\tilde{R}_s}\mathrm{d}W_s$, $I_{3,t} = \int_0^t \mathrm{e}^{-\tilde{R}_s}\mathrm{d}M_s$ and $I_{4,t} = \int_0^t \mathrm{e}^{-\tilde{R}_s}\mathrm{d}C_s$. By Lemma 3.5, the distribution of $\sup_{0\le t\le T} I_{4,t}$ belongs to the class $\mathrm{ERV}(-\alpha,-\beta)$ and

$$\mathbb{P}\left(\sup_{0\le t\le T} I_{4,t} > x\right) \sim \lambda^* \int_0^T \mathbb{P}\left(X^*\mathrm{e}^{-\tilde{R}_t} > x\right)\mathrm{d}t, \tag{4.3}$$

where X^*, independent of $\tilde{R}$, follows the distribution F as defined above. Note that the right-hand side of (4.3) with $x>0$ is identical to the right-hand side of (2.3). If

$$\mathbb{E}\left(\sup_{0\le t\le T} |I_{j,t}|^{\beta+\varepsilon}\right) < \infty, \qquad j=1,2,3, \tag{4.4}$$

then by Lemma 3.1(2) and Lemma 3.3, all terms except $\sup_{0\le t\le T} I_{4,t}$ appearing in the upper and lower bounds for Z_T^* in (4.2) are negligible and it follows from (4.3) that

$$\mathbb{P}\left(Z_T^* > x\right) \sim \mathbb{P}\left(\sup_{0\le t\le T} I_{4,t} > x\right) \sim \lambda \int_0^T \mathbb{P}\left(X\mathrm{e}^{-\tilde{R}_t} > x\right)\mathrm{d}t,$$

yielding relation (2.3). Therefore, it suffices to prove (4.4).

By Lemma 3.8, we have

$$\mathbb{E}\left(\sup_{0\le t\le T} \mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_t}\right) < \infty. \tag{4.5}$$

Then relation (4.4) with $j=1$ follows trivially from (4.5). Since $I_{2,t}$ is a continuous martingale and $[I_{2,t}, I_{2,t}] = \sigma_P^2 \int_0^t \mathrm{e}^{-2\tilde{R}_s}\mathrm{d}s$, we use Lemma 3.9(1) and relation (4.5) again to obtain that, for some constant $c_1>0$,

$$\mathbb{E}\left(\sup_{0\le t\le T} |I_{2,t}|^{\beta+\varepsilon}\right) \le c_1 \mathbb{E}\left(\int_0^T \mathrm{e}^{-2\tilde{R}_t}\mathrm{d}t\right)^{(\beta+\varepsilon)/2} \le c_1 T^{(\beta+\varepsilon)/2}\mathbb{E}\left(\sup_{0\le t\le T} \mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_t}\right) < \infty.$$

Similarly, by Lemma 3.9 and relation (4.5), it holds for some constant $c_2>0$ that

$$\mathbb{E}\left(\sup_{0\le t\le T} |I_{3,t}|^{\beta+\varepsilon}\right) \le c_2 \left(\int_{|x|\le 1} x^2 \nu_P(\mathrm{d}x)\right)^{(\beta+\varepsilon)/2} \mathbb{E}\left(\sup_{0\le t\le T} \mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_t}\right) < \infty.$$

4.2 Proof of Theorem 2.1(2)

Recall (1.3) with $T=\infty$, that is, $\psi(x,\infty) = \mathbb{P}\left(Z_\infty^* > x\right)$. The basic idea of our proof is to construct two discrete-time processes, fulfilling a certain recursive structure, whose limits serve as the stochastic upper and lower bounds, respectively, for the ultimate supremum of the discounted net loss process. This idea is from Grey (1994).

To derive an asymptotic upper bound for Z_∞^*, we observe that

$$Z_\infty^* \le Z_1^* + \mathrm{e}^{-\tilde{R}_1} \sup_{1\le t<\infty} \int_1^t \mathrm{e}^{-\left(\tilde{R}_s - \tilde{R}_1\right)}\mathrm{d}P_s \stackrel{\mathrm{d}}{=} Z_1^* + Z_\infty^* \mathrm{e}^{-\tilde{R}_1}, \tag{4.6}$$

where on the right-hand side Z_∞^* is independent of $\left(Z_1^*, \tilde{R}_1\right)$. Consider the stochastic difference equation

$$Q^* \overset{\mathrm{d}}{=} Z_1^* + Q^* \mathrm{e}^{-\tilde{R}_1}, \tag{4.7}$$

where on the right-hand side Q^* is independent of $\left(Z_1^*, \tilde{R}_1\right)$. By Theorem 2.1(1), the distribution of Z_1^* belongs to the class ERV$(-\alpha, -\beta)$ and

$$\mathbb{P}\left(Z_1^* > x\right) \sim \lambda \int_0^1 \mathbb{P}\left(X \mathrm{e}^{-\tilde{R}_t} > x\right) \mathrm{d}t. \tag{4.8}$$

By comparing (4.6) with (4.7) and applying Lemma 3.7, we have

$$\mathbb{P}\left(Z_\infty^* > x\right) \leq \mathbb{P}\left(Q^* > x\right) \sim \sum_{i=1}^\infty \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x\right), \tag{4.9}$$

where $\{(A_k, B_k), k \in \mathbb{N}\}$ is a sequence of i.i.d. copies of the random pair $\left(Z_1^*, \mathrm{e}^{-\tilde{R}_1}\right)$. It follows from (4.9) and (4.8) that, for arbitrarily fixed $\delta > 0$ and some large x_0,

$$\begin{aligned}
&\mathbb{P}\left(Z_\infty^* > x\right) \\
&\lesssim \sum_{i=1}^\infty \mathbb{P}\left(A_i \prod_{j=1}^{i-1} B_j > x, \prod_{j=1}^{i-1} B_j \leq \frac{x}{x_0}\right) + \sum_{i=1}^\infty \mathbb{P}\left(\prod_{j=1}^{i-1} B_j > \frac{x}{x_0}\right) \\
&\leq (1+\delta)\lambda \sum_{i=1}^\infty \int_{i-1}^i \mathbb{P}\left(X \mathrm{e}^{-\left(\tilde{R}_t - \tilde{R}_{i-1}\right)} \prod_{j=1}^{i-1} B_j > x\right) \mathrm{d}t + \left(\frac{x}{x_0}\right)^{-(\beta+\varepsilon)} \sum_{i=1}^\infty \mathbb{E}\left(\prod_{j=1}^{i-1} B_j^{\beta+\varepsilon}\right) \\
&= (1+\delta)\lambda \int_0^\infty \mathbb{P}\left(X \mathrm{e}^{-\tilde{R}_t} > x\right) \mathrm{d}t + \frac{(x/x_0)^{-(\beta+\varepsilon)}}{1 - \mathbb{E}\mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_1}}.
\end{aligned}$$

Since the last term above is negligible and δ can be arbitrarily small, we obtain

$$\psi(x, \infty) \lesssim \lambda \int_0^\infty \mathbb{P}\left(X \mathrm{e}^{-\tilde{R}_t} > x\right) \mathrm{d}t.$$

To derive an asymptotic lower bound, by Theorem 2.1(1) we have, for every $T > 0$,

$$\mathbb{P}\left(Z_\infty^* > x\right) \geq \mathbb{P}\left(Z_T^* > x\right) \sim \lambda \left(\int_0^\infty - \int_T^\infty\right) \mathbb{P}\left(X \mathrm{e}^{-\tilde{R}_t} > x\right) \mathrm{d}t. \tag{4.10}$$

By Lemma 3.2, it holds for every $b > 1$ and all large x that

$$\begin{aligned}
\int_T^\infty \frac{\mathbb{P}\left(X \mathrm{e}^{-\tilde{R}_t} > x\right)}{\mathbb{P}(X > x)} \mathrm{d}t &\leq b \int_T^\infty \mathbb{E}\left(\mathrm{e}^{-(\alpha-\varepsilon)\tilde{R}_t} \vee \mathrm{e}^{-(\beta+\varepsilon)\tilde{R}_t}\right) \mathrm{d}t \\
&\leq b \int_T^\infty \left(\mathrm{e}^{\varphi_{\tilde{R}}(\alpha-\varepsilon)t} + \mathrm{e}^{\varphi_{\tilde{R}}(\beta+\varepsilon)t}\right) \mathrm{d}t \\
&= b \left(\frac{\mathrm{e}^{\varphi_{\tilde{R}}(\alpha-\varepsilon)T}}{-\varphi_{\tilde{R}}(\alpha-\varepsilon)} + \frac{\mathrm{e}^{\varphi_{\tilde{R}}(\beta+\varepsilon)T}}{-\varphi_{\tilde{R}}(\beta+\varepsilon)}\right)
\end{aligned}$$

since $\varphi_{\tilde{R}}(\alpha-\varepsilon)<0$ and $\varphi_{\tilde{R}}(\beta+\varepsilon)<0$. This means that, as T becomes large, the second term on the right-hand side of (4.10) is negligible when compared with $\mathbb{P}(X>x)$, hence with $\int_0^\infty \mathbb{P}\left(X\mathrm{e}^{-\tilde{R}_t}>x\right)\mathrm{d}t$. It follows that

$$\psi(x,\infty)\gtrsim \lambda\int_0^\infty \mathbb{P}\left(X\mathrm{e}^{-\tilde{R}_t}>x\right)\mathrm{d}t.$$

4.3 Proof of Corollary 2.1

We start with (2.3), which holds for $T\in(0,\infty]$. By Lemma 3.2, it holds for every $b>1$ and all large x that

$$\frac{\mathbb{P}\left(X\mathrm{e}^{-\tilde{R}_t}>x\right)}{\mathbb{P}\left(X>x\right)}\leq b\mathbb{E}\left(\mathrm{e}^{-(\alpha-\varepsilon)\tilde{R}_t}\vee\mathrm{e}^{-(\alpha+\varepsilon)\tilde{R}_t}\right)\leq b\left(\mathrm{e}^{\varphi_{\tilde{R}}(\alpha-\varepsilon)t}+\mathrm{e}^{\varphi_{\tilde{R}}(\beta+\varepsilon)t}\right),$$

the right-hand side of which is integrable with respect to $\mathrm{d}t$ over $[0,\infty)$. Therefore, by the dominated convergence theorem,

$$\lim_{x\to\infty}\frac{\psi(x,T)}{\mathbb{P}\left(X>x\right)}=\lambda\int_0^T\lim_{x\to\infty}\frac{\mathbb{P}\left(X\mathrm{e}^{-\tilde{R}_t}>x\right)}{\mathbb{P}\left(X>x\right)}\mathrm{d}t=\lambda\int_0^T\mathbb{E}\mathrm{e}^{-\alpha\tilde{R}_t}\mathrm{d}t=\lambda\frac{1-\mathrm{e}^{\varphi_{\tilde{R}}(\alpha)T}}{-\varphi_{\tilde{R}}(\alpha)},$$

where the second step is due to the well-known Breiman's (1965) theorem.

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