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# ASYMPTOTIC SOLUTIONS OF RESONANT NONLINEAR SINGULARLY PERTURBED PROBLEMS IN THE CASE OF INTERSECTING EIGENVALUES OF THE LIMIT OPERATOR

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***Abstract.*** Lomov's regularization method is generalized to resonant, weakly nonlinear, singularly perturbed systems in the case of intersecting roots of the characteristic equation of the limit operator. For constructing asymptotic solutions, the regularization of the original problem by using normal forms developed by the authors is performed. In the absence of resonance, the regularizing normal form is linear, whereas in the presence of resonances, it is nonlinear. In this paper, the resonant case of a weakly nonlinear problem is considered. By using an algorithm of normal forms, we construct an asymptotic solution of any order (with respect to a parameter) and justify this algorithm.

***Keywords and phrases*:** singular perturbation, normal form, regularization, asymptotic convergence.

***AMS Subject Classification*:** 34E05, 34E20

**1. Introduction.** As was proposed by V. F. Safonov and A. A. Bobodzhsnov (see, e.g., [9]), Lomov's regularization method (see [5, 6]) can be generalized to resonant, weakly nonlinear, singularly perturbed systems in the case of intersecting roots of the characteristic equation of the limit operator $A(t)$ by regularizing the original problem based on normal forms. It is well known that if all points of the spectrum $\sigma(A(t)) = \{\lambda_j(t),\ j = \overline{1,n}\}$ are distinct for all values of the independent variable $t$ (i.e., in the case of stable spectrum of the limit operator), the problem can be regularized by integrals of points of the spectrum (see [5, p. 39-40]). In the case of intersecting roots of the characteristic equation of the operator $A(t)$, singularities of new types appear that cannot be described in terms of spectra. To examine such singularities, one must revise approaches to regularization. *In the case of stable spectrum* $\sigma(A(t))$ of weakly nonlinear systems, so-called identity and nonidentity resonances can be regularized by *normal forms* (see [9, Chap. 3]). The corresponding algorithm of constructing asymptotic solutions is called the *algorithm of normal forms*. This algorithm is a generalization of Lomov's regularization algorithm; it is based on regularization of singularly perturbed problems by a vector of regularizing variables satisfying a certain normal differential form. It turned out that this approach can also be applied to nonlinear, singularly perturbed problems in the case of intersecting roots of the characteristic equation of the operator $A(t)$. If resonances are absent, the regularizing normal form is linear (see [1]); in the presence of resonances it is nonlinear. In this paper, we consider the resonance case of the weakly nonlinear original problem. Using the algorithm of normal forms, we construct and justify an asymptotic solution of an arbitrary order with respect to the parameter $\varepsilon$.

**2. Notation and the statement of the problem.** Throughout the paper, we use the following notation. Row vectors are denoted by parentheses (e.g, $b = (b_1, \dots, b_r)$) and column vectors by curly braces (e.g., $a = \{a_1, \dots, a_r\}$), so that $a^T = (a_1, \dots, a_r)$). The asterisk means simultaneous transposition and conjugation: $b^* = (\bar{b}^T)$. For a multi-index $k = (k_1, \dots, k_n)$, we set $|k| = k_1 + \dots + k_n$. We denote by $\lambda(t)$ the row vector $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))$ and by $e_j$ the $j$th ort in the space $\mathbb{C}^n$ of complex-valued $n$-dimensional columns. Further, introduce the notation

$$(m, \lambda(t)) \equiv m_1\lambda_1(t) + \dots + m_n\lambda_n(t), \quad (m, u) \equiv m_1 u_1 + \dots + m_n u_n.$$

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The scalar product in the complex space $\mathbb{C}^n$ of $n$-dimensional column vectors (or row vectors) is introduced as usually: for any vectors $y = \{y_1, \dots, y_n\}$ and $z = \{z_1, \dots, z_n\}$ of the space $\mathbb{C}^n$, we set, by definition,

$$(y, z)_{\mathbb{C}^n} = \sum_{j=1}^{n} y_j \bar{z}_j;$$

in the sequel, we omit the subscript $\mathbb{C}^n$. Finally, we denote a $\lambda_j(t)$-eigenvector of the matrix $A(t)$ by $\varphi_j(t)$ (recall that $A(t)\varphi_j(t) \equiv \lambda_j(t)\varphi_j(t)))$ and the $i$th column of the matrix $[\Phi^*(t)]^{-1}$ by $\chi_i(t)$; here $\Phi(t) \equiv (\varphi_1(t), \dots, \varphi_n(t))$. Then $\chi_i(t)$ is a $\bar{\lambda}_i(t)$-eigenvector of the matrix $A^*(t)$ and $(\varphi_j(t), \chi_i(t)) = \delta_{ji}$, where $\delta_{ji}$, $i, j = \overline{1, n}$, is the Kronecker delta.

We consider the following nonlinear, singular perturbed problem:

$$\varepsilon \frac{dy}{dt} = A(t)y + \varepsilon f(t, y) + h(t), \quad y(0, \varepsilon) = y^0, \quad t \in [0, T], \tag{1}$$

where $y = \{y_1, \dots, y_n\}$ is an unknown vector-valued function, $A(t)$ is a given $(n \times n)$-matrix, $f(t, y) = \{f_1, \dots, f_n\}$ and $h(t) = \{h_1, \dots, h_n\}$ are given vector-valued functions, $y^0 = \{y_1^0, \dots, y_n^0\} \in \mathbb{C}^n$ is a given constant vector, and $\varepsilon$ is a small positive parameter. We assume that the following conditions are fulfilled:

(I) $h(t) \in C^\infty([0, T], \mathbb{C}^n)$, $A(t) \in C^\infty([0, T], \mathbb{C}^{n \times n})$, and $f(y, t)$ is a polynomial with respect to $y$, i.e.,

$$f(y, t) = \sum_{0 \le |m| \le N} f^{(m)}(t) y^m \equiv \sum_{m_1 + \dots + m_n = 0}^{N} f^{(m_1, \dots, m_n)}(t) y_1^{m_1} \dots y_n^{m_n},$$

with the coefficients $f^{(m)}(t) \in C^\infty([0, T], \mathbb{C}^n)$, $|m| = \overline{0, N}$, $N < \infty$;

(II) for all $t \in [0, T]$, the spectrum $\{\lambda_j(t)\}$ of the matrix $A(t)$ satisfies the following conditions:
  (a) $\lambda_1(t) - \lambda_2(t) \equiv tk(t)$, $k(t) \neq 0$, $\lambda_j(t) \neq 0$, $j = \overline{1, n}$;
  (b) $\lambda_i(t) \neq \lambda_j(t)$, $i \neq j$, $i, j = \overline{3, n}$; $\lambda_j(t) \neq \lambda_1(t)$, $\lambda_j(t) \neq \lambda_2(t)$, $j = \overline{3, n}$;
  (c) there exists a matrix $\Phi(t) \in C^\infty([0, T], \mathbb{C}^{n \times n})$ such that

$$\Phi^{-1}(t) A(t) \Phi(t) = \Lambda(t) \equiv \operatorname{diag}(\lambda_1(t), \dots, \lambda_n(t));$$

  (d) $\operatorname{Re} \lambda_j(t) \le 0$, $j = \overline{1, n}$;

(III) the spectrum $\sigma(A(t)) \equiv \{\lambda_j(t)\}$ is such that the equations

$$(m, \lambda(t)) \equiv m_1 \lambda_1(t) + \ldots + m_n \lambda_n(t) = \lambda_j(t), \quad |m| \ge 2, \tag{$*$}$$

for the unknown multi-index $m$ have solutions for all $t \in [0, T]$ and only for $j \in \{3, \dots, n\}$; moreover, the solutions $m$ of Eq. $(*)$ are independent of $t$ and have the following form:

$$\{m\} \equiv \{(m_1, \dots, m_n) = (0, 0, m_3, \dots, m_n)\}, \quad |m| = m_3 + \ldots + m_n \ge 2;$$

this means that the eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ do not involve in the resonance relations $(*)$.

The conditions (IIa) and (IIb) imply that only two points of the spectrum, namely, $\lambda_1(t)$ and $\lambda_2(t)$, mutually intersect at $t = 0$ and remain distinct for $t \in (0, T]$. Other points of the spectrum $\{\lambda_j(t)\}$ do not intersect anywhere. Under the conditions specified, one must construct a regularized asymptotic solution (see [5, 6]) of the problem (1) as $\varepsilon \to +0$.

For this problem, we develop an algorithm based on regularizing the system (1) by normal forms. In the case of stable spectrum $\sigma(A(t)) \equiv \{\lambda_j(t)\}$, the regularized normal form is linear and diagonal (see [9]). However, in the case where roots of the characteristic equation intersect (this case is determined by the conditions (II)), the normal form is not diagonal even in the case where resonances are absent (see [1]): it consists of two cells that form a quasi-diagonal matrix $\{A_1(\varepsilon), \Lambda_0\}$, where $\Lambda_0 = \operatorname{diag}(\lambda_3(t), \dots, \lambda_n(t))$ and $A_1(\varepsilon)$ is a second-order matrix polynomial with respect to $\varepsilon$. In the case of a resonance determined by the conditions (III), the normal form, in addition to a linear part, contains also a nonlinear part, which significantly complicates the development and justification of the algorithm. Note that in the linear case (i.e., for $f(t,y)\equiv 0$), the problem (1) was solved in [3] by a method that slightly differs from the method of normal forms (see also [4]).

**3. Regularization of the problem** (1)**. Solvability of the first iterative problem.** Assume that the conditions (I)–(III) are fulfilled. Since the spectrum $\sigma(A(t))$ of the matrix $A(t)$ is resonant (see the condition (III)), we regularize the problem (1) by using the vector $u=\{u_1,\dots,u_n\}$ of regularizing variables satisfying the nonlinear normal form

$$\varepsilon\frac{du}{dt}=\Lambda(t)u+\sum_{k=1}^{r+1}\varepsilon^k\Big(\mu_2^{(k)}e_1u_2+\mu_1^{(k)}e_2u_1\Big)+\sum_{k=1}^{r+1}\varepsilon^k\sum_{i=3}^{n}\sum_{m^i\in\Gamma_i}g_k^{(m^i)}(t)e_iu^{m^i},\quad u(0,\varepsilon)=\bar{1},\tag{2}$$

where $\Lambda(t)=\operatorname{diag}\big(\lambda_1(t),\dots,\lambda_n(t)\big)$, $\bar{1}=\{1,\dots,1\}$, $e_j=\big\{0,\dots 0,\underset{(j)}{1},0,\dots,0\big\}$ is the $j$th ort in $\mathbb{C}^n$,

$$\Gamma_i=\Big\{m^i\equiv\big(m_1^i,\dots,m_n^i\big):\ \big(m^i,\lambda(t)\big)\equiv\lambda_i(t),\ |m^i|\geq 2\ \forall t\in[0,T]\Big\},\quad i=\overline{3,n},$$

and the constants $\mu_j^{(k)}$ and the functions $g_k^{(m^i)}(t)$ will be specified in the process of constructing an asymptotic solution of the problem (1). Note that the *regularizing normal form* (2) (of order $r+1$) is not diagonal even in the linear case ($f(y,t)\equiv 0$) due to the instability of the spectrum $\{\lambda_j(t)\}$ (see the condition (IIa)) and the presence of resonances (the condition (III)). Instead of the problem (1), we consider the "extended problem"

$$\varepsilon\frac{\partial\tilde{y}}{\partial t}+\frac{\partial\tilde{y}}{\partial u}\Bigg[\Lambda(t)u+\sum_{k=1}^{r+1}\varepsilon^k\Big(\mu_2^{(k)}e_1u_2+\mu_1^{(k)}e_2u_1\Big)+\sum_{k=1}^{r+1}\varepsilon^k\sum_{i=3}^{n}\sum_{m^i\in\Gamma_i}g_k^{(m^i)}(t)e_iu^{m^i}\Bigg]$$
$$-A(t)\tilde{y}-\varepsilon f(y,t)=h(t),\quad \tilde{y}(t,u,\varepsilon)\big|_{t=0,\ u=\bar{1}}=y^0\tag{3}$$

for the function $\tilde{y}=\tilde{y}(t,u,\varepsilon)$ of the variables $t$, $u=\big(u_1,\dots,u_n\big)$, and $\varepsilon$. Clearly, if $\tilde{y}=\tilde{y}(t,u,\varepsilon)$ is a solution of the problem (3), then its restriction $y(t,\varepsilon)=\tilde{y}\big(t,u(t,\varepsilon),\varepsilon\big)$ to the solution $u=u(t,\varepsilon)$ of the regularizing normal form (2) is an exact solution of the original problem (1). Since the order of the system of differential equations (3) does not decrease for $\varepsilon=0$, we can write its solution as the series

$$\tilde{y}(t,u,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k y_k(t,u)\tag{4}$$

with respect to nonnegative powers of the parameter $\varepsilon$. Substituting the series (4) into (3) and equating the coefficients of the same powers of $\varepsilon$, we obtain the following iterative problems:

$$\begin{cases}\mathscr{L}y_0(t,u)\equiv\dfrac{\partial y_0}{\partial u}\Lambda(t)u-A(t)y_0=h(t),\\ y_0(0,\bar{1})=y^0;\end{cases}\tag{$4_0$}$$

$$\begin{cases}\mathscr{L}y_1(t,u)=-\dfrac{\partial y_0}{\partial t}-\dfrac{\partial y_0}{\partial u}\Big(\mu_2^{(1)}e_1u_2+\mu_1^{(1)}e_2u_1\Big)+\dfrac{\partial y_0}{\partial u}\displaystyle\sum_{i=3}^{n}\sum_{m^i\in\Gamma_i}g_1^{(m^i)}(t)e_iu^{m^i}+f(y_0,t),\\ y(0,\bar{1})=0;\end{cases}\tag{$4_1$}$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$
\begin{cases}
\mathscr{L}y_{k+1}(t,u) = -\dfrac{\partial y_k}{\partial t} - \dfrac{\partial y_0}{\partial u}\Big(\mu_2^{(k+1)}e_1u_2 + \mu_1^{(k+1)}e_2u_1\Big) - \dfrac{\partial y_0}{\partial u}\displaystyle\sum_{i=1}^{n}\sum_{m^i\in\Gamma_i} g_{k+1}^{(m^i)}(t)e_iu^{m^i} \\
\qquad - \displaystyle\sum_{s=1}^{k}\dfrac{\partial y_s}{\partial u}\Big(\mu_2^{(k+1-s)}e_1u_2 + \mu_1^{(k+1-s)}e_2u_1\Big) - \displaystyle\sum_{s=1}^{k}\dfrac{\partial y_s}{\partial u}\sum_{i=1}^{n}\sum_{m^i\in\Gamma_i} g_{k+1-s}^{(m^i)}(t)e_iu^{m^i} \\
\qquad + P_k\big(t, y_0, \dots, y_{k-1}\big), \\
y_{k+1}(0,\bar{1}) = 0, \quad k \geq 1,
\end{cases} \tag{$4_k$}
$$

where $P_k\big(t, y_0, \dots, y_k\big)$ is a certain polynomial of $y_1, \dots, y_{k-1}$ with coefficients depending on partial derivatives of the function $f(t,y)$ at the point $y = y_0(t,u)$; moreover, $P_k(t, y_0, \dots, y_k)$ is linear with respect to the last argument $y_k$.

For description of the solvability theory of iterative problems, we introduce the following notation.

**Definition.** A monomial $z^{(m^j)}(t)u^{m^j} \equiv z^{(m_1^j,\dots,m_n^j)}(t)u_1^{m_1^j}\cdots u_n^{m_n^j}$ is called the *$\lambda_j$-resonant monomial* if $|m^j| \geq 2$ and the equality $(m^j, \lambda(t)) = \lambda_j(t)$ holds for all $t \in [0,T]$. This monomial is said to be *orthogonalized* if for all $t \in [0,T]$, the following identity holds:

$$
\Big(z^{(m^j)}(t), \chi_j(t)\Big) \equiv 0;
$$

here $\chi_j(t)$ is a $\bar{\lambda}_j$-eigenvector of the matrix $A^*(t)$.

We define a solution of the problem $(4_k)$ in the class $U$ of vector-valued functions of the form

$$
\begin{gathered}
z(t,u) = z_0(t) + \sum_{j=0}^{n} z_j(t)u_j + \sum_{2\leq|m|\leq N_z} z^{(m)}(t)u^m, \\
z_0(t), z_j(t), z^{(m)}(t) \in C^\infty\big([0,T], \mathbb{C}^n\big), 2 \leq |m| \leq N_z < \infty,
\end{gathered} \tag{5}
$$

with orthogonalized resonant monomials $z^{(m^i)}(t)u^{m^i}$, $i = \overline{1,n}$; note that the degree $N_z$ of the polynomial $z(t,u)$ is not fixed; it depends on this polynomial. First, we examine its solvability in the class $U$ of the first iterative problem $(4_0)$. The following assertion holds.

**Theorem 1.** *Let the conditions* (I), (IIa)–(IIc), *and* (III) *be fulfilled. Then the system* $(4_0)$ *has a solution in the class* $U$*, which can be represented in the form*

$$
y_0(t,u) = \Phi(t)\mathfrak{A}(t)u + y_0^{(0)}(t) \equiv \sum_{j=1}^{n}\alpha_j(t)\varphi_j(t)u_j + y_0^{(0)}(t), \tag{6}
$$

*where* $\mathfrak{A}(t) = \operatorname{diag}\big(\alpha_1(t), \dots, \alpha_n(t)\big)$, $\Phi(t) \equiv \big(\varphi_1(t), \dots, \varphi_n(t)\big)$ *is the matrix constructed from the eigenvectors* $\varphi_j(t)$ *of the operator* $A(t)$, $y_0^{(0)}(t) \equiv -A^{-1}(t)h(t)$, *and* $\alpha_i(t) \in C^\infty\big([0,T], \mathbb{C}^1\big)$ *are arbitrary scalar functions.*

*Proof.* We search for a solution of the system $(4_0)$ in the form

$$
y_0(t,u) = y_0^{(0)}(t) + \sum_{j=1}^{n} y_0^{e_j}(t)u_i + \sum_{2\leq|m|\leq N_{y_0}} y_0^{(m)}(t)u^m; \tag{$6_0$}
$$

then we obtain the following equations for the coefficients of this sum :

$$
-A(t)y_0^{(0)}(t) = h(t), \tag{$7_0$}
$$

$$
\Big[\lambda_i(t)I - A(t)\Big]y_i^{e_i}(t) = 0, \quad i = \overline{3,n}, \tag{$7_i$}
$$

$$
\Big[\lambda_1(t)I - A(t)\Big]y_1^{e_1}(t) = 0, \tag{$7_1$}
$$

$$\Big[\lambda_2(t)I - A(t)\Big] y_2^{e_2}(t) = 0, \tag{$7_2$}$$

$$\Big[\big(m^i, \lambda(t)\big)I - A(t)\Big] y_0^{(m^i)}(t) = 0 \quad \forall m^i \in \Gamma_i, \quad 2 \le |m| \le N_{y_0}, \quad i = \overline{3,n}, \tag{$7_{m^i}$}$$

$$\Big[b\big(m, \lambda(t)\big)I - A(t)\Big] y_0^{(m)}(t) = 0 \quad \forall m \notin \bigcup_{i=1}^{n} \Gamma_i, \quad 2 \le |m| \le N_{y_0}. \tag{$7_m$}$$

Since $\det A(t) \ne 0$ and $\det\big[(m,\lambda(t))I - A(t)\big] \ne 0$ for all $t \in [0,T]$, $m \notin \bigcup_{i=1}^{n} \Gamma_i$, $2 \le |m| \le N_{y_0}$, each of the systems $(7_0)$ and $(7_m)$ possesses a unique solution $y_0^{(0)}(t) = -A^{-1}(t)h(t)$ (respectively, $y_0^{(m)}(t) \equiv 0$) of the class $C^\infty\big([0,T],\mathbb{C}^n\big)$. Consider the system $(7_{m^i})$ for fixed $i$. Since $m^i \in \Gamma_i$, we conclude that $(m^i,\lambda(t)) - \lambda_i(t) \equiv 0$ for all $t \in [0,T]$. Therefore, the system $(7_{m^i})$ has a solution in the form of a vector-valued function $\beta^i(t)\varphi_i(t)$, where $\beta^i(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$ are arbitrary scalar functions, $i = \overline{1,n}$. However, all resonant monomials in the solution $(6_0)$ must be orthogonalized, hence

$$\big(\beta^i(t)\varphi_i(t), \chi_i(t)\big) \equiv 0 \quad \Longleftrightarrow \quad \beta^i(t)\big(\varphi_i(t), \chi_i(t)\big) \equiv 0 \quad \Longleftrightarrow \quad \beta^i(t) \equiv 0$$

(since the systems $\{\varphi_j(t)\}$ and $\{\chi_i(t)\}$ are bi-orthonormed). Therefore, for all $t \in [0,T]$ we have $\beta^i(t)\varphi_i(t) \equiv 0$ and hence $y_0^{(m^i)}(t) \equiv 0$. Thus, the sum $(6_0)$ does not contain nonlinear terms, i.e., the solution of the system $(6_0)$ has the form

$$y_0(t,u) = y_0^{(0)}(t) + \sum_{j=1}^{n} y_0^{e_j}(t)u_i,$$

where $y_0^{(0)}(t) = -A^{-1}(t)h(t)$ and the function $y_0^{e_j}(t)$ have not been found. Now we calculate these functions.

For fixed $i \in \{3,\dots,n\}$, the systems $(7_i)$ have (see [8, p. 84-85]) infinite set of solutions that can be represented in the form $y_i^{(0)}(t) = \alpha_i(t)\varphi_i(t)$, where $\alpha_i(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$ are arbitrary scalar functions, $i = \overline{3,n}$. Now we consider the system $(7_1)$. We search for its solution in the form $y_1^{(0)}(t) = \Phi(t)\xi$, where $\xi = \{\xi_1,\dots,\xi_n\}$; then for the components of the vector $\xi$, we obtain the following equations:

$$0 \cdot \xi_1 = 0, \qquad \big(\lambda_1(t) - \lambda_2(t)\big)\xi_2 = 0, \qquad \big(\lambda_1(t) - \lambda_j(t)\big)\xi_j = 0, \quad j = \overline{3,n}. \tag{8}$$

The first equation in (8) has a solution in the form of an arbitrary function $\xi_1 = \alpha_1(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$; the last equations in (8) have unique trivial solutions $\xi_j \equiv 0$, $j = \overline{3,n}$. Since $\lambda_1(0) - \lambda_2(0) = 0$ and $\lambda_1(t) - \lambda_2(t) \ne 0$ for all $t \in (0,T]$, the second equation in (8) has solutions that can be represented in the form

$$\xi_2 = \xi_2(t) = \begin{cases} \gamma, & t = 0, \\ 0, & t \in (0,T], \end{cases}$$

where $\gamma$ is an arbitrary constant. Thus, the formal solution of the system $(7_1)$ has the form

$$y_1^{(0)}(t) = \alpha_1(t)\varphi_1(t) + \xi_2(t)\varphi_2(t).$$

Since we search for a solution of the system $(4_0)$ in the space $U$, we have

$$y_1^{(0)}(t) = \alpha_1(t)\varphi_1(t) + \xi_2(t)\varphi_2(t) \in C^\infty\big([0,T],\mathbb{C}^n\big).$$

Multiplying this equality scalarly by the vector $\chi_2(t)$, we obtain $\xi_2(t) = \big(y_1^{(0)}(t), \chi_2(t)\big)$; therefore, $\xi_2(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$. In particular, the function $\xi_2(t)$ must be continuous at the point $t = 0$, i.e., $\lim\limits_{t\to+0} \xi_2(t) = \xi_2(0)$. This implies $\gamma = 0$ and hence $\xi_2(t) \equiv 0$ for all $t \in [0,T]$ and the solution of the system $(7_1)$ has the from $y_1^{(0)}(t) = \alpha_1(t)\varphi_1(t)$. Similarly we can prove that $y_2^{(0)}(t) = \alpha_2(t)\varphi_2(t)$. Therefore, the solution of the system $(4_0)$ has the form (6). The theorem is proved. $\square$

Now we introduce the following notation: if $w(t,u)$ is a polynomial

$$w(t,u) = \sum_{|m|=0}^{N_w} w^{(m)}(t)u^m,$$

then we denote by $w^{(k)}(t,u)$ the sum of its terms such that $|m| = k$, i.e.,

$$w^{(k)}(t,u) = \sum_{|m|=k} w^{(m)}(t)u^m, \quad k = 0, 1, 2, \ldots, N_w,$$

and by $U^{(k)} \subset U$ the subspace of the class $U$ consisting of all such sums (we assume that the element $0 \equiv \sum\limits_{|m|=k} 0 \cdot u^m$ belongs to the space $U^{(k)}$). For each $t \in [0,T]$, we introduce the scalar product

$$\left\langle w^{(k)}(t,u), z^{(k)}(t,u) \right\rangle \equiv \left\langle \sum_{|m|=k} w^{(k)}(t)u^m, z^{(k)}(t)u^m \right\rangle$$
$$\stackrel{\text{def}}{=} \sum_{|m|=k} \left(w^{(k)}(t), z^{(k)}(t)\right) = \sum_{|m|=k} \left(w^{(k)}(t)\right)^T \cdot \overline{z^{(k)}(t)}$$

in the space $U^{(k)}$, where $(\cdot,\cdot)$ is the ordinary scalar product (for each $t \in [0,T]$) in the complex space $\mathbb{C}^n$. In $U^{(1)}$, consider the operator

$$\mathscr{L} \equiv \frac{\partial}{\partial u}\Lambda(t)u - A(t).$$

The conjugate operator has the form

$$\mathscr{L}^* \equiv \frac{\partial}{\partial u}\bar{\Lambda}(t)u - A^*(t);$$

the vector-valued functions $\nu_j(t,u) = \chi_j(t)u_j$, $j = \overline{1,n}$, form a basis of its kernel. We assume that the solution (6) of the system $(4_0)$ satisfy the initial condition $y_0(0,\bar{1}) = y^0$; then

$$\alpha_1(0)\varphi_1(0) + \ldots + \alpha_n(0)\varphi_n(0) = A^{-1}(0)h(0) + y^0,$$

which allows one to find $\alpha_j(0)$:

$$\alpha_j(0) = (A^{-1}(0)h(0) + y^0, \chi_j(0)), \quad j = \overline{1,n}.$$

Substituting the function (6) into $f(t, y_0)$, we obtain the polynomial (with respect to $u$)

$$f\big(t, y_0(t,u)\big) = f\big(t, y_0^{(0)}(t)\big) + \frac{\partial f(t, y_0^{(0)}(t))}{\partial y}\Phi(t)\,\mathrm{diag}\,\big(\alpha_1(t), \ldots, \alpha_n(t)\big)u + \ldots,$$

where the dots mean terms with $|m| \geq 2$ with respect to $u$; therefore,

$$f^{(1)}\big(t, y_0(t,u)\big) = \frac{\partial f\big(t, y_0^{(0)}(t)\big)}{\partial y}\Phi(t)\,\mathrm{diag}\,\big(\alpha_1(t), \ldots, \alpha_n(t)\big)u \equiv \sum_{j=1}^{n} \frac{\partial f\big(t, y_0^{(0)}(t)\big)}{\partial y}\varphi_j(t)\alpha_j(t)u_j.$$

We prove below (see Theorem 3, in which $H(t,u) \equiv h(t)$) that the system $(4_1)$ is solvable in the space $U$ if and only if the following identities hold:

$$\left\langle -\frac{\partial y_0^{(1)}}{\partial t} + f^{(1)}\big(t, y_0(t,u)\big),\ \chi_j(t)u_j \right\rangle \equiv 0 \quad \forall t \in [0,T], \quad j = \overline{1,n}.$$

This is an additional restriction for the solution of the problem $(4_0)$. Substituting the functions (6) and $f^{(1)}\big(t, y_0(t,u)\big)$ into this formula, we arrive at the identity

$$\left\langle -\sum_{j=1}^{n}\left[\frac{d}{dt}\big(\alpha_j(t)\varphi_j(t)\big)u_j + \frac{\partial f\big(t, y_0^{(0)}(t)\big)}{\partial y}\varphi_j(t)\alpha_j(t)u_j\right],\ \chi_j(t)u_j\right\rangle \equiv 0,$$

or

$$-\dot{\alpha}_j(t) + \left(\frac{\partial f\big(t, y_0^{(0)}(t)\big)}{\partial y}\varphi_j(t) - \dot{\varphi}_j(t),\ \chi_j(t)\right)\alpha_j(t) \equiv 0, \quad j=\overline{1,n}.$$

These identities are satisfied by a unique collection of functions $\alpha_j(t)$ with the initial conditions

$$\alpha_j(0) = \big(A^{-1}(0)h(0) + y^0,\ \chi_j(0)\big), \quad j=\overline{1,n},$$

namely,

$$\alpha_j(t) = \Big(A^{-1}(0)h(0) + y^0,\ \chi_j(0)\Big) \times \exp\left\{\int_0^t \left(\frac{\partial f\big(\theta, y_0^{(0)}(\theta)\big)}{\partial y}\varphi_j(\theta) - \dot{\varphi}_j(\theta),\ \chi_j(\theta)\right) d\theta\right\}, \quad j=\overline{1,n}. \tag{9}$$

Therefore, the solution (6) of the problem $(4_0)$ with the additional restriction is uniquely defined in the space $U$. We arrive at the following assertion.

**Theorem 2.** *Let the condition* (I), (IIa)–(IIc), *and* (III) *be fulfilled. Then the system* $(4_0)$ *with the additional conditions*

$$y_0(0,\bar{1}) = y^0, \quad \left\langle -\frac{\partial y_0^{(1)}}{\partial t} + f^{(1)}\big(t, y_0(t,u)\big),\ \nu_j(t,u)\right\rangle \equiv 0 \quad \forall t\in[0,T], \quad j=\overline{1,n}, \tag{10}$$

*has a unique solution in the class $U$, which can be represented in the form* (6), *where the scalar functions $\alpha_j(t)$ have the form* (9).

**4. Solvability of the iterative problems $(4_k)$ for $k\ge 1$.** We solve the iterative systems $(4_k)$ consecutively. Each of them can be represented in the form

$$\mathscr{L}y(t,u) = -\frac{\partial y_0}{\partial u}\Big(\mu_2 e_1 u_2 + \mu_1 e_2 u_1\Big) - \frac{\partial y_0}{\partial u}\sum_{i=3}^{n}\sum_{m^i\in\Gamma_i} g^{(m^i)}(t)e_i u^{m^i} + H(t,u), \tag{11}$$

where $y_0 = y_0(t,u)$ is the solution of the first iterative problem $(4_0)$, $\mu_1$ and $\mu_2$ are unknown numbers, and

$$H(t,u) = H_0(t) + \sum_{j=1}^{n} H_j(t)u_j + \sum_{|m|=2}^{N_H} H^{(m)}(t)$$

is a vector-valued function, which may involve nonorthogonalized resonant monomials (for this reason, in general, the right-hand side of the system (11) does not belong to $U$). The first thing to do is to answer the question: Can the right-hand side of the system (11) be included into the space $U$ by an appropriate choice of the function $g^{(m^i)}(t)$?

**Lemma.** *Let the conditions* (I), (IIa)–(IIc), *and* (III) *and the conditions* (10) *of Theorem* 2 *be fulfilled. Moreover, let $\alpha_j(0) = \big(y^0 - y_0^{(0)}(0), \chi_j(0)\big) \neq 0$, $j=\overline{1,n}$. Then there exists a unique collection of the $g^{(m^i)}(t) \in C^\infty\big([0,T], \mathbb{C}^1\big)$ satisfying the condition*

$$h(t,u) \equiv -\frac{\partial y_0}{\partial u}\sum_{i=3}^{n}\sum_{m^i\in\Gamma_i} g^{(m^i)}(t)e_i u^{m^i} + H(t,u) \in U. \tag{11a}$$

*Proof.* Taking into account the formula (6), we write the vector-valued function $h(t,u)$ from the right-hand side of the system (11), explicitly extracting resonant monomials:

$$h(t,u) = -\Phi(t)\mathfrak{A}(t)\sum_{i=3}^{n}\sum_{m^i\in\Gamma_i} g^{(m^i)}(t)e_i u^{m^i} + \sum_{|m|\geq 0} H^{(m)}(t)u^m$$
$$\equiv \sum_{i=3}^{n}\sum_{m^i\in\Gamma_i}\Big[-g^{(m^i)}(t)\alpha_i(t)\varphi_i(t) + H^{(m^i)}(t)\Big]u^{m^i} + \sum_{\substack{|m|\geq 0:\\ m\notin \bigcup_{i=3}^{n}\Gamma_i}} H^{(m)}(t)u^m.$$

The condition (11a) means that all resonant multi-indexes $m^i \in \Gamma_i$ satisfy the identities

$$\Big(-g^{(m^i)}(t)\alpha_i(t)\varphi_i(t) + H^{(m^i)}(t),\ \chi_j(t)\Big) \equiv 0 \quad \forall t\in[0,T],$$

which allow one to find the functions

$$g^{(m^i)}(t) = \alpha_i^{-1}(t)\Big(H^{(m^i)}(t),\ \chi_j(t)\Big), \quad m^i\in\Gamma_i, \quad i=\overline{3,n}. \tag{11b}$$

Thus, choosing the functions $g^{(m^i)}(t)$ in the form (11b), we embed the vector-valued function $h(t,u)$ (and hence the right-hand side of the system (11)) into the space $U$. The lemma is proved. □

**Theorem 3.** *Let the conditions* (I), (IIa)–(IIc), *and* (III), *and also the condition* (10) *of Theorem* 2 *be fulfilled. Moreover, let* $\alpha_j(0) = \big(y^0 - y_0^{(0)}(0), \chi_j(0)\big) \neq 0$, $j=\overline{1,n}$, *and the scalar functions* $g^{(m^i)}(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$ *have the form* (11b). *Then there exist unique constants* $\mu_1$ *and* $\mu_2$ *such that the system* (11) *is solvable in the space* $U$ *if and only if*

$$\langle H(t,u), \nu_j(t,u)\rangle \equiv 0 \quad \forall t\in[0,T], \quad j=\overline{1,n}. \tag{11*}$$

*Proof.* We write the right-hand side of the system (11) as follows:

$$-\frac{\partial y_0}{\partial u}\big(\mu_2 e_1 u_2 + \mu_1 e_2 u_1\big) - \frac{\partial y_0}{\partial u}\sum_{i=3}^{n}\sum_{m^i\in\Gamma_i} g^{(m^i)}(t)e_i u^{m^i} + H(t,u)$$
$$\equiv -\Phi(t)\mathfrak{A}(t)\big(\mu_2 e_1 u_2 + \mu_1 e_2 u_1\big) + \sum_{i=3}^{n}\sum_{m^i\in\Gamma_i}\Big[-g^{(m^i)}(t)\alpha_i(t)\varphi_i(t) + H^{(m^i)}(t)\Big]u^{m^i}$$
$$+ H_0(t) + \sum_{j=1}^{n} H_j(t)u_j + \sum_{\substack{|m|\geq 2:\ m\notin\Gamma_i,\\ i=\overline{3,n}}} H^{(m)}(t)u^m = -\alpha_1(t)\mu_2\varphi_1(t)u_2 - \alpha_2(t)\mu_1\varphi_2(t)u_1$$
$$+ \sum_{i=3}^{n}\sum_{m^i\in\Gamma_i}\Big[-g^{(m^i)}(t)\alpha_i(t)\varphi_i(t) + H^{(m^i)}(t)\Big]u^{m^i} + H_0(t) + \sum_{j=1}^{n} H_j(t)u_j \sum_{\substack{|m|\geq 2:\ m\notin\Gamma_i,\\ i=\overline{3,n}}} H^{(m)}(t)u^m.$$

We search for a solution of the system (11) in the following form:

$$y(t,u) = y_0(t) + \sum_{j=0}^{n} y_j(t)u_j + \sum_{2\leq|m|\leq N_y} y^{(m)}(t)u^m. \tag{12}$$

Substituting this sum into the system (11) and equating the coefficients of the same $u^m$, we obtain the following systems:

$$-A(t)y_0(t) = H_0(t), \tag{$12_0$}$$

$$\Big[\lambda_i(t)I - A(t)\Big]y_i(t) = H_i(t), \quad i=\overline{3,n}, \tag{$12_i$}$$

$$\Big[\lambda_1(t)I - A(t)\Big]y_1(t) = -\alpha_2(t)\mu_1\varphi_2(t) + H_1(t), \tag{12$_1$}$$

$$\Big[\lambda_2(t)I - A(t)\Big]y_2(t) = -\alpha_1(t)\mu_2\varphi_1(t) + H_2(t), \tag{12$_2$}$$

$$\Big[\big(m^i,\lambda(t)\big)I - A(t)\Big]z^{(m^i)}(t) = -g^{(m^i)}(t)\alpha_i(t)\varphi_i(t) + H^{(m^i)}(t) \quad \forall m^i \in \Gamma_i, \quad i = \overline{3,n}, \tag{12$_{m^i}$}$$

$$\Big[\big(m,\lambda(t)\big)I - A(t)\Big]z^{(m)}(t) = H^{(m)}(t) \quad \forall m \notin \bigcup_{i=3}^{n}\Gamma_i, \quad |m| \geq 2. \tag{12$_m$}$$

The system (12$_0$) has a unique solution $y_0(t) = -A^{-1}(t)H_0(t) \in C^\infty\Big([0,T],\mathbb{C}^n\Big)$. Due to the condition (III), this is also valid for the system (12$_m$). The solutions have the form $y^{(m)}(t) = \big[(m,\lambda(t))I - A(t)\big]^{-1}H^{(m)}(t)$, $2 \leq |m| \leq N_y$. The system (12$_i$) is solvable in the space $C^\infty\big([0,T],\mathbb{C}^n\big)$ if and only if the following identities hold:

$$\big(H_i(t),\chi_i(t)\big) \equiv 0 \quad \Longleftrightarrow \quad \langle H(t,u),\nu_i(t,u)\rangle \equiv 0, \quad i = \overline{3,n}. \tag{12$^*$}$$

The solutions of the system (12$_i$) (see [8, p. 84-85]) can we written as follows:

$$y_i(t) = \beta_i(t)\varphi_i(t) + \sum_{\substack{s=1,\\ s\neq i}}^{n}\frac{\big(H_i(t),\chi_s(t)\big)}{\lambda_i(t)-\lambda_s(t)}\varphi_s(t), \quad i = \overline{3,n}, \tag{13}$$

where $\beta_i(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$ are arbitrary functions, $i = \overline{3,n}$. Now we consider the system (12$_{m^i}$). It is solvable in the space $C^\infty\big([0,T],\mathbb{C}^n\big)$ if and only if the following identities hold:

$$\Big(-g^{(m^i)}(t)\alpha_i(t)\varphi_i(t) + H^{(m^i)}(t),\ \chi_i(t)\Big) \equiv 0, \quad i = \overline{3,n}.$$

These conditions are fulfilled automatically due to the choice of the functions $g^{(m^i)}(t)$ in the form (11b). To obtain solutions of the system (12$_{m^i}$) (an index $i \in \{3,\ldots,n\}$ is fixed), we perform the transformation $y^{(m^i)}(t) = \Phi(t)\xi^{(m^i)}(t)$. For the components $\xi_s^{(m^i)}(t)$ of the vector $\xi^{(m^i)}(t)$, we obtain the following equations:

$$\Big[b\big(m^i,\lambda(t)\big) - \lambda_i(t)\Big]\xi_i^{(m^i)}(t) = 0, \tag{13a}$$

$$\Big[\big(m^i,\lambda(t)\big) - \lambda_s(t)\Big]\xi_s^{(m^i)}(t) = \big(H^{(m^i)}(t),\ \chi_s(t)\big), \quad s \neq i, \quad s = \overline{1,n}. \tag{13b}$$

Since $m^i \notin \Gamma_s$ for $s \neq i$, we conclude that $\big(m^i,\lambda(t) - \lambda_s(t)\big) \neq 0$ for all $t \in [0,T]$, and Eqs. (13b) are uniquely solvable in the class $C^\infty\big([0,T],\mathbb{C}^1\big)$ and their solutions have the form

$$\xi_s^{(m^i)}(t) = \frac{\big(H^{(m^i)}(t),\chi_s(t)\big)}{\big(m^i,\lambda(t)\big)-\lambda_s(t)}\varphi_s(t), \quad s \neq i, \quad s = \overline{1,n}, \quad \forall t \in [0,T].$$

We have $\big(m^i,\lambda(t) - \lambda_i(t)\big) \equiv 0$ for all $t \in [0,T]$ since $m^i \in \Gamma_i$; therefore, an arbitrary function $\xi_i^{(m^i)}(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$ is a solution of Eq. (13a). Finally, the solution of the system (12$_{m^i}$) can be written in the form

$$y^{(m^i)}(t) = \xi_i^{(m^i)}(t)\varphi_i(t) + \sum_{\substack{s\neq i,\\ s=1}}^{n}\frac{\big(H^{(m^i)}(t),\ \chi_s(t)\big)}{\big(m^i,\lambda(t)\big)-\lambda_s(t)}\varphi_s(t).$$

It must be orthogonalized, i.e., $\big(y^{(m^i)}(t),\chi_i(t)\big) \equiv 0$ for all $t \in [0,T]$. Taking into account the fact that the systems of vectors $\{\varphi_i(t)\}$ and $\{\chi_j(t)\}$ are bi-orthonormed, we have

$$\big(y^{(m^i)}(t),\chi_i(t)\big) = \xi_i^{(m^i)}(t) \equiv 0 \quad \forall t \in [0,T],$$

which implies

$$y^{(m^i)}(t) = \sum_{\substack{s\neq i,\\ s=1}}^{n} \frac{\big(H^{(m^i)}(t),\ \chi_s(t)\big)}{\big(m^i, \lambda(t)\big) - \lambda_s(t)}\varphi_s(t),$$

i.e., a solution of the system $(12_{m^i})$ is defined in the class $C^\infty\big([0,T],\mathbb{C}^n\big)$ uniquely.

Now we consider the system $(12_1)$. We search for its solution in the form $y_1(t) = \Phi(t)\eta$. For the components of the vector $\eta = \{\eta_1,\dots,\eta_n\}$, we obtain the following equations:

$$\begin{cases} \big(\lambda_1(t)-\lambda_1(t)\big)\eta_1 = \big(-\alpha_2(t)\mu_1\varphi_2(t) + H_1(t), \chi_1(t)\big), \\ \big(\lambda_1(t)-\lambda_2(t)\big)\eta_2 = \big(-\alpha_2(t)\mu_1\varphi_2(t) + H_1(t), \chi_2(t)\big), \\ \big(\lambda_1(t)-\lambda_3(t)\big)\eta_3 = \big(-\alpha_2(t)\mu_1\varphi_2(t) + H_1(t), \chi_3(t)\big), \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \big(\lambda_1(t)-\lambda_n(t)\big)\eta_3 = \big(-\alpha_2(t)\mu_1\varphi_2(t) + H_1(t), \chi_n(t)\big). \end{cases}$$

Here we used the fact that

$$\Phi^{-1}(t)g(t) \equiv \Big\{\big(g(t),\chi_1(t)\big),\ \dots,\ \big(g(t),\chi_n(t)\big)\Big\}.$$

Since the systems of vectors $\{\varphi_j(t)\}$ and $\{\chi_i(t)\}$ are bi-orthonormed, we can rewrite the previous equations in the form

$$\begin{cases} 0\cdot\eta_1 = \big(H_1(t),\chi_1(t)\big), \\ \big(\lambda_1(t)-\lambda_2(t)\big)\eta_2 = -\alpha_2(t)\mu_1 + \big(H_1(t),\chi_2(t)\big), \\ \big(\lambda_1(t)-\lambda_3(t)\big)\eta_3 = \big(H_1(t),\chi_3(t)\big), \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \big(\lambda_1(t)-\lambda_n(t)\big)\eta_3 = \big(H_1(t),\chi_n(t)\big). \end{cases}$$

The first equation of this system is solvable if and only if

$$\big(H_1(t),\chi_1(t)\big) \equiv 0 \quad\iff\quad \langle H(t,u), \nu_1(t,u)\rangle \equiv 0 \quad \forall t\in[0,T], \tag{13*}$$

where $\eta_1 = \eta_1(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$ is an arbitrary function. The equations for $\eta_i$, $i=\overline{3,n}$, have unique solutions

$$\eta_i = \frac{\big(H_1(t),\chi_i(t)\big)}{\lambda_1(t)-\lambda_i(t)}, \quad i=\overline{3,n}. \tag{14}$$

For $t\in(0,T]$, the equation

$$\big(\lambda_1(t)-\lambda_2(t)\big)\eta_2 = -\alpha_2(t)\mu_1 + \big(H_1(t),\chi_2(t)\big)$$

has the following solution:

$$\eta_2 = \eta_2(t) = \frac{-\alpha_2(t)\mu_1 + \big(H_1(t),\chi_2(t)\big)}{\lambda_1(t)-\lambda_2(t)} = \frac{-\alpha_2(t)\mu_1 + \big(H_1(t),\chi_2(t)\big)}{tk(t)};$$

for $t=0$ it takes the form

$$0\cdot\eta_2(0) = -\alpha_2(0)\mu_1 + \big(H_1(0),\chi_2(0)\big),$$

and we obtain

$$\mu_1 = \frac{\big(H_1(0),\chi_2(0)\big)}{\alpha_2(0)}, \tag{15}$$

where $\eta_2(0)=\sigma$ is an arbitrary number and the solution of the equation

$$\big(\lambda_1(t)-\lambda_2(t)\big)\eta_2 = -\alpha_2(t)\mu_1 + \big(H_1(t),\chi_2(t)\big)$$

takes the form

$$\eta_2 = \eta_2(t) = \begin{cases} \dfrac{-\alpha_2(t)\mu_1 + \big(H_1(t), \chi_2(t)\big)}{tk(t)}, & t \in (0,T], \\ \sigma, & t = 0, \end{cases}$$
$$= \begin{cases} \dfrac{-\frac{\alpha_2(t)}{\alpha_2(0)}\big(H_1(0), \chi_2(0)\big) + \big(H_1(t), \chi_2(t)\big)}{t \cdot k(t)}, & t \in (0,T], \\ \sigma, & t = 0. \end{cases} \tag{16}$$

The vector-valued function

$$y_1(t) = \Phi(t)\eta(t) = \eta_2(t)\varphi_2(t) + \sum_{\substack{j=1,\\ j\neq 2}}^{n} \eta_j(t)\varphi_j(t)$$

belongs to the class $C^\infty\big([0,T], \mathbb{C}^n\big)$; therefore, $\eta_2(t) = \big(y_1(t), \chi_2(t)\big) \in C^\infty\big([0,T], \mathbb{C}^1\big)$. In particular, we obtain

$$\lim_{t\to+0} \eta_2(t) = \lim_{t\to+0} \frac{-\frac{\alpha_2(t)}{\alpha_2(0)}\big(H_1(0), \chi_2(0)\big) + \big(H_1(t), \chi_2(t)\big)}{t \cdot k(t)} = \sigma,$$

i.e., in (16) $\sigma$ is defined uniquely. Note that the formula $\eta_2(t)$ can be written as follows:

$$\eta_2(t) = \frac{-\frac{\alpha_2(t)}{\alpha_2(0)}\big(H_1(0), \chi_2(0)\big) + \big(H_1(t), \chi_2(t)\big)}{t \cdot k(t)} \quad \forall t \in [0,T],$$

assuming that this equality at the point $t = 0$ is meant in the limit sense. Thus, choosing the constant $\mu_2$ in the form (15), we find the solution of the system $(12_1)$ in the form

$$\begin{aligned} y_1(t) &= \Phi(t)\eta(t) \\ &= \eta_1(t)\varphi_1(t) + \frac{-\frac{\alpha_2(t)}{\alpha_2(0)}\big(H_1(0), \chi_2(0)\big) + \big(H_1(t), \chi_2(t)\big)}{t \cdot k(t)}\varphi_2(t) + \sum_{i=3}^{n} \frac{\big(H_1(t), \chi_i(t)\big)}{\lambda_1(t) - \lambda_i(t)}\varphi_i(t), \end{aligned} \tag{17}$$

where $\eta_1(t) \in C^\infty\big([0,T], \mathbb{C}^1\big)$ is an arbitrary function and all other terms are defined uniquely.

Now we consider the system $(12_2)$. Arguing similarly, we can prove that if the constant $\mu_2$ is chosen in the form

$$\mu_2 = \frac{\big(H_2(0), \chi_1(0)\big)}{\alpha_1(0)}, \tag{18}$$

then the system $(12_2)$ is solvable in the space $C^\infty\big([0,T], \mathbb{C}^n\big)$ if and only if the following identities hold:

$$\big(H_2(t), \chi_2(t)\big) \equiv 0 \quad \Longleftrightarrow \quad \langle H(t,u), \nu_2(t,u)\rangle \equiv 0 \quad \forall t \in [0,T]. \tag{18*}$$

Then the solution $(12_2)$ of the system has the form

$$\begin{aligned} y_2(t) &= \Phi(t)\varsigma(t) \\ &= \varsigma_2(t)\varphi_2(t) + \frac{-\frac{\alpha_1(t)}{\alpha_1(0)}\big(H_2(0), \chi_1(0)\big) + \big(H_2(t), \chi_1(t)\big)}{-t \cdot k(t)}\varphi_1(t) + \sum_{i=3}^{n} \frac{\big(H_2(t), \chi_i(t)\big)}{\lambda_2(t) - \lambda_i(t)}\varphi_i(t), \end{aligned} \tag{19}$$

where $\varsigma_2(t) \in C^\infty\big([0,T], \mathbb{C}^1\big)$ is an arbitrary function and all other terms are defined uniquely. Thus, there exist a unique collection of numbers $\mu_1$ and $\mu_2$ and functions $g^{(m^i)}(t)$ of the form (11b) such that the system (11) is solvable in the space $U$ if and only if the identities $(12^*)$, $(13^*)$, and $(18^*)$ hold simultaneously, i.e., the conditions $(11^*)$ hold. The theorem is proved. □

Now we clarify the condition of unique solvability of the iterative system (11). Consider the system (11) under the additional conditions

$$y(0,\bar{1}) = y_*,$$
$$\left\langle -\frac{\partial y}{\partial t} - \frac{\partial y_0}{\partial u}\Big(\hat{\mu}_2 e_1 u_2 + \hat{\mu}_1 e_2 u_1\Big) + G(t,u),\ \nu_j(t,u)\right\rangle \equiv 0 \quad \forall t \in [0,T], \quad j = \overline{1,n}, \tag{20}$$

where $\hat{\mu}_2$ and $\hat{\mu}_1$ are arbitrary constants, $G(t,u) \in U$ is a given vector-valued function, and $y_* \in \mathbb{C}^n$ is a given constant vector. The following assertion holds.

**Theorem 4.** *Assume that vector-valued function $H(t,u) \in U$ in the system* (11) *satisfies the orthogonality conditions* (11*) *and the conditions* (I), (IIa)–(IIc), *and* (III) *are fulfilled. Moreover, let $\alpha_j(0) = \big(y^0 - y_0^{(0)}(0), \chi_j(0)\big) \neq 0$, $j = 1,2$, the constants $\mu_1$ and $\mu_2$ are chosen in the form* (15) *and* (18), *respectively, and the scalar functions $g^{(m^i)}(t) \in C^\infty\big([0,T],\mathbb{C}^1\big)$ have the form* (11b). *Then the system* (11) *under the additional conditions* (20) *is uniquely solvable in the space $U$.*

*Proof.* Since the orthogonality conditions (11*) are fulfilled, the system (11) has a solution in the class $U$, which has the form of a vector-valued function (12) in which all terms $y_0(t)$, $y^{(m)}(t)$, $2 \le |m| \le N_y$, are defined uniquely; the coefficients $y_i(t)$ have the form (13) for $i = \overline{3,n}$ and the form (17) and (19) for $i = 1,2$. Now $\beta_i(t)$, $\eta_1(t)$, and $\varsigma_2(t)$ are unknown arbitrary functions of the class $C^\infty\big([0,T],\mathbb{C}^1\big)$ (all other terms in (17) and (19) are defined uniquely). Assuming that the solution (12) satisfies the initial condition $y(0,\bar{1}) = y_*$, we obtain the equality

$$\eta_1(0)\varphi_1(0) + \varsigma_2(0)\varphi_2(0) + \sum_{i=1}^{n} \beta_i(0)\varphi_i(0) = z_*,$$

where $z_* \in \mathbb{C}^n$ is a known constant vector. Multiplying scalarly this equality by $\chi_1(0)$, $\chi_2(0)$, and $\chi_i(0)$, $i = \overline{3,n}$, we find

$$\eta_1(0) = \big(z_*, \chi_1(0)\big), \quad \varsigma_2(0) = \big(z_*, \chi_2(0)\big), \quad \beta_i(0) = \big(z_*, \chi_i(0)\big), \quad i = \overline{3,n}. \tag{21}$$

Substituting the function (12) into the second condition (20), we obtain (similarly to the proof of Theorem 2) $n$ scalar independent linear ordinary differential equations for the functions $\eta_1(t), \varsigma_2(t), \beta_i(t)$, $i = \overline{3,n}$ (these equations are independent of the constants $\hat{\mu}_2$ and $\hat{\mu}_1$). Solving these equations with the initial conditions (21), we uniquely find the functions $\eta_1(t), \varsigma_2(t), \beta_i(t)$, $i = \overline{3,n}$, and hence construct a solution of the system (11) in the space $U$. The theorem is proved. □

**5. Asymptotic convergence of formal solutions of the problem** (1) **to the exact solution.** Let the conditions (I)–(III) be fulfilled. Then, due to Theorems 1–4, we can uniquely find solutions $y_0(t,u), \ldots, y_r(t,u)$ in the space $U$ for all iterative problems $(4_0)$–$(4_k)$ and construct the regularizing normal form (2) of order $r+1$. Under the conditions specified, we can construct the partial sum

$$S_r(t,u,\varepsilon) = \sum_{k=0}^{r} \varepsilon^k y_k(t,u)$$

of the series (4), where $y_k(t,u) \in U$. Then the regularizing normal form (2)

$$\begin{aligned}
\varepsilon\frac{du_1}{dt} &= \lambda_1(t)u_1 + \sum_{k=1}^{r+1} \varepsilon^k \mu_2^{(k)} u_2, & u_1(0,\varepsilon) &= 1,\\
\varepsilon\frac{du_2}{dt} &= \lambda_2(t)u_2 + \sum_{k=1}^{r+1} \varepsilon^k \mu_1^{(k)} u_1, & u_2(0,\varepsilon) &= 1,\\
\varepsilon\frac{du_i}{dt} &= \lambda_1(t)u_i + \sum_{k=1}^{r+1} \varepsilon^k \sum_{m^i \in \Gamma_i} g_k^{(m^i)}(t) u^{m^i}, & u_i(0,\varepsilon) &= 1, \quad i = \overline{3,n}
\end{aligned} \tag{22}$$

splits into two mutually independent differential systems, one of which is linear ($i = 1, 2$) and the other is nonlinear ($i = \overline{3, n}$). We impose the following additional condition:

(IV) the normal for (22) is globally solvable on the segment $[0, T]$ and its solution satisfies the estimates

$$\left|u_i(t,\varepsilon)\right| < 1 + \delta \quad \forall (t,\varepsilon) \in [0,T] \times (0,\varepsilon_0], \quad i = \overline{1,n}, \tag{23}$$

where $\varepsilon_0 > 0$ is sufficiently small and $\delta > 0$ is independent of $\varepsilon$,

Then we can prove that the restriction $y_{\varepsilon r}(t)$ of the partial sum $S_r(t, u, \varepsilon)$ of the series (4) to the solution $u = u(t, \varepsilon)$ of the regularizing form (22) of order $r + 1$ is a formal asymptotic solution of the problem (1), i.e., the following identity holds:

$$\varepsilon \frac{dy_{\varepsilon r}(t)}{dt} - A(t) y_{\varepsilon r}(t) - \varepsilon F\bigl(y_{\varepsilon r}(t), t\bigr) \equiv h(t) + \varepsilon^{r+1} R_r(t,\varepsilon), \quad y_{\varepsilon r}(0) = y^0, \tag{24}$$

where $\left\|R_r(t,\varepsilon)\right\|_{C[0,T]} \le \bar{R}_r$ and $\bar{R}_r > 0$ is a constant independent of $\varepsilon \in (0, \varepsilon_0]$. Using the identity (24) and the results of [8, p. 163–166], one can easily prove the following assertion.

**Theorem 5.** *Assume that the system* (1) *satisfies the conditions* (I), (IIa)–(IId), *and* (III). *Moreover, let* $\alpha_j(0) = \bigl(y^0 - y_0^{(0)}(0), \chi_j(0)\bigr) \neq 0$, $j = 1, 2$, *and the constants* $\mu_1^{(k)}$ *and* $\mu_2^{(k)}$ *and the functions* $g_k^{(m^i)}(t)$ *are chosen according to the solvability of the iterative problems* $(4_k)$ *in the space* $U$. *Then, under the condition* (IV), *the problem* (1) *is uniquely solvable in the space* $C^1\bigl([0,T], \mathbb{C}^\infty\bigr)$ *and its solution* $y(t,\varepsilon)$ *satisfies the estimate*

$$\Bigl\|y(t,\varepsilon) - y_{\varepsilon r}(t)\Bigr\|_{C[0,T]} \le C_r \varepsilon^{r+1}, \quad r = 0, 1, 2, \ldots,$$

*where the constant* $C_r > 0$ *is independent of* $\varepsilon \in (0, \varepsilon_0]$.

**Remark.** If the spectrum $\sigma(A(t)) = \{\lambda_j(t)\}$ is such that there exists a straight line $(\pi)$ independent of $t \in [0, T]$), and passing through the zero of the complex plane $\lambda$, and such that the spectrum $\{\lambda_j(t)\}$ lies on one side of $(\pi)$ and does not intersect with $(\pi)$, then the nonlinear part of the normal form (22) is triangular. In this case, one can easily prove that the inequalities (23) always hold.

**Example.** Consider the system (1) with the matrix

$$A(t) = \begin{pmatrix} -\sqrt{2} & \sqrt{2} - e^t\sqrt{2} & -\sqrt{2} + 2e^t\sqrt{2} - 1 & 0 \\ 0 & -e^t\sqrt{2} & -2 + 2e^t\sqrt{2} & 0 \\ 0 & 0 & -1 & 0 \\ 2 - \sqrt{2} & \sqrt{2} - 2 & 2 - \sqrt{2} & -2 \end{pmatrix}.$$

Its spectrum $\sigma(A(t)) = \bigl\{-\sqrt{2},\ -e^t\sqrt{2},\ -1,\ -2\bigr\}$ satisfies the conditions (I) and (IIa)–(IId) and lies to the left of the straight line $\operatorname{Re}\lambda < 0$. Therefore, we can apply the algorithm proposed.

Also, we note that the method of normal forms can be generalized to more complicated nonlinear singular perturbed problems (see, e.g., [2, 7]).

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