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Asymptotic spreading of KPP reactive fronts in incompressible space–time random flows

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Abstract

We study the asymptotic spreading of Kolmogorov–Petrovsky–Piskunov (KPP) fronts in space–time random incompressible flows in dimension $d > 1$. We prove that if the flow field is stationary, ergodic, and obeys a suitable moment condition, the large time front speeds (spreading rates) are deterministic in all directions for compactly supported initial data. The flow field can become unbounded at large times. The front speeds are characterized by the convex rate function governing large deviations of the associated diffusion in the random flow. Our proofs are based on the Harnack inequality, an application of the sub-additive ergodic theorem, and the construction of comparison functions. Using the variational principles for the front speed, we obtain general lower and upper bounds of front speeds in terms of flow statistics. The bounds show that front speed enhancement in incompressible flows can grow at most linearly in the root mean square amplitude of the flows, and may have much slower growth due to rapid temporal decorrelation of the flows.
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Résumé

On étudie le comportement asymptotique des solutions des équations de réaction–diffusion du type Kolmogorov–Petrovsky–Piskunov (KPP) avec convection stochastique en dimension $d > 1$. Dans le cas où l'écoulement est stationnaire et ergodique, nous démontrons que la solution forme un front qui se propage avec une vitesse déterministe. Les vitesses de propagation satisfont une formule variationnelle associée à un principe de grandes déviations pour un processus de diffusion en milieu aléatoire. Avec cette formule, nous obtenons quelques estimations de la vitesse. Les preuves sont basées sur une inégalité de type Harnack, le principe de maximum, et le théorème ergodique sous-additif.
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Keywords: Front propagation; KPP; Random drift; Large deviations

1. Introduction

Reaction–diffusion front propagation in incompressible space–time random flows is a fundamental subject in premixed turbulent combustion [6,34,28,20,32]. One challenging mathematical problem is to establish the propagation

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velocity of the front (large time asymptotic spreading rate) using the governing partial differential equations. Another mathematical problem is to characterize the propagation velocity in terms of flow statistics. Such a velocity is called the turbulent flame speed in combustion [28], and it is an upscaled quantity that depends on statistics of the random flows in a highly nontrivial manner. Due to the notorious closure problem in turbulence, the turbulent front speed has been approximated by ad hoc and formal procedures in combustion literature, such as various closures and renormalization group methods [27,34,7]. However, these methods are difficult to justify mathematically.

A pleasant surprise is that fronts governed by the Kolmogorov–Petrovsky–Piskunov (KPP) nonlinearity are in some sense solvable, and the front speeds have a well-defined variational characterization in the large time limit. This important mathematical property of KPP fronts has been analyzed for special temporally random flows (time random shear flows) [20,24,33] and spatially random environments [12,10,19,29]. There have been several studies of KPP fronts in periodic flows, for example see [10,4,21,11,8,3,26].

In this paper, we study KPP fronts propagating through space–time random incompressible flows. The flows can be unbounded in time, as for a Gaussian process. We establish the almost sure existence of propagating fronts which evolve from compactly supported initial data, and we derive a variational characterization for the front speeds. Using this characterization, we derive some estimates of the fronts speed. One can also use this characterization to numerically approximate the front speed, as presented separately in [25].

The governing equation for KPP reactive fronts is the reaction–diffusion–advection equation:

$$\partial_t u = \Delta u + V(x,t,\hat{\omega})\cdot\nabla u + f(u) \stackrel{\Delta}{=} \mathcal{L}u + f(u), \tag{1.1}$$

with smooth, compactly supported, nonnegative initial data $u(x,0,\hat{\omega}) = u_0(x)$, $0 \leqslant u_0 \leqslant 1$. The reaction function $f(u)$ is nonlinear and satisfies: $f \in C^1([0,1])$, $f(0) = f(1) = 0$, $f(u) > 0$ for $u \in (0,1)$, and $f(u) \leqslant uf'(0)$. For example, $f(u) = u(1-u)$. The value $u = 1$ corresponds to the hot or burned state in the combustion model, while $u = 0$ corresponds to the cold or unburned state, which is unstable.

The vector field $V(x,t,\hat{\omega})$ is defined over a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. We assume that:

(1) V is stationary with respect to shifts in x and t: there is a group of measure-preserving transformations $\tau_{(x,t)} : \hat{\Omega} \to \hat{\Omega}$ such that $V(x+h, t+r, \hat{\omega}) = V(x,t,\tau_{(h,r)}\hat{\omega})$, and τ acts ergodically on $\hat{\Omega}$.

(2) V is locally Hölder continuous, almost surely, in the sense that for each $T > 0$ there is $\alpha = \alpha(\hat{\omega}, T)$ such that

$$\left\| V(\cdot,\cdot,\hat{\omega}) \right\|_{C^\alpha(\mathbb{R}^d\times[0,T])} < \infty \tag{1.2}$$

holds for almost every $\hat{\omega} \in \hat{\Omega}$.

(3) V is divergence free, $\nabla\cdot V = 0$, in the sense of distribution, almost surely with respect to $\hat{P}$.

(4) V satisfies the moment condition:

$$\bar{V}_2 \stackrel{\Delta}{=} E_{\hat{P}}\Big[\sup_{\substack{t\in[0,1]\\ x\in\mathbb{R}^d}} \left|V(x,t)\right|^2\Big] < \infty. \tag{1.3}$$

The condition (4) means that $V(x,t,\hat{\omega})$ is uniformly bounded in x for each fixed t and $\hat{\omega}$. However, we do not require that $V(x,t,\cdot) \in L^\infty(\hat{\Omega})$, so that V may become unbounded as $t \to \infty$. The Hölder regularity condition (2) is satisfied by turbulent flows [20,32] and is a physical assumption for turbulent combustion problems [28,27,32].

For almost every $\hat{\omega}$, there exists a unique classical solution satisfying (1.1). Our main result is the following theorem regarding the almost-sure asymptotic behavior of the solution $u(x,t,\hat{\omega})$ as $t \to \infty$:

Theorem 1.1. *There is a convex open set $G \subset \mathbb{R}^d$ and a set of full measure $\hat{\Omega}_0 \subset \hat{\Omega}$, $\hat{P}(\hat{\Omega}_0) = 1$, such that the following limits hold for all $\hat{\omega} \in \hat{\Omega}_0$:*

$$\lim_{t\to\infty} \sup_{c\in F} u(ct,t) = 0 \tag{1.4}$$

for any closed set $F \subset \mathbb{R}^d \setminus \bar{G}$, and

$$\lim_{t\to\infty} \inf_{c\in K} u(ct,t) = 1 \tag{1.5}$$

for any compact set $K \subset G$.

Thus, the set $\{ct \in \mathbb{R}^d \mid c \in \partial G\}$, which is deterministic, represents the spreading interface in an asymptotic sense, made precise by (1.4) and (1.5). The set G may be characterized in the following way. Let $\phi(x,t,\hat{\omega}) \geqslant 0$ solve the advection–diffusion equation $\partial_t \phi = \mathcal{L}\phi$ with initial condition $\phi(x,0,\hat{\omega}) = \phi_0(x) \geqslant 0$, where $\phi_0(x)$ is smooth, deterministic, and compactly supported.

Theorem 1.2. *The limit*

$$\mu(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log \int_{\mathbb{R}^d} e^{\lambda\cdot x}\phi(x,t,\hat{\omega})\,dx = \lim_{t\to\infty} \frac{1}{t}\log E_{\hat{P}}\big[e^{\lambda\cdot x}\phi(x,t,\hat{\omega})\big] \tag{1.6}$$

exists almost surely with respect to $\hat{P}$. Moreover, $\mu(\lambda)$ is a finite, convex function of $\lambda \in \mathbb{R}^d$.

Now the characterization of G is given by the following theorem:

Theorem 1.3. *The set G described in Theorem* 1.1 *is given by*

$$G = \big\{c \in \mathbb{R}^d \bigm| H(c) \leqslant f'(0)\big\} \tag{1.7}$$

where $H(c) = \sup_{\lambda\in\mathbb{R}^d}(\lambda\cdot c - \mu(\lambda))$ and $\mu(\lambda)$ is defined as in Theorem 1.2. *It follows that the asymptotic front speed c^* in direction $e \in R^d$ is given by the variational formula*:

$$c^*(e) = \inf_{\lambda\cdot e>0} \frac{\mu(\lambda)+f'(0)}{\lambda\cdot e}. \tag{1.8}$$

For the KPP model, Theorems 1.1 and 1.3 address two open problems in turbulent combustion [28]: the existence of a well-defined turbulent flame speed and the precise analytical characterization of the turbulent flame speed. In Theorem 1.2, one may normalize ϕ so that ϕ is the density for a probability measure on $\mathbb{R}^d$, for each fixed $\hat{\omega}$, and the theorem characterizes the asymptotic behavior of the tails of the distribution (large deviations from the mean behavior) almost surely with respect to the measure $\hat{P}$ on the velocity field. The function H in Theorem 1.3 is the rate function that governs these large deviations.

The quantity $\mu(\lambda)$ has another characterization. Consider the function $\varphi^*(x,\tau;t,\hat{\omega})$ which solves the terminal value problem ($\tau \in (0,t)$):

$$\partial_\tau\varphi^* + \Delta\varphi^* - \big(V(x,\tau) - 2\lambda\big)\cdot\nabla\varphi^* + \big(|\lambda|^2 - \lambda\cdot V(x,\tau)\big)\varphi^* = 0, \tag{1.9}$$

with linear terminal data $\varphi^*(x,t;t,\hat{\omega}) \equiv 1$, $x\in\mathbb{R}^d$. We will show that $\varphi^*(x,0;t,\hat{\omega})$ grows exponentially in t with a rate equal to $\mu(\lambda)$:

Theorem 1.4. *If $\varphi^*(x,\tau;t,\hat{\omega})$ solves* (1.9) *with terminal data $\varphi^*(x,t,\hat{\omega}) \equiv 1$, then for any $r > 0$*

$$\lim_{t\to\infty}\sup_{|x|\leqslant rt}\left|\frac{1}{t}\log\varphi^*(x,0;t,\hat{\omega}) - \mu(\lambda)\right| = 0 \tag{1.10}$$

holds almost surely with respect to the measure $\hat{P}$.

The function $\mu(\lambda)$ is related to the effective Hamiltonian that arises from the theory of homogenization of "viscous" Hamilton–Jacobi equations in stationary ergodic media (see [16,17,19]). For V that depends only on x, Lions and Souganidis [19] showed that the front is governed by an effective Hamilton–Jacobi equation (see Section 9 of [19]). It turns out that $\mu(\lambda)$ in (1.6) is equal to an effective Hamiltonian $\overline{\mathcal{H}}(\lambda)$. To see this clearly, define the function $\eta^*(x,\tau;t,\hat{\omega}) = e^{\lambda\cdot y}\varphi^*(x,\tau;t,\hat{\omega})$ which satisfies $\partial_\tau\eta + \mathcal{L}^*\eta^* = 0$ for $\tau < t$ and terminal data $\eta^*(x,t;t,\hat{\omega}) = e^{\lambda\cdot y}$. Here $\mathcal{L}^*\eta^* = \Delta_x\eta^* - \nabla\cdot(V\eta^*)$ denotes the adjoint operator. For $\epsilon > 0$ and $T > 0$, define

$$\zeta^\epsilon(x,\tau;T,\hat{\omega}) = \epsilon\log\eta^*\big(\epsilon^{-1}x,\epsilon^{-1}\tau;\epsilon^{-1}T,\hat{\omega}\big).$$

Then ζ^ϵ solves the Hamilton–Jacobi equation

$$\partial_\tau\zeta^\epsilon + \epsilon\Delta\zeta^\epsilon + \big|\nabla\zeta^\epsilon\big|^2 - V\left(\frac{x}{\epsilon},\frac{\tau}{\epsilon},\hat{\omega}\right)\cdot\nabla\zeta^\epsilon = 0, \quad \tau\in[0,T) \tag{1.11}$$

with terminal data $\zeta^\epsilon(x,T;T,\hat\omega)=\lambda\cdot x$. For a velocity field $V(x,\tau,\hat\omega)$ which is uniformly bounded in τ (i.e. $V\in L^\infty(\hat\Omega;L^\infty(\mathbb{R}^d)))$, the result of Kosygina and Varadhan [17] implies that as $\epsilon\to 0$, the function ζ^ϵ converges locally uniformly to a function $\zeta^0(x,\tau;t)$ which solves an effective Hamilton–Jacobi equation $\partial_\tau\zeta^0(z,\tau;t)+\overline{\mathcal{H}}(\nabla\zeta^0)=0$ with the same terminal data. The effective Hamiltonian $\overline{\mathcal{H}}(\lambda)$ is a deterministic function. In particular, by choosing $T=1$, we see that

$$\overline{\mathcal{H}}(\lambda)=\lim_{\epsilon\to 0}\zeta^\epsilon(0,0;1,\hat\omega)=\lim_{\epsilon\to 0}\epsilon\log\eta^*\bigl(0,0;\epsilon^{-1},\hat\omega\bigr)=\mu(\lambda) \tag{1.12}$$

holds almost surely with respect to $\hat P$. Theorem 1.4 extends this connection to the case of velocity fields $V(x,t)$ which are not uniformly bounded in t, a case not covered by the results in [16,17], and [19].

We develop a new Eulerian approach to prove the results. The first step is to use the Harnack-type inequality of Krylov and Safonov to establish continuity estimates of the solution. One technical difficulty that arises is that the constants appearing in the Harnack inequality may be arbitrarily bad. However, we show that the constants are well-behaved "on average". We use this observation and the subadditive ergodic theorem to establish almost sure behavior of the tails of the linearized equation. To apply this to the solution of the nonlinear equation, we construct sub- and super-solutions and use the comparison principle. Our proof uses only the Harnack inequality and the comparison principle, and so applies readily to a large class of operators $\mathcal{L}$. In fact, one can see that all of the proofs may be modified slightly to treat the case that the diffusion is also variable. For example, a variant of Theorems 1.1–1.4 hold in the case that u is governed by an equation of the form

$$\partial_t u=\nabla\cdot\bigl(A(x,t,\hat\omega)\nabla u\bigr)+V(x,t,\hat\omega)\cdot\nabla u+f(u) \tag{1.13}$$

where $A(x,t,\hat\omega)=A_{ij}(x,t,\hat\omega)$ is random, positive-definite matrix function and uniformly $C^{1,\alpha}$. For clarity we concentrate on the case that A_{ij} is the identity.

Some previous analysis [12,10,24] of KPP fronts have been based on analysis of the associated Itô diffusion processes that play the role of characteristics in the Feynman–Kac formula for solutions of the linearized equation. This Lagrangian approach is particularly useful when there is either an explicit solution formula [24] or a hitting time characterization of the Itô paths in one space dimension [12,10]. In the present Eulerian approach, quantities like $\mu(\lambda)$ and H have a similar Lagrangian interpretation, and we utilize both the Eulerian and Lagrangian aspects to prove bounds on the front speeds.

The paper is organized as follows. In Section 2, we employ the Krylov–Safonov–Harnack inequality and the subadditive ergodic theorem to obtain the large deviation estimates for solutions to the linearized evolution and to identify the function $H(c)$. In Section 3, we construct sub- and super-solutions to show that the large deviation rate function H indeed defines the propagating interfaces in the large time limit. The proofs in this section are related to those in previous works [10,24]; the new twist is to rely on comparison functions instead of the associated Itô paths and the Feynman–Kac formula. In Section 4, we prove Theorems 1.2–1.4. We study the Lyapunov exponent μ, and establish its connection to the function H. The variational principle for the front speeds is given in terms of μ, which is easier to calculate and estimate than H. In Section 5, we prove upper and lower bounds on the front speeds. A Lagrangian method and random change of measure are used in a Feynman–Kac representation to deduce an upper bound of μ in terms of second order flows statistics. These bounds extend those on time random shear flows by the authors [24]. The bounds show that front speed enhancement in incompressible flows can grow at most linearly in the root mean square amplitude of the flows, and may have much slower growth due to rapid temporal decorrelations of flows. Conclusions are in Section 6, and acknowledgments are in Section 7.

2. Preliminary estimates

2.1. Harnack inequality

To prove Theorem 1.1, we will make use of the Harnack-type inequality proved by Krylov–Safonov [18]. First, we define $Q(\theta,R)=\{(x,t)\in\mathbb{R}^{n+1}\mid \max_i|x_i|\leqslant R,\ t\in(0,\theta R^2)\}$, and we state a well-known result of Krylov and Safonov:

Theorem 2.1. *(See Krylov–Safonov [18].) Let $\theta > 1$, $R \leqslant 2$, and $0 \leqslant \xi(x,t) \leqslant 1$. Suppose $\eta \in W^{1,2}_{d+1}(Q(\theta,R))$, $\eta \geqslant 0$, and $\partial_t \eta - \mathcal{L}\eta + \xi(x,t)\eta = 0$ in $Q(\theta,R)$. Suppose $\|V\|_{L^\infty(Q(\theta,R))} \leqslant 1$. Then there exists a constant $K_o > 0$ depending only on θ and the dimension such that*

$$\inf_{|x| \leqslant R/2} \eta\bigl(x, \theta R^2\bigr) \geqslant K_o \eta\bigl(0, R^2\bigr).$$

Remark 2.1. Throughout this paper, the constant θ from Theorem 2.1 will appear. Our arguments do not depend on the precise value of θ, and we will assume this constant is always fixed at $\theta = 2$.

We wish to apply this estimate to the function $u(x,t,\hat{\omega})$ and to the function $\varphi(x,t,\hat{\omega})$ defined by (1.9). As we have stated the theorem, the drift V and the source function ξ must be bounded uniformly in the region of interest. Although we are working with a drift for which individual realizations are not uniformly bounded for all $t > 0$, we may obtain a Harnack-type inequality for u by rescaling the solution and iteratively applying Theorem 2.1. The constants that appear in the resulting inequality may become arbitrarily large since V may not be uniformly bounded in t. However, in the next section we will show that the constants are well-behaved on average.

Suppose $\eta(x,t) \geqslant 0$ solves

$$\partial_t \eta = \Delta \eta + V(x,t) \cdot \nabla \eta + \xi(x,t)\eta$$

for $(x,t) \in Q(\theta,R)$, while V and ξ are not necessarily globally bounded. Then for $(x,t) \in Q(\theta,R)$ and $h \geqslant 0$ to be chosen, the function $\bar{\eta}(x,t) = e^{-ht}\eta(M^{-1}x, M^{-2}t)$ solves

$$\partial_t \bar{\eta}(x,t) = \Delta \bar{\eta}(x,t) + V_M(x,t) \cdot \nabla \bar{\eta}(x,t) + \xi_M(x,t)\bar{\eta}(x,t),$$

where $V_M(x,t) = M^{-1}V(M^{-1}x, M^{-2}t)$ and $\xi_M(x,t) = -h + M^{-2}\xi(M^{-1}x, M^{-2}t)$. If we choose the constant M to be

$$M = \max\Bigl(1, \sup_{(x,t)\in Q(\theta,R)} \bigl|V(x,t)\bigr|, \sup_{(x,t)\in Q(\theta,R)} \sqrt{2\bigl|\xi(x,t)\bigr|}\Bigr) \tag{2.1}$$

and set $h = 1/2$, then for any $(x,t) \in Q(\theta,R)$, we also have $(M^{-1}x, M^{-2}t) \in Q(\theta,R)$ and $|V_M(x,t)| \leqslant 1$. Also, $-1 \leqslant \xi_M(x,t) \leqslant 0$. Thus, Theorem 2.1 applies to $\bar{\eta}$:

$$\inf_{|x| \leqslant R/2} \bar{\eta}\bigl(x, \theta R^2\bigr) \geqslant K_o \bar{\eta}\bigl(0, R^2\bigr).$$

Therefore, for the original η we have

$$\inf_{|x| \leqslant R/2M} \eta\left(x, \theta \frac{R^2}{M^2}\right) \geqslant K_o e^{R^2(\theta-1)/2} \eta\left(0, \frac{R^2}{M^2}\right) \geqslant K_o \eta\left(0, \frac{R^2}{M^2}\right). \tag{2.2}$$

We now summarize these observations in a manner that will be convenient for our analysis. For $x \in \mathbb{R}^d$ and $t \geqslant 1$, let us define the cylinder set

$$Q'(x,t,\theta,R) = \Bigl\{(y,\tau) \in \mathbb{R}^{n+1} \bigm| \max_i |y_i - x_i| \leqslant R,\ \tau - t \in \bigl(-R^2, (\theta-1)R^2\bigr)\Bigr\}$$

and the constant

$$M(x,t,R,\theta) = \max\Bigl(1, \sup_{(y,\tau)\in Q'(x,t,\theta,R)} \bigl|V(y,\tau)\bigr|, \sup_{(y,\tau)\in Q'(x,t,\theta,R)} \sqrt{2\bigl|\xi(y,\tau)\bigr|}\Bigr), \tag{2.3}$$

which is a local upper bound on $|V|$ and $|\xi|$ over the cylinder set Q'. Theorem 2.1 and the above scaling analysis imply the following:

Corollary 2.1. *Let $\theta > 1$, $R \leqslant 2$. Let $M(x,t,R,\theta)$ be defined by (2.3). For any $M \geqslant M(x,t,R,\theta)$, let $\Delta t = (\theta-1)R^2/M^2$. Then*

$$\eta(x + \Delta x, t + \Delta t) \geqslant K_o \eta(x,t)$$

whenever $|\Delta x| \leqslant \frac{R}{2M}$, where $K_o = K_o(\theta)$ is the constant from Theorem 2.1.

Now we will use this estimate iteratively to relate $\eta(x_1,t_1)$ to $\eta(x_2,t_2)$ for two points x_1, x_2 and two different times $1 \leqslant t_1 < t_2 - 1$. We will derive a lower bound on $\inf_{y \in B_\delta(x_2)} \eta(y,t_2)$ in terms of $\sup_{y\in B_\delta(x_1)} \eta(y,t_1)$, where $B_\delta(x)$ denotes the ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^d$.

Let $c \in \mathbb{R}^d$ be defined by $c = (x_2 - x_1)/(t_2 - t_1)$, and let $\gamma(x_1,t_1;x_2,t_2)$ denote the set of points in $\mathbb{R}^{d+1}$ formed by the line segment with endpoints at (x_1,t_1) and (x_2,t_2). Define $T \subset \mathbb{R}^{d+1}$ to be the set

$$T = \bigcup_{s\in[0,t_2-t_1]} \big(B_\delta(x_1 + cs) \times (t_1 + s)\big). \tag{2.4}$$

This is a tubular region with the line segment γ as the central axis and radius δ. Now choose $R \leqslant 1$ small enough so that

$$|c| + 2\delta \leqslant \frac{1}{2R(\theta - 1)}.$$

Then define the constant

$$M_{x_1,t_1;x_2,t_2} = \sup_{(x,t)\in T} M(x,t,R,\theta) \tag{2.5}$$

with $M(x,t,R,\theta)$ given by (2.3). This constant bounds $|V(x,t,\hat{\omega})|$ and $\sqrt{|\xi|}$ over a neighborhood of the tube T.

Next, using $M = M_{x_1,t_1;x_2,t_2}$, let Δt be defined as in Corollary 2.1:

$$\Delta t = \frac{(\theta - 1)R^2}{(M_{x_1,t_1;x_2,t_2})^2}.$$

Let k be the ratio $k = (t_2 - t_1)/\Delta t$. By increasing M slightly, we may assume that k is an integer:

$$k = \frac{t_2 - t_1}{\Delta t} = \frac{(t_2 - t_1)(M_{x_1,t_1;x_2,t_2})^2}{(\theta - 1)R^2}. \tag{2.6}$$

Now suppose that $x_1' \in B_\delta(x_1)$ and $x_2' \in B_\delta(x_2)$. Define $y_j \in \mathbb{R}^d$ by

$$y_j = x_1' + \frac{x_2' - x_1'}{t_2 - t_1}(\Delta t) j, \quad j = 0,1,2,\ldots,k.$$

The set of points $\{(y_j, t_1 + j\Delta t)\}_{j=1}^k$ is contained in the tube T. Moreover, from our choice of R, we see that $|y_{j+1} - y_j| \leqslant \frac{R}{2M}$ for each j. Therefore, we can iteratively apply Corollary 2.1 k times to conclude that

$$\eta\big(y_{j+1}, t_1 + (j+1)\Delta t\big) \geqslant K_o \eta(y_j, t_1 + j\Delta t), \quad j = 0,\ldots,k-1,$$

and thus

$$\inf_{y\in B_\delta(x_2)} \eta(y,t_2) \geqslant K_o^k \sup_{y\in B_\delta(x_1)} \eta(y,t_1).$$

The constant K_o is the same constant from Corollary 2.1, depending only on θ. The integer k, however, depends on x_1, x_2, t_1, and t_2 through (2.5) and (2.6). By putting together the above analysis, we have the following lemma:

Lemma 2.1. *Fix $\theta > 1$. Let $\delta > 0$, and $x_1, x_2 \in \mathbb{R}^d$. Let t_1, t_2 satisfy $1 \leqslant t_1 < t_2 - 1$. Then*

$$\inf_{x\in B_\delta(x_2)} \eta(x,t_2) \geqslant K_o^k \sup_{y\in B_\delta(x_1)} \eta(y,t_1) \tag{2.7}$$

where K_o is the constant from Theorem 2.1, depending only on θ, and k is an integer bounded by

$$k \leqslant 5\theta^2 (t_2 - t_1)(M_{x_1,t_1;x_2,t_2})^2 \left(\frac{|x_2 - x_1|}{t_2 - t_1} + 2\delta\right)^2.$$

Although the constant K_o is universal, the integer k and the constant $M_{x_1,t_1;x_2,t_2}$ depend on the x_1,t_1,x_2,t_2 and on the realization of V. Where V is large, these constants also become large. However, when applying Lemma 2.1 we will use the stationarity and ergodicity of V to show that, on the average, the constants are not too bad.

2.2. Continuity estimates

In this section we derive a continuity estimate on the function $\log u(x,t)$ that holds asymptotically as $t \to \infty$. By the maximum principle, $u > 0$ for all (x,t), and we define $\xi(x,t,\hat\omega) = f(u(x,t,\hat\omega))/u(x,t,\hat\omega)$. Therefore, Eq. (1.1) may be written as

$$\partial_t u = \Delta u + V(x,t,\hat\omega)\cdot\nabla u + \xi(x,t,\hat\omega)u \tag{2.8}$$

where $\xi(x,t,\cdot) \in L^\infty(\hat\Omega; L^\infty(\mathbb{R}^{d+1}))$ and $\xi(x,t,\hat\omega) \in [0, f'(0)]$, almost surely with respect to $\hat P$. In fact, the regularity of u implies that $\xi(x,t,\hat\omega)$ is locally C^1, almost surely. For the following estimates, however, we assume only that $\xi(\cdot,\cdot,\hat\omega)$ is almost surely continuous and that

$$\bigl|\xi(x,t,\hat\omega)\bigr| \leqslant C\bigl(1+\bigl|V(x,t,\hat\omega)\bigr|\bigr) \tag{2.9}$$

for some deterministic constant C, $\hat P$-almost surely, for all (x,t).

Proposition 2.1. *Let $u(x,t,\hat\omega) > 0$ solve* (2.8) *such that $\xi(x,t,\hat\omega)$ satisfies* (2.9)*. There is a set of full measure $\hat\Omega_0 \subset \hat\Omega$, $\hat P(\hat\Omega_0) = 1$, such that the following holds*: *if $\gamma(t) \geqslant 0$ is any nondecreasing function satisfying $\limsup_{t\to\infty}\gamma(t)/t \leqslant \epsilon$, then for any $c \in \mathbb{R}^d$*

$$\liminf_{t\to\infty}\frac{1}{t}\Bigl(\log\inf_{|z|\leqslant\gamma(t)}u(ct+z,t) - \log\sup_{y\in B_\delta(c(t-\gamma(t)))}u\bigl(y,t-\gamma(t)\bigr)\Bigr) \geqslant -C\bigl(1+|c|+\delta\bigr)^2\epsilon(1+\bar V_2)$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\Bigl(\log\sup_{|z|\leqslant\gamma(t)}u(ct+z,t) - \log\inf_{y\in B_\delta(c(t+\gamma(t)))}u\bigl(y,t+\gamma(t)\bigr)\Bigr) \leqslant C\bigl(1+|c|+\delta\bigr)^2\epsilon(1+\bar V_2)$$

for all $\hat\omega \in \hat\Omega_0$. Here, $\bar V_2$ is defined by (1.3) *and $C = C(\theta)$ is a constant.*

To prove this continuity estimate we will make use of the following estimates on the growth of the vector field V as $t \to \infty$:

Lemma 2.2. *Almost surely with respect to $\hat P$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\sup_{\substack{t\in[j,j+1]\\ x\in\mathbb{R}^d}}\bigl|V(x,t,\hat\omega)\bigr|^2 = E_{\hat P}\Bigl[\sup_{\substack{t\in[0,1]\\ x\in\mathbb{R}^d}}\bigl|V(x,t,\hat\omega\bigr|^2\Bigr] = \bar V_2 < \infty. \tag{2.10}$$

Proof. Due to the moment bound (1.3), this follows from the ergodic theorem and the assumption that V is stationary and ergodic with respect to shifts in x and t. $\square$

Corollary 2.2. *There is a set of full measure $\hat\Omega_0 \subset \hat\Omega$, $\hat P(\hat\Omega_0) = 1$, such that*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=k_n}^{n-1}\sup_{\substack{t\in[j-1,j]\\ x\in\mathbb{R}^d}}|V|^2 \leqslant \epsilon\bar V_2 \tag{2.11}$$

whenever $\epsilon \in [0,1)$ and $\{k_n\}_{n=1}^\infty$ is a nondecreasing sequence of positive integers satisfying $k_n \leqslant n$ for all n and $\liminf_{n\to\infty}k_n/n \geqslant (1-\epsilon)$.

Proof. This follows from Lemma 2.2 and the fact that the number of terms in the sum grows more slowly than $O(\epsilon n)$. Specifically,

$$\frac{1}{n}\sum_{j=k_n}^{n-1}\sup_{\substack{t\in[j-1,j]\\ x\in\mathbb{R}^d}}|V|^2 = \frac{1}{n}\sum_{j=1}^{n-1}\sup_{\substack{t\in[j-1,j]\\ x\in\mathbb{R}^d}}|V|^2 - \Bigl(\frac{k_n-1}{n}\Bigr)\Bigl(\frac{1}{k_n-1}\Bigr)\sum_{j=1}^{k_n-1}\sup_{\substack{t\in[j-1,j]\\ x\in\mathbb{R}^d}}|V|^2$$

Now the result follows from Lemma 2.2 and the fact that $\liminf_{n\to\infty}(k_n - 1)/n \geqslant (1-\epsilon)$. $\square$

Proof of Proposition 2.1. We first prove the lower bound by a chaining argument. Let $\hat{\Omega}_0 \subset \hat{\Omega}$ be the set described in Corollary 2.2 with $\hat{P}(\hat{\Omega}_0) = 1$. Fix $c \in \mathbb{R}^d$ and suppose that $\limsup_{t\to\infty} \gamma(t)/t \leqslant \epsilon < 1$. Let $z_t \in \mathbb{R}^d$ satisfy $|z_t| \leqslant \gamma(t)$. Without loss of generality, we assume that $\gamma(t)$ takes values in $\mathbb{Z}$. For t sufficiently large, $t - \gamma(t) > 1$. Let $t_1 = t - \gamma(t)$, and $x_1 = ct_1 = ct - c\gamma(t)$. For $j = 2, \ldots, N_t = \gamma(t)$ define the points $(x_j, t_j) \in \mathbb{R}^{d+1}$ by

$$t_j = t_1 + j$$

and

$$x_j = \left(1 - \frac{j}{\gamma(t)}\right)x_1 + \frac{j}{\gamma(t)}(z_t + ct).$$

Notice that for $N_t = \gamma(t)$, $x_N = z_t + ct$, and that (x_j, t_j) is a sequence of equally spaced points in $\mathbb{R}^{d+1}$ along the line segment connecting (ct_1, t_1) to $(z_t + ct, t)$. Now we apply Lemma 2.1 for each pair of points (x_j, t_j), (x_{j+1}, t_{j+1}). Notice that

$$\left|\frac{x_{j+1} - x_j}{t_{j+1} - t_j}\right| = \left|\frac{z_t}{\gamma(t)} + c\right| \leqslant |c| + 1. \tag{2.12}$$

By applying Lemma 2.1 iteratively, we find that

$$\inf_{y\in B_\delta(z_t+ct)} u(y,t) \geqslant K_o^{k(t)} \sup_{y\in B_\delta(x_1)} u(y,t_1) \tag{2.13}$$

where $k(t) = \sum_{j=1}^{N_t} k_j$ and the numbers k_j are random variables bounded by

$$k_j \leqslant 5\theta^2(M_j)^2\left(\left|\frac{x_{j+1} - x_j}{t_{j+1} - t_j}\right| + 2\delta\right)^2 \leqslant 5\theta^2(M_j)^2\big(|c| + 1 + 2\delta\big)^2 \tag{2.14}$$

and the numbers M_j (also depending on $\hat{\omega}$) are

$$M_j = M_{x_j,t_j;x_{j+1},t_{j+1}}. \tag{2.15}$$

Although the choice of points (x_j, t_j) depends on z_t, the term $k = k(t)$ can be bounded, independently of the choice of z_t since

$$(M_{x_j,t_j;x_{j+1},t_{j+1}})^2 \leqslant C\Big(1 + \sup_{\substack{t\in[t_j-a,t_{j+1}+a]\\ x\in\mathbb{R}^d}} \big|V(x,t,\hat{\omega})\big|^2\Big) \tag{2.16}$$

for some integer $a \leqslant 5\theta^2$ (since $R \leqslant 1$). The right-hand side of (2.16) is now independent of the choice of z_t, and we can bound $\log(K_o^k)$ by

$$\begin{aligned}\log\big(K_o^{k(t)}\big) &\geqslant -\big|\log(K_o)\big|C_1 \sum_{j=1}^{N_t}(M_{x_j,t_j;x_{j+1},t_{j+1}})^2 \\ &\geqslant -\big|\log(K_o)\big|C_1C_2 \sum_{j=k_n}^{n-1}\Big(1 + \sup_{\substack{t\in[j-1,j]\\ x\in\mathbb{R}^d}} \big|V(x,t,\hat{\omega})\big|^2\Big)\end{aligned} \tag{2.17}$$

where $n \leqslant t + a + 1$ and $k_n \geqslant t - \gamma(t) - a - 1$ are integers satisfying $\liminf_{n\to\infty} k_n/n \geqslant 1 - \epsilon$ and $k_n \leqslant n$. The constant C_1 may be bounded uniformly by $C_1 \leqslant (5\theta^2(|c| + 1 + 2\delta)^2)$, and the constant C_2 depends only on the integer a (which depends only on θ). The right-hand side of (2.17) is independent of the choice of z_t, as long as $|z_t| \leqslant \gamma(t)$.

Inequalities (2.13) and (2.17) now imply that

$$\frac{1}{t}\Big(\log \inf_{|z|\leqslant\gamma(t)} u(ct+z,t) - \log \sup_{y\in B_\delta(ct_1)} u(y,t_1)\Big) \tag{2.18}$$

$$\geqslant -\big|\log(K_o)\big|C_1C_2\frac{1}{n}\sum_{j=k_n}^{n-1}\Big(1+\sup_{\substack{t\in[j-1,j]\\ x\in\mathbb{R}^d}}\big|V(x,t,\hat\omega)\big|^2\Big).$$

Now we apply Corollary 2.2 to the sum on the right-hand side to conclude that

$$\frac{1}{t}\Big(\log \inf_{|z|\leqslant\gamma(t)} u(ct+z,t) - \log \sup_{y\in B_\delta(ct_1)} u(y,t_1)\Big) \geqslant C_3\epsilon(1+\bar V_2)$$

holds for any $\hat\omega\in\hat\Omega_0$, where $\hat\Omega_0$ has full measure. The constant C_3 now satisfies $C_3\leqslant C_4(|c|+1+\delta)^2$ for some other constant C_4 depending only on θ. This proves the lower bound.

The upper bound can be proved by following the same argument, except that Lemma 2.1 is applied forward in time along points $(x_j,t_j)\in\mathbb{R}^{d+1}$ defined by

$$x_j=\left(1-\frac{j}{\gamma(t)}\right)(z_t+ct)+\frac{j}{\gamma(t)}\big(ct+c\gamma(t)\big),\quad t_j=t+j \tag{2.19}$$

for $j=1,\dots,N_t=\gamma(t)$. Thus, $(x_1,t_1)=(z_t+ct,t)$ and $(x_N,t_N)=(c(t+\gamma(t)),t+\gamma(t))$. The remaining details are the same as in the case of the lower bound. □

2.3. *Large deviation estimates*

For $\delta>0$, $x\in\mathbb{R}^d$, and $t\geqslant s\geqslant 0$, let $\phi(y,t;x,s)=\phi(y,t;x,s,\hat\omega)$ satisfy the advection–diffusion equation

$$\partial_t\phi=\Delta_y\phi+V\cdot\nabla\phi \tag{2.20}$$

for $t>s$ with the initial condition

$$\phi(y,s;x,s,\hat\omega)=\begin{cases}1 & y\in B_\delta(x),\\ 0 & \text{otherwise}\end{cases} \tag{2.21}$$

at time $t=s$, where $\delta>0$ is a fixed parameter. In this section we will derive tail estimates on ϕ that we will later use to bound the solution $u(x,t,\hat\omega)$. The main result of this section is the following:

Theorem 2.2. *There is a set of full measure $\hat\Omega_0\subset\hat\Omega$, $\hat P(\hat\Omega_0)=1$, and a convex function $H(c):\mathbb{R}^d\to[0,\infty)$ such that the following holds. For any open set $G\subset\mathbb{R}^d$,*

$$\liminf_{t\to\infty}\frac{1}{t}\log\inf_{z\in tG}\phi(z,t;0,0,\hat\omega)\geqslant-\inf_{c\in G^o}H(c) \tag{2.22}$$

and for any closed set $F\subset\mathbb{R}^d$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{z\in tF}\phi(z,t;0,0,\hat\omega)\leqslant-\inf_{c\in\bar F}H(c) \tag{2.23}$$

for all $\hat\omega\in\hat\Omega_0$.

The function H appearing here is the same H described in Theorem 1.3. Later in Section 4 we will show that this function H is characterized as in Theorems 1.2 and 1.3.

Remark 2.2. The function $\phi(x,t;0,0)$ depends on the parameter δ. However, using the stationarity of the field $V(x,t)$ and the linearity of the equation for $\phi(x,t;0,0)$, one can show that the function $H(c)$ is actually independent of δ and that Theorem 2.2 holds for any such ϕ with nonnegative, compactly supported initial data.

The proof of Theorem 2.2 will rely on the following lemma:

Lemma 2.3. *There is a set of full measure $\hat{\Omega}_0 \subset \hat{\Omega}$, $\hat{P}(\hat{\Omega}_0) = 1$, and a convex function $H(c) : \mathbb{R}^d \to [0, \infty)$ such that the following holds: If $\gamma(t) \geqslant 0$ is any nondecreasing function satisfying $\limsup_{t\to\infty} \gamma(t)/t \leqslant \epsilon$, then for any $c \in \mathbb{Q}^d$*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{|z|\leqslant\gamma(t)} \phi(ct + z, t; 0, 0) \leqslant C\big(1 + |c| + \delta\big)^2 \epsilon(1 + \bar{V}_2) - H(c), \tag{2.24}$$

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{|z|\leqslant\gamma(t)} \phi(ct + z, t; 0, 0) \geqslant -C\big(1 + |c| + \delta\big)^2 \epsilon(1 + \bar{V}_2) - H(c) \tag{2.25}$$

for all $\hat{\omega} \in \hat{\Omega}_0$. Here, $\bar{V}$ is defined by (1.3) *and $C = C(\theta)$ is a constant.*

Proof of Lemma 2.3. Define the family of functions

$$\phi^-(y, t; x, s) = \inf_{y' \in B_\delta(y)} \phi(y', t; x, s). \tag{2.26}$$

(For clarity we will suppress the dependence of ϕ and ϕ^- on $\hat{\omega}$.) By the maximum principle, it is easy to see that for any $x, y, z \in \mathbb{R}^d$ and $r < s < t$,

$$\phi^-(z, t; x, r) \geqslant \phi^-(y, s; x, r)\phi^-(z, t; y, s). \tag{2.27}$$

For $c \in \mathbb{R}^d$ fixed, define the random process $q_{m,n}(\hat{\omega}) = \log \phi^-(cm, m; cn, n, \hat{\omega})$ indexed by $m, n \in \mathbb{Z}$, $0 \leqslant m < n$. We observe that $q_{m,n}$ is stationary and superadditive:

$$\begin{aligned} &q_{m,n} \geqslant q_{m,k} + q_{k,n}, \quad \forall m < k < n, \\ &q_{m+r,n+r}(\hat{\omega}) = q_{m,n}\big(\tau_{(cr,r)}\hat{\omega}\big). \end{aligned} \tag{2.28}$$

We will show in Lemma 2.4,

$$E\big[|q_{0,n}|\big] < \infty \tag{2.29}$$

for all n. Therefore, from the ergodic theorem (e.g. [1]) it now follows that the limit

$$-H(c) \triangleq \lim_{n\to\infty} \frac{1}{n} q_{0,n} = \sup_{n>0} \frac{1}{n} q_{0,n} \leqslant 0 \tag{2.30}$$

exists almost surely and is nonrandom. The convexity of H follows from the subadditivity relationship (2.27), as in [24].

Lemma 2.4. *For any $c \in \mathbb{R}^d$, $\delta > 0$, and any integer $n \geqslant 1$, $E[|q_{o,n}|] < \infty$.*

Proof of Lemma 2.4. We will iteratively apply Lemma 2.1 to the function $\phi(y, t; 0, 0)$. First, we claim that

$$E\Big[\Big|\log \inf_{y\in B_\delta(c)} \phi(y, 1; 0, 0, \hat{\omega})\Big|\Big] < \infty. \tag{2.31}$$

(Here $t = 1$.) To prove this, consider the function $\rho(\lambda, t, \hat{\omega})$ defined by

$$\rho(\lambda, t, \hat{\omega}) = t|\lambda|^2 + \int_0^t \sup_{x\in\mathbb{R}^d} \big|\lambda \cdot V(x, s, \hat{\omega})\big|\, ds. \tag{2.32}$$

It is easy to verify that the function $\eta = e^{-\lambda\cdot x + \rho(\lambda,t)}$ satisfies $\partial_t \eta \leqslant \mathcal{L}\eta$ for all $t > 0$. So, for any $x, \lambda \in \mathbb{R}^d$, the maximum principle implies that $\phi(x, t; 0, 0) \leqslant e^{|\lambda|\delta} e^{-\lambda\cdot x + \rho(\lambda,t)}$. For $t = 1$, we may construct an upper bound on $\phi(x, t; 0, 0)$ using multiple such λ with $|\lambda| = 1$. This implies that

$$\int_{|x|\geqslant R} \phi(x, 1/2; 0, 0)\, dx \leqslant K e^{\bar{\rho}(1/2)} e^{-R} R^{d-1}$$

where $\bar{\rho}(1/2) = 1/2 + \int_0^{1/2} \sup_x |V(x,s)|\,ds$. Therefore, there is a constant K such that for $R > K + 4\bar{\rho}(1/2)$, the right-hand side is bounded by $1/2 \int \phi_0(x)\,dx > 0$, where $\phi_0(x) = \phi(x,0;0,0)$. From the incompressibility of $V(x,t,\hat{\omega})$, we see that the integral of ϕ is preserved for all $t > 0$. Thus

$$\int\limits_{|x|\leqslant R} \phi(x,1/2;0,0)\,dx \geqslant \frac{1}{2}\int \phi_0(x)\,dx$$

and therefore, $\sup_{|x|\leqslant R} \phi(x,1/2;0,0) \geqslant CR^{-d}\frac{1}{2}\int \phi_0(x)\,dx$. Lemma 2.1 now implies that

$$\inf_{|x|\leqslant R} \phi(x,1;0,0) \geqslant K_o^k C R^{-d}\frac{1}{2}\int \phi_0(x) \tag{2.33}$$

where k is bounded by

$$k \leqslant C_2\Bigl(1 + \sup_{\substack{x\in\mathbb{R}^d\\ t\in[0,3]}} \bigl|V(x,t)\bigr|\Bigr)^2. \tag{2.34}$$

Since the right-hand side of (2.34) is integrable with respect to $\hat{P}$, by assumption (1.3), the lower bound (2.33) implies (2.31).

Next, for any integer $j \geqslant 1$, define $x_j = cj$ and $t_j = j$, and let

$$M_j = M_{x_j,t_j;x_{j+1},t_{j+1}} \tag{2.35}$$

where $M_{x_j,t_j;x_{j+1},t_{j+1}}$ is given by (2.5). Now if we apply Lemma 2.1 iteratively, once at each of the $n-1$ intervals $[j, j+1]$, $j = 1,\dots,n-1$, we see that

$$\log \inf_{y\in B_\delta(cn)} \phi(y,n;0,0) \geqslant \log \sup_{y\in B_\delta(c(1))} \phi(y,1;0,0) + \log(K_o)\sum_{j=1}^{n} k_j \tag{2.36}$$

where the numbers k_j are bounded by

$$k_j \leqslant 5\theta^2(t_{j+1}-t_j)(M_j)^2\left(\left|\frac{x_{j+1}-x_j}{t_{j+1}-t_j}\right| + 2\delta\right)^2 = 5\theta^2(M_j)^2\bigl(|c| + 2\delta\bigr)^2(t_{j+1}-t_j).$$

The k_j are the exponents from estimate (2.7) when we replace $(x_1,t_1;x_2,t_2)$ by $(x_j,t_j;x_{j+1},t_{j+1})$.

Since each M_j is square integrable by assumption (1.3), it follows that the sum $\sum_{j=1}^{n} k_j$ is integrable. This implies that

$$E\bigl[|q_{o,n}|\bigr] = E\Bigl[\Bigl|\log \inf_{y\in B_\delta(cn)} \phi(y,n;0,0)\Bigr|\Bigr] < \infty$$

if (2.31) holds. This proves Lemma 2.4. □

So far we have shown that for a given $c \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{1}{n} q_{0,n} = -H(c) \tag{2.37}$$

holds almost surely with respect to $\hat{P}$, as n runs through the integers. Using (2.7), we see that for any $t \geqslant 1$

$$\inf_{\substack{y\in B_\delta(c(t+r))\\ r\in[1,2]}} \phi(y,t+r;0,0) \geqslant K_o^{k(t)} \sup_{y\in B_\delta(ct)} \phi(y,t;0,0) \tag{2.38}$$

for some number $k(t)$ that can be bounded by $k(t) \leqslant 10(\theta^2)(M_{ct,t;c(t+2),(t+2)})^2(|c|+\delta)^2$. However, this bound and (2.37) imply that both

$$\lim_{t\to\infty} \frac{1}{t} \log \inf_{y\in B_\delta(ct)} \phi(y,t;0,0) = -H(c), \tag{2.39}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\log\sup_{y\in B_\delta(ct)}\phi(y,t;0,0)=-H(c),\tag{2.40}$$

holds along continuous time provided that $\limsup_{t\to\infty}\frac{k(t)}{t}=0$. Since the random variable $M_t=M_{ct,t;c(t+2),(t+2)}$ is square integrable and stationary with respect to shifts in t, the ergodic theorem implies that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(M_n)^2=E\big[(M_1)^2\big]<\infty.$$

Therefore,

$$\limsup_{t\to\infty}\frac{k(t)}{t}=\lim_{t\to\infty}\frac{(M_t)^2}{t}=0$$

almost surely.

This proves Lemma 2.3 for $\gamma(t)\equiv\delta$ and $c\in\mathbb{R}^d$ fixed. For the general case with $\limsup_{t\to\infty}\gamma(t)/t\leqslant\epsilon$ and $|z_t|\leqslant\gamma(t)$, we may prove (2.24) and (2.25) by applying the continuity estimates in Proposition 2.1 to the function $\phi(y,t;0,0)$ (in this case, $\xi(x,t,\hat\omega)\equiv 0$). From the lower bound in Proposition 2.1, we see that there is a set $\hat\Omega_o$ of full measure such that

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\Big(\log\inf_{|z|\leqslant\gamma(t)}\phi(ct+z,t;0,0)-\log\sup_{y\in B_\delta(c(t-\gamma(t)))}\phi\big(y,t-\gamma(t);0,0\big)\Big)\\&\quad\geqslant -C\big(1+|c|+\delta\big)^2\epsilon(1+\bar V_2)\end{aligned}\tag{2.41}$$

holds for all $c\in\mathbb{R}^d$. From (2.40) and (2.41), it now follows that for any fixed $c\in\mathbb{R}^d$

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\log\inf_{|z|\leqslant\gamma(t)}\phi(ct+z,t;0,0)\\&\quad\geqslant -C\big(1+|c|+\delta\big)^2\epsilon(1+\bar V_2)+\liminf_{t\to\infty}\frac{(t-\gamma(t))}{t}\frac{1}{(t-\gamma(t))}\log\sup_{y\in B_\delta(c(t-\gamma(t)))}\phi\big(y,t-\gamma(t);0,0\big)\\&\quad\geqslant -C\big(1+|c|+\delta\big)^2\epsilon(1+\bar V_2)-H(c)\end{aligned}\tag{2.42}$$

holds almost surely with respect to $\hat P$. (Note that since $H(c)\geqslant 0$, we have discarded the extra $\epsilon H(c)$ term that comes from the factor $\gamma(t)/t$.) Similarly, the upper bound in Proposition 2.1 and (2.39) imply that for any fixed $c\in\mathbb{R}^d$

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\sup_{|z|\leqslant\gamma(t)}\phi(ct+z,t;0,0)\\&\quad\leqslant C\big(1+|c|+\delta\big)^2\epsilon(1+\bar V_2)+\limsup_{t\to\infty}\frac{(t+\gamma(t))}{t}\frac{1}{(t+\gamma(t))}\log\inf_{y\in B_\delta(c(t+\gamma(t)))}\phi\big(y,t-\gamma(t);0,0\big)\\&\quad\leqslant C\big(1+|c|+\delta\big)^2\epsilon(1+\bar V_2)-H(c).\end{aligned}\tag{2.43}$$

The subset of $\hat\Omega$ on which this convergence holds depends on c. However, by taking the countable union of all such subsets for $c\in\mathbb{Q}^d$, we obtain a set $\hat\Omega_0$, $\hat P(\hat\Omega_0)=1$, such that both (2.42) and (2.43) hold for all $c\in\mathbb{Q}^d$ and all $\hat\omega\in\hat\Omega_0$. This completes the proof of Lemma 2.3. □

Proof of Theorem 2.2. We first prove the upper bound (2.23). Suppose that F is compact. For any $\epsilon>0$, there is a finite set $\{c_j\}_{j=1}^N\subset\mathbb{Q}^d$, such that $F\subset\bigcup_{j=1}^N B_\epsilon(c_j)$. Therefore,

$$\sup_{z\in tF}\phi(z,t;0,0)\leqslant\sup_{j=1,\dots,N}\sup_{|z|\leqslant\epsilon t}\phi(c_jt+z,t;0,0).$$

Since N is finite, and F is compact, (2.24) now implies that

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{z\in tF}\phi(z,t;0,0)\leqslant\limsup_{t\to\infty}\frac{1}{t}\log\sup_{j=1,\dots,N}\sup_{|z|\leqslant\epsilon t}\phi(c_jt+z,t;0,0)$$
$$=-H(c)+\mathrm{O}(\epsilon).$$

So, for compact F, we obtain the upper bound (2.23) by letting $\epsilon\to 0$. The case of general closed F follows from Lemma 4.1.

The proof of the lower bound (2.22) is similar to the preceding argument, except that we invoke (2.25) instead of (2.24). This completes the proof of Theorem 2.2. □

3. Proof of Theorem 1.1

3.1. The upper bound (1.4)

The upper bound (1.4) of Theorem 1.1 follows easily form Theorem 2.2. Let $\delta>0$ be large enough so that the support of u_0 is contained in the ball $B_\delta(0)$. Then by the maximum principle,

$$u(y,t)\leqslant e^{tf'(0)}\phi(y,t;0,0)=e^{t(f'(0)+\frac{1}{t}\log\phi(y,t;0,0))}. \tag{3.1}$$

Let F be a closed set satisfying $F\subset\mathbb{R}^d\setminus\bar{G}$ where G is the bounded, convex set

$$G=\{c\in\mathbb{R}^d\mid H(c)\leqslant f'(0)\}.$$

Now, by Theorem 2.2,

$$\lim_{t\to\infty}\frac{1}{t}\log\sup_{c\in F}\phi(ct,t;0,0)<-f'(0). \tag{3.2}$$

Combining this with (3.1), we have $\lim_{t\to\infty}\sup_{c\in F}u(ct,t)=0$, which proves (1.4). □

3.2. The lower bound (1.5)

To prove the lower bound (1.5) we will use the following lower bound on the decay rate of the solution $u(x,t,\hat{\omega})$ beyond the front interface. This bound is modeled after a similar estimate of Freidlin in the case of steady, spatially periodic drift (see Lemma 3.3 of [10]), and it relies on the assumption that $f'(0)>0$, which holds for the KPP-type nonlinearity.

Lemma 3.1. *For any compact set $K\subset\{c\in\mathbb{R}^d\mid H(c)-f'(0)>0\}$,*

$$\liminf_{t\to\infty}\frac{1}{t}\log\inf_{c\in K}u(ct,t)\geqslant-\max_{c\in K}\bigl(H(c)-f'(0)\bigr) \tag{3.3}$$

holds almost surely with respect to the measure $\hat{P}$.

We will postpone the proof of Lemma 3.1 and conclude the proof of the lower bound (1.5). In the following step, we construct subsolutions and use a comparison argument to show that $u\nearrow 1$ behind the interface. For each $s\geqslant 0$, define the bounded convex set $\Gamma_s\subset\mathbb{R}^d$ by

$$\Gamma_s=\{c\in\mathbb{R}^d\mid H(c)\leqslant s\}. \tag{3.4}$$

Let $\epsilon_1>0$ and set $s_1=f'(0)-\epsilon_1$. For $h\in(0,1)$, we will show that

$$\lim_{t\to\infty}\inf_{c\in\Gamma_{s_1}}u(ct,t)\geqslant h \tag{3.5}$$

since ϵ_1 and h are arbitrarily chosen, this implies the lower bound (1.5).

Now we construct a subsolution to (2.8) to which we will compare u and obtain (3.5). Let $h\in(0,1)$ be fixed. Let us define the set

$$J_h(t)=\{x\in\mathbb{R}^d\mid u(x,t)<h\} \tag{3.6}$$

for each $t > 0$. The boundary of $J_h(t)$ (if there is a boundary) is the level set defined by $u(\cdot, t) = h$, and this level set must be bounded, by the established upper bound on u. For $\epsilon_2 > 0$, let $s_2 = f'(0) + \epsilon_2$. Let $J^1(t)$ and $J^2(t)$ denote the sets

$$J^1(t) = J_h(t) \cap t\Gamma_{s_1}, \qquad J^2(t) = J_h(t) \cap t\Gamma_{s_2}.$$

Notice that these sets are bounded at each t, and that $J^1(t) \subset J^2(t)$ for all t whenever the sets are nonempty, since $\Gamma_{s_1} \subset \Gamma_{s_2}$. Lemma 3.1 and the maximum principle imply that we can take ϵ_2 sufficiently small and $t_0 > 0$ sufficiently large so that

$$\inf_{c \in \Gamma_{s_2}} u(ct, t) \geqslant e^{-t2\epsilon_2} \tag{3.7}$$

for all $t \geqslant t_0$. Thus, $\inf_{x \in J^2(t)} u(x, t) \geqslant e^{-t2\epsilon_2}$ also holds for $t \geqslant t_0$.

Let us define the positive number $\xi_h = \inf_{u \in (0,h]} f(u)/u$. Thus, $\xi_h \to f'(0)$ as $h \to 0$. For given $h \in (0, 1)$, t_0, and a parameter $\kappa \in (0, 1)$ to be chosen, we will compare the solution $u(x, t)$ with a function $\psi(x, t; t_0)$ of the form

$$\psi(x, t; t_0) = h\phi(x, t; t_0) - g_0 e^{-\xi_h(t-t_0)} u(x, t). \tag{3.8}$$

We will compare $u(x, t)$ and $\psi(x, t; t_0)$ for $x \in J^2(t)$ and $t \in [t_0, (1 + \kappa)t_0]$. The family of functions $\phi(x, t; t_0)$ will be chosen to satisfy the following properties:

$$\begin{aligned}
&\text{(i)}\ \ \partial_t \phi \leqslant \mathcal{L}\phi \text{ for all } x \in \mathbb{R}^d \text{ and } t > t_0.\\
&\text{(ii)}\ \ \phi(x, t; t_0) \leqslant 1, \text{ for all } (x, t).\\
&\text{(iii)}\ \ \phi(x, t; t_0) \leqslant 0 \text{ for all } x \in t\partial\Gamma_{s_2} \text{ and } t \in [t_0, (1 + \kappa)t_0].\\
&\text{(iv)}\ \ \lim_{t_0 \to \infty} \inf_{c \in \Gamma_{s_1}} \phi\big(c(1 + \kappa)t_0, (1 + \kappa)t_0; t_0\big) = 1.
\end{aligned} \tag{3.9}$$

The constant g_0 will be positive. We choose the constant ϵ_2 (appearing in (3.7)) sufficiently small so that $2\epsilon_2 < \xi_h \kappa$. Thus, ϵ_2 and Γ_{s_2} depend on the choice of κ and h. Then we set $g_0 = h e^{2\epsilon_2 t_0}$.

A straightforward calculation using property (i) shows that $\psi(x, t; t_0)$ satisfies

$$\begin{aligned}
\partial_t \psi &\leqslant \mathcal{L}\psi + \xi\psi - \xi h\phi + \xi_h g_0 e^{-\xi_h(t-t_0)}\\
&= \mathcal{L}\psi + \xi\psi - h\phi(\xi - \xi_h) - \xi_h \psi
\end{aligned} \tag{3.10}$$

for $t \geqslant t_0$. For any $x \in J_h(t)$, $\xi(x, t) \geqslant \xi_h > 0$, by definition of ξ_h. Also, since $u > 0$ and $g_0 > 0$, (3.8) implies that $\phi(x, t) > 0$ wherever $\psi(x, t) \geqslant 0$. So, if $x \in J_h(t)$ and $\psi(x, t) \geqslant 0$, (3.10) implies that ψ must satisfy the inequality

$$\partial_t \psi \leqslant \mathcal{L}\psi + \xi\psi \tag{3.11}$$

at the point (x, t). So, the function ψ is a subsolution to the equation solved by u in the region of interest.

The function ψ also takes values less than $u(x, t)$ on the parabolic boundary of the region of interest. If the boundary $\partial J_h(t)$ is nonempty and $x \in \partial J_h(t)$, then $u(x, t) = h \geqslant h\phi(x, t) \geqslant \psi(x, t)$. Since $g_0 > 0$, $\psi(x, t; t_0) \leqslant 0 < u(x, t)$ for all $x \in t\partial\Gamma_{s_2}$ and $t > t_0$. Moreover, by the choice of g_0 and $\phi(x, t; t_0) \leqslant 1$, ψ satisfies

$$\psi(x, t_0; t_0) \leqslant u(x, t_0), \quad \forall x \in J^2(t_0), \tag{3.12}$$

since u satisfies the lower bound (3.7).

Inequality (3.11) holds if $x \in J^2(t)$ and $\psi(x, t) \geqslant 0$. Since $u > 0$ and $\partial_t u = \mathcal{L}u + \xi u$, the maximum principle implies that $u(x, t) \geqslant \psi(x, t; t_0)$ for all $x \in J^2(t)$ and $t \in [t_0, (1 + \kappa)t_0]$. From (3.9) and the definition of ψ we see that

$$\lim_{t_0 \to \infty} \inf_{x \in J^1(t)} \psi\big(x, (1 + \kappa)t_0; t_0\big) = h. \tag{3.13}$$

Here we have used the fact that $2\epsilon_2 < \xi_h \kappa$. Since $u(x, t) \geqslant h$ for all $x \in (J_h(t))^C$, the limit (3.13) now implies that

$$\lim_{t_0 \to \infty} \inf_{c \in \Gamma_{s_1}} u\big(c(1 + \kappa)t_0, (1 + \kappa)t_0\big) \geqslant h. \tag{3.14}$$

This is equivalent to the desired bound (3.5).

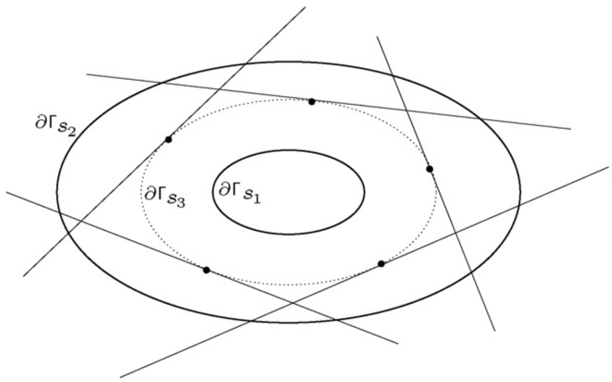

Fig. 1. The convex sets $\partial\Gamma_{s_1}$, $\partial\Gamma_{s_2}$, and $\partial\Gamma_{s_3}$. The points represent c_j, $j = 1, \ldots, N_c$, and the lines represent the sets $\{c \mid \lambda_j \cdot (c - c_j) = 0\}$. The region bounded by these line segments represents $\bigcap_j \{c \mid \lambda_j \cdot (c - c_j) > 0\}$.

Therefore, to complete the proof, we must construct the function $\phi(x, t; t_0)$ satisfying the desired properties. Set $\epsilon_3 = (\epsilon_1)/2$ and $s_3 = f'(0) - \epsilon_3$, so that $\Gamma_{s_1} \subset \Gamma_{s_3} \subset \Gamma_{s_2}$. Since Γ_{s_1}, Γ_{s_2} and Γ_{s_3} are convex, we can choose finite sets $\{c_j\}_{j=1}^{N_c} \subset \Gamma_{s_3}$ and $\lambda_j \subset \mathbb{R}^d$ such that both

$$\Gamma_{s_1} \subset \bigcap_{j=1}^{N_c} \{c \in \mathbb{R}^d \mid \lambda_j \cdot (c - c_j) > 0\} \tag{3.15}$$

and

$$\operatorname{dist}\left(\partial\Gamma_{s_2}, \bigcap_j \{c \in \mathbb{R}^d \mid \lambda_j \cdot (c - c_j) > 0\}\right) > 0 \tag{3.16}$$

are satisfied. Notice that properties (3.15) and (3.16) depend on the orientation of the λ_j but not on the magnitude of the λ_j. Also, notice that the sets $\partial\Gamma_{s_1}$ and $\partial\Gamma_{s_2}$ are both bounded away from the set $\partial\Gamma_{s_3}$ by a distance that is independent of κ. The sets Γ_{s_1} and Γ_{s_3} (and the vectors $\{c_j\}$, $\{\lambda_j\}$) do not depend on κ. The sets Γ_{s_1}, Γ_{s_2}, and Γ_{s_3} are depicted in Fig. 1.

Now for fixed t_0, let $x_j = c_j t_0$, and consider the function $\phi(x, t; t_0)$ defined by

$$\phi(x, t; t_0) = 1 - \sum_{j=1}^{N_c} e^{-\lambda_j \cdot (x - x_j) - \bar{\rho}(\lambda_j) t_0 + \rho(\lambda_j, t)} \tag{3.17}$$

where the function $\rho(\lambda, t, \hat{\omega})$ is defined by (2.32), and

$$\bar{\rho}(\lambda_j) = |\lambda_j|^2 + E\Big[\sup_{x \in \mathbb{R}^d} \big|\lambda_j \cdot V(x, 0, \hat{\omega})\big|\Big]. \tag{3.18}$$

It is easy to verify that $\partial_t \phi \leqslant \mathcal{L}\phi$ for all $t > t_0$. Thus, property (i) holds. Clearly property (ii) is satisfied, as well.

Now we verify properties (iii) and (iv) for $\phi(x, t; t_0)$. Since the sets $\partial\Gamma_{s_1}$ and $\partial\Gamma_{s_2}$ are both bounded away from the set $\partial\Gamma_{s_3}$ by a distance that is independence of κ, it follows from (3.15) and (3.16) that for κ sufficiently small there exists $\delta_1 > 0$ such that

$$\inf_{j \in \{1, \ldots, N_c\}} \inf_{c \in \Gamma_{s_1}} \lambda_j \cdot \left(c - \frac{c_j}{(1 - \kappa)}\right) > \delta_1 \tag{3.19}$$

is satisfied and such that

$$\inf_{c \in \Gamma_{s_2}} \sup_{j \in \{1, \ldots, N_c\}} -\lambda_j \cdot \left(c - \frac{c_j}{(1 - \kappa)}\right) > \delta_1 \tag{3.20}$$

is also satisfied.

From the ergodic theorem, we see that

$$\lim_{t\to\infty} \frac{1}{t}\rho(\lambda_j, t) = \bar{\rho}(\lambda)$$

holds almost surely with respect to $\hat{P}$. Define $R(\lambda, t) = |\bar{\rho}(\lambda) - \frac{1}{t}\rho(\lambda, t)|$, so that $|R(\lambda, t)| \to 0$ as $t \to \infty$, $\hat{P}$-almost surely.

Now by (3.19), we find that for each $j = 1, \dots, N_c$,

$$\begin{aligned}
&\sup_{c\in\Gamma_{s_1}} \frac{1}{(1+\kappa)t_0} \log e^{-\lambda_j\cdot(c(1+\kappa)-c_j)t_0-\bar{\rho}(\lambda_j)t_0+\rho(\lambda_j,(1+\kappa)t_0)} \\
&\quad\leqslant \left(\frac{\kappa}{1+\kappa}\right)\bar{\rho}(\lambda_j) + \sup_{c\in\Gamma_{s_1}} -\left(\lambda_j\cdot\left(c - \frac{c_j}{(1+\kappa)}\right)\right) + \left|R_j\big((1+\kappa)t_0\big)\right| \\
&\quad= \left(\frac{\kappa}{1+\kappa}\right)\bar{\rho}(\lambda_j) - \inf_{c\in\Gamma_{s_1}} \left(\lambda_j\cdot\left(c - \frac{c_j}{(1+\kappa)}\right)\right) + \left|R_j\big((1+\kappa)t_0\big)\right| \\
&\quad\leqslant \left(\frac{\kappa}{1+\kappa}\right)\bar{\rho}(\lambda_j) - \delta_1 + \left|R_j\big((1+\kappa)t_0\big)\right|.
\end{aligned} \tag{3.21}$$

Thus, by taking κ smaller, the right-hand side of (3.21) can be made negative, for all j, for t_0 sufficiently large. Therefore, returning to (3.17) we see that

$$\lim_{t_0\to\infty} \inf_{c\in\Gamma_{s_1}} \phi\big(c(1+\kappa)t_0, (1+\kappa)t_0; t_0\big) = 1.$$

This establishes (3.9).

Similarly, using (3.20) one can establish property (iii), as follows. We now find that

$$\begin{aligned}
&\inf_{\beta\in[0,\kappa]} \inf_{c\in\Gamma_{s_2}} \sup_j \frac{1}{(1+\beta)t_0} \log e^{-\lambda_j\cdot(c(1+\beta)-c_j)t_0-\bar{\rho}(\lambda_j)t_0+\rho(\lambda_j,(1+\beta)t_0)} \\
&\quad\geqslant \inf_{\beta\in[0,\kappa]} \inf_{c\in\Gamma_{s_2}} \sup_j \left(-\lambda_j\cdot\left(c - \frac{c_j}{(1+\beta)}\right) + \bar{\rho}(\lambda_j)\left(\frac{\beta}{1+\beta}\right)\right) - \sup_{\beta\in[0,\kappa]} \sup_j \left|R_j\big((1+\beta)t_0\big)\right|.
\end{aligned} \tag{3.22}$$

Using (3.20) and the fact that

$$\lim_{t_0\to\infty} \sup_{\beta\in[0,\kappa]} \sup_j \left|R\big(\lambda_j, (1+\beta)t_0\big)\right| = 0,$$

we may take κ sufficiently small and t_0 sufficiently large to make the right-hand side of (3.22) strictly positive. Then, returning to (3.17) we see that

$$\limsup_{t_0\to\infty} \sup_{\beta\in[0,\kappa]} \sup_{c\in\Gamma_{s_1}} \phi\big(c(1+\beta)t_0, (1+\beta)t_0; t_0\big) = -\infty. \tag{3.23}$$

This establishes property (iii). Having verified all the necessary properties for the family of functions $\phi(x, t; t_0)$, this completes the proof of the lower bound (1.5). □

Proof of Lemma 3.1. For $c \in \mathbb{R}^d$, $t - 1 \geqslant s \geqslant 0$ given, and $b > 0$ to be chosen, we define an auxiliary quantity $\phi_b^-(t; s, c)$ as follows. First, let $z_0 = c(t - s)/2$. Now, we will fix $b > c/2$ sufficiently large so that the ball $B_{b(t-s)}(z_0)$ contains both $\overline{B_\delta(cs)}$ and $\overline{B_\delta(ct)}$. For $z \in B_{b(t-s)}(z_0)$ and $\tau \in (s, t]$, let $\tilde{\phi}(z, \tau; s, t, c)$ satisfy

$$\partial_\tau \tilde{\phi} = \Delta_z \tilde{\phi} + V(z, \tau)\cdot\nabla\tilde{\phi} \tag{3.24}$$

with the initial condition

$$\tilde{\phi}(z, s; s, t, c) = \begin{cases} 1 & z \in B_\delta(cs), \\ 0 & \text{otherwise} \end{cases} \tag{3.25}$$

at time $\tau = s$, and Dirichlet boundary condition $\tilde{\phi}(z, \tau; s, t, c) = 0$ for $z \in \partial B_{b(t-s)}(z_0)$. Now define $\phi_b^-(t; s, c)$ by

$$\phi_b^-(t; s, c) = \inf_{y\in B_\delta(ct)} \tilde{\phi}(y, t; s, t, c). \tag{3.26}$$

Notice that the only difference between $\phi_b^-(t;s,c)$ and $\phi^-(ct,t;cs,s)$ (defined by (2.26)) is the Dirichlet boundary condition used in the definition of $\phi_b^-(t;s,c)$.

We will now make use of the following fact, which we prove later:

Theorem 3.1. *There is a set of full measure $\hat{\Omega}_0 \subset \hat{\Omega}$, $\hat{P}(\hat{\Omega}_0) = 1$, such that the following holds. For any $c \in \mathbb{Q}^d$ there is $b > 0$ sufficiently large so that for any $\kappa \in (0,1]$,*

$$\liminf_{t\to\infty} \frac{1}{\kappa t} \log \phi_b^-\bigl(t;(1-\kappa)t,c\bigr) = H(c) \tag{3.27}$$

for all $\hat{\omega} \in \hat{\Omega}_0$. The function $H(c)$ is the same as in Theorem 2.2 and Lemma 2.3.

Now we finish the proof Lemma 3.1. Pick $c \in K \cap \mathbb{Q}^d$. Thus, $H(c) > f'(0)$. Now take $b > 1 + |c|$ sufficiently large, as required by Theorem 3.1. The upper bound (1.4) on $u(x,t)$ implies that we may take $\kappa \in (0,1)$ sufficiently small and t sufficiently large so that

$$\xi(x,s) \geqslant \xi_h, \quad \forall x \in B_{b\kappa t}\left(\left(1 - \frac{\kappa}{2}\right)ct\right),\ s \in \bigl[(1-\kappa)t, t\bigr]. \tag{3.28}$$

The maximum principle implies that

$$\inf_{z\in B_\delta(ct)} u(z,t) \geqslant \bigl(e^{\kappa t \xi_h}\phi_b^-\bigl(t;(1-\kappa)t,c\bigr)\bigr) \inf_{y\in B_\delta(c(1-\kappa)t)} u\bigl(y,(1-\kappa)t\bigr). \tag{3.29}$$

We already know that

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{z\in B_\delta(ct)} u(z,t) \geqslant \liminf_{t\to\infty} \frac{1}{t} \log \inf_{z\in B_\delta(ct)} \phi(z,t;0,0),$$

which is finite since it is bounded below by $-H(c)$. Therefore, (3.29) and Theorem 3.1 imply that

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t} \log \inf_{z\in B_\delta(ct)} u(z,t) &\geqslant \xi_h + \liminf_{t\to\infty} \frac{1}{t} \log \phi_b^-\bigl(t;(1-\kappa)t,c\bigr)\\ &= \xi_h - H(c).\end{aligned}$$

Since the left-hand side is independent of h, we now let $h \to 0$ so that $\xi_h \to f'(0)$. Therefore,

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{z\in B_\delta(ct)} u(z,t) \geqslant f'(0) - H(c). \tag{3.30}$$

To finish the proof, we apply the continuity estimate of Proposition 2.1. For $\gamma(t) = \epsilon t$, the lower bound of Proposition 2.1 implies that

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t} \log \inf_{|z|\leqslant \epsilon t} u(ct+z,t) &\geqslant \liminf_{t\to\infty} \frac{1}{t} \log \inf_{y\in B_\delta(c(1-\epsilon)t} u\bigl(y,(1-\epsilon)t\bigr) - C\bigl(1+|c|+\delta\bigr)^2\epsilon\bigl(1+\|\xi\|_\infty+\bar{V}_2\bigr)\\ &= f'(0) - H(c) - \mathrm{O}(\epsilon).\end{aligned} \tag{3.31}$$

The last equality follows from (3.30).

Now we proceed as in the proof of Theorem 2.2. Since K is compact, we can pick $\epsilon > 0$ and a finite set $\{c_j\}_{j=1}^N \subset \mathbb{Q}^d$, such that

$$K \subset K'(\epsilon) \triangleq \bigcup_{j=1}^N B_\epsilon(c_j)$$

while ϵ is small enough so that $H(c) < f'(0) - \epsilon/2$ for all $c \in K'(\epsilon)$. Therefore,

$$\inf_{z\in tK} u(z,t) \geqslant \inf_{j=1,\dots,N} \inf_{|z|\leqslant\epsilon t} u(c_j t + z, t).$$

Since N is finite, and K is compact, (3.31) now implies the result (3.3). $\square$

Proof of Theorem 3.1. The only difference between $\tilde{\phi}(z,\tau;s,t,c)$ and $\phi(z,\tau;cs,s)$ (defined by (2.20)) is the Dirichlet boundary condition in the definition of $\tilde{\phi}$. Therefore, the maximum principle implies that for $\tau \in [s,t]$ and $z \in B_b(z_0)$ $\tilde{\phi}(z,\tau;s,t,c) \leqslant \phi(z,\tau;cs,s)$. For given $s<t$, let $\pi(z,\tau;s,t,c)$ be defined by

$$\pi(z,\tau;s,t,c)=\phi(z,\tau;cs,s)-\tilde{\phi}(z,\tau;s,t,c) \tag{3.32}$$

for $\tau \in (s,t]$ and $z \in B_b(z_0)$, $z_0=c(t-s)/2$. Then $\pi(z,\tau;s,t,c)$ satisfies $\partial_\tau \pi = \mathcal{L}\pi$ with

$$\begin{aligned} &\pi(z,s;s,t,c)=0, \quad \forall z \in \overline{B_b(z_0)},\\ &0<\pi(z,\tau;s,t,c)<1, \quad \forall z \in \partial B_b(z_0),\ \tau \in (s,t]. \end{aligned} \tag{3.33}$$

Now we choose $s=(1-\kappa)t$, and we claim that

$$\lim_{b\to\infty}\limsup_{t\to\infty}\frac{1}{\kappa t}\log\sup_{z\in B_\delta(ct)}\pi\big(z,t;(1-\kappa)t,t,c\big)=-\infty. \tag{3.34}$$

However, we already know that

$$\liminf_{t\to\infty}\frac{1}{\kappa t}\log\inf_{z\in B_\delta(ct)}\phi\big(z,t;c(1-\kappa)t,(1-\kappa)t\big)=-H(c)>-\infty. \tag{3.35}$$

Since $\pi(z,\tau;(1-\kappa)t,t,c)>0$ for all t, the combination of (3.34), (3.35) and the definition of π imply Theorem 3.1.

We prove the claim (3.34) for $\kappa=1$. The proof in the case $\kappa<1$ is similar. We compare $\pi(z,\tau;0,t,c)$ with a function $\eta(z,\tau)$ of the form

$$\eta(z,\tau)=\sum_{j=1}^{N}e^{-\lambda_j\cdot(z-z_j)+\rho(\lambda_j,\tau)} \tag{3.36}$$

where $\rho(\lambda_j,\tau)$ is defined by (3.17) (here, $t_0=(1-\kappa)t=0$). The function $\eta(z,\tau)$ satisfies $\partial_\tau\eta \geqslant \mathcal{L}\eta$. Next, we choose b, x_j and λ_j and use the maximum principle to show that $\eta(z,\tau)\geqslant\pi(z,\tau;0,t,c)$ for all $\tau>0$ whenever t and b are sufficiently large. The constructed function $\eta(z,\tau)$ depends on t, c, and b, but for clarity we suppress this dependence in the notation.

We choose $b>10(1+|c|)$. By choosing z_j in the set $\partial B_{bt/2}(z_0)$, we have $B_\delta(ct)\subset B_{bt/4}(z_0)$ so that

$$\inf_j\inf_{z\in B_\delta(ct)}\frac{|z-z_j|}{t}\geqslant b/4. \tag{3.37}$$

We choose the $\lambda_j\in\mathbb{R}^d$ independently of t so that $|\lambda_j|=1$ and

$$\inf_j\inf_{z\in B_{bt/4}(z_0)}\lambda_j\cdot\frac{(z-z_j)}{|z-z_j|}>Cb \tag{3.38}$$

and

$$\inf_{z\in\partial B_{bt}(z_0)}\inf_j-\lambda_j\cdot\frac{(z-z_j)}{|z-z_j|}>Cb \tag{3.39}$$

hold for all $t>1$, for some constant $C>0$.

Clearly $\eta(z,\tau)>0$ for all $z\in\mathbb{R}^d$, $\tau\geqslant 0$. Moreover, for b sufficiently large and t sufficiently large, $\eta(z,\tau)>1$ for all $z\in\partial B_{bt}(z_0)$. This follows from (3.39) since

$$\inf_{z\in\partial B_{bt}(z_0)}\inf_j e^{-\lambda_j\cdot(z-z_j)+\rho(\lambda_j,\tau)}\geqslant e^{Cbt+\rho(\lambda,\tau)}\geqslant e^{Cbt}. \tag{3.40}$$

So, we can take t sufficiently large so that the right-hand side of (3.40) is greater than 1 for all $\tau\in[0,t]$.

For any $z\in B_\delta(ct)\subset B_{bt/4}(z_0)$, (3.38) implies that

$$\begin{aligned}\frac{1}{t}\log\eta(z,t)&\leqslant N\max_{j=1,\dots,N}-\lambda_j\cdot\frac{(z-z_j)}{t}+\frac{\rho(\lambda_j,t)}{t}\\ &\leqslant N\max_{j=1,\dots,N}-Cb+\frac{\rho(\lambda_j,t)}{t}.\end{aligned}$$

Since $\lim_{t\to\infty} \rho(\lambda_j, t)/t = \bar\rho(\lambda_j)$ is finite, this implies that

$$\limsup_{b\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log \sup_{z\in B_\delta(ct)} \eta(z,t) = -\infty.$$

This implies the claim (3.34) since $\pi(z,t) \leqslant \eta(z,t)$ for $z \in B_\delta(ct)$. This completes the proof of Theorem 3.1. □

4. The Lyapunov exponent

In this section we prove Theorems 1.2, 1.3, and 1.4. For $\lambda \in \mathbb{R}^d$, let $\varphi = \varphi_\lambda$ be defined by (1.9) with $\varphi_\lambda(x,0) \equiv 1$. If $\eta_\lambda(x,t) = e^{-\lambda\cdot x}\varphi_\lambda(x,t)$, then η_λ solves

$$\partial_t(\eta_\lambda) = \Delta\eta_\lambda + V\cdot\nabla\eta_\lambda \tag{4.1}$$

with initial data $\eta_\lambda = e^{-\lambda\cdot x}$. When the dependence of η_λ and φ_λ on λ is clear from the context, we will just write η and φ respectively.

Lemma 4.1. *For any $c \in \mathbb{R}^d$,*

$$\phi(ct,t;0,0,\hat\omega) \leqslant \exp\left(-t\frac{(\bar V_t(\hat\omega) - |c| + \delta/t)^2}{4}\right) \tag{4.2}$$

for all $t > 0$, where $\bar V_t(\hat\omega) = \frac{1}{t}\int_0^t \sup_{y\in\mathbb{R}^d} |V(y,s,\hat\omega)|\,ds$ and $\lim_{t\to\infty} \bar V_t = E[\sup_x |V(x,0,\hat\omega)|]$ almost surely.

Proof. By the maximum principle, $\phi(x,t;0,0) \leqslant \eta_\lambda(x,t)e^{|\lambda|\delta}$. Therefore,

$$\phi(ct,t;0,0) \leqslant \bigl(\varphi_\lambda(ct,t)e^{-\lambda\cdot ct+|\lambda|\delta}\bigr). \tag{4.3}$$

By Grownwall's inequality, it is easy to see that

$$\sup_{x\in\mathbb{R}^d} \varphi_\lambda(x,t) \leqslant \exp\left(|\lambda|^2 t + \int_0^t \sup_{y\in\mathbb{R}^d} \bigl|\lambda\cdot V(y,s,\hat\omega)\bigr|\,ds\right) \tag{4.4}$$

so that by choosing $\lambda = r\frac{c}{|c|}$, we have

$$\begin{aligned}\phi(ct,t;0,0) &\leqslant e^{-\lambda\cdot ct+|\lambda|^2 t+\int_0^t \sup_{y\in\mathbb{R}^d} |\lambda\cdot V(y,s,\hat\omega)|\,ds+|\lambda|\delta} \\ &\leqslant e^{t(r^2+r(\bar V_t-|c|+\delta/t))}.\end{aligned} \tag{4.5}$$

The result follows by optimizing (4.5) over r. □

Proof of Theorem 1.3. We have already established that (1.7) holds. It remains to prove that H is characterized by

$$H(c) = \sup_{\lambda\in\mathbb{R}^d} \bigl(c\cdot\lambda - \mu(\lambda)\bigr). \tag{4.6}$$

Let $\phi(x,t) = \phi(x,t;0,0)$, and consider the family of probability measures P_t on $\mathbb{R}^d$ (for fixed $\hat\omega$) defined by

$$P_t(A) = \frac{1}{Z_t}\int_A \phi(ct,t)\,dc, \tag{4.7}$$

where Z_t is the normalizing constant $Z_t = \int_{\mathbb{R}^d} \phi(ct,t)\,dc = \frac{1}{t^d}\int_{\mathbb{R}^d} \phi(x,0)\,dx$. Using Theorem 2.2, one can show that (almost surely with respect to $\hat P$) the family of measures P_t satisfy a large deviation principle with rate function $H(c)$. Let $F(c) = \lambda\cdot c$. Then using Lemma 4.1, one can show that

$$\lim_{L\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log \int_{F(c)\geqslant L} e^{t\lambda\cdot c} P_t(dc) = -\infty.$$

Now, by Varadhan's Theorem (see [30], Section 3) the limit

$$\lim_{t\to\infty}\frac{1}{t}\log E_{P_t}\left[e^{tF(c)}\right]=\sup_{c\in\mathbb{R}^d}\left(F(c)-H(c)\right) \tag{4.8}$$

holds. Hence,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log\int_{\mathbb{R}^d}e^{t\lambda\cdot c}\phi(ct,t)\,dc&=\lim_{t\to\infty}\frac{1}{t}\log\int_{\mathbb{R}^d}e^{\lambda\cdot x}\phi(x,t)\,dx\\&=\sup_{c\in\mathbb{R}^d}\left(\lambda\cdot c-H(c)\right).\end{aligned} \tag{4.9}$$

The convexity and super-linearity of $H(c)$ now imply that

$$H(c)=\sup_{\lambda\in\mathbb{R}^d}\left(c\cdot\lambda-\mu(\lambda)\right) \tag{4.10}$$

where $\mu(\lambda)$ is defined by the almost sure limit

$$\mu(\lambda)=\lim_{t\to\infty}\frac{1}{t}\log\int_{\mathbb{R}^d}e^{\lambda\cdot x}\phi(x,t)\,dx. \qquad\Box$$

Proof of Theorem 1.4. Observe that the function $\varphi_\lambda=\eta_\lambda(x,t)e^{\lambda\cdot x}$ and the function $\phi(x,t)e^{\lambda\cdot x}$ solve the same equation (1.9) with different initial data, since η_λ and ϕ solve the same equation.

Let $\eta^*(y,s;t)$ solve the adjoint equation $\partial_s\eta^*+\mathcal{L}^*\eta^*=0$ for $s\in(0,t)$ with terminal data $\eta^*(y,t;t)=e^{\lambda\cdot y}$. Let $\varphi^*_\lambda(y,s;t)=e^{-\lambda\cdot y}\eta^*_\lambda(y,s;t)$. The function $\varphi^*(y,t;t)$ solves

$$\partial_s\varphi^*+\Delta_y\varphi^*-\left(V(y,s)-2\lambda\right)\cdot\nabla_y\varphi^*+\left(|\lambda|^2-\lambda\cdot V(y,s)-\nabla\cdot V\right)\varphi^*=0 \tag{4.11}$$

for $s\in(0,t)$ with terminal data $\varphi^*(y,t;t)\equiv 1$. Using the fact that $\eta^*(x,t;t)=e^{\lambda\cdot x}$, the equations satisfied by ϕ and η^*, and integration by parts, we see that for each $t>0$,

$$\begin{aligned}\int_{\mathbb{R}^d}\phi(x,t)e^{\lambda\cdot x}\,dx&=\int_{\mathbb{R}^d}\phi(x,t)\eta^*_\lambda(x,t;t)\,dx\\&=\int_{\mathbb{R}^d}\phi(y,0)\eta^*_\lambda(y,0;t)\,dy\\&=\int_{\mathbb{R}^d}\phi_0(y)\varphi^*_\lambda(y,0;t)e^{\lambda\cdot y}\,dy.\end{aligned} \tag{4.12}$$

Since $\phi_0(y)$ is compactly supported,

$$K_1\inf_{y\in B_\delta(0)}\varphi^*_\lambda(y,0;t)\leqslant\int_{\mathbb{R}^d}\phi_0(y)\varphi^*(y,0;t)e^{\lambda\cdot y}\,dy\leqslant K_2\sup_{y\in B_\delta(0)}\varphi^*_\lambda(y,0;t)$$

for some constants K_1, K_2. This implies that

$$\limsup_{t\to\infty}\frac{1}{t}\log\inf_{y\in B_\delta(0)}\varphi^*(y,0;t)\leqslant\mu(\lambda)\leqslant\liminf_{t\to\infty}\frac{1}{t}\log\sup_{y\in B_\delta(0)}\varphi^*(y,0;t). \tag{4.13}$$

Then by applying Harnack estimates to the function φ^*_λ, as in Proposition 2.1, we can use (4.13) to show that

$$\limsup_{t\to\infty}\frac{1}{t}\log\inf_{y\in B_\delta(0)}\varphi^*(y,0;t)=\mu(\lambda)=\liminf_{t\to\infty}\frac{1}{t}\log\sup_{y\in B_\delta(0)}\varphi^*(y,0;t).$$

This and the stationarity of V with respect to x implies that, for any $x\in\mathbb{R}^d$,

$$\lim_{t\to\infty}\sup_{y\in B_\delta(x)}\left|\frac{1}{t}\log\varphi^*(y,0;t)-\mu(\lambda)\right|=0 \tag{4.14}$$

holds almost surely with respect to $\hat{P}$. Finally, the convergence stated in Theorem 1.4 follows from (4.14) by applying Harnack type estimates (as in the proof of Lemma 2.3) to the function φ^* and proceeding just as in the proof of Theorem 2.2. □

5. Bounds on front speeds

In this section, we prove lower and upper bounds of c^* in terms of statistics of V and the front speed c_0 in the absence of advection. We define c_0 to be the front speed corresponding to $V \equiv 0$.

Proposition 5.1. *Suppose V is divergence free and mean zero: $E[V^{(j)}] = 0$ for $j = 1, \dots, d$. The front speed c^* satisfies the upper bound:*

(1) $c^*(e) \leqslant c_0 + E_{\hat{P}}[\|V\|_{L^\infty_x}]$, *implying at most linear growth in $\delta \gg 1$ if V is scaled according to $V \mapsto \delta V$.*

If $V(x,t)$ is uniformly bounded, then c^* also satisfies the lower bound

(2) $c^*(e) \geqslant c_0$.

Proof. Consider the function $\varphi^*(x, \tau; t, \hat{\omega})$ which solves the terminal value problem (1.9). The maximum principle implies that the function φ^* is bounded by

$$\varphi^*(x, 0; t, \hat{\omega}) \leqslant e^{\rho(t,\lambda,\hat{\omega})} = e^{t|\lambda|^2 + \int_0^t \sup_x |\lambda \cdot V(x,s)|\, ds}. \tag{5.1}$$

Therefore,

$$\begin{aligned} \mu(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log \varphi^*(x, 0; t) &\leqslant |\lambda|^2 + \lim_{t\to\infty} \frac{1}{t} \int_0^t \sup_x \bigl|\lambda \cdot V(x,s)\bigr|\, ds \\ &= |\lambda|^2 + E\bigl[\sup_x \bigl|\lambda \cdot V(x,0)\bigr|\bigr] \\ &\leqslant |\lambda|^2 + |\lambda| E\bigl[\sup_x \bigl| \cdot V(x,0)\bigr|\bigr]. \end{aligned} \tag{5.2}$$

Letting $\lambda_e = \lambda \cdot e$, we have:

$$c^*(e) \leqslant \inf_{\lambda_e > 0} \frac{\lambda_e^2 + f'(0) + \lambda_e E_{\hat{P}}[\|V\|_{L^\infty_x}]}{\lambda_e} = c_0 + E_{\hat{P}}\bigl[\|V\|_{L^\infty_x}\bigr]. \tag{5.3}$$

For (2), consider the function $\zeta(x,\tau) = \log \varphi^*(x,\tau) - |\lambda|^2(t-\tau)$ which satisfies

$$\partial_\tau \zeta + \Delta\zeta + |\nabla\zeta|^2 - \bigl(V(x,\tau) - 2\lambda\bigr) \cdot \nabla\zeta - \lambda \cdot V(x,\tau) = 0, \tag{5.4}$$

with terminal data $\zeta(x,t) \equiv 0$. For $R > 0$, let $g(x)$ be a smooth cutoff function satisfying $0 \leqslant g(x) \leqslant 1$ for all x, $g(x) = 0$ for $|x| > R$, $g(x) = 1$ for $|x| \leqslant R - 1$, and $\|\nabla g\|_\infty + \|\Delta g\|_\infty \leqslant K$. Multiplying by g and integrating over $\mathbb{R}^d$ and $[0,t]$ we have

$$\begin{aligned} 0 \leqslant{}& \frac{1}{t|B_R|} \int_0^t \int_{B_R} \zeta \Delta g \, dx\, dt + \frac{1}{t|B_R|} \int_0^t \int_{B_R} \zeta (V - 2\lambda) \cdot \nabla g \, dx\, dt \\ &+ \frac{1}{t|B_R|} \int_0^t \int_{B_R} \lambda \cdot V g \, dx\, dt + \frac{1}{t|B_R|} \int_{B_R} \zeta(x,0) g \, dx. \end{aligned} \tag{5.5}$$

Since V is uniformly bounded, it is easy to see that $0 \leqslant |\zeta(x,\tau;t)| \leqslant K_3(t-\tau)$ for some constant K_3 sufficiently large. This and the fact that Δg is supported in the set $R-1 \leqslant |x| \leqslant R$ imply that

$$\frac{1}{t|B_R|}\int_0^t\int_{B_R} |\zeta||\Delta g|\,dx\,dt \leqslant \frac{K_1 t|B_R \setminus B_{R-1}|}{|B_R|} \leqslant \frac{tK_2}{R}. \tag{5.6}$$

Similarly,

$$\frac{1}{t|B_R|}\int_0^t\int_{B_R} |\zeta|\big|(V-2\lambda)\big||\nabla g|\,dx\,dt \leqslant \frac{K_3 t|B_R \setminus B_{R-1}|}{2|B_R|} \leqslant \frac{K_4 t}{R}. \tag{5.7}$$

For $\epsilon > 0$, let $R = R(t) = 2K_4 t/\epsilon$, so that the right hand sides of (5.6) and (5.7) are bounded by $O(\epsilon)$ for t sufficiently large. By Theorem 1.4,

$$\lim_{t\to\infty}\left|\frac{1}{t|B_R|}\int_{B_R} \zeta(x,0)g\,dx - \mu(\lambda) + \lambda^2\right| = 0. \tag{5.8}$$

Moreover, the ergodic theorem implies that

$$\lim_{t\to\infty}\frac{1}{t|B_R|}\int_0^t\int_{B_R} \lambda\cdot Vg\,dx\,dt = E[\lambda\cdot V] = 0. \tag{5.9}$$

Therefore (5.5)–(5.8) and our choice of R imply that $\mu(\lambda) \geqslant |\lambda|^2 - \mathrm{O}(\epsilon)$. Letting $\epsilon \to 0$ we have $\mu(\lambda) \geqslant |\lambda|^2$ and

$$c^*(e) \geqslant \inf_{\lambda\cdot e>0}\frac{|\lambda^2| + f'(0)}{\lambda\cdot e} = c_0.$$

This proves (2). □

The above proposition extends similar bounds in the time random shear flows [24] as well as bounds for deterministic, periodic flows. For example, if the velocity field is periodic, mean-zero, and divergence-free, then it is known that the KPP front speed can only be enhanced by the flow and that the enhancement can be at most linear with respect to $\|V\|_\infty$ (see Refs. [4,8]). Numerical computation of c^* in randomly perturbed cellular flows by the authors [25] suggest that $c^* \sim \mathrm{O}(\delta^p)$ at large δ may occur for any exponent $p \in (0,1)$, when V is scaled according to $V \mapsto \delta V$. So the above bounds are optimal in time random incompressible flows. The other type of bound on c^* for δV with Gaussian statistics in time is obtained in Theorem 5 of [24], namely $c^* \leqslant c_0\sqrt{1+\delta^2 p_1}$, where p_1 is the integral of correlation function. We give an extension of such bound for nonshear space time random flows next.

Remark 5.1. The following computation is formal, but illustrative. A velocity field that is white-noise in time, could be incorporated rigorously through a term of the form $V\cdot\nabla u \circ dW$ in the original equation (1.1), where $\circ$ denotes the Stratonovich integral. Although this scenario does not fall within our assumptions on V given in the introduction, the following computation illustrates the difficulty in estimating c^* when the velocity V is correlated in time.

Proposition 5.2. *Suppose in that V has the form*

$$V(x,t,\hat{\omega}) = \sum_k X_k(x)F_k(t,\hat{\omega}) \tag{5.10}$$

where $\{X_k(x)\}$ are periodic or almost-periodic, divergence free fields and $\{F_k\}$ are white-noise processes in time, so that the covariance matrix function is:

$$\Gamma_{ij} = \Gamma_{ij}(x_1,x_2,t_1-t_2) = E_{\hat{P}}\big[V^{(i)}(x_1,t_1)V^{(j)}(x_2,t_2)\big] \leqslant p_1\delta_0(t_1-t_2)A_{ij}(x_1,x_2),$$

where δ_0 is the standard delta function centered at zero, p_1 is a constant. Then $c^* \leqslant c_0\sqrt{1+C_2p_1}$, where C_2 depends only on the dimension d and $f'(0)$.

Proof. The Feynman–Kac formula for φ^* of Eq. (1.9) gives:

$$\varphi^*(x,0) = E\big[e^{-\lambda\cdot\int_0^t V(Z^\lambda,s)\,ds}\big]e^{|\lambda|^2 t},$$

where Z^λ is the diffusion process obeying the Itô equation:

$$dZ^\lambda(s) = \big(V\big(Z^\lambda, s\big) - 2\lambda\big)\,ds + \sqrt{2}\,dW(s), \quad s \in [0,t],$$

$Z^\lambda(0) = z$, $W(s) = \{W^i(s)\}_{i=1}^d$ a d-dimensional Wiener process. Changing measure by the Girsanov Theorem ([14], Theorem 5.1) yields the following representation of φ^*:

$$E\left[\exp\left\{-\lambda\sqrt{2}\cdot W(t) + \sqrt{2}\sum_{i=1}^d \int_0^t V^{(i)}\big(W_z(r), r\big)\,dW^{(i)}(r) - \frac{1}{2}\int_0^t \big\|V\big(W_z(s),s\big)\big\|^2\,ds\right\}\right], \tag{5.11}$$

where $W_z(s) = z + W(s)$, E is expectation with respect to W. It follows that:

$$\varphi^* \leqslant E\left[\exp\left\{-\lambda\sqrt{2}\cdot W(t) + \sqrt{2}\sum_{i=1}^d \int_0^t V^{(i)}\big(W_z(r), r\big)\,dW^{(i)}(r)\right\}\right],$$

and

$$E_{\hat{P}}\varphi^* \leqslant E\left[e^{-\lambda\sqrt{2}\cdot W(t)}E_{\hat{P}}\left[\exp\left\{\sqrt{2}\sum_{i=1}^d \int_0^t V^{(i)}\big(W_z(r), r\big)\,dW^{(i)}(r)\right\}\right]\right]. \tag{5.12}$$

Notice that inside the inner expectation (with $W_z(r)$ fixed), the sum of stochastic integrals is a linear combination of Gaussian variables. In other words, the inner expectation is over a log-normal variable, and so:

$$E_{\hat{P}}\varphi^* \leqslant E\left[\exp\left\{-\lambda\sqrt{2}\cdot W(t) + \int_0^t\int_0^t \sum_{ij} \Gamma_{ij}\big(W(s), W(\tau), s, \tau\big)\,dW^{(i)}(s)\,dW^{(j)}(\tau)\right\}\right]. \tag{5.13}$$

As V is white in time, e.g. $\Gamma_{ij} = A_{ij}(x_1,x_2)p_1\delta_0(t_1 - t_2)$, the integral in (5.13) is bounded from above by $p_1 C_1 \int_0^t \|dW(s)\|^2$. The right-hand side expectation of (5.13) is bounded from above by:

$$E\left[\exp\left\{\int_0^t p_1 C_1 \big\|dW(s)\big\|^2 - \sqrt{2}\lambda\cdot dW(s)\right\}\right] = \prod_{j=1}^N \prod_{l=1}^d E\big[\exp\big\{p_1 C_1\big(dW^{(l)}(s)\big)^2 - \sqrt{2}\lambda^{(l)}\,dW^{(l)}\big\}\big], \tag{5.14}$$

where $dW^{(l)}$ is the Wiener increment over interval of length t/N. We have used independence of Wiener increments in each component and among components. The last expression of (5.14) can be calculated explicitly, and equals upon taking the limit $N \to \infty$:

$$\exp\big\{|\lambda|^2 t + p_1 d C_1 t\big\}.$$

It follows that:

$$\begin{aligned}\mu &= \lim_{t\to\infty} \frac{1}{t} E_{\hat{P}} \log\varphi^* \\ &\leqslant \lim_{t\to\infty} \frac{1}{t}\log E_{\hat{P}}\varphi^* \leqslant |\lambda|^2 + C_1 d p_1 \end{aligned} \tag{5.15}$$

or

$$c^* \leqslant 2\sqrt{f'(0) + C_1 d p_1} = c_0\sqrt{1 + C_2 p_1}. \qquad \square \tag{5.16}$$

Remark 5.2. If V is Gaussian but nonwhite in time, the $p_1\delta_0$ in the upper bound of the covariance matrix function is replaced by a nonnegative L^1 function with integral equal to p_1. The estimate of the right-hand side expectation of (5.13) will be more complicated. One may write the double integral into discrete sums, and carry out a direct evaluation. It is interesting to establish a similar result. Inequality (5.16) implies that rapid temporal decorrelation can reduce speed enhancement, as known for temporally random shear flows [24] among other time dependent flows in the literature [2,5,9,15,22].

6. Conclusions

A new Eulerian method is developed to prove the large time asymptotic spreading of KPP reactive fronts in incompressible space–time random flows in several space dimensions. The random flows are mean zero, stationary, ergodic, and can be unbounded in time as long as the moment condition (1.3) is satisfied. The flow field is locally Hölder continuous, which is the case for turbulent flow fields [20,32]. The large time front speed is almost surely deterministic and obeys a variational principle in terms of the Legendre dual of the large deviation rate function. This addresses the existence of a turbulent flame speed for KPP fronts, a long standing open problem in turbulent combustion [28].

A variational principle for the front speeds lead to analytical bounds that reveal upper and lower limits of speed enhancement in incompressible flows. In future work, it will be interesting to further relax the moment condition (1.3), so the flow field can be unbounded in space as well. Another open question is to study non-KPP reactive fronts in random flows [23], and to show that KPP front speeds qualitatively agree with non-KPP ones as seen in many deterministic front problems [3,4,8,13,22,31,35].

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