Asymptotic Stability Analysis of 2-D Discrete State Space Systems with Singular Matrix

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Abstract: - The aim of this paper is to establish, on the basis of Lagrange method for solving partial difference equations, conditions under an asymptotic stability analysis procedure to investigate conditions for the existence of a solution to 2-D (two dimensional) discrete system whose state space representation is composed by a non-singular matrix. To accomplish it, the concept of generalized inverse of matrices and Jordan canonical transformation are applied on the original system and then Lagrange solutions to the transformed systems are pursued. Once the conditions are determined on the grounds of the transformed system and the existence conditions of solutions for this system is accomplished, the conditions for the original system is obtained by back transformation. A numerical example is given to show how the procedure works.

Key-Words: - 2-D systems, analysis, asymptotic stability, Lagrange method

1 Introduction

The pioneering research on the stability of 2-D discrete systems described by the state space state space model paralleling the ordinary 1-D discrete system model that appears in engineering goes back to the early 1960s when the concept of multivariable positive real functions was introduced to study the stability of electrical networks with varying parameters [1]. This framework was then further extended and generalized to allow one to investigate more methodically the stability of these kinds of systems with two as well as more indices [2]. In this scope, basically two state space models, which are equivalent to each other, stood out in automatic control, digital signal processing and other related fields of engineering as frame of references for studying the stability and control design of 2-D discrete systems; namely, FM [3] and Roesser [4] models.

As far as the tools for delving into the stability of 2-D discrete systems are concerned, z-transform was, and still is to some extent the most widely adopted formalism [5] [6]. As for the ordinary 1-D systems case, this approach has provided a wide range of techniques and apparatus for testing the stability of systems. However, when there are multiple state variables composing the state space model that depend on the same index, say *i*, the analysis is in general neither straightforward or explicitly achievable. To cope with it the energy methods [7], which is essentially a generalization of Lyapunov based procedures developed in the ambit of 1-D systems, was intensively used in the 1980s. These in turn eventually evolved into linear matrix inequalities methods in the 1990s and early 2000s (see for example [8] and references therein). Powerful mechanisms in their own rights, they provide only sufficient conditions to inspect the stability of systems.

In order to deal with this shortcoming, Izuta [9-12] have investigate the solutions to 2-D discrete control systems of Roesser type from the Lagrange solution method perspective. Briefly, the philosophy is to transform the system matrix into a diagonal matrix by means of control feedback design and then determine the stability conditions under which the overall feedback control systems is asymptotically stable in the sense of Lagrange method for solving partial difference equations. This scheme makes it clear not only the role of the eigenvalues of the system matrix but also provides an explicit solution, when it exists. Yet, recently the authors [13] focused on 2-D systems with singular system matrix and established asymptotic stability conditions for the case in which the eigenvalues of the non-singular matrix block composing the original system matrix are all mutually different.

Motivated by the investigations so far, this paper aims to extend the previous paper [13] in the sense that, we aim here to establish an analysis method to check the existence of a solution. In addition, unlike considering the similarity transformation, the Jordan canonical form is exploited to embrace a larger class of matrix blocks. In a few words, the original system is re-arranged in a suitable format such that there will be a non-singular matrix block at the upper left corner of the system matrix; then, Jordan canonical transformation is applied on the sub-system defined by this matrix block. Thus since this procedure allows us to establish the conditions for the existence of a Lagrange solution to the transformed sub-system, this solution, if any, will in turn lead to an asymptotically stable solution to the original system.

Finally, this paper is organized as follows. Section 2 states the definitions relied on in the sequel, section 3 writes down the mathematical tools needed to unfold the suggested theoretical reasoning, and section 4 presents the results, and section 5 shows a numerical example.

2 Preliminaries and Problem

In this section the definitions and concepts required hereafter are presented. Firstly, consider the system as follows.

Definition 1. Let the 2-D discrete system be described by the following real valued state space model with indices being natural numbers.

$$\begin{bmatrix} x_1(i+1,j) \\ \vdots \\ x_k(i+1,j) \\ x_{k+1}(i,j+1) \\ \vdots \\ x_n(i,j+1) \end{bmatrix} = \overbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}^{A_{nxn}} \begin{bmatrix} x_1(i,j) \\ \vdots \\ x_k(i,j) \\ x_{k+1}(i,j) \\ \vdots \\ x_n(i,j) \end{bmatrix} \quad (1)$$

$$\text{rank}\,(A_{nxn}) = p, \qquad 1 \le p \le n$$

where the over-braced $A_{n\,x\,n}$ stands for short notation of the matrix under it.

Secondly, the definition of stability given next is the multidimensional version of the one used to study systems of difference equations [14].

Definition 2. System (1) is asymptotically stable as far as its solutions $x_*(i, j)$'s (for all $_*$) fulfill the following condition.

$$\lim_{(i+j)\to\infty} |x_*(i,j)| \to \infty, \text{for all } * = 1, \dots, n \quad (2)$$

Thirdly, the theoretical framework relies on the Lagrange method for solving partial difference equations, which consists basically in defining candidate solutions and then testing them for stability conditions. If those are satisfied then the candidates are in fact a solution to the system. This corresponds to the exponential functions that are assumed as solutions to partial differential equations in continuous settings.

Definition 3. A general form of a non-null Lagrange candidate solution is given by the following equation.

$$x(i,j) = \textstyle\sum_{k=1}^{n} I_k\, \alpha_k^i \beta_k^j \quad (3)$$

I_k's stand for the initial values and α_k's and β_k's are non-null real valued numbers.

Taking these into consideration, the problem to be tackled in this paper is as follows.

Problem. To establish, on the basis of Lagrange method for solving partial difference equations, conditions under which there exists an asymptotically stable solution to 2-D discrete systems.

3 Mathematical Facts

In this section, we present the mathematical tools that will be used thereafter.

First of all, note that algebra of matrices allows one to rewrite systems of equations with singular matrix in such a form that the top left matrix block of the resulting matrix is non-singular so that the system described by equation (1) can be written to have its non-singular matrix portion expressed as a matrix block as follows.

$$\begin{bmatrix} \mathbf{x_u(i+1,j+1)} \\ \mathbf{x_v(i+1,j+1)} \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} \mathbf{A}_{pxp} & \mathbf{B}_{px(n-p)} \\ \mathbf{C}_{(n-p)xp} & \mathbf{C}_{(n-p)xp}\mathbf{A}_{pxp}^{-1}\mathbf{B}_{px(n-p)} \end{bmatrix}}^{\bar{A}} \begin{bmatrix} \mathbf{x_u(i,j)} \\ \mathbf{x_v(i,j)} \end{bmatrix} \quad (4)$$

in which $\mathbf{A}_{pxp}$ is a non-singular matrix block composing matrix $\mathbf{A}_{nxn}$ and such that

$$\text{rank}\left(\mathbf{A}_{pxp}\right) = p, \qquad 1 \le p \le n$$

holds and the vectors are defined as given hereafter.

$$\mathbf{x_u(i+1,j+1)} \stackrel{\text{def}}{=} \begin{bmatrix} x_1(i+1,j) \\ \vdots \\ x_r(i+1,j) \\ x_{r+1}(i,j+1) \\ \vdots \\ x_p(i,j+1) \end{bmatrix}$$

$$\mathbf{x_u(i,j)} \stackrel{\text{def}}{=} \begin{bmatrix} x_1(i,j) \\ \vdots \\ x_r(i,j) \\ x_{r+1}(i,j) \\ \vdots \\ x_p(i,j) \end{bmatrix}$$

$$\mathbf{x_v}(\mathbf{i+1},\mathbf{j+1}) \stackrel{\text{def}}{=} \begin{bmatrix} x_{p+1}(i+1,j) \\ \vdots \\ x_s(i+1,j) \\ x_{s+1}(i,j+1) \\ \vdots \\ x_s(i,j+1) \end{bmatrix}$$

$$\mathbf{x_v}(\mathbf{i},\mathbf{j}) \stackrel{\text{def}}{=} \begin{bmatrix} x_{p+1}(i,j) \\ \vdots \\ x_s(i,j) \\ x_{s+1}(i,j) \\ \vdots \\ x_s(i,j) \end{bmatrix}$$

It is worth noting that nevertheless, for the sake of clarity and without loss of generality, the vector $\mathbf{x_u(i+1,j+1)}$ has the entries defined by the state variables written sequentially from $x_1(i+1,j)$ to $x_r(i+1,j)$ followed by variables from $x_{r+1}(i,j+1)$ to $x_p(i,j+1)$, in practice the computations lead in general to a different ordering with the state variables appearing in a mixed up fashion rather than aggregated as written here. In addition, should it be the case, the state variables composing the entries of vector $\mathbf{x_u(i,j)}$ will also follow the corresponding arrangement. The same remarks apply to vectors $\mathbf{x_v(i+1,j+1)}$ and $\mathbf{x_v(i,j)}$.

Apropos, another mathematical concept that plays a key role in this paper is that of generalized inverse of matrices [15], which assigned on matrix $\bar{A}$ in equation (4) renders straightforwardly the generalized inverse matrix G fulfilling the equality $\bar{A}G\bar{A}=\bar{A}$ and whose explicit expression is given by the following equation.

$$G = \begin{bmatrix} A^{-1}_{pxp} & 0_{px(n-p)} \\ 0_{(n-p)xp} & 0_{(n-p)x(n-p)} \end{bmatrix} \qquad (5)$$

On focusing on this generalized inverse matrix, system in equation (4) is solvable if and only if the following equality stands up.

$$\bar{A}G\begin{bmatrix} \mathbf{x_u}(\mathbf{i+1},\mathbf{j+1}) \\ \mathbf{x_v}(\mathbf{i+1},\mathbf{j+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{x_u}(\mathbf{i+1},\mathbf{j+1}) \\ \mathbf{x_v}(\mathbf{i+1},\mathbf{j+1}) \end{bmatrix} \qquad (6)$$

From which it is a simple matter of algebraic calculation to reach to the equation given by

$$\mathbf{x_v}(\mathbf{i+1},\mathbf{j+1}) = CA^{-1}\mathbf{x_u}(\mathbf{i+1},\mathbf{j+1}) \qquad (7)$$

which means that the vector $\mathbf{x_v(i,j)}$ is determined as far as the vector $\mathbf{x_u(i,j)}$ is established. As a matter of fact, in order to carry out this task, note first that a particular solution to system (4) is accomplished by the following system.

$$G\begin{bmatrix} \mathbf{x_u}(\mathbf{i+1},\mathbf{j+1}) \\ \mathbf{x_v}(\mathbf{i+1},\mathbf{j+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{x_u}(\mathbf{i},\mathbf{j}) \\ \mathbf{x_v}(\mathbf{i},\mathbf{j}) \end{bmatrix} \qquad (8)$$

thus it turns out that $\mathbf{x_u(i,j)}$ is computed from

$$\mathbf{x_u}(\mathbf{i+1},\mathbf{j+1}) = A\mathbf{x_u}(\mathbf{i},\mathbf{j}) \qquad (9)$$

In other words, equations (7) and (9) provide a particular solution to system (4), which will allow one to draw conclusions regarding the existence of asymptotically stable Lagrange solutions.

4 Results

In this section, we present the main results of this paper. In fact, they are gathered in the following theorem.

Theorem 1. Consider system (1) and its particular solution given by equations (7) and (9). In addition, assume that matrix A in equation (9) can be transformed into a Jordan canonical form. Then there exists an asymptotically stable Lagrange solution to system (1) if the initial values of the solutions to the system can be set to meet certain conditions as well as the eigenvalues of matrix A are all non-null numbers with absolute values less than unit

Proof. Since on solving equation (9) for $\mathbf{x_u(i,j)}$, $\mathbf{x_v(i,j)}$ is promptly obtained and consequently a solution to system (1) is achieved, we focus specifically on this problem in what follows. For, consider the transformation applied on (9) given by

$$\mathbf{z}(\mathbf{i},\mathbf{j}) = T^{-1}\mathbf{x_u}(\mathbf{i},\mathbf{j}) \qquad (10)$$

T: a non-singular matrix composed by generalized eigenvectors of matrix A.

This procedure leads to the Jordan canonical form as follows.

$$T^{-1}\mathbf{x_u}(\mathbf{i+1},\mathbf{j+1}) = T^{-1}ATT^{-1}\mathbf{x_u}(\mathbf{i},\mathbf{j}) \qquad (11)$$

which by means of (10) reads

$$\mathbf{z}(\mathbf{i+1},\mathbf{j+1}) = \overbrace{\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_t \end{bmatrix}}^{J_{pxp}} \mathbf{z}(\mathbf{i},\mathbf{j}) \qquad (12)$$

in which matrix J_{pxp} is square matrix of rank p and its composing Jordan block J_i is a matrix given by

$$J_i = \lambda_i \qquad (13)$$

or

$$J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & & \lambda_i & 1 & \\ & & & & \ddots \\ & & & & & \lambda_i \end{bmatrix} \qquad (14)$$

λ_i : eigenvalues of matrix A

Note that equation (14) has multiple rows and the last one is the diagonal entry alone and the previous entries have always two values: an eigenvalue on the diagonal line and number 1 at the right column.

Despite a little simplified but without loss of generality, a solution to (12) is established by basically considering the following cases.

Case 1: let J_i be given by λ_i then the possible combinations are as follows.

$$z_s(i+1,j) = \lambda_s z_s(i,j) \quad (15)$$

or

$$z_t(i,j+1) = \lambda_t z_t(i,j) \quad (16)$$

to which the Lagrange candidate solutions are taken to be

$$z_s(i,j) = K_s \alpha_s^i \beta_s^j \quad (17)$$

and

$$z_t(i,j) = K_t \alpha_t^i \beta_t^j \quad (18)$$

with K_s and K_t being the initial conditions. Now substituting (17) into (15) and (18) into (16) lead to

$$K_s \alpha_s^{i+1} \beta_s^j = K_s \lambda_s \alpha_s^i \beta_s^j \quad (19)$$

and

$$K_t \alpha_t^i \beta_t^{j+1} = K_t \lambda_t \alpha_t^i \beta_t^j \quad (20)$$

from which we obtain

$$\alpha_s = \lambda_s, \text{ and } \beta_t = \lambda_t \quad (21)$$

meaning that (15) and (16) are asymptotically stable as far as λ_s and λ_t are non-null real numbers with absolute values less than unit and α_s and β_t are chosen in the same way.

Case 2: let J_i a non-diagonal Jordan matrix with the diagonal entry given by λ_i and (12) have the following entries.

$$z_s(i+1,j) = \lambda_s z_s(i,j) + z_r(i,j) \quad (22)$$
$$z_r(i+1,j) = \lambda_r z_r(i,j) \quad (23)$$

or

$$z_r(i,j+1) = \lambda_r z_r(i,j) \quad (24)$$

From case 1, z_r is given by

$$z_r(i,j) = K_r \lambda_r^i \beta_r^j \quad (25)$$

or

$$z_r(i,j) = K_r \alpha_r^i \lambda_r^j \quad (26)$$

Moreover, assuming the Lagrange solution

$$z_s(i,j) = K_s \alpha_s^i \beta_s^j \quad (27)$$

equation (22) becomes

$$K_s \alpha_s^{i+1} \beta_s^j = K_s \alpha_s^i \beta_s^j + K_r \lambda_r^i \beta_r^j \quad (28)$$

or

$$K_s \alpha_s^{i+1} \beta_s^j = K_s \alpha_s^i \beta_s^j + K_r \alpha_r^i \lambda_r^j \quad (29)$$

which can be written as

$$K_s(\alpha_s - 1)\alpha_s^i \beta_s^j = K_r \lambda_r^i \beta_r^j \quad (30)$$

or

$$K_s(\alpha_s - 1)\alpha_s^i \beta_s^j = K_r \alpha_r^i \lambda_r^j \quad (31)$$

and since (30) and (31) must hold for all the values of indices i and j, we conclude that we get a solution if we can define non-null α_s and β_s with absolute values less than unit along with initial conditions such that the following are fulfilled.

$$\alpha_s = \lambda_s = 1 + \frac{K_r}{K_s}, \text{ and } \beta_s = \beta_r \quad (32)$$

or

$$\alpha_s = \alpha_r = 1 + \frac{K_r}{K_s}, \text{ and } \beta_s = \lambda_r \quad (33)$$

Case 3: dual equations to those given in case 2 are given by

$$z_s(i,j+1) = \lambda_s z_s(i,j) + z_r(i,j) \quad (34)$$
$$z_r(i,j+1) = \lambda_r z_r(i,j) \quad (35)$$

or

$$z_r(i+1,j) = \lambda_r z_r(i,j) \quad (36)$$

As in the previous case, z_r and z_s are given by

$$z_r(i,j) = K_r \lambda_r^i \beta_r^j \quad (37)$$

or

$$z_r(i,j) = K_r \alpha_r^i \lambda_r^j \quad (38)$$

and

$$z_s(i,j) = K_s \alpha_s^i \beta_s^j \quad (39)$$

Thus, it turns out that equation (34) translates into the following expression.

$$K_s \alpha_s^i \beta_s^{j+1} = K_s \alpha_s^i \beta_s^j + K_r \alpha_r^i \lambda_r^j \quad (40)$$

or

$$K_s \alpha_s^i \beta_s^{j+1} = K_s \alpha_s^i \beta_s^j + K_r \lambda_r^i \beta_r^j \quad (41)$$

written the other way

$$K_s(\beta_s - 1)\alpha_s^i \beta_s^j = K_r \alpha_r^i \lambda_r^j \quad (42)$$

or

$$K_s(\beta_s - 1)\alpha_s^i \beta_s^j = K_r \lambda_r^i \beta_r^j \quad (43)$$

and analogous to the previous case, the conditions to be satisfied are

$$\alpha_s = \alpha_r, \text{and } \beta_s = \lambda_r = 1 + \frac{K_r}{K_s}, \qquad (44)$$

or

$$\alpha_s = \lambda_r, \text{and } \beta_s = \beta_r = 1 + \frac{K_r}{K_s}, \qquad (45)$$

All in all, no matter the case, the Lagrange solution to equation (12) and the Jordan canonical transformation (10) written as $\mathbf{x_u}$(i,j)=$\mathbf{Tz}$(i,j) yield the solution $\mathbf{x_u}$(i,j), which has the general format expressed by the following equation.

$$x_*(i,j) = \sum_g I_{*g}\,\lambda_{*g}^i \beta_{*g}^j + \sum_h I_{*h}\,\alpha_{*h}^i \lambda_{*h}^j \qquad (46)$$

in which the terms I's are constant values resulting from the computations. Furthermore, (46) gives a solution to equation (7), and consequently the claim of theorem follows. □

Remark 1: For the sake of simplicity, in cases 2 and 3, only Jordan block of size of 2 was considered. However care should be taken for blocks of greater sizes because, in general, they are solvable only if the entries of the non-diagonal Jordan canonical form block with 1's in its off diagonal positions, which are the rows of the Jordan block not including the last one, are given by either $z_r(i+1,j) = \lambda_r z_r(i,j) + z_s(i,j)$ along with $z_r(i+1,j) = \lambda_s z_r(i,j) + z_t(i,j)$, or $z_r(i,j+1) = \lambda_r z_r(i,j) + z_s(i,j)$ along with $z_r(i,j+1) = \lambda_s z_r(i,j) + z_t(i,j)$. Yet if z_r is the (n-1)-th row of the block, then the n-th row corresponding to $z_s(i,j)$ can be a vector depending on either index.

5 Numerical Example

Hereafter we present a numerical example to show how the analysis procedure works.

Consider a randomly generated system be already written as in equation (4) and given by the following expression.

$$\begin{bmatrix} x_1(i+1,j) \\ x_3(i+1,j) \\ x_2(i,j+1) \\ x_4(i,j+1) \end{bmatrix} = \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix} \begin{bmatrix} x_1(i,j) \\ x_3(i,j) \\ x_2(i,j) \\ x_4(i,j) \end{bmatrix} \qquad (47)$$

in which

$$A = \begin{bmatrix} 0.0271647 & 0.5249944 & 0.1728464 \\ -0.0956705 & 0.1749889 & 0.1456927 \\ 0.0271647 & 0.2749944 & 0.0978464 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.3333333 \\ 0.5555556 \\ 0.4117647 \end{bmatrix} \qquad (48)$$

$$C = \begin{bmatrix} 0.2222222 & 0.3333333 & 0.2173913 \end{bmatrix}$$

Note that matrix A has only one eigenvalue (λ), namely λ=0.1, whose algebraic multiplicity is 3 and geometric multiplicity given by the dimension of kernel of matrix (A-λI), which is given by the dimension of matrix A minus the rank of (A-λI), is 1. Moreover the geometric multiplicity says that there is only one Jordan block corresponding to this eigenvalue.

Now as far as the existence of a solution to (47) is concerned, it can be solved if and only if

$$x_4(i,j+1) = CA^{-1} \begin{bmatrix} x_1(i,j) \\ x_3(i,j) \\ x_2(i,j) \end{bmatrix} \qquad (49)$$

for a particular solution given by

$$\begin{bmatrix} x_1(i+1,j) \\ x_3(i+1,j) \\ x_2(i,j+1) \end{bmatrix} = A \begin{bmatrix} x_1(i,j) \\ x_3(i,j) \\ x_2(i,j) \end{bmatrix}. \qquad (50)$$

Now, in order to obtain a solution to (50), we consider the linear space basis transformation given by the following equation.

$$\begin{bmatrix} z_1(i,j) \\ z_3(i,j) \\ z_2(i,j) \end{bmatrix} = T^{-1} \begin{bmatrix} x_1(i,j) \\ x_3(i,j) \\ x_2(i,j) \end{bmatrix}. \qquad (51)$$

with T being a matrix composed by the generalized eigenvectors of matrix A and chosen here to be

$$T = \begin{bmatrix} -0.0635263 & -0.1728464 & 1.4191176 \\ 0.0059247 & -0.1456927 & -0.1323529 \\ -0.0447647 & 0.0021536 & 0.0 \end{bmatrix}$$

Thus, (50) and (51) lead to

$$\begin{bmatrix} z_1(i+1,j) \\ z_3(i,j+1) \\ z_2(i,j+1) \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} z_1(i,j) \\ z_3(i,j) \\ z_2(i,j) \end{bmatrix} \qquad (52)$$

so, assuming the following Lagrange solution for $z_2(i,j)$

$$z_2(i,j) = K_2 \alpha_2^i \beta_2^j \qquad (53)$$

gives

$$K_2 \alpha_2^i \beta_2^{j+1} = \lambda\, K_2\, \alpha_2^i \beta_2^j \qquad (54)$$

hence

$$\beta_2 = \lambda \qquad (55)$$

which means that β_2 must be set to be λ, which in turn must be a non-null value of absolute value less than unit. As for $z_3(i,j)$ we have that

$$z_3(i+1,j) = \lambda\; z_3(i,j) + z_2(i,j) \qquad (56)$$

and letting

$$z_3(i,j) = K_3 \alpha_3^i \beta_3^j \qquad (57)$$

which with equation (56) provides

$$K_3(\alpha_3 - \lambda)\alpha_3^i\beta_3^j = K_2\alpha_2^i\beta_2^j \quad (58)$$

since this equality must hold for all the indices i and j, the corresponding terms must be equal. In other words, $\alpha_3=\alpha_2$, $\beta_3=\beta_2$. Moreover, there is a solution as far as β_3 given by the following equation has absolute value less than unit.

$$\alpha_3 = \lambda + \frac{K_2}{K_3} \quad (59)$$

Finally, let us tackle $z_1(i,j)$ which is given by

$$z_1(i+1,j) = \lambda z_1(i,j) + z_3(i,j) \quad (60)$$

for which, similarly to the previous cases, the following Lagrange solution is chosen

$$z_1(i,j) = K_1\alpha_1^i\beta_1^j \quad (61)$$

Now from (57), (60) and (61) we have

$$K_1(\alpha_1 - \lambda)\alpha_1^i\beta_1^j = K_3\alpha_3^i\beta_3^j \quad (62)$$

and following the previous reasoning we conclude that $\alpha_1=\alpha_3$, $\beta_1=\beta_3$ along with

$$\alpha_1 = \lambda + \frac{K_3}{K_1} \quad (63)$$

It turns out that since $\beta_1=\beta_3=\beta_2$ and β_3 is established by means of equation (59), the other β's are achieved in a quite natural way. On the other hand, for α's, despite their equality, they must satisfy equations (55) and (63), which lead to

$$\frac{K_2}{K_3} = \frac{K_3}{K_1} \quad (64)$$

subject to the following condition in order to yield an asymptotically stable solution.

$$\left|\frac{K_2}{K_3}\right| = \left|\frac{K_3}{K_1}\right| < 1 \quad (65)$$

which will hold as far as we can set adequately the initial values, which in practical terms are, if possible, carried out by means of a solution to (47). As a matter of fact, a solution to (52) is promptly accomplished by (53), (57) and (61) all with a common α as well as β followed by substitution into (51) not only to compute a particular solution to (47), but also to pursue $x_4(i,j)$ in equation (49).

References:

[1] H. Ozaki and T. Kasami, Positive real functions of several variables and their applications to variable network, *IRE Trans. on Circuit Theory*, CT-7, 1960, pp. 251-260.

[2] H. G. Ansell, On certain two variable generalizations of circuit theory, with applications to networks of transmission lines and lumped reactances, *IRE Trans. on Circuit Theory*, CT-11, 1964, pp. 214-223.

[3] E. Fornasini and G. Marchesini, Doubly indexed dynamical systems: State space models and structural properties, *Math. Syst. Th.*, 12, 1978, pp. 59-72.

[4] R. Roesser, A discrete state-space model for image processing, *IEEE Trans. Automat. Contr.*, AC-20, 1975, pp. 1-10.

[5] E. I. Juri, Stability of multidimensional scalar and matrix polynomials, *Proc. IEEE*, 66, 1978, pp. 1018-1047.

[6] N. K. Bose, Applied multdimensional systems theory, *Van Nostradand Reinhold Co.*, 1982.

[7] W. S. Lu and A. Antoniou, Two-Dimensional Digital Filters, *New York:Marcel Dekker*, 1992.

[8] G. Izuta, Stability and disturbance attenuation of 2-D discrete delayed systems via memory state feedback controller, *Int. J. Gen. Systems*, 36-3, 2007, pp. 263-280.

[9] G. Izuta, Stability analysis of 2-D discrete systems on the basis of Lagrange solutions and doubly similarity transformed systems, *Proc. 35th conf. IEEE IES*, 2010, pp. 1756-1761.

[10] G. Izuta, On the asymptotic stability analysis of a certain type of discrete time 3-d linear systems, *Proc. ICINCO 2014*, pp. 665-670.

[11] G. Izuta and T. Nishikawa, An observer controller design method for 2-D discrete control systems, *Proc. IEEE Int. Conf. on Information and Automation*, 2015, pp. 1337-1343.

[12] G. Izuta, Existence conditions of asymptotically stable 2-D feedback control systems on the basis of block matrix diagonalization, *Proc. ICINCO 2016*, pp. 665-670.

[13] G. Izuta, Asymptotic stability of partial difference equations systems with singular matrix, *Proc. 2nd APAS*, 2017, in press.

[14] A. J. Jerri, Linear difference equations with discrete transform methods, *Kluwer Acad. Pub.*, 1996.

[15] C. R. Rao, and S. K. Mitra, Generalized inverse of matrices and its applications, *Wiley*, 1971.

[16] N. E. Mastorakis, I. F. Gonos, and M. N. S. Swamy, Stability of multidimensional systems using genetic algorithms, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications,* 2003, pp. 962 - 965, Volume: 50, Issue: 7

[17] N. E. Mastorakis; M. N. S. Swamy, A new method for computing the stability margin of two-dimensional continuous systems, *IEEE Transactions on Circuits and Systems I:*

Fundamental Theory and Applications, pp. 869 - 872, Year: 2002, Volume: 49, Issue: 6

[18] N. E. Mastorakis, New necessary stability conditions for 2-D systems, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications,* pp. 1103 - 1105, Year: 2000, Volume: 47, Issue: 7