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ASYMPTOTIC STABILITY AND SPIRALING PROPERTIES FOR SOLUTIONS OF STOCHASTIC EQUATIONS(1)

BY

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ABSTRACT. We consider a system of Itô equations in a domain in R^d. The boundary consists of points and closed surfaces. The coefficients are such that, starting for the exterior of the domain, the process stays in the exterior. We give sufficient conditions to ensure that the process converges to the boundary when $t \to \infty$. In the case of plane domains, we give conditions to ensure that the process "spirals"; the angle obeys the strong law of large numbers.

Introduction. In a previous work [4] we have investigated the behavior of solutions of linear stochastic differential equations when $t \to \infty$. The purpose of the present work is to extend the results of [4] to nonlinear equations. Specifically we shall consider a Markov process on R^l defined by the stochastic equations

$$dx_i = \sum_{s=1}^{n} \sigma_{is}(x)\,dw^s + b_i(x)\,dt \qquad (1 \le i \le l,\ 1 \le s \le n),$$

$$x_i(0) = x_i$$

together with a "stable manifold" ∂G. The set G will consist of a finite number of points together with a finite number of closed domains. The coefficients σ_{is}, b_i are such that if the process starts on ∂G then it stays forever on ∂G.

Our first result (Theorem 1.1) gives a set of sufficient conditions for the nonattainability of G, starting from the exterior. If G consists of points and convex bodies, it suffices that the normal components of the diffusion and the drift vanish on ∂G; in general we need to impose an additional "convexity" relation between ∂G, the drift and the diffusion coefficients to ensure the nonattainability of ∂G.

The next result (Theorem 2.1) gives sufficient conditions that $x(t) \to \partial G$ when $t \to \infty$. This theorem contains local stability conditions (near ∂G and near ∞) reminiscent of the linear case [4], as well as a certain nondegeneracy condition. None of these conditions can be relaxed.

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The techniques used to prove both the nonattainability and the stability theorems involve construction of certain comparison functions which generalize, respectively, $r^{-\epsilon}$ and $\log r$, used in the linear case.

In §§3–5 we construct "exact" comparison functions to prove that when $l = 2$, $x(t)$ "spirals" at a linear rate when $t \to \infty$. (In the linear case we were able to prove this result more directly by probabilistic methods.) Our method of proof differentiates strongly between the cases of degenerate and nondegenerate tangential diffusion. In §§3,4 we deal with the special case where G is a point; the nondegenerate case is treated in §3, and the degenerate case is treated in §4. Finally, the general case is treated in §5.

1. Nonattainability of the boundary. Consider a system of l stochastic differential equations

$$dx_i = \sum_{s=1}^{n} \sigma_{is}(x)\,dw^s + b_i(x)\,dt \qquad (1 \le i \le l) \tag{1.1}$$

where $w^1(t), \cdots, w^n(t)$ are independent Brownian motions. We shall assume

(A) The functions $\sigma_{is}(x)$, $b_i(x)$ $(1 \le i \le l,\ 1 \le s \le n)$ are uniformly Lipschitz continuous on R^l and

$$|\sigma_{is}(x)| + |b_i(x)| \le K(1 + |x|) \qquad (x \in R^l) \tag{1.2}$$

for some constant K.

Let $G_1, \cdots, G_k$ be mutually disjoint sets in R^l; for $1 \le j \le k_0$, G_j consists of one point z_j, and, for $k_0 + 1 \le j \le k$, G_j is a bounded closed domain with C^3 boundary ∂G_j. If G_j consists of one point z_j, we set $\partial G_j = \{z_j\}$. Let $\rho_j(x)$ be the distance function $\rho_j(x) = \operatorname{dist}(x, G_j)$ defined for $x \notin \operatorname{int} G_j$, and let

$$\hat{G}_{j,\epsilon} = \{x;\ x \notin \operatorname{int} G_j,\ \rho_j(x) \le \epsilon\} \qquad (\epsilon > 0),$$

$$\hat{G}_\epsilon = \bigcup_{j=1}^{k} \hat{G}_{j,\epsilon}.$$

If G_j is a closed domain, then $\rho_j(x)$ is a C^2 function in $\hat{G}_{j,\epsilon}$ provided ϵ is sufficiently small. If G_j is a point z_j, then $\rho_j(x)$ is a C^∞ function for $x \ne z_j$.

Set $(a_{ij}) = \sigma\sigma^*$, $\sigma = (\sigma_{is})$, $\sigma^* =$ transpose of σ, and let $b = (b_1, \cdots, b_l)$. Let $\nu = (\nu_1, \cdots, \nu_l)$ be the outward normal to ∂G_h if G_h is a closed domain. We assume

(B) If $1 \le h \le k_0$ then $b_i(z_h) = 0$, $\sigma_{is}(z_h) = 0$ for $1 \le i \le l$, $1 \le s \le n$. If $k_0 + 1 \le h \le k$ then

$$\sum_{i,j=1}^{l} a_{ij}\nu_i\nu_j = 0 \quad \text{on } \partial G_h, \tag{1.3}$$

$$(1.4) \qquad (b, \nu) + \frac{1}{2} \sum_{i,j=1}^{l} a_{ij} \frac{\partial^2 \rho_b}{\partial x_i \partial x_j} \geq 0 \quad \text{on } \partial G_b.$$

Note that (1.3) means that $\sum_{s=1}^{n} (\sum_{i=1}^{l} \sigma_{is}\nu_i)^2 = 0$. Hence $|\sum_i \sigma_{is}\nu_i| = 0$. Since $\nu_i = \partial\rho_b/\partial x_i$ on ∂G_b, it follows that

$$\sum_{i=1}^{l} \sigma_{is} \frac{\partial \rho_b(x)}{\partial x_i} = O(\rho_b(x)) \quad \text{as } \rho_b(x) \to 0 \quad (1 \leq s \leq n).$$

Taking squares we get

$$(1.5) \qquad \sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial \rho_b(x)}{\partial x_i} \frac{\partial \rho_b(x)}{\partial x_j} \leq C_0[\rho_b(x)]^2 \qquad (x \in \hat{G}_{b,\epsilon_0})$$

for ϵ_0 sufficiently small, where C_0 is a positive constant. The condition (1.4) implies that

$$(1.6) \qquad \sum_{i=1}^{l} b_i(x) \frac{\partial \rho_b(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial^2 \rho_b(x)}{\partial x_i \partial x_j} \geq -C_1\rho_b(x) \qquad (x \in \hat{G}_{b,\epsilon_0})$$

where C_1 is a positive constant.

Suppose $\sigma_{ij} \in C^1$ in $\hat{G}_{b,\epsilon_0}$. If (1.3) holds then $\sum_i \sigma_{is}\nu_i = 0$, i.e., the vectors $T_s = (\sigma_{1s}, \cdots, \sigma_{ls})$ are tangent to ∂G_b. Since the function $\sum_j \sigma_{js}\nu_j$ vanishes on ∂G_b, it follows that its derivative with respect to T_s also vanishes on ∂G_b, so that

$$\sum_s \sum_i \sigma_{is} \frac{\partial}{\partial x_i} \sum_j \sigma_{js} \frac{\partial \rho_b}{\partial x_j} = 0 \quad \text{on } \partial G_b.$$

This leads to

$$\sum_{i,j=1}^{l} a_{ij} \frac{\partial \rho_b}{\partial x_i \partial x_j} = - \sum_{i,j=1}^{l} \frac{\partial x_{ij}}{\partial x_j} \nu_i \quad \text{on } \partial G_b.$$

Hence (1.4) is then equivalent to

$$\sum_{i=1}^{l} \left[b_i - \frac{1}{2} \sum_{j=1}^{l} \frac{\partial a_{ij}}{\partial x_j} \right] \nu_i \geq 0 \quad \text{on } \partial G_b.$$

In what follows we shall take ϵ_0 so small that $\hat{G}_{b;\epsilon_0} \cap \hat{G}_{r,\epsilon_0} = \emptyset$ if $b \neq r$.

Denote by $\tilde{G}$ the complement of $\bigcup_{j=1}^{k} G_j$. Its boundary $\partial \tilde{G}$ is the union $\bigcup_{j=1}^{k} \partial G_j$.

Theorem 1.1. *Let* (A), (B) *hold, and let* $x(t)$ *be any solution of* (1.1) *with* $x(0) \in \tilde{G}$. *Then* $P\{\exists t > 0;\ x(t) \in \partial\tilde{G}\} = 0$.

Proof. Let $R(x)$ be a function defined in the closure of $\tilde{G}$, C^2 in $\tilde{G}$, such that

$$R(x) = \begin{cases} \rho_j(x) & \text{if } x \in \hat{G}_{j,\epsilon_0} \quad (1 \le j \le k), \\ |x| & \text{if } |x| > M, \end{cases} \tag{1.7}$$

and $\epsilon_0 \le R(x) \le M$ elsewhere; M is chosen so large that $\hat{G}_{\epsilon_0} \subset \{x;\ |x| < M\}$. Let $V(x) = 1/[R(x)]^\epsilon$ for some $\epsilon > 0$. Introducing

$$Lu \equiv \frac{1}{2} \sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{l} b_i(x) \frac{\partial u}{\partial x_i}$$

we have

$$\begin{aligned} LV &= -\epsilon R^{-\epsilon-1} \sum_i b_i \frac{\partial R}{\partial x_i} + \frac{1}{2} \sum_{i,j} a_{ij} \left\{ \epsilon(\epsilon+1) R^{-\epsilon-2} \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j} - \epsilon R^{-\epsilon-1} \frac{\partial^2 R}{\partial x_i \partial x_j} \right\} \\ &= V \left\{ -\frac{\epsilon}{R} \sum_i b_i \frac{\partial R}{\partial x_i} + \frac{1}{2} \sum_{i,j} \frac{a_{ij}}{R^2} \left[\epsilon(\epsilon+1) \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j} - \epsilon R \frac{\partial^2 R}{\partial x_i \partial x_j} \right] \right\} \end{aligned} \tag{1.8}$$

Using (1.5), (1.6) in $\hat{G}_{b,\epsilon_0}$, if G_b is a closed domain, and the boundedness of the functions

$$\frac{\partial R}{\partial x_i}, \quad R \frac{\partial^2 R}{\partial x_i \partial x_j}, \quad \frac{a_{ij}}{R^2}, \quad \frac{b_i}{R} \tag{1.9}$$

in $\hat{G}_{b,\epsilon_0}$, if G_b is a point, we deduce that the coefficient of V on the right-hand side of (1.8) is bounded above by a constant, say μ_0, if $x \in \hat{G}_{\epsilon_0}$. For $|x| > M$, the functions in (1.9), as well as b_i/R, are still bounded. Hence the coefficient of V in (1.8) is bounded above by some constant μ, throughout the whole set $\tilde{G}$. Thus, we have $LV(x) \le \mu V(x)$ in $\tilde{G}$, $V(x) \to \infty$ if $\operatorname{dist}(x, \partial\tilde{G}) \to 0$.

Introduce the hitting time of $\partial\tilde{G}$,

$$T = \begin{cases} \inf\{t > 0;\ x(t) \in \partial\tilde{G}\}, \\ \infty \quad \text{if no such } t \text{ exists.} \end{cases}$$

Analogously define hitting times T_p with respect to $1/p$-neighborhoods of $\partial\tilde{G}$ $(p = 1, 2, \cdots)$. By the proof of Theorem 1.1 in [3] it follows that

$$E\{e^{-\mu T_p} \chi_B\} \to 0 \quad \text{if } p \to \infty, \tag{1.10}$$

where B is the set where $\inf_{t>0} R(x(t)) = 0$. It follows that $E\{e^{-\mu T} \chi_B\} = 0$, i.e., $T = \infty$ a.s. on B. This completes the proof.

Remark 1. The condition (1.3) means that the "radial" diffusion vanishes on ∂G_b. The condition (1.4) is a "convexity" condition on ∂G_b with respect to the diffusion matrix and the drift. It is elementary to verify that the matrix $(\partial^2 \rho_b / \partial x_i \partial x_j)$ is a positive matrix on ∂G_b whenever G_b is a convex body. [A matrix (b_{ij}) is called *positive* if $\sum b_{ij} x_i x_j \ge 0$ for any real numbers x_i.] Since, on ∂G_b,

$$\sum_{i,j} a_{ij}\frac{\partial^2\rho_b}{\partial x_i\partial x_j} = \sum_{i,j}\frac{\partial^2\rho_b}{\partial x_i\partial x_j}\left(\sum_r \sigma_{ir}\sigma_{jr}\right) = \sum_r\sum_{i,j}\frac{\partial^2\rho_b}{\partial x_i\partial x_j}\sigma_{ir}\sigma_{jr},$$

we conclude that (1.4) holds whenever $(b, \nu) \geq 0$ and $(\partial^2\rho_b/\partial x_i\partial x_j)$ is a positive matrix; in particular, whenever $(b, \nu) \geq 0$ and G_b is a convex body.

Remark 2. The condition (1.4) is essential for the validity of Theorem 1.1. In fact, let y be a point on a hypersurface ∂G_b and let V be an open neighborhood of y. Suppose (1.3) holds on $V \cap \partial G_b$, and

$$(b, \nu) + \frac{1}{2}\sum_{i,j=1} a_{ij}\frac{\partial^2\rho_b}{\partial x_i\partial x_j} < 0 \quad \text{on } V \cap \partial G_b.$$

For $x \in V \cap \tilde{G}$, denote by $p(x)$ the probability that $x(t)$ exits $V \cap \tilde{G}$ by hitting ∂G_b, given that $\tilde{x}(0) = x$. Then, as proved by Pinsky [7], not only is $\tilde{p}(x)$ positive for $x \in V \cap \tilde{G}$, x near y, but also $p(x) \to 1$ if $x \to y$, $x \in V \cap \tilde{G}$.

2. **Stability.** We now turn to the question of asymptotic stability when $t \to \infty$. As in the linear case [4], to prove asymptotic stability it suffices to construct a solution of $Lf \leq -\nu$ with certain auxiliary properties. If $f(x) = \Phi(R(x))$, a short calculation yields

$$\begin{aligned} Lf(x) &= \tfrac{1}{2}\mathfrak{A}\Phi''(R(x)) + \mathfrak{B}\Phi'(R(x)) \equiv \mathfrak{L}\Phi \\ &= \tfrac{1}{2}\mathfrak{A}[\Phi''(R(x)) + \Phi'(R(x))/R(x)] + R(x)Q(x)\Phi'(R(x)), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \mathfrak{A} &= \sum_{i,j} a_{ij}(x)\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j}, \\ \mathfrak{B} &= \sum_i b_i(x)\frac{\partial R}{\partial x_i} + \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2 R}{\partial x_i\partial x_j}, \\ Q &= (\mathfrak{B} - \mathfrak{A}/2R)/R. \end{aligned} \tag{2.2}$$

Suppose $\theta(r)$ $(0 \leq r < \infty)$ is a continuous function satisfying

$$Q(x) \leq \theta(R(x)) \tag{2.3}$$

with

$$\theta_0 = \lim_{r\to 0}\theta(r) < 0, \qquad \theta_\infty = \lim_{r\to\infty}\theta(r) < 0. \tag{2.4}$$

The condition $\theta_0 < 0$ can be realized if and only if

$$\overline{\lim_{0<\rho_b(x)\to 0}}\; Q(x) < 0 \qquad (1 \leq b \leq k). \tag{2.5}$$

The condition $\theta_\infty < 0$ can be realized if and only if

$$\overline{\lim}_{|x|\to\infty} Q(x) < 0. \tag{2.6}$$

Thus, if (2.5), (2.6) hold then there exists a continuous function $\theta(r)$ satisfying (2.3), (2.4).

If (2.5) holds for $k_0 + 1 \le b \le k$ then from (1.5) it follows that $\overline{\lim}(\mathcal{B}/R) < \infty$ as $\rho_b(x) \searrow 0$. Hence $\mathcal{B} \le 0$ on ∂G_b. Combining this with (1.4) we conclude that

$$(b, \nu) + \frac{1}{2} \sum_{i,j=1}^{l} a_{ij} \frac{\partial^2 \rho_b}{\partial x_i \partial x_j} = 0 \quad \text{on } \partial G_b \quad (k_0 + 1 \le b \le k).$$

We shall need the following assumptions:

(C) Denote by $\widetilde{G}_\eta$ $(\eta > 0)$ the set of all points with $\eta < R(x) < 1/\eta$. Then

$$\begin{cases} \displaystyle\sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j} > 0 & \text{if } x \in \widetilde{G}_\eta,\ \nabla_x R(x) \ne 0; \\ \displaystyle\sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial^2 R(x)}{\partial x_i \partial x_j} < 0 & \text{if } x \in \widetilde{G}_\eta,\ \nabla_x R(x) = 0, \end{cases} \tag{2.7}$$

where η is such that $\theta(r) < 0$ if $r \le \eta$ or if $r \ge 1/\eta$.

(D) (i) the functions $\sigma_{ij}(x)$ are twice continuously differentiable if $0 \le R(x) \le \eta$, or if $R(x) \ge 1/\eta$; (ii) the functions

$$\partial a_{ij}/\partial x_j, \quad \partial^2 a_{ij}/\partial x_i \partial x_j, \quad \partial b_i/\partial x_i$$

are uniformly Hölder continuous on compact subsets, and

$$\sum_{i,j} \left| \frac{\partial a_{ij}}{\partial x_j} \right| \le C, \quad \sum_{i,j} \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} - \sum_i \frac{\partial b_i}{\partial x_i} \le C$$

where C is a constant.

The following result shows that the condition (C) is satisfied whenever (a_{ij}) is nondegenerate outside $\bigcup_j G_j$.

Lemma 2.1. *If $n \ge 2$ and $(a_{ij}(x))$ is positive definite for $x \notin \hat{G}_\epsilon$, $|x| < 1/\epsilon$, where ϵ is sufficiently small, then the condition* (C) *is satisfied for some choice of R.*

Proof. By the proof of the Schoenflies' theorem [5] there is a diffeomorphism $y = f(x)$ of the exterior of $\bigcup_{j=1}^k G_j$ onto the exterior of $\bigcup_{j=1}^k G'_j$ in R^l, where $G'_1, \cdots, G'_{k_0}$ are points situated on the y_1-axis and $G'_{k_0+1}, \cdots, G'_k$ are balls with centers on the y_1-axis; the center of G'_j lies to the left of the center of G'_{j+1}. Furthermore, this diffeomorphism preserves the distance functions (to $\bigcup_j G_j$ and to $\bigcup_j G'_j$) as long as the distance is sufficiently small. Suppose for simplicity that $k_0 = 0$, $k = 2$. Denote by $(c_1, 0, \cdots, 0)$ the midpoint of the segment connecting the center $(\alpha_1, 0, \cdots, 0)$ of G'_1 to the center $(\alpha_2, 0, \cdots, 0)$ of G'_2. Construct a positive C^2 function $\phi(y')$ (where $y' = (y_2, \cdots, y_l)$) on the plane $y_1 = c_1$, which increases radially, with grad $\phi(y') \neq 0$ if $y' \neq 0$, such that $\partial\phi/\partial y_i = 0$, $\partial^2\phi/\partial y_i \partial y_j = 0$ $(2 \leq i, j \leq l)$ at $y' = 0$. Construct also a C^2 function $\psi(y_1)$, positive for $\alpha_1 < y_1 < \alpha_2$, such that $\psi(y_1) = |y_1 - \alpha_j|$ for y_1 near α_j, and such that $\psi'(y_1) \neq 0$ if $y_1 \neq c_1$ and $\psi(c_1) = \phi(0, \cdots, 0)$, $\psi'(c_1) = 0$, $\psi''(c_1) < 0$.

We now construct a C^2 positive function $\lambda(y)$ for $y \notin (G'_1 \cup G'_2)$, $|y| < R_0$ (R_0 large) which extends the functions ϕ, ψ and the distance function from $G'_1 \cup G'_2$ (as long as the distance is sufficiently small). This function is to satisfy

$$\operatorname{grad} \lambda(y) \neq 0 \quad \text{if } y \neq (c_1, 0, \cdots, 0),$$

$$\operatorname{grad} \lambda(y) = 0, \qquad \frac{\partial^2 \lambda(y)}{\partial y_i \partial y_j} = 0 \quad \text{if } (i, j) \neq (1, 1),$$

$$\frac{\partial^2 \lambda(y)}{\partial y_1^2} < 0 \quad \text{at } y = (c_1, 0, \cdots, 0).$$

The construction of such a function $\lambda(y)$ can be accomplished by introducing a family of curves $\gamma_{y'}$ connecting $(\alpha_1, 0, \cdots, 0)$ to $(\alpha_2, 0, \cdots, 0)$ and intersecting the plane $y_1 = c_1$ orthogonally at (c_1, y'). $\lambda(y)$ is defined along $\gamma_{y'}$ such that its tangential derivative vanishes only at $y_1 = c_1$.

Let B_0 be a ball $\{x: |x| < R^*\}$ containing $G_1 \cup G_2$. Choose R_0 so large that the image of $B = B_0 - (G_1 \cup G_2)$ under the diffeomorphism $y = f(x)$ is contained in the ball $\{y; |y| < R_0\}$. Define $R(x) = \lambda(f(x))$ for $x \in B$. Clearly $\nabla_x R(x) \neq 0$ if $x \in B$, $x \neq x^*$ where $f(x^*)$ is the point $(c_1, 0, \cdots, 0)$. Furthermore, as easily seen,

$$\sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial^2 R}{\partial x_i \partial x_j} < 0 \quad \text{at } x = x^*.$$

Now extend $R(x)$ as a positive C^2 function in $R^l - (G_1 \cup G_2)$ such that $R(x) = |x|$ for all $|x|$ sufficiently large, and such that $\nabla_x R(x) \neq 0$ if $|x| > R^*$. This completes the proof of the lemma in case $k_0 = 0$, $k = 2$. The proof for general k_0, k is similar.

Theorem 2.2. *Let* (A), $(\underset{\sim}{\mathrm{B}})$, (2.5), (2.6) *and* (C), (D) *hold, and let* $x(t)$ *be any solution of* (1.1) *with* $x(0) \in \tilde{G}$. *Then*

$$P\left\{\lim_{t\to\infty} \operatorname{dist}(x(t), \partial\tilde{G}) = 0\right\} = 1.$$

Proof. Suppose first that the $\sigma_{ij}(x)$ belong to $C^2(R^l)$. Let $E = \{x \in \tilde{G}_\eta;\ \nabla_x R(x) = 0\}$. E is a compact set. On E,

$$Q(x) = \frac{1}{2R} \sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial^2 R}{\partial x_i \partial x_j} < 0$$

by the second inequality of (2.7). Hence there is a small neighborhood E_0 of E, whose closure is in $\tilde{G}_\eta$, such that $Q(x) < 0$ if $x \in E_0$. Let $r_1 = \eta$, $r_2 = 1/\eta$. From the first inequality of (2.7),

$$\sum_{i,j=1}^{l} a_{ij}(x) \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j} \geq \alpha R^2 \quad \text{if } r_1 \leq R(x) \leq r_2,\ x \notin E_0 \tag{2.8}$$

where α is a positive constant. We shall construct a function $\Phi(r)$ whose second derivative $\Phi''(r)$ has jump discontinuities at the points r_1, r_2, and is otherwise continuous, such that

2(a) $\Phi'(r) > 0$;

2(b) $\mathcal{L}\Phi(R(x)) < -\nu$ if $R(x) \neq r_1, r_2$; ν positive constant;

2(c) $\lim_{r\to 0} \Phi(r) = -\infty$;

2(d) $r\Phi'(r)$ is bounded, $0 < r < \infty$.

Let

$$\mu(r) = \exp\left\{\int_1^r \frac{1 + 2\theta(s)/\alpha}{s}\, ds\right\} \tag{2.9}$$

and define $\psi(r)$ in $r_1 \leq r \leq r_2$ by

$$\mu(r)\psi'(r) = \frac{2\nu}{\alpha} \int_r^{r_2+1} \frac{\mu(s)}{s^2}\, ds, \qquad \psi(1) = 0.$$

Then $\psi'(r) > 0$ if $r_1 \leq r \leq r_2$. Also,

$$\psi''(r) + (1 + 2\theta(r)/\alpha)\, \psi'/r = -2\nu/\alpha r^2. \tag{2.10}$$

On the other hand, from (2.1), (2.8) we have, for $x \in \tilde{G}_\eta - E_0$,

$$L\psi(R(x)) = \frac{1}{2}\mathfrak{A}\left[\psi'' + \left(1 + \frac{2R^2 Q}{\mathfrak{A}}\right)\frac{\psi'}{R}\right] \leq \frac{1}{2}\mathfrak{A}\left[\psi'' + \left(1 + \frac{2\theta(R)}{\alpha}\right)\frac{\psi'}{R}\right]$$

where the argument in ψ, ψ', ψ'' is $R(x)$; here we have used (2.8). By (2.10) the quantity in the last brackets is $\leq -2\nu/(\alpha R^2)$. Hence

$$L\psi(R(x)) \leq \tfrac{1}{2}\mathfrak{A}(-2\nu/\alpha R^2) = -\mathfrak{A}\nu/\alpha R^2.$$

Application of (2.8) once more shows that $L\psi(R(x)) \leq -\nu$ for $x \in \tilde{G}_\eta - E_0$. If $x \in E_0$, then

$$\psi''(R(x)) + \psi'(R(x))/R(x) < 0$$

by (2.10) [since we may assume that $\theta(r) > 0$ if $r = R(x)$, $x \in E_0$]. Recalling that

$\psi'(R(x)) > 0$, $Q(x) < 0$ on the closure of E_0, we conclude, by (2.1), that $L\psi(R(x)) \leq -\nu_0 < 0$ if $x \in E_0$. Designating $\min(\nu, \nu_0)$ by ν, we get the inequality $L\psi(R(x)) \leq -\nu$ throughout $\tilde{G}_\eta$.

Define

$$\Phi(r) = \begin{cases} A_1 \log r + B_1 & \text{if } 0 < r < r_1, \\ \psi(r) & \text{if } r_1 \leq r \leq r_2, \\ A_2 \log r + B_2 & \text{if } r_2 < r < \infty, \end{cases} \tag{2.11}$$

and choose the constants A_i, B_i so that $\Phi(r)$ and $\Phi'(r)$ are continuous at r_1, r_2. Since $\psi'(r) > 0$ in $r_1 \leq r \leq r_2$, the constants A_i are positive. But then 2(a) holds. The conditions 2(c), 2(d) are also obviously satisfied. Finally, 2(b) was already proved above for $r_1 \leq r \leq r_2$. Its validity for $r < r_1$ and for $r > r_1$ follows from (2.1) and the fact that $\theta(r) \leq \mu < 0$ if $r \leq r_1$ or if $r \geq r_1$.

Let $\Gamma^m(r)$ $(m = 1, 2, \cdots)$ be a continuous function such that $\Gamma^m(r) = \Phi''(r)$ if $|r - r_i| > 1/m$ $(i = 1, 2)$, $\Gamma^m(r)$ is bounded independently of m, when $|r - r_i| < 1/m$, and

$$\int_a^b \Gamma^m(r)\,dr = \int_a^b \Phi''(r)\,dr \qquad (a = r_i - 1/m,\ b = r_i + 1/m;\ i = 1, 2).$$

Define

$$\Phi^m(r) = \Phi(1) + \Phi'(1)(r-1) + \int_1^r \int_1^r \Gamma^m(s)\,ds\,dr.$$

Then $(\Phi^m)'(r) = \Phi'(r)$, $(\Phi^m)''(r) = \Phi''(r)$ if $|r - r_i| > 1/m$ $(i = 1, 2)$ and $(\Phi^m)''(r)$ is bounded independently of m when $|r - r_i| < 1/m$. Finally,

$$|\Phi^m(r) - \Phi(r)| \leq C/m \quad \text{for all } r > 0,$$

where C is a constant independent of m. For any small δ, $0 < \delta < r_1$, define

$$\Phi_\delta(r) = \begin{cases} \Phi(r) & \text{if } r \geq \delta, \\ \log\delta + (r-\delta)/\delta - \tfrac{1}{2}(r-\delta)^2/\delta^2 & \text{if } 0 < r < \delta, \end{cases}$$

and

$$\Phi_\delta^m(r) = \begin{cases} \Phi^m(r) & \text{if } r \geq \delta, \\ \log\delta + (r-\delta)/\delta - \tfrac{1}{2}(r-\delta)^2/\delta^2 & \text{if } 0 < r < \delta. \end{cases}$$

Let $R_\delta(x)$ be a positive C^2 function in the whole space, coinciding with $R(x)$ if $R(x) > \delta$.

Let

$$L_\epsilon u \equiv \frac{1}{2}\sum_{i,j=1}^{l} (a_{ij}(x) + \epsilon\delta_{ij})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{l} b_i(x)\frac{\partial u}{\partial x_i} \tag{2.12}$$

where $\epsilon > 0$, and let $\sigma^\epsilon(x)$ be a matrix such that $\sigma^\epsilon(\sigma^\epsilon)^* = (a_{ij} + \epsilon\delta_{ij})$. We can choose σ^ϵ to be uniformly Lipschitz continuous on compact subsets [2]. Denote by $x^\epsilon(t)$ the solution of the stochastic equation (1.1) when the σ_{ij} are replaced by the σ^ϵ_{ij}.

Since $\Phi^m_\delta(r)$ and $R_\delta(x)$ are C^2 functions, we can apply Itô's formula:

$$\Phi^m_\delta(R_\delta(x^\epsilon(t))) - \Phi^m_\delta(R_\delta(x(0)))$$

$$(2.13) \qquad = \sum_{i,j} \int_0^t (\Phi^m_\delta)'(R_\delta(x^\epsilon(s))) \frac{\partial R_\delta(x^\epsilon(s))}{\partial x_i} \sigma^\epsilon_{ij}(x^\epsilon(s))\,dw^j + \int_0^t L_\epsilon \Phi^m_\delta(R_\delta(x^\epsilon(s)))\,ds.$$

Our assumptions on a_{ij}, b_i (in particular, assumption (D) (ii)) are such that a result of Aronson and Besala [1] ensures the existence of a fundamental solution $K_\epsilon(x, t, y)$ for the uniformly parabolic operator (with, generally, unbounded coefficients) $L_\epsilon - \partial/\partial t$. Hence, letting $d\mu = dP \times dt$, we have

$$\mu\{(\omega, s);\ r_i - 1/m < R_\delta(x^\epsilon(s)) < r_i + 1/m,\ 0 \le s \le 1\}$$

$$(2.14) \qquad = \int_0^t ds \int_{|R_\delta(\xi) - r_i| < 1/m} K_\epsilon(x(0), s, \xi)\,d\xi \to 0 \quad \text{if } m \to \infty.$$

We used here the fact that the measure of the set $\{\xi;\ |R_\delta(\xi) - r_i| < 1/m\}$ converges to zero if $m \to \infty$. This is certainly true if r_1 and $1/r_2$ are sufficiently small, which may be assumed.

Computing $L_\epsilon \Phi^m_\delta(R_\delta(x^\epsilon(s)))$ in a manner analogous to (2.1), and using the definitions of Φ^m_δ and R_δ, we find that $L_\epsilon \Phi^m_\delta(R_\delta(x^\epsilon(s)))$ is bounded uniformly with respect to m. We can then use (2.14) to conclude [by the Lebesgue bounded convergence theorem] that, as $m \to \infty$, the second integral on the right-hand side of (2.13) converges in L^2 to $\int_0^t L_\epsilon \Phi_\delta(R_\delta(x^\epsilon(s)))\,ds$. Similarly, the stochastic integral on the right-hand side of (2.13) is convergent in probability to

$$\sum_{i,j} \int_0^t (\Phi_\delta)'(R_\delta(x^\epsilon(s))) \frac{\partial R_\delta(x^\epsilon(s))}{\partial x_i} \sigma^\epsilon_{ij}(x^\epsilon(s))\,dw^j.$$

We conclude that

$$\Phi_\delta(R_\delta(x^\epsilon(t))) - \Phi_\delta(R_\delta(x(0)))$$

$$(2.15) \qquad = \sum_{i,j} \int_0^t \Phi'_\delta(R_\delta(x^\epsilon(s))) \frac{\partial R_\delta(x^\epsilon(s))}{\partial x_i} \sigma^\epsilon_{ij}(x^\epsilon(s))\,dw^j + \int_0^t L_\epsilon \Phi_\delta(R_\delta(x^\epsilon(s)))\,ds.$$

We now need the relation

$$(2.16) \qquad \sup_{0 \le s \le t} |x^\epsilon(s) - x(s)| \to 0 \quad \text{in probability, as } \epsilon \to 0.$$

To verify it, notice by [2] that, as $\epsilon \to 0$, (2.17) $$\sigma^\epsilon(x) \to \sigma(x) \quad \text{uniformly on compact sets;}$$

here we use the assumption that $\sigma_{ij} \in C^2(R^l)$. Hence, by a standard argument [5, p. 52], for any $T > 0$,

(2.18) $$\sup_{0\le t\le T} E|x^\epsilon(t) - x(t)|^2 \to 0 \quad \text{if } \epsilon \to 0.$$

We can write

$$x^\epsilon(t) - x(t) = \int_0^t [b(x^\epsilon(s)) - b(x(s))]\,ds$$

(2.19) $$+\int_0^t [\sigma^\epsilon(x^\epsilon(s)) - \sigma(x^\epsilon(s))]\,dw(s) + \int_0^t [\sigma(x^\epsilon(s)) - \sigma(x(s))]\,dw(s)$$
$$\equiv A_\epsilon(t) + B_\epsilon(t) + C_\epsilon(t).$$

If we denote by μ the Lebesgue measure on $(0, T)$, then

$$(P\times\mu)\{|x^\epsilon(s)| > R\} \le \frac{1}{R^2}\int_0^T E|x^\epsilon(s)|^2\,ds \le \frac{C}{R^2} \to 0$$

if $R \to \infty$, where C is a constant independent of ϵ. Hence, by (2.17), for any $\delta > 0$,

(2.20) $$(P\times\mu)\{|\sigma^\epsilon(x^\epsilon(s)) - \sigma(x^\epsilon(s))| > \delta\} \to 0 \quad \text{if } \epsilon \to 0.$$

We can now show that

(2.21) $$E\int_0^T |\sigma^\epsilon(x^\epsilon(s)) - \sigma(x^\epsilon(s))|^2\,ds \to 0 \quad \text{if } \epsilon \to 0.$$

Indeed, by (2.20), the integrand converges to zero in measure $P\times\mu$. The integrand is also uniformly integrable, since

$$|\sigma^\epsilon(x^\epsilon(s)) - \sigma(x^\epsilon(s))|^4 \le C(1 + |x^\epsilon(s)|^4)$$

and (by [5, p. 48])

$$E\int_0^T |x^\epsilon(s)|^4\,ds \le C$$

(C constant independent of ϵ). Hence (2.21) follows.

Now, for the Itô integral $\int_0^t f(s)\,dw(s)$ we have [5]

$$E\sup_{0\le t\le T}\left|\int_0^t f(s)\,dw(s)\right|^2 \le 4E\int_0^T |f(s)|^2\,ds.$$

Using this with $f(s) = \sigma^\epsilon(x^\epsilon(s)) - \sigma(x^\epsilon(s))$ we get, upon using (2.21),

$$E\sup_{0\le t\le T}|B_\epsilon(t)|^2 \to 0 \quad \text{if } \epsilon \to 0.$$

Recalling (2.17), we can similarly show that

$$E \sup_{0\le t\le T} |C_\epsilon(t)|^2 \to 0 \quad \text{if } \epsilon \to 0.$$

The same assertion also holds for $A_\epsilon(t)$. Hence, (2.19) gives

$$E \sup_{0\le t\le T} |x^\epsilon(t) - x(t)|^2 \to 0 \quad \text{if } \epsilon \to 0,$$

and (2.16) follows.

From (2.16), we deduce, for a sequence $\{\epsilon'\}$,

$$P\{x^{\epsilon'}(s) \to x(s) \text{ uniformly in } s,\ 0 \le s \le t\} = 1 \quad \text{if } \epsilon' \to 0.$$

Hence, by Theorem 1.1, for almost all ω, if δ is sufficiently small, say $\delta \le \delta^*(\omega)$, then $\inf_{0\le s\le t} R(x^{\epsilon'}(s)) > \delta$ for all ϵ' sufficiently small, so that

$$\overline{\lim_{\epsilon'\to 0}} \int_0^t L_{\epsilon'}\Phi_\delta(R_\delta(x^{\epsilon'}(s)))\,ds \le -\nu t. \tag{2.22}$$

As for the stochastic integral we have, for fixed δ: if $\epsilon' \to 0$ then

$$\int_0^t \Phi_\delta'(R_\delta(x^\epsilon(s)))\frac{\partial R_\delta(x^\epsilon(s))}{\partial x_i}\sigma_{ij}(x^\epsilon(s))\,dw^j$$

$$\to \int_0^t \Phi_\delta'(R_\delta(x(s)))\frac{\partial R_\delta(x(s))}{\partial x_i}\sigma_{ij}(x(s))\,dw^j \tag{2.23}$$

$$= \int_0^t \Phi'(R(x(s)))\frac{\partial R(x(s))}{\partial x_i}\sigma_{ij}(x(s))\,dw^j$$

in probability; for a subsequence $\{\epsilon''\}$ of $\{\epsilon'\}$ the convergence is a.s. Hence, if δ is any one of the numbers $1/p$ $(p = 1, 2, \cdots)$ then (2.17) holds for all $\omega \in \Omega_0$ where $P(\Omega_0) = 1$, Ω_0 independent of p, where ϵ varies over a suitable sequence.

In the definition of $\delta^*(\omega)$ given above we can take the values of δ^* to be $1/p$ $(p = 1, 2, \cdots)$. Denote by A_p the set of points ω with $\delta^*(\omega) = 1/p$. If $\omega \in A_p \cap \Omega_0$, then (2.22) holds with $\delta = 1/p$, and (2.23) holds with $\delta = 1/p$ (where the convergence is at the point ω). Since $P[(\bigcup_p A_p) \cap \Omega_0] = 1$, we conclude that a.s.

$$\Phi(R(x(t))) - \Phi(R(x(0))) \le \sum_{i,j}\int_0^t \Phi'(R(x(s)))\frac{\partial R(x(s))}{\partial x_i}\sigma_{ij}(x(s))\,dw^j - \nu t. \tag{2.24}$$

In deriving (2.24) we have assumed that $\sigma_{ij} \in C^2(R^l)$. If this assumption is not satisfied, we approximate the σ_{ij} uniformly by σ_{ij}^k which belong to $C^2(R^l)$ and for which the assumptions (A)–(D), (2.5), (2.6) hold. In view of (D)(i), we can take $\sigma_{ij}^k = \sigma_{ij}$ if $R(x) > 1 + 1/\eta$ or if $R(x) < \eta/2$. If we apply (2.24) to σ_{ij}^k (with $x(t) = x^k(t)$) and take $k \to \infty$, we obtain (2.24).

We can now easily complete the proof of Theorem 2.2. First,

$$\sum_j \left| \sum_i \Phi'(R(x)) \frac{\partial R}{\partial x_i} \sigma_{ij}(x) \right|^2 = (\Phi'(R))^2 \sum a_{ij}(x) \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j}$$

is a bounded function in $\tilde{G}$. Hence by Lemma 1.3 of [3]

$$\frac{1}{t} \sum_{i,j} \int_0^t \Phi'(R(x(s))) \frac{\partial R(x(s))}{\partial x_i} \sigma_{ij}(x(s))\, dw^j \to 0$$

a.s. as $t \to \infty$. From (2.24) we then conclude that

$$\overline{\lim_{t\to\infty}} \frac{\Phi(R(x(t)))}{t} \leq -\nu. \tag{2.25}$$

This implies that $\Phi(R(x(t))) \to -\infty$ if $t \to \infty$. Hence, $R(x(t)) \to 0$ if $t \to \infty$ a.s. This completes the proof of Theorem 2.2.

Remark 1. The inequality (2.25) implies that

$$\operatorname{dist}(x(t), \partial\tilde{G}) \leq A e^{-\nu' t}$$

for any $0 < \nu' < \nu$, where A is a random variable.

Remark 2. The differentiability assumptions made in (D)(ii) can be weakened, if we redefine L_ϵ (see (2.12)) by

$$L_\epsilon u \equiv \frac{1}{2} \sum_{i,j=1}^{l} (a_{ij}^\epsilon(x) + \epsilon\delta_{ij}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{l} b_i^\epsilon(x) \frac{\partial u}{\partial x_i}$$

where the a_{ij}^ϵ, b_i^ϵ are smooth functions that converge to a_{ij}, b_i in an appropriate manner.

Remark 3. Theorems 1.1, 2.2 can be extended to the case where some of the domains G_h $(k_0 + 1 \leq h \leq k)$ have piecewise C^3 boundary and are convex. For simplicity take $G = G_1$, $k_0 + 1 = k = 1$. We assume

(G) G is a bounded, closed and convex domain with piecewise C^3 boundary ∂G.

By ∂G being piecewise C^3 we mean the following: ∂G can be triangulated by means of C^3 surfaces (with boundary) $\Gamma_{l-j,i}$ of dimension $l-j$, $1 \leq j \leq l$; $\Gamma_{0,i}$ being points, i.e., vertices of ∂G. One can then show that the distance function $R(x)$ if C^1 and piecewise C^2 in $\hat{G}_{\epsilon_0}$ (for some $\epsilon_0 > 0$). The set of discontinuities Σ of the second derivatives of $R(x)$ divides $\hat{G}_{\epsilon_0}$ into regions $\Omega_{l-j,i}$ bounded by some hypersurface of Σ, by the outer boundary of $\hat{G}_{\epsilon_0}$, and by $\Gamma_{l-j,i}$. We replace the condition (B) by

$(\hat{\mathrm{B}})$ $b = 0$, $\sigma = 0$ at the vertices $\Gamma_{0,i}$. On each $\Gamma_{l-j,i}$ $(1 \leq j < l)$ (1.3), (1.4) hold for all the normals ν to $\Gamma_{l-j,i}$ pointing into $\Omega_{l-j,i}$.

Theorem 1.1′. *Let* (G), (A), $(\hat{B})$, (D) *hold. Then the assertion of Theorem* 1.1 *is valid.*

Proof. For any small $\delta > 0$, let $R_\delta(x) = R(x)$ if $R(x) > \delta$, and $R_\delta(x)$ is positive C^1 and piecewise C^2 in R^l. Let $R_\delta^m(x)$ be a mollifier of $R_\delta(x)$, obtained by convolving $R_\delta(x)$ with $\rho_{1/m}(x)$, where $\rho_\epsilon(x) = 0$ if $|x| \geq \epsilon$, $\rho_\epsilon(x) = \gamma \exp[\epsilon^2/(|x|^2 - \epsilon^2)]$ if $|x| < \epsilon$, $\int \rho_\epsilon(x)dx = 1$. One can verify that $R_\delta^m(x) \to R_\delta(x)$, $D_x R_\delta^m(x) \to D_x R_\delta(x)$ for all x, and $D_x^2 R_\delta^m(x) \to D_x^2 R_\delta(x)$ if $R(x) > \delta$, $x \notin \Sigma$. Further, $|D_x^2 R_\delta^m(x)| \leq C$, C constant independent of m. We now modify the proof of Theorem 1.1. First we apply Itô's formula to $e^{-\mu t} V_\delta^m(x^\eta(t))$ ($x^\eta(t)$ is the solution of (1.1) with σ replaced by σ^η; $\sigma^\eta(\sigma^\eta)^* = (a_{ij} + \eta\delta_{ij})$ as in [3]. Then we let $m \to \infty$, using the fact that L_η has a fundamental solution. Finally we take $\eta \to 0$. This leads to (1.10).

Theorem 2.2 also extends to the case where (G) holds and (B) is replaced by $(\hat{B})$. In the proof we use (2.13) with R_δ replaced by R_δ^m.

Note that the convexity of ∂G is actually required only in a neighborhood of the set where the boundary is not C^3 (so as to ensure that $R(x)$ is in C^1).

Remark 4. The conditions (2.5), (2.6) and (C) are (essentially) necessary for the validity of the assertion of Theorem 2.1. In fact, if in (2.5) the inequality is reversed at G_1 [G_1 consisting of one point] then $\rho_1(x(t))$ may not converge to 0 a.s. (compare the linear case [3]). A similar remark applies to (2.6). Finally, regarding (C), if for instance $b_i \equiv 0$, $\sigma_{ir} \equiv 0$ in an open set Ω outside G, then $x(t)$ will not leave Ω, so that the assertion of Theorem 2.1 will not hold. This remark applies also when the b_i do not vanish identically in Ω, but there is an integral manifold of $\dot{x} = b(x)$ in Ω.

Remark 5. The condition (D)(ii) was needed only in order to ensure the existence of the fundamental solution $K_\epsilon(x, t, s)$. If $a_{ij}(x)$, $b_i(x)$ are bounded functions then, since (A) holds, the existence of the fundamental solution follows from the general theory of parabolic equations [3] (without assuming (D)(ii)). However, we do not consider here the case of bounded a_{ij}, b_i, for the condition (2.6) cannot hold in this case.

Application. Let L be the elliptic operator associated with the diffusion process (1.1), and consider the Cauchy problem

$$\frac{\partial u}{\partial t} = Lu \quad \text{if } x \in R^l,\ t > 0, \tag{2.26}$$
$$u(0, x) = f(x) \quad \text{if } x \in R^l.$$

Suppose $f(x)$ is continuous and bounded. Then a solution of (2.26) is given by $u(t, x) = Ef(\xi_x(t))$ where $\xi_x(t)$ is the solution of the stochastic system (1.1) with the initial condition $\xi_x(0) = x$. Set $c_j = f(z_j)$ if $1 \leq j \leq k_0$, and suppose $f = c_j$

(c_j constant) on ∂G_j if $k_0 + 1 \le j \le k$. Then, under the conditions of Theorem 2.1,

$$\lim_{t\to\infty} u(t, x) = \sum_{j=1}^{k} c_j p_j(x) \quad \left(p_j(x) \ge 0, \sum_{j=1}^{k} p_j(x) = 1\right) \tag{2.27}$$

where $p_j(x)$ is the probability that $\rho_j(\xi_x(t)) \to 0$ as $t \to \infty$.

The assumption (D) made in Theorem 2.2 is superfluous. Indeed, this condition was used only in proving (2.24). As indicated in a forthcoming paper by one of us (A.F.), (2.24) can be proved (more simply) without assuming (D). This is also a consequence of a new treatment of stability problems by one of us (M.P.).

The condition (2.6) made in Theorem 2.2 can be replaced by a weaker condition, e.g.

$$B + \lambda \mathfrak{A}/R \le -\alpha/R^{\lambda} \quad \text{for some } \lambda > -1. \tag{*}$$

Indeed, just modify the definition of $\Phi(r)$ in (2.11), taking

$$\Phi(r) = A_2 r^{1+\lambda} + B_2,$$

and observe that (*) implies that $L\Phi(R) \le -\nu$ if $R(x)$ is large (ν positive constant). Further refinements of (2.6) will appear in a forthcoming paper by one of us (M.P.).

3. **Angular behavior in the case $l = 2$; case of a point.** We now consider the case $l = 2$ and propose to study the rotation properties of $x(t)$. We introduce polar coordinates (r, ϕ) by $x = r\cos\phi$, $y = r\sin\phi$. The stochastic differentials dr, $d\phi$ may be formally computed by

$$dr = r_x dx + r_y dy + \tfrac{1}{2} r_{xx} a_{11} dt + r_{xy} a_{12} dt + \tfrac{1}{2} r_{yy} a_{22} dt,$$

$$d\phi = \phi_x dx + \phi_y dy + \tfrac{1}{2}\phi_{xx} a_{11} dt + \phi_{xy} a_{12} dt + \tfrac{1}{2}\phi_{yy} a_{22} dt.$$

Noting that

$$\phi_x = -\frac{\sin\phi}{r}, \quad \phi_y = \frac{\cos\phi}{r},$$

$$\phi_{xx} = \frac{2\sin\phi\cos\phi}{r^2}, \quad \phi_{xy} = \frac{\sin^2\phi - \cos^2\phi}{r^2}, \quad \phi_{yy} = -\frac{2\sin\phi\cos\phi}{r^2},$$

$$r_x = \cos\phi, \quad r_y = \sin\phi,$$

$$r_{xx} = \frac{\sin^2\phi}{r}, \quad r_{xy} = -\frac{\sin\phi\cos\phi}{r}, \quad r_{yy} = \frac{\cos^2\phi}{r},$$

we have

$$dr = \sum_{s=1}^{n} \tilde{\sigma}_s(r, \phi) dw^s + \tilde{b}(r, \phi) dt, \qquad d\phi = \sum_{s=1}^{n} \tilde{\tilde{\sigma}}_s(r, \phi) dw^s + \tilde{\tilde{b}}(r, \phi) dt \tag{3.1}$$

where

$$\tilde{\sigma}_s(r, \phi) = \sigma_{1s} \cos \phi + \sigma_{2s} \sin \phi,$$

$$\tilde{b}(r, \phi) = b_1 \cos \phi + b_2 \sin \phi + \frac{1}{2r} \langle a(x)\lambda^{\perp}, \lambda^{\perp} \rangle,$$

$$\tilde{\tilde{\sigma}}_s(r, \phi) = -\frac{\sin \phi}{r} \sigma_{1s} + \frac{\cos \phi}{r} \sigma_{2s},$$

$$\tilde{\tilde{b}}(r, \phi) = -\frac{\sin \phi}{r} b_1 + \frac{\cos \phi}{r} b_2 - \frac{1}{r^2} \langle a(x)\lambda, \lambda^{\perp} \rangle;$$

here $\lambda = (\cos \phi, \sin \phi)$, $\lambda^{\perp} = (-\sin \phi, \cos \phi)$ and $\langle a(x)\mu, \nu \rangle = \sum a_{ij}(x)\mu_i\nu_j$ ($\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2)$). We now assume

$$\sigma_{is}(x) = \sum_{j=1}^{2} \sigma_{is}^j x_j + \epsilon_{is}(x), \qquad \frac{\epsilon_{is}(x)}{|x|} \to 0 \quad \text{if } |x| \to 0,$$

(E)

$$b_i(x) = \sum_{j=1}^{2} b_i^j x_j + \tilde{\epsilon}_i(x), \qquad \frac{\tilde{\epsilon}_i(x)}{|x|} \to 0 \quad \text{if } |x| \to 0,$$

where σ_{is}^j, b_i^j are constants.

This implies that the stochastic differential equations (3.1) have the form

(3.2)

$$dr = r\left[\sum_{s=1}^{n} \tilde{\sigma}_s(\phi)dw^s + \tilde{b}(\phi)dt\right] + \left[\sum_{s=1}^{n} R_s dw^s + R_0 dt\right],$$

$$d\phi = \left[\sum_{s=1}^{n} \tilde{\tilde{\sigma}}_s(\phi)dw^s + \tilde{\tilde{b}}(\phi)dt\right] + \left[\sum_{s=1}^{n} \Theta_s dw^s + \Theta_0 dt\right]$$

where $R_s = o(r)$, $\Theta_s = o(1)$ $(0 \le s \le n)$ when $r \to 0$, uniformly for $0 \le \phi \le 2\pi$.

Now let $y(t) = (r(t), \phi(t))$ be the diffusion process defined by the solution of the stochastic differential equation (3.1) with $r(0) > 0$. By the method used to prove Theorem 1.1, the solution never leaves the half-plane $(0, \infty) \times (-\infty, \infty)$. Define $x(t) = (x_1(t), x_2(t))$ where $x_1(t) = r(t) \cos \phi(t)$, $x_2(t) = r(t) \sin \phi(t)$. By the method used to prove Theorem 2.1 of [3], we deduce

Theorem 3.1. $\{x(t), t \ge 0\}$ *is a diffusion process which can be obtained as a solution of* (1.1).

This theorem allows us to study the algebraic angle $\phi(t)$ as one component of a Markov process, rather than as a multivalued function of $x(t)$. In what follows we shall compare $\phi(t)$ with the solution of the single stochastic equation

$$d\phi = \sigma(\phi)dw + b(\phi)dt \tag{3.3}$$

where

$$\sigma(\phi) = \sqrt{\sum_{s=1}^{n} (\tilde{\tilde{\sigma}}_s(\phi))^2}, \qquad b(\phi) = \tilde{\tilde{b}}(\phi).$$

Theorem 3.2. *Assume that* (A)–(D) *and* (2.5), (2.6) *hold with* $k = k_0 = 1$, $G_1 = \{0\}$. *Assume also that* (E) *holds and that* $\sigma(z) > 0$ *for all real* z. *Let*

$$\Lambda \equiv \int_0^{2\pi} \frac{b(z)}{\sigma^2(z)}\, dz > 0.$$

Then

$$P\left\{\lim_{t\to\infty} \frac{\phi(t)}{t} = c\right\} = 1 \tag{3.4}$$

where c *is a positive constant. If* $\Lambda < 0$, *the conclusion holds with* c *negative.*

Proof. For the proof, it suffices to find a function f such that

3(a) $\frac{1}{2}\sigma^2(\phi)f''(\phi) + b(\phi)f'(\phi) = 1$ $(-\infty < \phi < \infty)$,

3(b) $\lim_{\phi\to\infty} f(\phi)/\phi = 1/c$ (c positive constant),

3(c) f' and f'' are bounded,

3(d) f' is bounded below by a positive constant.

Indeed, if f is such a function, then by Itô's formula,

$$f(\phi(t)) = f(\phi(0)) + \sum_s \int_0^t \tilde{\tilde{\sigma}}_s(r, \phi) f'(\phi)\, dw^s$$
$$+ \int_0^t \left[\frac{1}{2} \sum_s (\tilde{\tilde{\sigma}}_s(r, \phi))^2 f''(\phi) + \tilde{\tilde{b}}(r, \phi) f'(\phi)\right] dr. \tag{3.5}$$

Since $|\tilde{\tilde{\sigma}}_s(r(t), \phi(t))f'(\phi(t))| \le \text{const}$, Lemma 1.3 of [3] gives

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_s \tilde{\tilde{\sigma}}_s(r, \phi) f'(\phi)\, dw^s = 0 \quad \text{a.s.} \tag{3.6}$$

We now consider the integrand of the second integral on the right-hand side of (3.5). Given $\epsilon > 0$, let $r_0 > 0$ be such that

$$\left|\sum_{s=1}^{n} (\tilde{\tilde{\sigma}}_s(r, \phi))^2 - \sigma^2(\phi)\right| < \epsilon, \qquad |\tilde{\tilde{b}}(r, \phi) - b(\phi)| < \epsilon$$

for $0 < r < r_0$. Let $T_\epsilon = \sup\{t > 0;\ r(t) > r_0\}$. By Theorem 2.2, $T_\epsilon < \infty$ a.s. For $t > T_\epsilon$ we have by 3(a)

$$\left|\frac{1}{2} \sum_s (\tilde{\tilde{\sigma}}_s(r(t), \phi(t)))^2 f''(\phi(t)) + \tilde{\tilde{b}}(r(t), \phi(t)) f'(\phi(t)) - 1\right| < 2\epsilon K$$

where K is a common bound on f' and f'', given by 3(c). Combining this with (3.6), it follows from (3.5) that

$$\overline{\lim_{t\to\infty}} \frac{f(\phi(t))}{t} \le 1 + 2\epsilon K, \qquad \underline{\lim}_{t\to\infty} \frac{f(\phi(t))}{t} \ge 1 - 2\epsilon K.$$

This implies that a.s. $\lim_{t\to\infty} f(\phi(t))/t = 1$; in particular, $\phi(t) \to \infty$ if $t \to \infty$. Invoking condition 3(b), we then get

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to\infty} \frac{\phi(t)}{f(\phi(t))} \frac{f(\phi(t))}{t} = c,$$

which completes the proof of the theorem, subject to the construction of f.

To construct f, let

$$\beta(x) = \exp\left\{2 \int_0^x \frac{b(\phi)}{\sigma^2(\phi)}\, d\phi\right\}, \qquad f(x) = \int_0^x \frac{1}{\beta(z)} \int_{-\infty}^z \frac{2\beta(\phi)}{\sigma^2(\phi)}\, d\phi.$$

Clearly f satisfies 3(a). Since $\Lambda > 0$ we may write

$$2 \int_0^x \frac{b(z)}{\sigma^2(z)}\, dz = 2\Lambda \frac{x}{2\pi} + m(x)$$

where

$$m(x) = 2 \int_0^{x - [x/2\pi]2\pi} \frac{b(z)}{\sigma^2(z)}\, dz - 2\Lambda\left(\frac{x}{2\pi} - \left[\frac{x}{2\pi}\right]\right)$$

is a 2π-periodic function. Thus $\beta(x) = \exp\{\lambda x + m(x)\}$ $(\lambda = \Lambda/\pi)$. Hence we have

$$(3.7)\quad \begin{aligned} f'(x) &= \frac{2}{\beta(x)} \int_{-\infty}^x \frac{\beta(z)}{\sigma^2(z)}\, dz = \int_{-\infty}^x \frac{\exp\{\lambda z + m(z) - \lambda x - m(x)\}}{\sigma^2(z)}\, dz \\ &= \int_{-\infty}^{\infty} \frac{\exp\{-\lambda u + m(x-u) - m(x)\}}{\sigma^2(x-u)}\, du \qquad (u = x - z). \end{aligned}$$

Denote the last integral by $G(x)$. Since m and σ^2 are 2π-periodic, the same is true of $G(x)$. We conclude that

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\int_0^x G(z)\, dz}{x} = \frac{1}{2\pi} \int_0^{2\pi} G(z)\, dz.$$

This proves 3(b). The condition 3(c) follows immediately from (3.7) and the differential equation 3(a). Finally, condition 3(d) follows from the positivity of $f'(x)$ for any x (by (3.7)) and the asymptotic relation (3.7), noting that the integral on the right (denoted above by $G(x)$) is both 2π-periodic and positive function. Having proved 3(a)–3(d), the proof of Theorem 3.2 is complete.

4. **Case of a point, continued.** In this section we continue the analysis of §3 in case the condition $\sigma(z) > 0$ imposed in Theorem 3.2 is not satisfied, i.e., in case the angular diffusion is degenerate. We shall need the condition:

(E′) The condition (E) holds, and, for some $\bar{\epsilon} > 0$,

$$(4.1)\qquad \sum_{s=1}^{n} [\tilde{\partial}_s(r,\phi)]^2 = \sum_{s=1}^{n} [\tilde{\partial}_s(\phi)]^2[1+\eta(r,\phi)] \qquad (0 \le r \le \bar{\epsilon})$$

where $\eta(r, \phi) \to 0$ if $r \to 0$, uniformly with respect to ϕ.

Theorem 4.1. *Assume that* (A)–(D) *and* (2.5), (2.6) *hold with* $k = k_0 = 1$, $G_1 = \{0\}$. *Assume also that* (E′) *holds, that* $\sigma(z) \not\equiv 0$, $\sigma(z)$ *is not everywhere positive, and that* $b(z) > 0$ ($b(z) < 0$) *whenever* $\sigma(z) = 0$. *Then*

$$P\left\{\lim_{t\to\infty} \frac{\phi(t)}{t} = c\right\} = 1$$

where c *is a positive (negative) constant.*

Proof. It suffices to prove the theorem in case $b(z) < 0$ whenever $\sigma(z) = 0$. Denote by x_k $(k = \pm 1, \pm 2, \cdots)$ the zeros of $\sigma(z)$, so enumerated that $x_{k+1} > x_k$ for all k. Note that $\sigma(z)$ vanishes to a finite order at each point x_k.

Lemma 4.2. *There exists a function* f *with a periodic, positive and continuous derivative* f' *and with second derivative* f'' *existing for all* $x \ne x_k$ $(-\infty < k < \infty)$ *such that*

$$(4.2)\qquad \frac{1}{2}\sigma^2(x) f''(x) + b(x) f'(x) = -1 \quad \text{if } x \ne x_k.$$

Further, $\lim_{x\to x_k} \sigma^2(x) f''(x) = 0$ *and* $\lim_{x\to-\infty} f(x)/x$ *exists and is positive.*

Proof. Let

$$\beta(x) = \exp\ 2\left\{\int_{e_1}^{x} \frac{b(u)}{\sigma^2(u)}\, du\right\} \qquad (x_1 < x < x_2)$$

where e_1 is a point in (x_1, x_2). By the assumptions on b and σ, $\beta(x) < \exp\{-K/|x - x_2|^{\gamma}\}$ $(K > 0,\ \gamma > 0)$. Hence $\int^{x_2} \beta(u)/\sigma^2(u)\, du < \infty$. We define

$$f'(x) = \frac{1}{\beta(x)} \int_x^{x_2} \frac{2\beta(u)}{\sigma^2(u)}\, du, \qquad f(e_1) = 0.$$

Clearly $(\beta f')' = -2\beta/\sigma^2$, and hence

$$f'' + \frac{2b}{\sigma^2} f' = -\frac{2}{\sigma^2} \quad \text{for } x_1 < x < x_2.$$

By l'Hospital's rule,

$$\lim_{x\to x_2} f'(x) = \lim_{x\to x_2} \frac{-2\beta(x)/\sigma^2(x)}{\beta'(x)} = -\frac{1}{b(x_2)}.$$

Similarly, $\lim_{x\to x_1} f'(x) = -1/b(x_1)$.

In the interval (x_2, x_3) we define $f'(x)$ by the formula

$$f'(x) = \frac{1}{\beta(x)} \int_x^{x_3} \frac{2\beta(u)}{\sigma^2(u)} du,$$

where β is now defined by $\beta(x) = \exp\{\int_{e_2}^x 2b(u)/\sigma^2(u)\, du\}$ for some e_2 in the interval (x_2, x_3). We define $f(x)$ uniquely in (x_2, x_3) by setting $f(x_2 + 0) = f(x_2 - 0)$. Inductively we can thus extend f' and f to the whole line, preserving the condition $\lim_{x\to x_i} b(x)f'(x) = -1$ and the continuity of f. From the differential equation (4.2) for f' we deduce that $\lim_{x\to x_i} \sigma^2(x)f''(x) = 0$. Next, f' is positive and 2π-periodic. Finally, since

$$f(x) = f(e_1) + \int_{e_1}^x f'(u)du = \sum_1^{[x/2\pi]} \int_0^{2\pi} f'(u)du + O(1),$$

we have

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \frac{f(2\pi) - f(0)}{2\pi} > 0.$$

Assume now that (4.1) holds for all $r \geq 0$. Then we have

Lemma 4.3. *f satisfies*

$$\left| f(\phi(t)) - f(\phi(0)) - \sum_{s=1}^n \int_0^t f'(\phi(\tau))\tilde{\sigma}_s(r(\tau), \phi(\tau))\, dw^s + t \right| \tag{4.3}$$
$$\leq K \int_0^t |\hat{\eta}(r(\tau), \phi(\tau))|\, ds$$

where K is a constant, and $\hat{\eta}(r, \phi) \to 0$ if $r \to 0$, uniformly with respect to ϕ.

Note that if $\hat{\eta}(r, \phi) \equiv 0$ then this reduces to Itô's formula.

Proof. We shall apply Itô's formula to a regularization of f, and go to the limit.

Given $\epsilon > 0$ and a positive integer m, let

$$\beta_\epsilon(x) = \exp\left\{2 \int_0^x \frac{b(u)}{\sigma^2(u) + \epsilon} du\right\} \quad (x_{-m} \leq x \leq x_m),$$

(4.4)

$$f'_{\epsilon,m}(x) = \frac{1}{\beta_\epsilon(x)} \left[\int_x^{x_m} \frac{2\beta_\epsilon(y)}{\sigma^2(y) + \epsilon} dy + \frac{\beta_\epsilon(x_m)}{b(x_m)}\right].$$

Lemma 4.4. *For any x, $x_{-m} \leq x \leq x_m$, $f'_{\epsilon,m}(x) \to f'(x)$ as $\epsilon \to 0$.*

Proof. First we will show that convergence holds for $x_{m-1} \leq x \leq x_m$. Note that

(4.5)
$$\frac{\beta_\epsilon(y)}{\beta_\epsilon(x)} = \exp\left\{2\int_x^y \frac{b(u)}{\sigma^2(u)+\epsilon}\,du\right\} \qquad (x \le y \le x_m).$$

If $\delta > 0$, $\beta_\epsilon(y)/\beta_\epsilon(x) \to \exp\{2\int_x^y b(u)/\sigma^2(u)\,du\}$ boundedly for $x \le y \le x_m - \delta$, and hence

(4.6)
$$2\int_x^{x_m-\delta} \frac{\beta_\epsilon(y)}{\beta_\epsilon(x)} \frac{dy}{\sigma^2(y)+\epsilon} \to 2\int_x^{x_m-\delta} \frac{\beta(y)dy}{\beta(x)\sigma^2(y)} \qquad (\epsilon \to 0).$$

Also,

(4.7)
$$\frac{\beta_\epsilon(x_m)}{\beta_\epsilon(x)} = \exp\left\{2\int_x^{x_m} \frac{b(u)du}{\sigma^2(u)+\epsilon}\right\} \to 0 \quad \text{if } \epsilon \to 0.$$

Suppose we show that, given any $\gamma > 0$, there exists $\delta = \delta(\gamma)$ such that

(4.8)
$$\overline{\lim_{\epsilon\to 0}} \int_{x_m-\delta}^{x_m} \frac{\beta_\epsilon(y)}{\beta_\epsilon(x)} \frac{dy}{\sigma^2(y)+\epsilon} \le \gamma.$$

Then, by combining this with (4.6), (4.7) we conclude that $\overline{\lim}_{\epsilon\to 0} |f'_{\epsilon,m}(x) - f'(x)| \le 2\gamma$. Since γ is arbitrary, we get

(4.9)
$$\lim_{\epsilon\to 0} f'_{\epsilon,m}(x) = f'(x) \qquad (x_{m-1} < x < x_m).$$

For the purpose of proving (4.8), we may assume, for simplicity, that $x = 0$, $x_m = 1$. From (4.5),

$$\frac{\beta_\epsilon(y)}{\beta_\epsilon(x)} \le \exp\left\{\left[\int_0^\theta + \int_\theta^y\right]\left(\frac{2b(u)}{\sigma^2(u)+\epsilon}\right)du\right\}$$
$$\le K_1 \exp\left[\int_\theta^y \frac{2b(u)}{\sigma^2(u)+\epsilon}\,du\right] \le K_1 \exp\left[-K_2\int_\theta^y \frac{du}{\sigma^2(u)+\epsilon}\right]$$

where the K_i are positive constants, and θ is chosen so that $b(y) < 0$ for $\theta \le y \le 1$. Hence,

$$\int_{x_m-\delta}^{x_m} \frac{\beta_\epsilon(y)}{\beta_\epsilon(x)} \frac{dy}{\sigma^2(y)+\epsilon} \le K_1 \int_{1-\delta}^1 \frac{\exp\left[-K_2\int_\theta^y \frac{du}{\sigma^2(u)+\epsilon}\right]}{\sigma^2(y)+\epsilon}\,dy$$
$$= K_3\left\{\exp\left[-K_2\int_\theta^{1-\delta}\frac{du}{\sigma^2+\epsilon}\right] - \exp\left[-K_2\int_\theta^1 \frac{du}{\sigma^2+\epsilon}\right]\right\}.$$

When $\epsilon \to 0$, $\int_\theta^1 du/(\sigma^2(u)+\epsilon) \to \infty$ and hence the second term can be ignored. As for the first term, we have

$$\lim_{\epsilon\to 0}\int_\theta^{1-\delta}\frac{du}{\sigma^2(u)+\epsilon}=\int_\theta^{1-\delta}\frac{du}{\sigma^2(u)}.$$

By the Lipschitz continuity of σ, $\int_\theta^1 du/\sigma^2(u)=\infty$. Hence, given any $M>0$, we can find $\delta=\hat{\delta}(M)>0$ so that $\int_\theta^{1-\delta} du/\sigma^2(u)>M$. We then have

$$\overline{\lim_{\epsilon\to 0}}\exp\left[-K_2\int_\theta^{1-\delta}\frac{du}{\sigma^2(u)+\epsilon}\right]\le \exp[-K_2M].$$

Hence, given $\gamma>0$, if M is chosen so that $K_3\exp[-K_2M]\le\gamma$ then (4.8) holds (with $\delta(\eta)=\hat{\delta}(M)$).

To prove the convergence (4.9) at $x=x_{m-1}$, we can write, for any $\delta>0$,

$$f'_{\epsilon,m}(x_{m-1})=\int_{x_{m-1}}^{x_m}\frac{2}{\sigma^2(y)+\epsilon}\exp\left[2\int_{x_{m-1}}^{y}\frac{b(u)du}{\sigma^2(u)+\epsilon}\right]dy+\frac{\beta_\epsilon(x_m)}{\beta_\epsilon(x_{m-1})b(x_m)}$$

$$=\int_{x_{m-1}}^{x_{m-1}+\delta}\frac{2}{\sigma^2(y)+\epsilon}\exp\left[2\int_{x_{m-1}}^{y}\frac{b(u)du}{\sigma^2(u)+\epsilon}\right]dy$$

$$+O(e^{-K/\sqrt{\epsilon}})+\frac{\beta_\epsilon(x_m)}{\beta_\epsilon(x_{m-1})b(x_m)}$$

since the exponent tends to zero exponentially fast when $y\ge x_{m-1}+\delta$. If we let $F_\epsilon(y)=2\int_{x_{m-1}}^{y} b(u)/(\sigma^2(u)+\epsilon)\,du$, we have

$$f'_{\epsilon,m}(x_{m-1})=\int_{x_{m-1}}^{x_{m-1}+\delta}\frac{F'_\epsilon(y)}{b(y)}\exp[F_\epsilon(y)]\,dy+O(e^{-K/\sqrt{\epsilon}})+\frac{\beta_\epsilon(x_m)}{\beta_\epsilon(x_{m-1})b(x_m)}$$

provided δ is chosen so small that $b(y)<0$ for $x_{m-1}\le y\le x_{m-1}+\delta$. If we now integrate by parts, we get

$$f'_{\epsilon,m}(x_{m-1})=-\frac{1}{b(x_{m-1})}+\frac{\exp[F_\epsilon(x_{m-1}+\delta)]}{b(x_{m-1}+\delta)}$$

$$+\int_{x_{m-1}}^{x_{m-1}+\delta}\exp[F_\epsilon(y)]\frac{b'(y)}{b^2(y)}\,dy+O(e^{-K/\sqrt{\epsilon}})+\frac{\beta_\epsilon(x_m)}{\beta_\epsilon(x_{m-1})b(x_m)}.$$

The integrated term tends to zero by the bounded convergence theorem. Likewise $\exp[F_\epsilon(x_{m-1}+\delta)]\to 0$ if $\epsilon\to 0$. Hence

$$\lim_{\epsilon\to 0} f'_{\epsilon,m}(x_{m-1})=-\frac{1}{b(x_{m-1})}=f'(x_{m-1}).$$

Finally, we recall that $f'_{\epsilon,m}(x_m) = -1/b(x_m) = f'(x_m)$. We have therefore completed the proof of (4.9) for $x_{m-1} \le x \le x_m$.

Consider now the general case, and let $x_{m-k-1} \le x < x_{m-k}$. Rewrite (4.4):

$$f'_{\epsilon,m}(x) = \frac{1}{\beta_\epsilon(x)}\left[\int_x^{x_{m-k}} \frac{2\beta_\epsilon(y)}{\sigma^2(y)+\epsilon}\,dy + \frac{\beta_\epsilon(x_{m-k})}{b(x_{m-k})}\right]$$

$$+ \sum_{j=1}^{k} \frac{1}{\beta_\epsilon(x)}\left[\int_{x_{m-j-1}}^{x_{m-j}} \frac{2\beta_\epsilon(y)}{\sigma^2(y)+\epsilon}\,dy\right] + \frac{\beta_\epsilon(x_m)}{\beta_\epsilon(x)b(x_m)} - \frac{\beta_\epsilon(x_{m-k})}{\beta_\epsilon(x)b(x_{m-k})}$$

$$\equiv I_\epsilon + \sum_{j=1}^{k} II_\epsilon^{(j)} + III_\epsilon - IV_\epsilon.$$

By the previous argument, $\lim_{\epsilon\to 0} I_\epsilon = f'(x)$. To estimate II_ϵ^j, write

$$II_\epsilon^{(j)} = \frac{\beta_\epsilon(x_{m-j})}{\beta_\epsilon(x)}\left[\frac{1}{\beta_\epsilon(x_{m-j})}\int_{x_{m-j}}^{x_{m-j-1}} \frac{2\beta_\epsilon(y)}{\sigma^2(y)+\epsilon}\,dy\right].$$

When $\epsilon \to 0$, the factor in brackets tends to $f'(x_{m-j})$ (uniformly with respect to m, j); hence it is bounded by $2\sup|f'|$ if ϵ is sufficiently small (independently of m, j). On the other hand,

$$\frac{\beta_\epsilon(x_{m-j})}{\beta_\epsilon(x)} = \exp\left[2\int_x^{x_{m-j}} \frac{b(u)}{\sigma^2(u)+\epsilon}\,du\right] \le \exp[-K_4/\sqrt{\epsilon}].$$

Hence,

$$\sum_{j=1}^{k} II_\epsilon^{(j)} \le K_5 n \exp[-K_4/\sqrt{\epsilon}] \le \frac{K_5}{\epsilon}\exp[-K_4/\sqrt{\epsilon}].$$

Similarly,

$$III_\epsilon \le K_6 \exp[-K_4/\sqrt{\epsilon}], \qquad IV_\epsilon \le K_6 \exp[-K_4/\sqrt{\epsilon}].$$

Putting all these estimates together gives the conclusion (4.9) for $x_{m-k-1} \le x < x_{m-k}$.

From the proof of Lemma 4.4 we see that $|f'_{\epsilon,m}(x)| \le C$ if $x_{-m} \le x \le x_m$ where the constant C is independent of m. Further, for fixed x, $x_{-k} \le x \le x_k$, as $\epsilon \to 0$, $f'_{\epsilon,m}(x) \to f'(x)$ for any $m \ge k$, where the convergence is uniform with respect to m. Hence, taking $m = [1/\epsilon]$ and denoting the corresponding function $f'_{\epsilon,m}$ by f'_ϵ, we conclude

Lemma 4.4′. *On any compact set of the real line, $f'_\epsilon(x) \to f'(x)$ boundedly, as $\epsilon \to 0$.*

We extend each $f_\epsilon(x)$ as a C^2 bounded function on the whole line. The first derivatives of these extended functions converge to $f'(x)$ boundedly on every compact subset of the real line.

If we define $f_\epsilon(0) = f(0)$ for all ϵ, then also $f_\epsilon(x) \to f(x)$ boundedly on compact subsets, as $\epsilon \to 0$.

We use the notation of §3, and set

$$\tilde{\tilde{\sigma}}_s^\epsilon(r, \phi) = \sqrt{[\tilde{\sigma}_s(r,\phi)]^2 + \epsilon}.$$

Denote by $(r^\epsilon(t), \phi^\epsilon(t))$ the solution of (3.1) when $\tilde{\sigma}_s$ is replaced by $\tilde{\tilde{\sigma}}_s^\epsilon$. Since the equation for dr^ϵ is the same as for dr, $r^\epsilon(t) > 0$ for all $t \geq 0$. Application of Itô's formula yields

$$f_\epsilon(\phi^\epsilon(t)) = f_\epsilon(\phi^\epsilon(0)) + \sum_{s=1}^n \int_0^t \tilde{\tilde{\sigma}}_s^\epsilon(r^\epsilon(r), \phi^\epsilon(r)) f_\epsilon'(\phi^\epsilon(r))\, dw^s(r)$$

$$(4.10) \qquad + \int_0^t \left\{ \frac{1}{2} \sum_{s=1}^n [\tilde{\tilde{\sigma}}_s^\epsilon(r^\epsilon(r), \phi^\epsilon(r))]^2 f_\epsilon''(\phi^\epsilon(r)) + \tilde{b}(r^\epsilon(r), \phi^\epsilon(r)) f_\epsilon'(\phi^\epsilon(r)) \right\} dr.$$

When $\epsilon \to 0$ the stochastic integrals converge, by Lemma 4.4′, to

$$\sum_s \int_0^t \tilde{\sigma}_s(r(r), \phi(r)) f'(\phi(r))\, dw^s(r).$$

The Lebesgue integral differs from the corresponding integral, obtained upon replacing the $\tilde{\tilde{\sigma}}_s^\epsilon(r, \phi)$ by $\tilde{\sigma}_s(\phi) + \epsilon$, by

$$\sum_s \int_0^t [\tilde{\sigma}_s^2(r, \phi) - \tilde{\sigma}_s^2(\phi)] f_\epsilon''(\phi)\, dr,$$

where (r, ϕ) stands for $(r^\epsilon(r), \phi^\epsilon(r))$. But by the condition (4.1), this last expression is bounded by $\int_0^t \eta(r^\epsilon, \phi^\epsilon)\sigma^2(\phi^\epsilon) f_\epsilon''(\phi^\epsilon)\, dr$. From the differential equation for f_ϵ we have $|\sigma^2 f_\epsilon''| \leq (\epsilon + \sigma^2)|f_\epsilon''| = |1 + bf_\epsilon'| \leq K_7$. Hence, the Lebesgue integral on the right-hand side of (4.10) is equal to

$$\int_0^t \{[\tfrac{1}{2}\sigma^\epsilon(\phi^\epsilon) + \epsilon] f_\epsilon'' + \tilde{b}(r^\epsilon, \phi^\epsilon) f_\epsilon'\}\, dr + \theta K_7 \int_0^t |\eta(r^\epsilon, \phi^\epsilon)|\, dr$$

for some θ, $|\theta| \leq 1$. Using the fact that $\tfrac{1}{2}(\sigma^2 + \epsilon) f_\epsilon'' + bf_\epsilon' = -1$ and taking $\epsilon \to 0$ in (4.10), we obtain the assertion (4.3) with $\hat{\eta}(r, \phi) = \eta(r, \phi) + |\tilde{b}(r, \phi) - b(\phi)|$.

Completion of the proof of Theorem 4.1. Consider first the case that (4.1) holds for all $r > 0$. Since $r(t) \to 0$ if $t \to \infty$, the condition (4.1) yields

$$\frac{1}{t} \int_0^t |\eta(r(s), \phi(s))|\, ds \to 0 \quad \text{if } t \to \infty.$$

Further, (3.6) holds for the present case (with the same proof). Hence, dividing both sides of (4.3) by t and letting $t \to \infty$, we get $\lim_{t\to\infty} f(\phi(t))/t = -1$ a.s. Recalling that $\lim_{x\to-\infty} f(x)/x$ exists and is a positive number, we deduce that $\lim_{t\to\infty} \phi(t)/t$ exists a.s. and is a negative number.

In case (4.1) holds only for $0 \le r \le \bar{\epsilon}$, let $R(r)$ be a C^∞ function, $R(r) = 1$ if $r < \bar{\epsilon}/2$, $R(r) = 0$ if $r > \bar{\epsilon}$. By Itô's formula we have

$$\begin{aligned} f_\epsilon(\phi^\epsilon(t))R(r^\epsilon(t)) &= f_\epsilon(\phi(0))R(r(0)) \\ &+ \sum_{s=1}^n \int_0^t [f_\epsilon(\phi^\epsilon(\tau))R'(r^\epsilon(\tau))\,\tilde{\sigma}_s(r^\epsilon(\tau), \phi^\epsilon(\tau)) \\ &\qquad + f'_\epsilon(\phi^\epsilon(\tau))R(r^\epsilon(\tau))\tilde{\tilde{\sigma}}_s(r^\epsilon(\tau), \phi^\epsilon(\tau))]\,dw^s(\tau) \\ &+ \int_0^t \left\{\frac{1}{2}\sum_{s=1}^n [\tilde{\tilde{\sigma}}_s(r^\epsilon(\tau), \phi^\epsilon(\tau))]^2 f''_\epsilon(\phi^\epsilon(\tau)) \right. \\ &\qquad \left. + f'_\epsilon(\phi^\epsilon(\tau))\tilde{\tilde{b}}(r^\epsilon(\tau), \phi^\epsilon(\tau))\right\} R(r^\epsilon(\tau))\,d\tau \\ &+ \int_0^t \frac{1}{2}\left\{\sum_{s=1}^n [\tilde{\sigma}_s(r^\epsilon(\tau), \phi^\epsilon(\tau))]^2 R''(r^\epsilon(\tau)) \right. \\ &\qquad \left. + R'(r^\epsilon(\tau))\tilde{b}(r^\epsilon(\tau), \phi^\epsilon(\tau))\right\} f_\epsilon(\phi^\epsilon(\tau))\,d\tau \\ &+ \int_0^t \sum_{s=1}^n \{R'(r^\epsilon(\tau))f'_\epsilon(\phi^\epsilon(\tau))\tilde{\sigma}_s(r^\epsilon(\tau), \phi^\epsilon(\tau))\tilde{\tilde{\sigma}}_s(r^\epsilon(\tau), \phi^\epsilon(\tau))\}\,d\tau. \end{aligned}$$

Set $\bar{T} = \sup\{t;\ r(t) > \bar{\epsilon}/2\}$. By Theorem 2.2, $\bar{T} < \infty$ a.s. If we write each $\int_0^t$ as $\int_0^{\bar{T}} + \int_{\bar{T}}^t$ and proceed as before, we conclude that (4.3) holds in the modified form

$$\left| f(\phi(t)) - f(\phi(0)) - \sum_{s=1}^n \int_0^t f'(\phi(t))\tilde{\tilde{\sigma}}_s(r(\tau), \phi(\tau))\,dw^s(\tau) + t \right| \tag{4.11}$$
$$\le K \int_0^t \eta(r(\tau), \phi(\tau))\,d\tau + C,$$

where C is a.s. finite valued random variable. We can now proceed as before to show that $\lim_{t\to\infty} \phi(t)/t = c$ a.s., $c > 0$.

We have succeeded in eliminating condition (4.1) in Theorem 4.1. This proof will appear in a forthcoming paper by one of us (M.P.).

5. Angular behavior in the general case $l = 2$. We shall consider in this section the general case $l = 2$. We first treat the case where $k = 1$ and G_1 is a closed unit disc with center (0, 0). If we introduce polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, we get the equations in (3.1) with $\tilde{\sigma}_s$, $\tilde{b}$, $\tilde{\tilde{\sigma}}_s$, $\tilde{\tilde{b}}$ defined as before. We assume

(E_c) σ_{is} and b_i are continuously differentiable near $|x| = 1$.

The condition (1.3) translates into $\tilde{\sigma}_s(1, \phi) = 0$; coupled with $(b, \nu) = \sum b_i x_i = 0$ on $|x| = 1$, it implies that $\tilde{b}(1, \phi) = 0$. Hence we may rewrite (3.1) in the form

$$
(5.1)\qquad
\begin{aligned}
dr &= (r-1)\left[\sum_{s=1}^{n} \tilde{\sigma}_s(\phi)dw^s + \tilde{b}(\phi)dt\right] + \left[\sum_{s=1}^{n} R_s dw^s + R_0 dt\right],\\
d\phi &= \left[\sum_{s=1}^{n} \tilde{\tilde{\sigma}}_s(\phi)dw^s + \tilde{\tilde{b}}(\phi)dt\right] + \left[\sum_{s=1}^{n} \Theta_s dw^s + \Theta_0 dt\right]
\end{aligned}
$$

where $R_s = o(r-1)$, $R_0 = o(r-1)$, $\Theta_s = o(1)$, $\Theta_0 = o(1)$ when $r \searrow 1$, and $\tilde{\sigma}_s(\phi)$, $\tilde{b}(\phi)$, $\tilde{\tilde{\sigma}}_s(\phi)$, $\tilde{\tilde{b}}(\phi)$ are (2π)-periodic continuous functions which are not necessarily trigonometric polynomials. We adhere to the notation $\sigma(\phi) = \{\sum_{s=1}^n (\tilde{\tilde{\sigma}}_s(\phi))^2\}^{1/2}$, $b(\phi) = \tilde{\tilde{b}}(\phi)$.

Let $y(t) = (r(t), \phi(t))$ be the solution of (5.1) with $r(0) > 1$, and set $x_1(t) = r(t) \cos \phi(t)$, $x_2(t) = r(t) \sin \phi(t)$. By the remarks of §3 (Theorem 3.1), $x(t) = (x_1(t), x_2(t))$ is a solution of the original system (1.1).

Theorem 5.1. *Assume that* (A)–(D) *and* (2.5), (2.6) *hold with* $k = 1$, $G_1 = \{x; |x| \le 1\}$. *Assume also that* (E_c) *holds and that* $\sigma(z) > 0$ *for all real* z. *Let* $\Lambda = 2\int_0^{2\pi} b(z)/\sigma^2(z)\,dz > 0$. *Then*

$$P\left\{\lim_{t\to\infty} \frac{\phi(t)}{t} = c\right\} = 1$$

where c *is a positive constant. If* $\Lambda < 0$, *the conclusion holds with* c *negative.*

The proof of this result is entirely parallel to the proof of Theorem 3.2, and we omit the details. It suffices to remark only that $\sigma(z)$ may no longer be a trigonometric polynomial, but this does not affect the construction of f satisfying 3(a)–3(d).

Consider next the degenerate case, and assume

($\tilde{E}_c$) $\sigma(z)$ is not everywhere positive, $\sigma(z) \not\equiv 0$, and $\sigma(z)$ has no zeros of infinite order.

(E'_c) The condition (E_c) holds, and, for some $\bar{\epsilon} > 0$,

$$\sum_{s=1}^{n} [\tilde{\tilde{\sigma}}_s(r, \phi)]^2 = \sum_{s=1}^{n} [\tilde{\tilde{\sigma}}(\phi)]^2[1 + \eta(r, \phi)] \qquad (1 \le r \le 1 + \bar{\epsilon})$$

where $\eta(r, \phi) \to 0$ if $r \to 1$, uniformly with respect to ϕ.

Theorem 5.2. *Assume that* (A)–(D) *and* (2.5), (2.6) *hold with* $k = 1$, $G_1 = \{x; |x| \le 1\}$. *Assume also that* ($\tilde{E}_c$), (E'_c) *hold, and that* $b(z) > 0$ ($b(z) < 0$) *whenever* $\sigma(z) = 0$. *Then* $P\{\lim_{t\to\infty} \phi(t)/t = c\} = 1$ *where* c *is a positive (negative) constant.*

In order to treat a general domain G_1 we shall transform to the case of a circle. In a neighborhood of G_1 we can introduce new variables $y_1 = (1+\rho)\cos(2\pi s/L)$, $y_2 = (1+\rho)\sin(2\pi s/L)$ where the "polar coordinates" (ρ, s) are defined by

$$x_1 = f(s) + \rho \dot{g}(s), \quad x_2 = g(s) - \rho \dot{f}(s), \tag{5.2}$$

$0 \le s \le L$, $0 \le \rho \le \rho_0$ and $\dot{f}^2 + \dot{g}^2 = 1$; L is the length of the boundary ∂G_1. By means of Schoenflies' theorem [6] we can extend this mapping to a diffeomorphism from G^c (the complement of G) onto the set $\{y: |y| > 1\}$. The stochastic differentials $d\rho$, ds can be computed in the form

$$d\rho = \sum_{r=1}^{n} \tilde{\sigma}_r dw^r + \tilde{b}\,dt, \quad d\phi = \sum_{r=1}^{n} \tilde{\tilde{\sigma}}_r dw^r + \tilde{\tilde{b}}\,dt \quad (\phi = 2\pi s/L). \tag{5.3}$$

To compute $\tilde{\sigma}_r$, $\tilde{\tilde{\sigma}}_r$, $\tilde{b}$, $\tilde{\tilde{b}}$ we compute dx_i from (5.2) and then compare with the expression for dx_i from (1.1). After some calculation, we arrive at the formulas

$$\begin{aligned} &\frac{L}{2\pi}\tilde{\tilde{\sigma}}_r(\rho, \phi) = [\dot{f}\sigma_{1r} + \dot{g}\sigma_{2r}]/[1 - \rho(\dot{g}\ddot{f} - \dot{f}\ddot{g})] \\ &\frac{L}{2\pi}\tilde{\tilde{b}}(0, \phi) = (\dot{f}b_1 + \dot{g}b_2) - (\ddot{g} \; -\ddot{f})\begin{pmatrix} \Sigma\sigma_{1r}^2 & \Sigma\sigma_{1r}\sigma_{2r} \\ \Sigma\sigma_{1r}\sigma_{2r} & \Sigma\sigma_{2r}^2 \end{pmatrix}\begin{pmatrix} \dot{f} \\ \dot{g} \end{pmatrix}, \end{aligned} \tag{5.4}$$

in agreement with the formulas in case of a point or a circle (see (3.1)).

Corollary 5.3. *Let* $\sigma(\phi) = \{\sum_{r=1}^n (\tilde{\tilde{\sigma}}_r(0, \phi))^2\}^{1/2}$, $b(\phi) = \tilde{\tilde{b}}(0, \phi)$, *where* $\tilde{\tilde{\sigma}}_r$, $\tilde{\tilde{b}}$ *are defined by* (5.4). *Then the statements of Theorems* 5.1, 5.2 *remain true for the present case.*

The assertion $\phi(t)/t \to c$ a.s. can be stated in the following form: Denote by $(\rho(t), s(t))$ the position of the solution $x(t)$ near the boundary ∂G_1, where $s(t)$ is the "algebraic" length. [If a point moves along ∂G_1 so that its argument increases (decreases) by 2π, its "algebraic" length increases (decreases) by L.] Then $s(t)/t \to c'$ a.s., where c' is a constant.

Suppose finally that there are k disjoint sets $G_1, \cdots, G_k$ as in §§1, 2. Then, on the set where $\rho_j(x(t)) \to 0$ we can apply (with trivial changes) the analysis of §§3, 4 if $1 \le j \le k_0$ and of the present section if $k_0 + 1 \le j \le k$. Thus if the conditions of Theorems 3.2 or 4.1 are satisfied, for a particular G_j, $1 \le j \le k_0$, then $\phi(t)/t \to c$ (c constant) for almost all ω for which $\rho_j(x(t)) \to 0$. Similarly, Corollary 5.3 can be applied for a particular G_j ($k_0 + 1 \le j \le k$) on the set where $\rho_j(x(t)) \to 0$.

Remark. Corollary 5.3 extends to the case where G_1 is a star domain with piecewise C^3 boundary, provided ∂G_1 is locally convex near the vertices (so that $\rho_1(x)$ is in C^1). Indeed, let $r = g(\phi)$ be the equation for ∂G_1 (we assume that G_1 is a star domain with respect to the origin). Define

$$\sigma(\phi) = \left\{ \sum_{s=1}^{n} [\tilde{\tilde{\sigma}}_s(g(\phi), \phi)]^2 \right\}^{1/2}, \qquad b(\phi) = \tilde{\tilde{b}}(g(\phi), \phi)).$$

Note that $t \to \infty$,

$$\tilde{\tilde{\sigma}}_s(r(t), \phi(t)) - \tilde{\tilde{\sigma}}_s(g(\phi(t)), \phi(t)) \to 0, \qquad \tilde{\tilde{b}}(r(t), \phi(t)) - \tilde{\tilde{b}}(g(\phi(t)), \phi(t)) \to 0,$$

by Theorem 2.1 extended to the present G_1 (see Remark 3, §2). Hence the assertion of Corollary 5.3 (with the present $\sigma(\phi)$, $b(\phi)$) remain true; the proof being similar to the proofs of Theorems 5.1, 5.2.

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