# Asymptotic stability of a solution of an autonomous system in R2, consisting of subsystems

*Citation for published version (APA):*
Heuvel, van den, P. J. (1980). *Asymptotic stability of a solution of an autonomous system in R2, consisting of subsystems*. (Memorandum COSOR; Vol. 8009). Technische Hogeschool Eindhoven.

*Document status and date:*
Published: 01/01/1980

*Document Version:*
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

*Please check the document version of this publication:*

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• The final author version and the galley proof are versions of the publication after peer review.
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Download date: 25. Aug. 2022

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Memorandum COSOR 80-09

# Asymptotic stability of a solution of an autonomous system in $\mathbb{R}^2$, consisting of subsystems

by

Paul van den Heuvel

Eindhoven, July 1980

The Netherlands

# ASYMPTOTIC STABILITY OF A SOLUTION OF AN AUTONOMOUS SYSTEM IN $\mathbb{R}^2$, CONSISTING OF SUBSYSTEMS

by

Paul van den Heuvel

## Abstract

In this paper a generalization is proved of a theorem by Laroque (1979). This theorem asserts that if an autonomous system $\dot{x} = F(x)$ consists of linear subsystems defined on cones in $\mathbb{R}^2$ and if the function $F(x)$ is continuous, then the origin is an asymptotically stable solution of the system, if the subsystems are asymptotically stable in $\mathbb{R}^2$. It is shown that the linearity restrictions in the theorem of Laroque can be relaxed in a neighbourhood of the equilibrium.

## 1. INTRODUCTION

Recently a number of papers have appeared in economic literature dealing with the equilibria of systems, consisting of several subsystems.
The location of these equilibria in state space is such that there are several domains in a neighbourhood of the equilibrium, on which different adjustment equations are valid. If the resulting systems are denoted by $\overset{\circ}{x}(t) = F(x(t),t)$, then the functions in the right-hand sides of these equations are continuous but not differentiable on the boundaries of the domains. Laroque (1979) proved the asymptotic stability of such a system in $\mathbb{R}^2$, if the domains are cones and the subsystems are asymptotically stable linear autonomous first order systems.
The goal of this paper is to establish a similar theorem in which the restriction of linearity is relaxed with respect to both the domains and the subsystems.
In section 2 some well-known definitions and theorems are given concerning stability of the total system. Furthermore a brief description is given of Laroque's theorem.
The main theorem is the subject of Section 3.

## 2. SOME STABILITY PROPERTIES

In the sequel we will restrict ourselves to autonomous systems in $\mathbb{R}^2$, i.e. systems of the form

$$(1) \qquad \overset{\circ}{x} = F(x);\ F(0) = 0$$

where $x \in \mathbb{R}^2$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$.
By

$$f(x) = o(\|g(x)\|) \qquad (x \to 0)$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$, is meant that

$$\lim_{x\to 0} \frac{\|f(x)\|}{\|g(x)\|} = 0 \ .$$

(see de Bruijn (1961)).
We will start with the definition of (asymptotic) stability and some stability theorems, which can be found for instance in Wilson (1971).

Definition 1

Let 0 be an equilibrium of system (1) and let $x(t,x_0)$ denote the solution of the initial value problem

$$\overset{\circ}{x} = F(x)$$

$$x(0) = x_0 \ .$$

0 is a stable equilibrium if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $a \in \mathbb{R}^2$

$$\|a\| < \delta \Rightarrow \forall t \geq 0 : \|x(t,a)\| < \varepsilon \ .$$

If 0 is a stable equilibrium and there is an $\eta > 0$ such that

$$\|a\| < \eta \Rightarrow \lim_{t\to\infty} \|x(t,a)\| = 0$$

then 0 is called an asymptotically stable equilibrium.

### Definition 2

A function $V : \mathbb{R}^2 \to \mathbb{R}$ is a *Liapunov function* of system (1) if in some neighbourhood of 0

(i) $V(x)$ is continuous

(ii) $V(0) = 0$, $V(x) > 0$ for $x \neq 0$

(iii) $V(x)$ is decreasing on any solution path of system (1).

We recall the following well-known results.

### Property 1

If there is a Liapunov function of system (1), the origin is an asymptotically stable solution of system (1).

### Property 2

The eigenvalues of the matrix A have negative real parts if and only if the origin is an asymptotically stable solution of the system

$$\overset{\circ}{x} = Ax$$

Note that Property 2 implies that in $\mathbb{R}^2$ asymptotic stability is equivalent with $\operatorname{tr} A < 0$ and $\det A > 0$ .

### Property 3 (Poincaré-Liapunov)

If $f(x) = \sigma(\|x\|)$ $(x \to 0)$ and the origin is an asymptotically stable equilibrium of

$$\overset{\circ}{x} = Ax$$

it is also an asymptotically stable solution of

$$\overset{\circ}{x} = Ax + f(x) \ .$$

The following property is proved in Laroque (1979) (:Proposition 3.4).

Property 4

Let $C_i$, $i = 1,\ldots,n$ be closed cones in $\mathbb{R}^2$ with vertices in the origin, with disjoint interiors and such that

$$\bigcup_{i=1}^{n} C_i = \mathbb{R}^2$$

Let the numbering of the cones around the origin be clockwise, with $C_0 := C_n$.

If the systems

$$\overset{\circ}{x} = A_i x \ , \quad i = 1,\ldots,n$$

defined on the whole $\mathbb{R}^2$, have the origin as an asymptotically stable equilibrium and if $G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(x) := A_i x \quad \text{for } x \in \{0\} \cup [C_i \backslash C_{i-1}]$$

is continuous, then the origin is an asymptotically stable solution of the system

$$(2) \qquad \overset{\circ}{x} = G(x) \ .$$

The proof of Laroque's result consists of two parts.
First he considers the case in which there exists a real eigenvector in one of the cones, let us say $C_i$. He shows that trajectories can not pass this eigenvector, so any trajectory stays ultimately in the cone $C_i$ and converges to the origin.
The second part of the proof concerns the case, where there does not exist such an eigenvector. Then the function L(x) defined by

$$L(x) := \det [x, G(x)]$$

is non-zero for $x \neq 0$. On account of the continuity of G(x) in system (2), L(x) is continuous and it can be proved now that

$$\overset{\circ}{L}(x) L(x) < 0$$

on a trajectory. Therefore $[L(x)]^2$ is a Liapunov function.

## 3. THE MAIN THEOREM

In this section the following notations and assumptions are valid.
Let $c_i$, $i = 1,\dots,n$ be vectors in $\mathbb{R}^2$ with $\|c_i\| = 1$, no two of them equal ~~independent~~. The numbering of these vectors is clockwise with regard to the origin~~al~~ and $c_0 := c_n$.
Let there be given n curves represented by

$$x = h_i(t) = c_i t + \sigma(t) \quad (t \downarrow 0) \quad i = 1,\dots,n\ .$$

The set $B_\rho$ is defined as

$$B_\rho := \mathrm{cl}\ B(0;\rho)\ .$$

It is assumed $\rho$ is sufficiently small to let the ball $B_\rho$ be devided by the curves $x = h_i(t)$ in n disjoint subsets.
The set

$$S_i,\ i = 1,\dots,n$$

is defined as the closure of such a subset of $B_\rho$, which has the boundaries $z = h_{i-1}(t)$ and $x = h_i(t)$ (and part of the boundary of $B_\rho$).
We define $S_0 := S_n$ .

Let for $i = 1,\dots,n$, $A_i$ be a $2 \times 2$ matrix and $f_i : B_\rho \to \mathbb{R}^2$. The function $F : B_\rho \to \mathbb{R}^2$ is defined by

$$F(x) := A_i x + f_i(x) \quad \text{for} \quad x \in \{0\} \cup [S_i \backslash S_{i-1}]\ .$$

The following system is investigated.

$$(3) \qquad \overset{\circ}{x} = F(x)\ , \quad x \in B_\rho\ .$$

Assumption 1

For $i = 1,\dots,n$ the function $f_i$ is continuously differentiable on $S_i$ and $f_i(x) = \sigma(\|x\|) \quad (x \to 0)$.

Assumption 2

The function F is continuous on $B_\rho$ . It can be proved that under assumption 1 and 2 the function F is locally Lipschitz continuous on $B_\rho$, which implies that any initial value problem of system (3) has a unique solution (cf. Wilson, 1971, p. 247).

It will be shown (see Lemma 1) that the continuity of the right-hand side is maintained, if both the subsystems and the domains are linearized. As a consequence of this property the same Liapunov function, that plays a role in Laroque's proof, can be applied. We will also have to show that linearization of the domains gives rise to differences, that are $\sigma(x)$. This linearization is handled in Lemma 2.
Henceforth the following definition is applied.

Definition 3

The tangent cone $C(S_i)$ of $S_i$, $i = 1,\dots,n$ is defined by the closed cone (not necessarily convex) generated by the tangent vectors of curves in $S_i$.

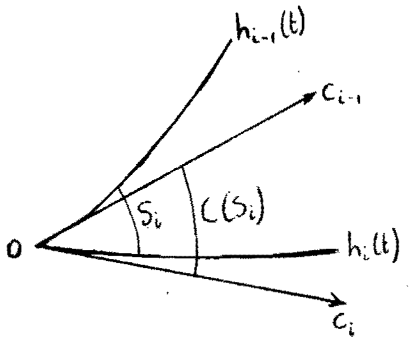

Figure 1
tangent cone $C(S_i)$ of $S_i$

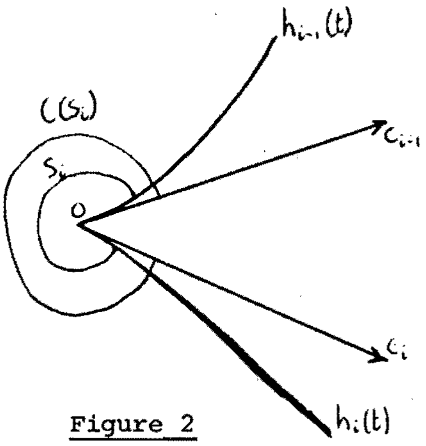

Figure 2
tangent cone $C(S_i)$ of $S_i$

In the Figures 1 and 2 examples of tangent cones are depicted.

With relation to the tangent cones the following lemma can be proved.

### Lemma 1

Under Assumption 1 the function $G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(x) := A_i x \quad \text{for} \quad x \in \{0\} \cup [C(S_i) \setminus C(S_{i-1})]$$

is continuous on $\mathbb{R}^2$.

### Proof

It suffices to prove that the function $G(x)$ is continuous on the common boundary of $C(S_1)$ and $C(S_2)$.
The common boundary of $C(S_1)$ and $C(S_2)$ is given by

$$x = tc_1 \ .$$

Since $F(x)$ is continuous on $x = h_1(t) = c_1 t + \sigma(t) \ (t \downarrow 0)$, we have

$$(A_1 - A_2)\ (c_1 t + \sigma(t)) = f_2[h_1(t)] - f_1[h_1(t)] = O(t \quad (t \downarrow 0)$$

From these equations it follows that

$$t(A_1 - A_2)c_1 = \sigma(t) \quad (t \downarrow 0) \ .$$

Hence

$$(A_1 - A_2)c_1 = \sigma(1) \quad (t \downarrow 0)$$

and this can only be true if

$$(A_1 - A_2)c_1 = 0 \ .$$

This equation implies that the function $G(x)$ is continuous on the boundary given by $x = tc_1$. (In a similar way it can be proved that this function is continuous anywhere).

□

It follows from the lemma that G(x) is equal to

$$G(x) = A_i x \qquad \text{for} \qquad x \in C(S_i) \ .$$

**Lemma 2**

Let the function $g_i(x)$ be defined on $C(S_i) \cap B_\rho$ by

$$g_i(x) := F(x) - A_i x \qquad \text{for} \qquad x \in C(S_i) \cap B_\rho$$

where F(x) is defined in the same way as in system (3). Then under Assumption 2

$$g_i(x) = \sigma(\|x\|) \qquad (x \to 0) \ .$$

**Proof**

The proof will be given for i = 1. Note that $g_1(x) - f_1(x)$ can only be nonzero if

$$x \in C(S_1) \cap \text{int } S_2 \qquad \text{or} \qquad x \in C(S_1) \cap \text{int } S_n \ .$$

Let t be defined by the property that $tc_1$ is the orthogonal projection of x onto the ray generated by $c_1$. Then if $x \to 0$ and $x \in C(S_1) \cap \text{int } S_2$ it is easily seen that

$$\frac{\|x - tc_1\|}{t} \to 0 \qquad (t \downarrow 0) \ .$$

Hence

$$x = tc_1 + \sigma(\|x\|) \qquad (x \to 0) \ .$$

Application of Lemma 1 yields:

$$(A_1 - A_2)x = t(A_1 - A_2)c_1 + \sigma(\|x\|) = 0 + \sigma(\|x\|) = \sigma(\|x\|) \qquad (x \to 0) \ .$$

Therefore, for $x \in C(S_1) \cap \text{int } S_2$

$$g_1(x) = (A_2 - A_1)x + f_2(x) = \sigma(\|x\|) \qquad (x \to 0)$$

from which the assertion follows. □

It will be convenient to introduce the following definitions

$$q := x'x$$

is the square of the Euclidian norm of the vector x.

$$P := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

We will use a similar function as Laroque (see Section 2).
Let $L : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$L(x) := \det\,(x \quad A_i x) \qquad \text{for} \qquad x \in \{0\} \cup [C(S_i) \backslash C(S_{i-1})] \; .$$

$$(i = 1,\ldots,n) \; .$$

It can easily be seen that L(x) can be written as follows.

$$L(x) = x'P'A_i x \qquad \text{for} \qquad x \in C(S_i) \qquad (i = 1,\ldots,n) \; .$$

Furthermore L(x) is continuous on $\mathbb{R}^2$.

Lemma 3

Under Assumption 1

$$\mathring{L}(x) = (\text{tr } A_i)L(x) + \sigma(q) \qquad (q \downarrow 0) \qquad \text{for} \qquad x \in C(S_i) \; .$$

Proof

For $x \in C(S_i)$ we have, using the notation of Lemma 2

$$\begin{aligned}\mathring{L}(x) &= (\mathring{x})'P'A_i x + x'P'A_i\mathring{x} \\ &= [A_i x + g_i(x)]'P'A_i x + x'PA_i[A_i x + g_i(x)] \\ &= x'A_i P'A_i x + x'P'A_i^2 x + [g_i(x)]'P'A_i x + x'P'A_i g_i(x) \ .\end{aligned}$$

The first term equals zero. To the second term the Cayley-Hamilton property

$$A_i^2 - (\mathrm{tr}\ A_i)A_i + (\det A_i)I = 0$$

can be applied. The third and fourth term are $\sigma(\|x\|^2)$ $(x \to 0)$ (cf. Lemma 2) and therefore they are $\sigma(q)$ $(q \downarrow 0)$. Hence

$$\begin{aligned}\mathring{L}(x) &= (\mathrm{tr}\ A_i)x'P'A_i x + (\det A_i)x'P'x + o(q) \qquad (q \downarrow 0) \\ &= (\mathrm{tr}\ A_i)L(x) + 0 + \sigma(q) \qquad (q \downarrow 0) \ . \qquad \square\end{aligned}$$

Lemma 4

Let $y_0 \in C(S_i)$ with $\|y_0\| = 1$ be an eigenvector of $A_i$ with eigenvalue $\lambda_0$. For $x \in C(S_i)$

$$\mathring{q} = 2x'A_i x + \sigma(q) \qquad (q \downarrow 0) \ .$$

There is a positive $\rho$ and an open cone $K$ such that $y_0 \in K$ and

$$\mathring{q} < 0 \qquad \text{for} \qquad x \in K \cap B_\rho \ .$$

Proof

For $x \in C(S_i)$ we have

$$\begin{aligned}\mathring{q} &= 2x'\mathring{x} \\ &= 2x'[A_i x + g_i(x)] \ .\end{aligned}$$

Using Lemma 2,

$$g_i(x) = \sigma(\|x\|) \qquad (x \to 0)$$

$$2x'g_i(x) = \sigma(q) \qquad (q \downarrow 0) \; .$$

Hence for $x \in C(S_i)$

$$\mathring{q} = 2x'A_ix + \sigma(q) \qquad (q \downarrow 0) \; . \tag{4}$$

Choose $\varepsilon$ such that $\varepsilon \in (0,-\lambda_0)$, then the set K defined by

$$K := \{x \in \mathbb{R}^2 \mid x'A_ix < - \varepsilon\, x'x\}$$

contains $y_0$. K is an open cone.
For sufficiently small $\rho$ for any $x \in K \cap B_\rho$

$$\mathring{q} < - \varepsilon\, q < 0 \; . \tag{5}$$

□

Remark:
The result of Lemma 4 is also valid if $y_0$ lies on the common boundary of $C(S_i)$ and $C(S_{i-1})$ $(i = 1,\dots,n)$.

For the time being it is assumed that there is only one such eigenvector as in Lemma 4 (with eigenvalue $\lambda_0$).
We define

$$m := \min \{[L(x)]^2 \mid \|x\| = 1 \;,\; x \notin K\} \tag{6}$$

where K is defined as in the proof of Lemma 4.
Since the set $\{x \mid \|x\| = 1 \;,\; x \notin K\}$ is compact and the function $L(x)$ is continuous, this minimum exists. For $x \notin K$ the inequality $[L(x)]^2 > 0$ holds hence

$$m > 0 \; .$$

Define the real number $\alpha$ by

$$(7) \qquad \alpha := (\max_j \operatorname{tr} A_j)m/2\cancel{\lambda_0}\mu$$

where m is defined as in (6) and $\mu = \max_j \|A_j\|$
Since $m > 0$ , $\lambda_0 < 0$ and for each j the inequality $\operatorname{tr} A_j < 0$ holds, $\alpha$ is positive.
Now the following theorem can be proved.

## Theorem 1

Let Assumptions 1 and 2 hold. If the origin is an asymptotically stable solution of the systems

$$\overset{\circ}{x} = A_i x \qquad \text{for} \qquad x \in \mathbb{R}^2 \qquad (i = 1,\dots,n)$$

then the function $V : B_\rho \to \mathbb{R}$ defined by

$$V(x) := [L(x)]^2 + \alpha(x'x)^2$$

where $\alpha$ is defined by (7), is a Liapunov function of system (3).

## Proof

The function V(x) is continuous. Furthermore $V(0) = 0$ and

$$V(x) > 0 \qquad \text{for} \qquad x \neq 0 \ .$$

Along a solution path the following equation holds.

$$\overset{\circ}{V}(x) = 2L(x)\overset{\circ}{L}(x) + 2\alpha q\overset{\circ}{q} \ .$$

The Lemma's 3 and 4 imply that for $x \in C(S_i) \cap B_\rho$

$$\overset{\circ}{V}(x) = 2(\operatorname{tr} A_i)\ [L(x)]^2 + 2\alpha q x' A_i x + \sigma(q^2) \qquad (q \downarrow 0) \ .$$

From Lemma 4 we know that if $\rho$ is sufficiently small the inequality (5) holds for $K \cap B_\rho$ .
Since also $\mathrm{tr}\, A_i < 0$

$$\overset{\circ}{V}(x) < 0 \qquad \text{for} \qquad x \in K \cap B_\rho \ .$$

For $x \notin K$ we have

$$|x'A_ix| \leqslant \overset{\mu\|x\|^2}{\cancel{|\lambda_0 x'x| = -\lambda_0 q}} .$$

Hence

$$2\alpha q x'A_ix \leqslant -2\alpha q^2 \overset{\mu}{\cancel{\lambda_0}} = -(\max_j \mathrm{tr}\, A_j) m q^2 \ .$$

By definition of m and q

$$mq^2 \leq [L(x)]^2 \ .$$

Therefore for $x \in C(S_i) \setminus K$

$$\overset{\circ}{V}(x) < (2\, \mathrm{tr}\, A_i - \max_j \mathrm{tr}\, A_j) mq^2 + o(q^2) \qquad (q \downarrow 0)$$

and $\overset{\circ}{V}(x)$ is negative for sufficiently small q.
It can be concluded that V(x) is a Liapunov function of system (3). □

If there are several real eigenvectors such as in Lemma 4, Theorem 1 holds, provided that we take for $\lambda_0$ the minimum of the corresponding eigenvalues.

Theorem 1 and Property 1 immediately imply:

## Theorem 2

Let Assumptions 1 and 2 hold. If the origin is an asymptotically stable equilibrium of the systems

$$\overset{\circ}{x} = A_ix \qquad \text{for} \qquad x \in \mathbb{R}^2 \qquad (i = 1,\dots,n)$$

then the origin is an asymptotically stable equilibrium of system (3).

## Acknowledgement

The author would like to thank Jos Brands, Jan de Jong and especially Malo Hautus for their valuable comments.

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