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Research Article

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Asymptotic stability of the time-changed stochastic delay differential equations with Markovian switching

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Abstract: The aim of this work is to study the asymptotic stability of the time-changed stochastic delay differential equations (SDDEs) with Markovian switching. Some sufficient conditions for the asymptotic stability of solutions to the time-changed SDDEs are presented. In contrast to the asymptotic stability in existing articles, we present the new results on the stability of solutions to time-changed SDDEs, which is driven by time-changed Brownian motion. Finally, an example is given to demonstrate the effectiveness of the main results.

Keywords: asymptotic stability, time-changed stochastic delay differential equations, time-changed Brownian motions, Markovian switching

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1 Introduction

The research on stochastic differential equations (SDEs) plays an important role in modeling dynamic system areas, such as physics, economics and finance, biological and so forth. Recently, the qualitative study of the solution of SDEs has received much attention. Particularly, the stability of SDEs has been considered widely by many researchers [1–4]. It is well known that time delay is unavoidable in practice, then the corresponding stochastic delay differential equations (SDDEs) are used more widely in systems. It considers the effects of past behaviors imposed to the current status. The stability results of SDDEs we have mentioned here can be found in [5–8]. The delay term has main influence on the stability of SDDEs. It could be regarded as a perturbation to the stable systems, or may be the delay part has a stabilizing effect as well [8]. Jump system is a new type of SDE with Markovian switching [9–12]. In practice, the system can switch from one mode to another randomly, and the switching between the modes is governed by a Markov process. SDDE with Markovian switching is a kind of hybrid system, including both the logical switching mode and the state of system. It is used widely in many applied areas such as neural networks, traffic control model, and so on.

Very recently, Chlebak et al. [13] considered a sub-diffusion process in Hilbert space and the associated fractional Fokker-Planck-Kolmogorov equations. The process is connected with a limit process arising from continuous-time random walks. In fact, the limit process is a time-changed Lévy process, which is the first

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hitting time process of certain stable subordinator (see [14,15] for details). The existence and stability of SDEs driven by time-changed Brownian motion attracted lot of attention. Wu [16] established the time-changed Itô formula of time-changed SDE, and the stability analysis is investigated. Subsequently, Nane and Ni [17,18] established the Itô formula for time-changed Lévy noise, then discussed the asymptotic stability and path stability for the solution of time-changed SDEs with jump, respectively. And in [19], we considered the exponential stability for the time-changed stochastic functional differential equations with Markov switching.

Motivated strongly by the above, in this paper, we will study the stability of time-changed SDDEs with Markovian switching. By applying the time-changed Itô formula and Lyapunov function, we present the LaSalle-Type theorem [6,12] of the time-changed SDDEs with Markovian switching. More precisely, we consider the following SDDEs driven by time-changed Brownian motions:

$$\mathrm{d}x(t) = \rho(t, E_t, r(t), x(t), x(t-\tau))\mathrm{d}t + f(t, E_t, r(t), x(t), x(t-\tau))\mathrm{d}E_t + g(t, E_t, r(t), x(t), x(t-\tau))\mathrm{d}B_{E_t} \quad (1.1)$$

on $t \geq 0$ with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where ρ, f, g are appropriately specified later.

In the remaining parts of this paper, further needed concepts and related background are presented in Section 2. In Section 3, the main stability results of the time-changed SDDEs with Markovian switching are given. Finally, an example is given to illustrate the effectiveness of the main results.

2 Preliminary

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, P)$ be a complete probability space with the filtration $\{\mathscr{F}\}_{t\geq 0}$, which satisfies the usual conditions (i.e., $\{\mathscr{F}\}_{t\geq 0}$ is right continuous and $\mathscr{F}$ contains all the P-null sets in $\mathscr{F}$). Let $\{U(t)\}_{t\geq 0}$ be a right continuous with left limit (RCLL) increasing Lévy process that is called a subordinator. In particular, a β-stable subordinator is a strictly increasing process denoted by $U_\beta(t)$ and characterized by Laplace transform

$$E[\exp(-sU_\beta(t))] = \exp(-ts^\beta), \quad s > 0,\ \beta \in (0, 1).$$

For an adapted β-stable subordinator $U_\beta(t)$, define its generalized inverse as

$$E_t := E_t^\beta = \inf\{s > 0 : U_\beta(s) > t\},$$

which is called the first hitting time process. And E_t is continuous since $U_\beta(t)$ is strictly increasing.

Let B_t be a standard Brownian motion independent of E_t, define the filtration as

$$\mathscr{F}_t = \bigcap_{s>t}\{\sigma[B_r : 0 \leq r \leq s] \vee \sigma[E_r : r \geq 0]\},$$

where $\sigma_1 \vee \sigma_2$ denotes the σ-algebra generated by the union of σ-algebras σ_1 and σ_2. It concludes that the time-changed Brownian motion B_{E_t} is a square integrable martingale with respect to the filtration $\{\mathscr{F}_{E_t}\}_{t\geq 0}$. And its quadratic variation satisfies $\langle B_{E_t}, B_{E_t}\rangle = E_t$ [20].

Let $B_{E_t} = (B^1_{E_t}, \dots, B^m_{E_t})$ be an m-dimensional Brownian motion defined on the probability space. If $K \subset \mathbb{R}^n$, let $\mathrm{d}(x, K) = \inf_{y\in K}|x - y|$ be the distance from $x \in \mathbb{R}^n$ to K. If w is a real-valued function on $\mathbb{R}^n$, then its kernel is expressed by $\ker(w) = \{x \in \mathbb{R}^n : w(x) = 0\}$. We also denote by $L^1(\mathbb{R}_+, \mathbb{R}_+)$ the family of all functions $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty r(t)\mathrm{d}t < \infty$, $\int_0^\infty r(t)\mathrm{d}E_t < \infty$ and $\int_0^\infty r(t)\mathrm{d}U(t) < \infty$.

Let $r(t)$, $t \geq 0$ be a right continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} r_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + r_{ij}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, γ_{ij} is the transition rate from i to j if $i \neq j$ and $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assume that the Markov chain $r(t)$ is independent of Brownian motion, it is well known that almost each sample path of $r(t)$ is a right-continuous step function.

Let $C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times S; \mathbb{R}_+)$ denote the family of all continuous nonnegative functions $V(t, E_t, x(t), i)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times S$, such that for each $i \in S$, they are continuously once differentiable in t and E_t and twice differentiable in x. For each $V \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times S; \mathbb{R}_+)$, define the Itô operator $L_lV : (\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times S) \to \mathbb{R}$, $(l = 1, 2)$ by

$$L_1V(t, E_t, x, y, i) = V_t(t, E_t, x, i) + V_x(t, E_t, x, i)\rho(t, E_t, x, y, i) + \sum_{j=1}^{N}\gamma_{ij}V(t, E_t, x, j)$$

and

$$L_2V(t, E_t, x, y, i) = V_{E_t}(t, E_t, x, i) + V_x(t, E_t, x, i)f(t, E_t, x, y, i) + \frac{1}{2}\text{trace}[g^T V_{xx}(t, E_t, x, i)g(t, E_t, x, y, i)],$$

where

$$V_t(t, E_t, x, i) = \frac{\partial V(t, E_t, x, i)}{\partial t}, \quad V_{E_t}(t, E_t, x, i) = \frac{\partial V(t, E_t, x, i)}{\partial E_t},$$
$$V_x(t, E_t, x, i) = \left(\frac{\partial V(t, E_t, x, i)}{\partial x_1}, \frac{\partial V(t, E_t, x, i)}{\partial x_2}, \dots, \frac{\partial V(t, E_t, x, i)}{\partial x_n}\right)^T$$

and

$$V_{xx}(t, E_t, x, i) = \left(\frac{\partial^2 V(t, E_t, x, i)}{\partial x_l \partial x_m}\right)_{n\times n}.$$

At first, we introduce the important generalized time-changed Itô formula convenient for the subsequent stochastic calculation.

Lemma 2.1. (The generalized time-changed Itô formula) [19] *Suppose that $U_\beta(t)$ is a β-stable subordinator and E_t is its associated inverse subordinator. Let $x(t)$ be a $\mathscr{F}_{E_t}$ adapted process defined in* (1.1). *If $V : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times S \to \mathbb{R}$ is a $C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times S; \mathbb{R})$ function, then with probability one*

$$\begin{aligned} V(t, E_t, x(t), r(t)) = V(0, 0, x_0, r(0)) &+ \int_0^t L_1V(s, E_s, x(s), x(s-\tau), r(s))ds \\ &+ \int_0^t L_2V(s, E_s, x(s), x(s-\tau), r(s))dE_s \\ &+ \int_0^t V_x(s, E_s, x(s), r(s))g(s, E_s, x(s), x(s-\tau), r(s))dB_{E_s} \\ &+ \int_0^t\int_R [V(s, E_s, x(s), i_0 + h(r(s), l)) - V(s, E_s, x(s), r(s))]\mu(ds, dl), \end{aligned}$$

where $\mu(ds, dl) = \nu(ds, dl) - m(dl)ds$ is a martingale measure.

In this paper, the following hypothesis is imposed on the coefficients ρ, f and g.

(H1) Both $\rho, f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^{n\times m}$ are Borel-measurable functions. They satisfy the local Lipschitz condition. That is, for each $k = 1, 2, \dots$, there is $c_k > 0$ such that

$$|\rho(t,E_t,x,y,i)-\rho(t,E_t,\bar{x},\bar{y},i)|\vee|f(t,E_t,x,y,i)-f(t,E_t,\bar{x},\bar{y},i)|$$
$$\vee|g(t,E_t,x,y,i)-g(t,E_t,\bar{x},\bar{y},i)|\le c_k(|x-\bar{x}|+|y-\bar{y}|)$$

for all $t\ge0$, $i\in S$ and $x,y,\bar{x},\bar{y}\in\mathbb{R}^n$ with $|x|\vee|y|\vee|\bar{x}|\vee|\bar{y}|\le k$. Moreover,

$$\sup\{|\rho(t,E_t,i,0,0)|\vee|f(t,E_t,i,0,0)|\vee|g(t,E_t,i,0,0)|:t\ge0\}<\infty.$$

(H2) If $x(t)$ is an RCLL and $\mathscr{F}_{E_t}$-adapted process, then

$$\rho(t,E_t,x(t),x(t-\tau),r(t)),f(t,E_t,x(t),x(t-\tau),r(t)),g(t,E_t,x(t),x(t-\tau),r(t))\in\mathcal{L}(\mathscr{F}_{E_t}),$$

where $\mathcal{L}(\mathscr{F}_{E_t})$ denotes the class of RCLL and $\mathscr{F}_{E_t}$-adapted process.

We will need the useful semimartingale convergence theorem, which is cited here as a lemma.

Lemma 2.2. (Semimartingale convergence theorem [3]) *Let $\{A_t\}_{t\ge0}$ and $\{U_t\}_{t\ge0}$ be two continuous adapted increasing processes with $A_0=U_0=0$ a.s. Let $\{M_t\}_{t\ge0}$ be a real-valued condition local martingale with $M_0=0$ a.s. Let ξ be a nonnegative $\mathscr{F}_0$-measurable random variable. Define*

$$X_t=\xi+A_t-U_t+M_t\quad\text{for }t\ge0.$$

If X_t is nonnegative, then

$$\left\{\lim_{t\to\infty}A_t<\infty\right\}\subset\left\{\lim_{t\to\infty}X_t\text{ exists and is finite}\right\}\cap\left\{\lim_{t\to\infty}U_t<\infty\right\}\text{ a.s.},$$

where $B\subset D$ a.s. means $P(B\cap D^c)=0$. In particular, if $\lim_{t\to\infty}A_t<\infty$ a.s., then for almost all $\omega\in\Omega$,

$$\lim_{t\to\infty}X_t(\omega)\text{ exists and is finite},\ \text{ and }\lim_{t\to\infty}U_t(\omega)<\infty.$$

3 Main results and discussion

In this section, we aim to establish the stability results of the system equation.

Theorem 3.1. *Let conditions* (H1) *and* (H2) *hold. Assume that there are functions $V\in C^{1,1,2}(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n\times S,\mathbb{R}_+)$, $r_{ij}\in L^1(\mathbb{R}_+,\mathbb{R}_+)$ and $w_{ij}\in C(\mathbb{R}^n,\mathbb{R}_+)$, $i,j=1,2$ such that*

$$L_1V(t,E_t,x,y,i)\le r_{11}(t)+r_{12}(E_t)-w_{11}(x)+w_{12}(y),\tag{3.1}$$

$$L_2V(t,E_t,x,y,i)\le r_{21}(t)+r_{22}(E_t)-w_{21}(x)+w_{22}(y),\tag{3.2}$$

$$w_{i1}(x)\ge w_{i2}(x),\quad i=1,2\tag{3.3}$$

and

$$\lim_{|x|\to\infty}\inf_{0\le t<\infty}V(t,E_t,x,i)=\infty,\tag{3.4}$$

where $(t,E_t,x,y,i)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\times S$. Then $\mathrm{Ker}(w_1-w_2)\neq\emptyset$ and

$$\lim_{t\to\infty}\mathrm{d}(x(t;\xi),\mathrm{Ker}(w_1-w_2))=0\text{ a.s.}\tag{3.5}$$

for every $\xi\in C^b_{\mathscr{F}_0}([-\tau,0];\mathbb{R}^n)$, where $w_i=w_{1i}+w_{2i}$, $i=1,2$.

To prove this result, let us present an existence lemma at first.

Lemma 3.1. *Under the conditions of* Theorem 3.1, *for any initial data $\{x(\theta):-\tau\le\theta\le0\}=\xi\in C^b_{\mathscr{F}_0}([-\tau,0];\mathbb{R}^n)$, equation* (1.1) *has a unique global solution.*

Proof. Under conditions (H_1) and (H_2), equation (1.1) has a unique maximal local solution $x(t)$ on $t \in [-\tau, \sigma_\infty)$ for any initial data $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where σ_∞ is the explosion time [15,21]. So we only need to show that $\sigma_\infty = \infty$ a.s. For any $k \ge 1$, define the following stopping time

$$\tau_k = \sigma_\infty \wedge \inf\{t \in [[0, \sigma_\infty[[: |x(t)| \ge k\}.$$

By the generalized time-changed Itô formula in Lemma 2.1,

$$\begin{aligned}&\mathbb{E}V\big(t \wedge \tau_k, E_{t\wedge\tau_k}, x(t \wedge \tau_k), r(t \wedge \tau_k)\big)\\ &= \mathbb{E}V(0, 0, x(0), i_0) + \mathbb{E}\int_0^{t\wedge\tau_k} L_1V(s, E_s, x(s), x(s-\tau), r(t))\mathrm{d}s + \mathbb{E}\int_0^{t\wedge\tau_k} L_2V(s, E_s, x(s), x(s-\tau), r(t))\mathrm{d}E_s.\end{aligned}$$

By using conditions (3.1) and (3.2), we can see that

$$\begin{aligned}\int_0^{t\wedge\tau_k} L_1V(s, E_s, x(s), x(s-\tau), r(s))\mathrm{d}s &\le \int_0^{t\wedge\tau_k} [r_{11}(s) + r_{12}(E_s) - w_{11}(x(s)) + w_{12}(x(s-\tau))]\mathrm{d}s\\ &\le \int_0^{t} [r_{11}(s) + r_{12}(E_s)]\mathrm{d}s + \int_{-\tau}^{0} w_{12}(x(s))\mathrm{d}s - \int_0^{t\wedge\tau_k} [w_{11}(x(s)) - w_{12}(x(s))]\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\int_0^{t\wedge\tau_k} L_2V(s, E_s, x(s), x(s-\tau), r(s))\mathrm{d}E_s &\le \int_0^{t\wedge\tau_k} [r_{21}(s) + r_{22}(E_s) - w_{21}(x(s)) + w_{22}(x(s-\tau))]\mathrm{d}E_s\\ &\le \int_0^{t} [r_{21}(s) + r_{22}(E_s)]\mathrm{d}E_s + \int_{-\tau}^{0} w_{22}(x(s))\mathrm{d}E_s - \int_0^{t\wedge\tau_k} [w_{21}(x(s)) - w_{22}(x(s))]\mathrm{d}E_s,\end{aligned}$$

where we extend $E(s)$ to $[-\tau, 0)$ by setting $E(s) = E(0)$, then

$$\begin{aligned}&\mathbb{E}V\big(t \wedge \tau_k, E_{t\wedge\tau_k}, x(t \wedge \tau_k), r(t \wedge \tau_k)\big)\\ &= \mathbb{E}V(0, 0, x(0), i_0) + \mathbb{E}\int_0^{t\wedge\tau_k} L_1V(s, E_s, x(s), x(s-\tau), r(s))\mathrm{d}s + \mathbb{E}\int_0^{t\wedge\tau_k} L_2V(s, E_s, x(s), x(s-\tau), r(s))\mathrm{d}E_s\\ &\le \mathbb{E}V(0, 0, x(0), i_0) + \int_0^{t} [r_{11}(s) + r_{12}(E_s)]\mathrm{d}s + \int_0^{t} [r_{21}(s) + r_{22}(E_s)]\mathrm{d}E_s + \int_{-\tau}^{0} w_{12}(\xi(\theta))\mathrm{d}\theta + \int_{-\tau}^{0} w_{22}(\xi(s))\mathrm{d}E_s := A.\end{aligned}$$

This yields that

$$P(\tau_k \le t) \le \frac{A}{\inf_{|x|\ge k, t\ge 0, i\in S} V(t, E_t, x, i)}.$$

Taking $k \to \infty$, we can see from (3.4) that $P(\tau_\infty \le t) = 0$. Since t is arbitrary, it follows that $P(\tau_\infty = \infty) = 1$, this completes the proof. □

Now, let us prove our main results.

Proof of Theorem 3.1. We divide the proof into three steps.
Step 1. For any ξ and i_0 we write $x(t; i_0, \xi) = x(t)$ for simply. It is well known that a continuous time Markov chain $r(t)$ with generator $\Gamma = \{\gamma_{ij}\}_{N\times N}$ can be expressed as a stochastic integral with respect to a Poisson

random measure. In fact, let Δ_{ij} be consecutive, left closed, right open intervals of the real line each having length γ_{ij} such that

$$\Delta_{12} = [0, \gamma_{12}), \quad \Delta_{13} = [\gamma_{12}, \gamma_{12} + \gamma_{13}), \quad \dots, \quad \Delta_{1N} = \left[\sum_{j=2}^{N-1} \gamma_{1j}, \sum_{j=2}^{N} \gamma_{1j}\right),$$
$$\Delta_{21} = \left[\sum_{j=2}^{N} \gamma_{1j}, \sum_{j=2}^{N} \gamma_{1j} + \gamma_{21}\right), \quad \Delta_{23} = \left[\sum_{j=2}^{N} \gamma_{1j} + \gamma_{21}, \sum_{j=2}^{N} \gamma_{1j} + \gamma_{21} + \gamma_{23}\right), \quad \dots,$$
$$\Delta_{2N} = \left[\sum_{j=2}^{N} \gamma_{1j} + \sum_{j=1, j\neq 2}^{N-1} \gamma_{2j}, \sum_{j=2}^{N} \gamma_{1j} + \sum_{j=1, j\neq 2}^{N} \gamma_{2j}\right)$$

and so on. Define the function $h : S \times \mathbb{R} \to \mathbb{R}$ by

$$h(i, y) = \begin{cases} j - i, & \text{if } y \in \Delta_{ij}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $dr(t) = \int_R h(r(t-), y)\nu(dt, dy)$ with initial condition $r(0) = r_0$, where $\nu(dt, dy)$ is a Poisson random measure with intensity $dt \times m(dy)$, where m is the Lebesgue measure on $\mathbb{R}$. In what follows, we use the generalized time-changed Itô formula in Lemma 2.1: if $V \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times S; \mathbb{R}_+)$, then for any $t \geq 0$,

$$\begin{aligned} V(t, E_t, x(t), r(t)) = V(0, 0, x_0, r(0)) &+ \int_0^t L_1 V(s, E_s, x(s), x(s-\tau), r(s))ds \\ &+ \int_0^t L_2 V(s, E_s, x(s), x(s-\tau), r(s))dE_s \\ &+ \int_0^t V_x(s, E_s, x(s), r(s))g(s, E_s, x(s), x(s-\tau), r(s))dB_{E_s} \\ &+ \int_0^t \int_R [V(s, E_s, x(s), i_0 + h(r(s), l)) - V(s, E_s, x(s), r(s))]\mu(ds, dl), \end{aligned}$$

where $\mu(ds, dl) = \nu(ds, dl) - m(dl)ds$ is a martingale measure. By using conditions (3.1) and (3.2), we can see that

$$\begin{aligned} V(t, E_t, x(t), r(t)) \leq V(0, 0, \xi(0), r(0)) &+ \int_{-\tau}^0 w_{12}(\xi(s))ds + \int_{-\tau}^0 w_{22}(\xi(s))dE_s + \int_0^t [r_{11}(s) + r_{12}(E_s)]ds \\ &+ \int_0^t [r_{21}(s) + r_{22}(E_s)]dE_s - \int_0^t [w_{11}(x(s)) - w_{12}(x(s))]ds \\ &- \int_0^t [w_{21}(x(s)) - w_{22}(x(s))]dE_s + \int_0^t V_x(s, E_s, x(s))g(s, E_s, x(s), x(s-\tau), r(s))dB_{E_s} \\ &+ \int_0^t \int_R [V(s, E_s, x(s), i_0 + h(r(s), l)) - V(s, E_s, x(s), r(s))]\mu(ds, dl). \end{aligned}$$

Applying Lemma 2.2, the semimartingale convergence theorem yields that

$$\limsup_{t\to\infty} V(t, E_t, x(t), r(t)) < \infty \text{ a.s.} \tag{3.6}$$

Moreover,

$$\mathbb{E}\int_0^t [w_{11}(x(s)) - w_{12}(x(s))]\mathrm{d}s + \mathbb{E}\int_0^t [w_{21}(x(s)) - w_{22}(x(s))]\mathrm{d}E_s$$

$$\leq \mathbb{E}\left(V(0, 0, \xi(0), r(0)) + \int_{-\tau}^0 w_{12}(\xi(s))\mathrm{d}s + \int_{-\tau}^0 w_{22}(\xi(s))\mathrm{d}E_s\right) + \int_0^\infty [r_{11}(s) + r_{12}(E_s)]\mathrm{d}s + \int_0^\infty [r_{21}(s) + r_{22}(E_s)]\mathrm{d}E_s$$

$$< \infty.$$

Taking $t \to \infty$ we obtain that

$$\mathbb{E}\int_0^\infty [w_{11}(x(s)) - w_{12}(x(s))]\mathrm{d}s < \infty, \quad \mathbb{E}\int_0^\infty [w_{21}(x(s)) - w_{22}(x(s))]\mathrm{d}E_s < \infty,$$

this means

$$\int_0^\infty [w_{11}(x(s)) - w_{12}(x(s))]\mathrm{d}s < \infty, \quad \int_0^\infty [w_{21}(x(s)) - w_{22}(x(s))]\mathrm{d}E_s < \infty \quad \text{a.s.} \tag{3.7}$$

Step 2. If we set $w = w_1 - w_2$, where $w_i = w_{1i} + w_{2i}$, $i = 1, 2$. Clearly, $w \in C(\mathbb{R}^n, \mathbb{R}_+)$. It is straightforward to see from (3.7) that

$$\liminf_{t\to\infty} w(x(t)) = 0 \quad \text{a.s.} \tag{3.8}$$

We now claim that

$$\lim_{t\to\infty} w(x(t)) = 0 \quad \text{a.s.} \tag{3.9}$$

If it is false, then

$$P\left\{\limsup_{t\to\infty} w(x(t)) > 0\right\} > 0;$$

therefore, there exists a number $\varepsilon > 0$ such that

$$P(\Omega_1) \geq 3\varepsilon, \tag{3.10}$$

where

$$\Omega_1 = \left\{\limsup_{t\to\infty} w(x(t)) > 2\varepsilon\right\}.$$

Since E_t is continuous, by means of (3.6) and the continuity of both the solution $x(t)$ and the function $V(t, E_t, x, r(t))$ [6,12], we can see that

$$\sup_{-\tau\leq t<\infty} V(t, E_t, x(t), r(t)) < \infty \quad \text{a.s.}$$

Define the function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\mu(k) = \inf_{|x|\geq k, 0\leq t<\infty, i\in S} V(t, E_t, x, i) \quad \text{for } k \geq 0.$$

Clearly $\mu(|x(t)|) \leq V(t, E_t, x(t), r(t))$, hence,

$$\sup_{-\tau\leq t<\infty} \mu(|x(t)|) \leq \sup_{-\tau\leq t<\infty} V(t, E_t, r(t), x(t)) < \infty \quad \text{a.s.}$$

On the other hand, by (3.4) we have

$$\lim_{k\to\infty} \mu(k) = \infty.$$

Therefore, it follows that

$$\sup_{-\tau\le t<\infty} |x(t)| < \infty \quad \text{a.s.} \tag{3.11}$$

Since the initial data ξ is bounded, we can find a positive number h sufficiently large, which depends on ε, satisfying $|\xi(\theta)| < h$ for all $-\tau \le \theta \le 0$ almost surely, while

$$P(\Omega_2) \ge 1 - \varepsilon, \tag{3.12}$$

where

$$\Omega_2 = \left\{ \sup_{-\tau\le t<\infty} |x(t)| < h \right\}.$$

It is easy to see from (3.10) and (3.12) that

$$P(\Omega_1 \cap \Omega_2) \ge 2\varepsilon. \tag{3.13}$$

In what follows, we define a sequence of stopping times,

$$\sigma_1 = \inf\{t \ge 0 : w(x(t)) \ge 2\varepsilon\}, \quad \sigma_{2i} = \inf\{t \ge \sigma_{2i-1} : w(x(t)) \ge \varepsilon\},\ \ i = 1, 2, \dots,$$
$$\sigma_{2i+1} = \inf\{t \ge \sigma_{2i} : w(x(t)) \ge 2\varepsilon\}, \quad \tau_h = \inf\{t \ge 0 : |x(t)| \ge h\},\ \ i = 1, 2, \dots.$$

Throughout this paper we set $\inf\varnothing = \infty$, note from (3.8) and the definitions of Ω_i $(i = 1, 2)$ that if $\omega \in \Omega_1 \cap \Omega_2$, then

$$\tau_h(\omega) = \infty \quad \text{and} \quad \sigma_i(\omega) < \infty \quad \text{for all } i \ge 1. \tag{3.14}$$

Let I be the indicator function of set A, by means of the fact that $\sigma_{2i} < \infty$ whenever $\sigma_{2i-1} < \infty$ and (3.8), we obtain that

$$\begin{aligned} \infty &> \mathbb{E}\int_0^\infty w(x(t))\mathrm{d}t \\ &\ge \sum_{i=1}^\infty \mathbb{E}\left[I_{\{\sigma_{2i-1}<\infty,\sigma_{2i}<\infty,\tau_h=\infty\}} \int_{\sigma_{2i-1}}^{\sigma_{2i}} w(x(t))\mathrm{d}t \right] \\ &\ge \varepsilon \sum_{i=1}^\infty \mathbb{E}[I_{\{\sigma_{2i-1}<\infty,\tau_h=\infty\}}(\sigma_{2i} - \sigma_{2i-1})]. \end{aligned} \tag{3.15}$$

On the other hand, by the hypothesis (H_1), there exists a constant K_h such that

$$|\rho(t_1, t_2, x, y, i)| \vee |f(t_1, t_2, x, y, i)| \vee |g(t_1, t_2, x, y, i)| \le K_h,$$

whenever $|x| \vee |y| \le h$ and $t > 0$. By the Hölder inequality and the martingale property of the indefinite Itô integral [3], one can compute that

$$\begin{aligned} &\mathbb{E}[I_{\tau_h\wedge\sigma_{2i-1}<\infty} \sup_{0\le t\le T} |x(\tau_h \wedge (\sigma_{2i-1} + t)) - x(\tau_h \wedge \sigma_{2i-1})|^2] \\ &\le\ 3\mathbb{E}\left[I_{\tau_h\wedge\sigma_{2i-1}<\infty} \sup_{0\le t\le T} \left| \int_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+t)} \rho(s, E_s, x(s), x(s-\tau), r(s))\mathrm{d}s \right|^2 \right] \\ &\quad + 3\mathbb{E}\left[I_{\tau_h\wedge\sigma_{2i-1}<\infty} \sup_{0\le t\le T} \left| \int_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+t)} f(s, E_s, x(s), x(s-\tau), r(s))\mathrm{d}E_s \right|^2 \right] \\ &\quad + 3\mathbb{E}\left[I_{\tau_h\wedge\sigma_{2i-1}<\infty} \sup_{0\le t\le T} \left| \int_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+t)} g(s, E_s, x(s), x(s-\tau), r(s))\mathrm{d}B_{E_s} \right|^2 \right] \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
&\le 3T\mathbb{E}\left[I_{\tau_h\wedge\sigma_{2i-1}<\infty}\int\limits_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+T)}|\rho(s,E_s,x(s),x(s-\tau),r(s))|^2\mathrm{d}s\right]\\
&\quad+3T\mathbb{E}\left[I_{\tau_h\wedge\sigma_{2i-1}<\infty}\int\limits_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+T)}|f(s,E_s,x(s),x(s-\tau),r(s))|^2\mathrm{d}E_s\right]\\
&\quad+12\mathbb{E}\left[I_{\tau_h\wedge\sigma_{2i-1}<\infty}\int\limits_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+T)}|g(s,E_s,x(s),x(s-\tau),r(s))|^2\mathrm{d}E_s\right]\\
&\le 3K_h^2\mathbb{E}\left[T^2+(T+4)\Big(E_{\sigma_{2i-1}+T}-E_{\sigma_{2i-1}}\Big)\right].
\end{aligned}
$$

Since $w(\cdot)$ is continuous in $\mathbb{R}^n$, it must be uniformly continuous in the closed ball $\bar{S}_h = \{x \in \mathbb{R}^n : |x| \le h\}$. We can therefore choose $\delta = \delta(\varepsilon) > 0$ so small that

$$
|w(x) - w(y)| < \varepsilon \quad \text{whenever } |x - y| < \delta, x, y \in \bar{S}_h. \tag{3.17}
$$

Furthermore, since E_t is continuous at $t = \sigma_{2i-1}$, we choose $T = T(\varepsilon, \delta, h) > 0$ sufficiently small such that

$$
3K_h^2\mathbb{E}[T^2 + (T+4)(E_{\sigma_{2i-1}+T} - E_{\sigma_{2i-1}})] < \delta^2\varepsilon.
$$

It follows from (3.16) that

$$
P(\{\tau_h\wedge\sigma_{2i-1}<\infty\}\cap\{\sup_{0\le t\le T}|x(\tau_h\wedge(\sigma_{2i-1}+t))-x(\tau_h\wedge\sigma_{2i-1})|\ge\delta\})\le\frac{3K_h^2\mathbb{E}[T^2+(T+4)(E_{\sigma_{2i-1}+T}-E_{\sigma_{2i-1}})]}{\delta^2}<\varepsilon.
$$

This yields that

$$
\begin{aligned}
&P\left(\{\sigma_{2i-1}<\infty,\tau_h=\infty\}\cap\left\{\sup_{0\le t\le T}|x(\sigma_{2i-1}+t)-x(\sigma_{2i-1})|\ge\delta\right\}\right)\\
&\quad= P\left(\{\tau_h\wedge\sigma_{2i-1}<\infty,\tau_h=\infty\}\cap\left\{\sup_{0\le t\le T}|x(\tau_h\wedge(\sigma_{2i-1}+t))-x(\tau_h\wedge\sigma_{2i-1})|\ge\delta\right\}\right)\\
&\quad\le P\left(\{\tau_h\wedge\sigma_{2i-1}<\infty\}\cap\left\{\sup_{0\le t\le T}|x(\tau_h\wedge(\sigma_{2i-1}+t))-x(\tau_h\wedge\sigma_{2i-1})|\ge\delta\right\}\right)\le\varepsilon.
\end{aligned}
$$

Recalling (3.17), we further compute

$$
\begin{aligned}
&P\left(\{\sigma_{2i-1}<\infty,\tau_h=\infty\}\cap\left\{\sup_{0\le t\le T}|w(x(\sigma_{2i-1}+t))-w(x(\sigma_{2i-1}))|<\varepsilon\right\}\right)\\
&\ge P\left(\{\sigma_{2i-1}<\infty,\tau_h=\infty\}\cap\left\{\sup_{0\le t\le T}|x(\sigma_{2i-1}+t)-x(\sigma_{2i-1})|<\delta\right\}\right)\\
&\ge P\{\sigma_{2i-1}<\infty,\tau_h=\infty\}-P(\{\sigma_{2i-1}<\infty,\tau_h=\infty\}\cap\left\{\sup_{0\le t\le T}|x(\sigma_{2i-1}+t)-x(\sigma_{2i-1})|\ge\delta\right\}\Bigg)\\
&\ge 2\varepsilon-\varepsilon=\varepsilon.
\end{aligned}
\tag{3.18}
$$

Set

$$
\bar{\Omega}_i = \left\{\sup_{0\le t\le T}|w(x(\sigma_{2i-1}+t))-w(x(\sigma_{2i-1}))|<\varepsilon\right\},
$$

noting that

$$
\sigma_{2i}(\omega)-\sigma_{2i-1}(\omega)\ge T \text{ if } \omega\in\{\sigma_{2i-1}<\infty,\tau_h=\infty\}\cap\bar{\Omega}_i,
$$

we can derive from (3.15) and (3.18) that

$$\begin{aligned}\infty > \varepsilon \sum_{i=1}^{\infty} \mathbb{E}[I_{\{\sigma_{2i-1}<\infty,\tau_h=\infty\}}(\sigma_{2i}-\sigma_{2i-1})] \\ \geq \varepsilon \sum_{i=1}^{\infty} \mathbb{E}[I_{\{\sigma_{2i-1}<\infty,\tau_h=\infty\}\cap\overline{\Omega}_i}(\sigma_{2i}-\sigma_{2i-1})] \\ \geq \varepsilon T \sum_{i=1}^{\infty} P(\{\sigma_{2i-1}<\infty,\tau_h=\infty\}\cap\overline{\Omega}_i) \\ \geq \varepsilon T \sum_{i=1}^{\infty} \varepsilon = \infty,\end{aligned}$$

it is a contradiction. So (3.9) must be hold.

Step 3. We first show that $\mathrm{Ker}(w) \neq \varnothing$. Note from (3.9) and (3.11) that there exists a $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that

$$\lim_{t\to\infty} w(x(t,\omega)) = 0 \quad \text{and} \quad \sup_{0\leq t<\infty} |x(t,\omega)| < \infty \quad \text{for all } \omega \in \Omega_0. \tag{3.19}$$

For any $\omega \in \Omega_0$, since $\{x(t,\omega)\}_{t\geq 0}$ is bounded, there exists an increasing sequence $\{t_i\}_{i\geq 1}$ such that $\{x(t_i,\omega)\}_{i\geq 1}$ converges to some $y \in \mathbb{R}^n$. Hence,

$$w(y) = \lim_{t\to\infty} w(x(t_i,\omega)) = 0, \tag{3.20}$$

which implies $y \in \mathrm{Ker}(w)$, that is, $\mathrm{Ker}(w) \neq \varnothing$.

Now, we claim that

$$\lim_{t\to\infty} \mathrm{d}(x(t,\omega),\mathrm{Ker}(w)) = 0 \quad \text{for all } \omega \in \Omega_0. \tag{3.21}$$

If it is not true, there exist some $\overline{\omega} \in \Omega_0$ such that

$$\limsup_{t\to\infty} \mathrm{d}(x(t,\overline{\omega}),\mathrm{Ker}(w)) > 0;$$

therefore, there is a subsequence $\{x(t_i,\overline{\omega})\}_{i\geq 1}$ of $\{x(t,\overline{\omega})\}_{t\geq 0}$ such that

$$\mathrm{d}(x(t_i,\overline{\omega}),\mathrm{Ker}(w)) \geq \varepsilon, \quad \forall i \geq 1,$$

for some $\varepsilon > 0$. Since $\{x(t_i,\overline{\omega})\}_{i\geq 1}$ is bounded, we can find a subsequence $\{x(\bar{t}_i,\overline{\omega})\}_{i\geq 1}$, which converges to z. Clearly, $z \in \mathrm{Ker}(w)$, that is, $w(z) > 0$. However, from (3.19) we can see that

$$w(z) = \lim_{i\to\infty} w(x(\bar{t}_i,\overline{\omega})) = 0,$$

which contradicts with $w(z) > 0$. Hence, (3.21) holds and the required assertion (3.5) follows since $P(\Omega_0) = 1$. This completes the proof. □

Corollary 3.1. *Let conditions* (H_1) *and* (H_2) *hold. Assume that there are functions* $V \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times S, \mathbb{R}_+)$, $r_{ij} \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ *and* $w_{ij} \in C(\mathbb{R}^n, \mathbb{R}_+)$, $i, j = 1, 2$ *such that*

$$\begin{aligned} L_1V(t,E_t,x,y,i) &\leq r_{11}(t) + r_{12}(E_t) - w_{11}(x) + w_{12}(y), \\ L_2V(t,E_t,x,y,i) &\leq r_{21}(t) + r_{22}(E_t) - w_{21}(x) + w_{22}(y), \\ w_{i1}(x) &> w_{i2}(x), \quad \forall x \neq 0, \ i = 1, 2 \end{aligned} \tag{3.22}$$

and

$$\lim_{|x|\to\infty} \inf_{0\leq t<\infty} V(t,E_t,x,i) = \infty.$$

Then

$$\lim_{t\to\infty} x(t;\xi) = 0 \quad \text{a.s.} \tag{3.23}$$

for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

Proof. By condition (3.22), $x \notin \mathrm{Ker}(w_1 - w_2)$ if $x \neq 0$. On the other hand, from Theorem 2.1 we obtain that $\mathrm{Ker}(w_1 - w_2) \neq \varnothing$, then it must be $\mathrm{Ker}(w_1 - w_2) = \{0\}$, and $\lim_{t\to\infty} x(t;\xi) = 0$ a.s. immediately. □

4 Controllability of linear stochastic differential system

Let us consider the following linear SDE with delay:

$$\begin{aligned} \mathrm{d}x(t) = {} & [A(r(t))x(t) + C(r(t))x(t-\tau) + D(r(t))u(t)]\mathrm{d}t + [F(r(t))x(t) + G(r(t))x(t-\tau)]\mathrm{d}E_t \\ & + \sum_{k=1}^{m} [M_k(r(t))x(t) + N_k(r(t))x(t-\tau)]\mathrm{d}B_k(E_t) \end{aligned} \tag{4.1}$$

on $t \geq 0$ with the initial data $\xi = \{x(\theta) : -\tau \leq \theta \leq 0\} \in C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $r(0) = i_0$. Here u is an $\mathscr{F}_t$-measurable and $\mathbb{R}^p$-value control law. For each $r(t) = i \in S$, we write $A(i) = A_i$. A_i, C_i, F_i, G_i, M_{ki}, N_{ki} are all $n \times n$ constant matrices and D_i is an $n \times p$ matrix.

The aim of this study is to design a delay-independent feedback controller with the form $u(t) = H(r(t))x(t)$, such that the following closed-loop system of (4.1)

$$\begin{aligned} \mathrm{d}x(t) = {} & [A(r(t))x(t) + C(r(t))x(t-\tau) + D(r(t))H(r(t))x(t)]\mathrm{d}t + [F(r(t))x(t) + G(r(t))x(t-\tau)]\mathrm{d}E_t \\ & + \sum_{k=1}^{m} [M_k(r(t))x(t) + N_k(r(t))x(t-\tau)]\mathrm{d}B_k(E_t) \end{aligned}$$

becomes almost surely asymptotically stable. Here for each mode $r(t) = i \in S$, $H(i) = H_i$ is a $p \times n$ matrix.

Lemma 4.1. (The Schur complement [22]) *Let M, N, R be constant matrices with appropriate dimensions such that $R = R^T > 0$ and $M = M^T$. Then $M + NR^{-1}N^T < 0$ iff*

$$\begin{bmatrix} M & N \\ N^T & -R \end{bmatrix} < 0.$$

(*Here, as usual, by $R = R^T$ we mean R is a symmetric matrix while by $R > 0$ or $R < 0$ we mean R is a positive-definite or negative matrix, respectively*.)

Theorem 4.1. *If the following linear matrix inequalities* (*LMIs*)

$$\begin{bmatrix} P_{1i} & Q_iC_i \\ C_i^TQ_i & -I \end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix} P_{2i} & Q_iG_i \\ G_i^TQ_i & -I \end{bmatrix} < 0, \quad i \in S$$

have the solutions Q_i and H_i such that $Q_i = Q_i^T > 0$, where I is the $n \times n$ identity matrix and

$$P_{1i} = Q_iA_i + A_i^TQ_i + Q_iD_iH_i + (D_iH_i)^TQ_i + \sum_{l=1}^{N} \gamma_{il}Q_l + I,$$

$$P_{2i} = Q_iF_i + F_i^TQ_i + 2\sum_{k=1}^{m} M_{ki}^TQ_iM_{ki} + 2\sum_{k=1}^{m} N_{ki}^TQ_iN_{ki} + I.$$

Then system (4.1) *is almost surely asymptotically stable with the controller $u(t) = H(r(t))x(t)$.*

Proof. Let $V(x, i) = x^T Q_i x$, then by the generalized time-changed Itô formula in Lemma 2.1,

$$L_1 V(x, y, i) = 2x^T Q_i (A_i + D_i H_i)x + 2x^T Q_i C_i y + x^T \sum_{l=1}^{N} \gamma_{il} Q_l x,$$

$$L_2 V(x, y, i) = 2x^T Q_i [F_i x + G_i y] + \sum_{k=1}^{m} [M_{ki} x + N_{ki} y]^T Q_i [M_{ki} x + N_{ki} y].$$

Note that

$$2x^T QAx = x^T [QA + A^T Q]x, \quad 2x^T QDHx = x^T [QDH + (DH)^T Q]x.$$

From the following inequality

$$(y^T - x^T QC)(y^T - x^T QC)^T \geq 0,$$

it follows that

$$2x^T QCy = x^T QCy + y^T C^T Qx \leq y^T y + x^T QCC^T Qx.$$

Then

$$\begin{aligned} L_1 V &\leq x^T [Q_i A_i + A_i^T Q_i]x + x^T [Q_i D_i H_i + (D_i H_i)^T Q_i]x + x^T Q_i C_i C_i^T Q_i x + y^T y + x^T \sum_{l=1}^{N} \gamma_{il} Q_l x \\ &\leq -w_{11i}(x) + w_{12}(y), \end{aligned}$$

where

$$w_{11i}(x) = x^T [-P_{1i} - Q_i C_i C_i^T Q_i + I]x, \quad w_{12}(x) = x^T x.$$

By Lemma 4.1, $P_{1i} + Q_i C_i C_i^T Q_i < 0$, it means that, $-P_{1i} - Q_i C_i C_i^T Q_i > 0$, so we obtain that

$$w_{11i}(x) \geq x^T x = w_{12}(x) > 0, \quad \forall x \neq 0.$$

Let $w_{11}(x) = \min_{i \in S} w_{11i}(x)$, clearly $w_{11}(x) \geq w_{12}(x)$ for $x \neq 0$. Furthermore, note that

$$[Mx + Ny]^T Q[Mx + Ny] \leq 2x^T M^T QMx + 2y^T N^T QNy,$$

then

$$\begin{aligned} L_2 V &\leq x^T [Q_i F_i + F_i^T Q_i]x + y^T y + x^T Q_i G_i G_i^T Q_i x + 2x^T \sum_{k=1}^{m} M_{ki}^T Q_i M_{ki} x + 2y^T \sum_{k=1}^{m} N_{ki}^T Q_i N_{ki} y \\ &\leq -w_{21i}(x) + w_{22i}(y), \end{aligned}$$

where

$$w_{21i}(x) = x^T \left[-P_{2i} - Q_i G_i G_i^T Q_i + I + 2\sum_{k=1}^{m} N_{ki}^T Q_i N_{ki} \right] x$$

and

$$w_{22i}(x) = x^T \left[I + 2\sum_{k=1}^{m} N_{ki}^T Q_i N_{ki} \right] x.$$

By Lemma 4.1, $P_{2i} + Q_i G_i G_i^T Q_i < 0$, in other words, $-P_{2i} - Q_i G_i G_i^T Q_i > 0$, so we obtain that

$$w_{21i}(x) \geq x^T \left[I + 2\sum_{k=1}^{m} N_{ki}^T Q_i N_{ki} \right] x = w_{22i}(x) > 0, \quad \forall x \neq 0.$$

Let $w_{21}(x) = \min_{i \in S} w_{21i}(x)$, $w_{22}(x) = \max_{i \in S} w_{22i}(x)$, then it is clear that $w_{21}(x) \geq w_{22}(x)$ for $x \neq 0$. Therefore, the system is almost surely asymptotically stable with the controller designed above. □

Remark. If the controller $u(t)$ applies to the time-changed term $\mathrm{d}E_t$, then system (4.1) becomes the following system:

$$\begin{aligned} \mathrm{d}x(t) &= [A(r(t))x(t) + C(r(t))x(t-\tau)]\mathrm{d}t + [F(r(t))x(t) + G(r(t))x(t-\tau) + D(r(t))u(t)]\mathrm{d}E_t \\ &\quad + \sum_{k=1}^{m}[M_k(r(t))x(t) + N_k(r(t))x(t-\tau)]\mathrm{d}B_k(E_t). \end{aligned} \tag{4.2}$$

In this case, we can also consider the asymptotic stability by means of the LMIs just renew P_{1i} and P_{2i} as follows:

$$\begin{aligned} P_{1i} &= Q_iA_i + A_i^TQ_i + \sum_{l=1}^{N}\gamma_{il}Q_l + I, \\ P_{2i} &= Q_iF_i + F_i^TQ_i + Q_iD_iH_i + (D_iH_i)^TQ_i + 2\sum_{k=1}^{m}M_{ki}^TQ_iM_{ki} + 2\sum_{k=1}^{m}N_{ki}^TQ_iN_{ki} + I. \end{aligned}$$

In what follows, we shall give an example to show the aforementioned results.

Example 4.1. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Let $B(t)$ be a one-dimensional Brownian motion independent of $r(t)$. Now, let us consider the following two-dimensional SDDE with Markovian switching as

$$\mathrm{d}x(t) = [A(r(t))x(t) + u(t)]\mathrm{d}t + F(r(t))x(t)\mathrm{d}E_t + N(r(t))x(t-\tau)\mathrm{d}B_{E_t}, \quad t \geq 0, \tag{4.3}$$

where

$$\begin{aligned} A_1 &= \begin{pmatrix} -1 & 1 \\ -2 & -3 \end{pmatrix}, \quad F_1 = \begin{pmatrix} -2 & 3 \\ -3 & -12 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -2 & 0 \\ -2 & -3 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

Next, let us design a feedback control $u(t) = H(r(t))x(t)$ in order to guarantee the stability of system (4.3). To the end, we set

$$\begin{aligned} P_{1i} &= Q_iA_i + A_i^TQ_i + Q_iH_i + H_i^TQ_i + \sum_{l=1}^{2}\gamma_{il}Q_l + I, \\ P_{2i} &= Q_iF_i + F_i^TQ_i + 2N_i^TQ_iN_i + I. \end{aligned}$$

It is easy to verify that the LMIs

$$\begin{bmatrix} P_{1i} & Q_iC_i \\ C_i^TQ_i & -I \end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix} P_{2i} & Q_iG_i \\ G_i^TQ_i & -I \end{bmatrix} < 0, \quad i \in S$$

have the solution

$$Q_1 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} -3 & 4 \\ 4 & -9 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} -2 & 1 \\ -3 & -1 \end{pmatrix}.$$

Obviously, Q_1 and Q_2 are positive-definite and symmetric matrices. By Theorem 4.1, we can see that equation (4.3) is almost surely asymptotically stable with the controller $u(t) = H(r(t))x(t)$. When the initial condition $x(t) = col[1.1\sin(t), 1.8\cos(t)]$ $(t \in [-\tau; 0])$, $\tau = 2$, $r(0) = 1$, Figure 1 shows the asymptotic behavior in almost sure sense of the global solution for (4.3).

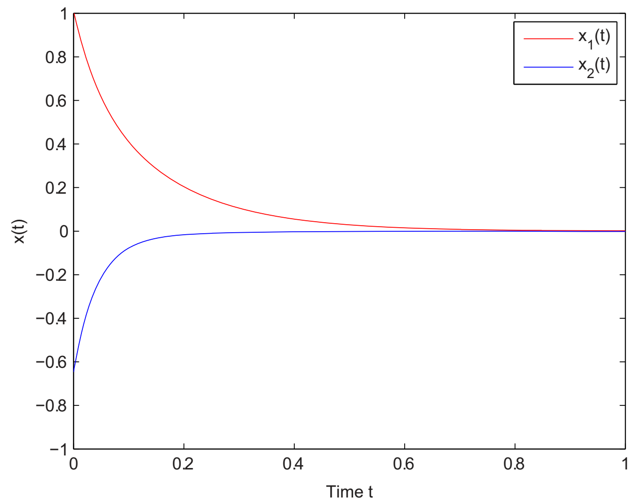

Figure 1: Asymptotic behavior in almost sure sense of the solution for equation (4.3).

5 Conclusions

The SDDEs driven by time-changed Brownian motions is a new research area for recent years. In this work, we have considered the asymptotic stability of the time-changed SDDEs with Markovian switching, by expanding the time-changed Itô formula and the time-changed semi-martingale convergence theorem. Our result generalizes that of SDDEs in the literature. Due to the more construction of SDDEs with time change than the usual SDDEs, our result is not a trivial generalization.

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