Asymptotic stability of Toda lattice solitons

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Abstract

We establish an asymptotic stability result for Toda lattice soliton solutions, by making use of a linearized Bäcklund transformation whose domain has codimension one. Combining a linear stability result with a general theory of nonlinear stability by Friesecke and Pego for solitary waves in lattice equations, we conclude that all solitons in the Toda lattice are asymptotically stable in an exponentially weighted norm. In addition, we determine the complete spectrum of an operator naturally associated with the Floquet theory for these lattice solitons.

1 Introduction

In this article we establish an asymptotic stability result for all 1-soliton solutions to the Toda lattice equations

$$\ddot{Q}_n = e^{-(Q_n - Q_{n-1})} - e^{-(Q_{n+1}-Q_n)}, \qquad n \in \mathbb{Z}. \tag{1}$$

Here $\dot{} = d/dt$. Let $P_n = \dot{Q}_n$ and $R_n = Q_{n+1} - Q_n$. The Toda lattice is an integrable system with Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2} P_n^2 + V(R_n) \right), \qquad V(R) = e^{-R} - 1 + R. \tag{2}$$

In terms of $\mathbf{P} = (P_n)_{n\in\mathbb{Z}}$, $\mathbf{Q} = (Q_n)_{n\in\mathbb{Z}}$, $\mathbf{R} = (R_n)_{n\in\mathbb{Z}}$ and $\mathbf{U} = {}^t(\mathbf{R}, \mathbf{P})$, the governing equations can be rewritten in the form (see [4])

$$\frac{d\mathbf{U}}{dt} = JH'(\mathbf{U}), \qquad J = \begin{pmatrix} 0 & e^{\partial} - 1 \\ 1 - e^{-\partial} & 0 \end{pmatrix} \tag{3}$$

where H' is the Fréchet derivative of H in $l^2 \times l^2$, and e^{∂} is the shift operator given by $e^{\partial}\mathbf{R} = (R_{n+1})_{n\in\mathbb{Z}}$. Here l^2 is the Hilbert space of complex sequences $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ equipped with norm $\|\mathbf{x}\| = \left(\sum_{n\in\mathbb{Z}} |x_n|^2\right)^{1/2}$.

The Toda lattice has a well-known two-parameter family of right-moving solitary waves (1-solitons) $\left\{\mathbf{Q}_c(t+\delta) \;\middle|\; c > 1,\ \delta \in \mathbb{R}\right\}$, where

$$\mathbf{Q}_c(t) = \left(\widetilde{Q}_c(n - ct)\right)_{n\in\mathbb{Z}}, \qquad \widetilde{Q}_c(x) = \log \frac{\cosh\{\kappa(x-1)\}}{\cosh \kappa x}, \tag{4}$$

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with $\kappa = \kappa(c)$ the unique positive solution of $\sinh\kappa/\kappa = c$. Regarding the question of stability for these waves, two things are worth pointing out. In general, stability of solitons is not automatic in integrable systems. For example, solitons in the "good" Boussinesq equation are unstable if the traveling speed is sufficiently slow ([2]), and line solitons for Kadomtsev-Petviashvili equation (KP-I) in $2+1$ dimensions are unstable to long-wave transverse perturbations [17, 1, 11]. Moreover, for lattice equations such as the Toda lattice, it does not appear possible to study stability by using variational methods based on the Vakhitov-Kolokolov condition, such as the theory of Grillakis, Shatah and Strauss [9]. Such methods are based on characterizing traveling waves as critical points of a time-invariant energy-momentum functional, but the existence of momentum functionals is usually due to the continuous translational invariance of the Hamiltonian, which does not hold in the discrete setting here. This differs from discrete nonlinear Schrödinger lattice equations, for which charge (l^2-norm) is conserved (see e.g. [13]).

Instead, we will study the stability of Toda lattice solitons by using the nonlinear stability theory of [6], which is based on obtaining suitable conditional asymptotic stability estimates for a linearized problem. We will write $e^{-\mathbf{x}} = (e^{-x_n})_{n\in\mathbb{Z}}$ and $\mathbf{xy} = (x_n y_n)_{n\in\mathbb{Z}}$ for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$. For $a \in \mathbb{R}$, we denote by l^2_a the Hilbert space of complex sequences equipped with the weighted norm

$$\|\mathbf{x}\|_{l^2_a} = \|e^{a\mathbf{n}}\mathbf{x}\| = \left(\sum_{n\in\mathbb{Z}} e^{2an}|x_n|^2\right)^{1/2}, \qquad \mathbf{n} = (n)_{n\in\mathbb{Z}}. \tag{5}$$

Then $\langle \mathbf{x}, \mathbf{y}\rangle := \sum_{n\in\mathbb{Z}} x_n \overline{y_n}$ is well-defined whenever $\mathbf{x} \in l^2_a$ and $\mathbf{y} \in l^2_{-a}$. For $\mathbf{u} = (u_{1,n}, u_{2,n})_{n\in\mathbb{Z}} \in l^2_a \times l^2_a$ and $\mathbf{v} = (v_{1,n}, v_{2,n})_{n\in\mathbb{Z}} \in l^2_{-a} \times l^2_{-a}$, we use the same notation

$$\langle \mathbf{u}, \mathbf{v}\rangle := \sum_{n\in\mathbb{Z}} \left(u_{1,n}\overline{u_{2,n}} + v_{1,n}\overline{v_{2,n}}\right).$$

In $l^2_{-a} \times l^2_{-a}$ with $a > 0$, the operator J has a bounded inverse, given by

$$J^{-1} = \begin{pmatrix} 0 & \sum_{k=-\infty}^{0} e^{k\partial} \\ \sum_{k=-\infty}^{-1} e^{k\partial} & 0 \end{pmatrix}.$$

Let $\mathbf{P}_c = \dot{\mathbf{Q}}_c$, $\mathbf{R}_c = (e^\partial - 1)\mathbf{Q}_c$ and $\mathbf{U}_c = {}^t(\mathbf{R}_c, \mathbf{P}_c)$. Writing $\mathbf{U} = \mathbf{U}_c + \mathbf{u}$ and linearizing (3), we get

$$\frac{d\mathbf{u}}{dt} = JH''(\mathbf{U}_c(t))\mathbf{u}. \tag{6}$$

For a class of lattice equations that includes the Toda lattice equations (3), Friesecke and Pego have proved [6, Theorem 1.1] that solitary waves are asymptotically stable in a weighted norm (5), provided that two conditions hold: first, the nondegeneracy condition

$$\frac{d}{dc}H(\mathbf{U}_c) \neq 0, \tag{7}$$

and second, the following exponential linear stability property, for some a with $0 < a < a_c := 2\kappa$:

(L) Every solution of (6) in $l^2_a \times l^2_a$, that satisfies the secular term condition

$$\langle \mathbf{u}, J^{-1}\dot{\mathbf{U}}_c\rangle = \langle \mathbf{u}, J^{-1}\partial_c \mathbf{U}_c\rangle = 0 \tag{8}$$

for some (hence every) $t \in \mathbb{R}$, decays exponentially in the weighted norm, in the sense that there exist positive constants K and β independent of $\mathbf{u}$, such that whenever $t \geq s$,

$$\|e^{a(\mathbf{n}-ct)}\mathbf{u}(t)\| \leq Ke^{-\beta(t-s)}\|e^{a(\mathbf{n}-cs)}\mathbf{u}(s)\|. \tag{9}$$

Our main result is that both of these conditions hold for *all* Toda-lattice 1-solitons, of arbitrarily large amplitude and for arbitrary $a \in (0, a_c)$. (The results of [7] and [8] imply that both conditions hold for waves of small amplitude for $a > 0$ sufficiently small.) In particular, we find

$$H(\mathbf{U}_c) = \sinh 2\kappa - 2\kappa, \tag{10}$$

(as asserted by Toda [14], see Lemma 4 below) and it follows $(d/dc)H(\mathbf{U}_c) > 0$ for all $c > 1$, so that (7) holds. Second, we shall check that (L) holds, by using a linearized Bäcklund transformation which turns out to be well-defined, not for all solutions of (6), but *exactly for solutions of* (6) *that satisfy the secular term condition in* (8). The transformation also allows us to identify precisely the optimal β in (L).

As a consequence of (7), (L) and the main theorem of [6], we have the following asymptotic stability result for the family of Toda 1-solitons.

Theorem 1. *Let $c > 1$, $0 < a < 2\kappa(c)$ and $\beta = ca - 2\sinh(a/2)$. Then for every $\beta' \in (0, \beta)$, there exist positive numbers δ_0 and C such that, if for some $t_0 \in \mathbb{R}$ we have*

$$\delta := \|\mathbf{U}(0) - \mathbf{U}_c(-t_0)\|^2 + \|e^{a(\mathbf{n}+ct_0)}(\mathbf{U}(0) - \mathbf{U}_c(-t_0))\| \leq \delta_0,$$

then the solution to (3) *satisfies, for every $t \geq t_0$,*

$$\|\mathbf{U}(t) - \mathbf{U}_{c_*}(t - t_*)\| \leq C\sqrt{\delta},$$
$$\|e^{a(\mathbf{n}-c_*(t-t_*))}(\mathbf{U}(t) - \mathbf{U}_c(t - t_*))\| \leq C\delta e^{-\beta'(t-t_0)},$$

where $c_* > 1$ and t_* are constants with $|c - c_*| + |t_0 - t_*| \leq C\delta$.

In Section 2 below, we verify (10) and (L) using a Bäcklund transformation as we have mentioned. In Section 3, we extend this analysis to obtain more complete spectral information regarding a linear operator naturally associated with the linearized evolution equation (6). Equation (6) is non-autonomous, but admits a type of Floquet theory, due to the fact that solitary waves on a lattice are really time-periodic solutions up to a shift. If one looks for solutions of (6) having the form of "traveling Floquet modes"

$$\mathbf{u}(t) = \left(e^{\lambda t}W(n - ct)\right)_{n\in\mathbb{Z}}, \tag{11}$$

then one requires that the function $W : \mathbb{R} \to \mathbb{R}^2$ satisfies

$$L_c W = \lambda W, \tag{12}$$

where

$$L_c = c\partial_x + J\begin{pmatrix} e^{-\widetilde{R}_c(x)} & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{R}_c(x) = \widetilde{Q}_c(x+1) - \widetilde{Q}_c(x).$$

As shown in [7], the linear stability condition (L) is equivalent to a pair of conditions that relate to the spectrum of L_c, regarded as a closed operator on $L^2(\mathbb{R}; e^{2ax}dx)^2$ with domain $H^1(\mathbb{R}; e^{2ax}dx)^2$. In Section 3, we identify the entire spectrum of L_c for $0 < a < 2\kappa$, showing that it consists only of essential spectrum (determined in [7] by Fourier analysis) and the eigenvalues $2\pi icn$ for $n \in \mathbb{Z}$ that are naturally associated with tangent vectors to the manifold of traveling-wave states.

Theorem 2. *Let $c > 1$ and suppose $0 < a < 2\kappa(c)$. Then the spectrum of L_c in $L^2(\mathbb{R}; e^{2ax}dx)^2$ consists of essential spectrum, given by*

$$\sigma_{ess}(L_c) = \{c(ik - a) \pm 2\sinh((ik - a)/2) \mid k \in \mathbb{R}\}$$

and contained in the left half of the complex plane, and point spectrum $\sigma_p(L_c) = 2\pi ic\mathbb{Z}$. Each eigenvalue has geometric multiplicity one and algebraic multiplicity two.

As shown in [7, Theorem 4.4], given $c > 1$, $0 < a < a_c = 2\kappa$ and (7), the part of this assertion regarding the location and multiplicity of eigenvalues in the closed right half plane is *equivalent* to the linear stability condition (L). In addition, however, Theorem 2 shows that there are no other eigenvalues anywhere.

We remark that by the theory of [7], the spectral stability and linear stability condition (L) are properties of lattice solitary waves that are robust under perturbations that are small in a suitable sense. As a consequence, for Fermi-Pasta-Ulam lattices with smoothly perturbed Toda potentials, solitary waves sufficiently close to Toda solitons will be asymptotically stable by the theory of [6].

2 Bäcklund transformation and linear stability

First, we establish a decay estimate corresponding to (9) for the system linearized about zero,

$$\frac{d\mathbf{u}}{dt} = JH''(\mathbf{0})\mathbf{u}. \tag{13}$$

Lemma 3. *Let $a > 0$ and $c > 1$ be constants and let $\beta = ca - 2\sinh(a/2)$. Then there exists a positive constant K such that, for any solution $\mathbf{u}(t)$ to* (13) *and any $t \geq s$,*

$$\|e^{a(\mathbf{n}-ct)}\mathbf{u}(t)\|_{l^2} \leq Ke^{-\beta(t-s)}\|e^{a(\mathbf{n}-cs)}\mathbf{u}(s)\|_{l^2}.$$

Note that $\beta > 0$ if and only if $a < 2\kappa$.

Proof. Let $\mathbf{u_a}(t) = (u_{a,n}(t))_{n\in\mathbb{Z}} := (e^{a(n-ct)}u_n(t))_{n\in\mathbb{Z}}$. Then

$$\frac{d\mathbf{u_a}}{dt} = \begin{pmatrix} -ca & e^{\partial - a} - 1 \\ 1 - e^{-\partial + a} & -ca \end{pmatrix}\mathbf{u_a}.$$

Now, we put $u_a(t, x) = u_{a,n}(t)$ for $x \in [n, n+1)$ and thus extend $\mathbf{u_a}(t)$ to a piecewise constant function on $\mathbb{R}$. Obviously,

$$\|u_a(t, \cdot)\|_{L^2(\mathbb{R})} = \|\mathbf{u_a}(t)\|_{l^2}. \tag{14}$$

Taking the Fourier transform of u_a, we have

$$\frac{\partial \hat{u}_a}{\partial t}(t,\xi) = A(\xi)\hat{u}_a, \qquad A(\xi) = \begin{pmatrix} -ca & e^{i\xi - a} - 1 \\ 1 - e^{-i\xi + a} & -ca \end{pmatrix}.$$

Let $\mu_\pm(\xi) = -ca \pm 2\sinh(\frac{1}{2}(i\xi - a))$ and

$$P(\xi) = \begin{pmatrix} e^{(i\xi - a)/2} & 1 \\ 1 & -e^{(-i\xi + a)/2} \end{pmatrix}, \quad D(\xi) = \begin{pmatrix} \mu_+(\xi) & 0 \\ 0 & \mu_-(\xi) \end{pmatrix}.$$

Then $P(\xi)^{-1}AP(\xi) = D(\xi)$. Observe that

$$\mu_\pm(\xi) = -ca \mp 2\cos\frac{\xi}{2}\sinh\frac{a}{2} \pm 2i\sin\frac{\xi}{2}\cosh\frac{a}{2}, \tag{15}$$

hence $\sup_{\xi\in\mathbb{R}} \Re\mu_\pm(\xi) = -\beta$. Since $P(\xi)$ and $P(\xi)^{-1}$ are uniformly bounded with respect to $\xi \in \mathbb{R}$, there exists $K > 0$ such that for any $t \geq s$,

$$\|\hat{u}_a(t,\cdot)\|_{L^2} \leq Ke^{-\beta(t-s)}\|\hat{u}_a(s,\cdot)\|_{L^2}.$$

Using Plancherel's identity and (14), we have

$$\|\mathbf{u_a}(t)\|_{l^2} \leq Ke^{-\beta(t-s)}\|\mathbf{u_a}(s)\|_{l^2}.$$

This completes the proof. □

The Toda lattice admits a Bäcklund transformation determined by the equations

$$\begin{aligned} \dot{Q}_n + e^{-(Q'_n - Q_n)} + e^{-(Q_n - Q'_{n-1})} &= \alpha, \\ \dot{Q}'_n + e^{-(Q'_n - Q_n)} + e^{-(Q_{n+1} - Q'_n)} &= \alpha, \end{aligned} \tag{16}$$

where α is a constant [16]. Presuming (16) holds, if $\mathbf{Q}(t) = (Q_n(t))_{n\in\mathbb{Z}}$ is a solution to (1), then $\mathbf{Q'}(t) = (Q'_n(t))_{n\in\mathbb{Z}}$ becomes a solution to (1) and vice-versa (see [3, 15]). In particular, the Bäcklund transformation connects the zero solution to 1-solitons: if $\mathbf{Q'}(t) = (Q'_n(t))_{n\in\mathbb{Z}} \equiv \mathbf{0}$ and $\alpha = 2\cosh\kappa$, then

$$\dot{Q}_n(t) + e^{Q_n(t)} + e^{-Q_n(t)} = 2\cosh\kappa, \tag{17}$$

$$e^{-Q_{n+1}(t)} + e^{Q_n(t)} = 2\cosh\kappa, \tag{18}$$

whence $\mathbf{Q}(t) = \mathbf{Q}_c(t+\delta)$, where $c = \sinh\kappa/\kappa$ and $\delta \in \mathbb{R}$ is an arbitrary constant independent of t. At this point it is convenient to establish (10).

Lemma 4. *Let $\kappa > 0$ and $c = \sinh\kappa/\kappa$. Then $H(\mathbf{U}_c) = \sinh 2\kappa - 2\kappa$.*

Proof. Since $H(\mathbf{U}_c(t))$ does not depend on t,

$$H(\mathbf{U}_c) = \quad c\int_0^{1/c} H(\mathbf{U}_c(t))\,dt = \int_{\mathbb{R}} \left(\tfrac{c^2}{2}(\partial_x\widetilde{Q}_c(x))^2 + V(\widetilde{R}_c(x))\right)dx.$$

By (17) and (18) we have

$$c\partial_x\widetilde{Q}_c(x) = e^{\widetilde{Q}_c(x)} + e^{-\widetilde{Q}_c(x)} - 2\cosh\kappa, \qquad ce^{\widetilde{Q}_c(x)}\partial_x\widetilde{Q}_c(x) = 1 - e^{-\widetilde{R}_c(x)}.$$

Using the above, we compute

$$\frac{c^2}{2}\int_{\mathbb{R}}(\partial_x\widetilde{Q}_c(x))^2 \quad = c\int_{\mathbb{R}}\left(\cosh\left(\widetilde{Q}_c(x)\right)-\cosh\kappa\right)\partial_x\widetilde{Q}_c(x)\,dx$$
$$= \sinh 2\kappa - \tfrac{2\sinh^2\kappa}{\kappa},$$

and

$$\int_{\mathbb{R}}\left(e^{-\widetilde{R}_c(x)}-1\right)dx = \frac{2\sinh^2\kappa}{\kappa}.$$

By Fubini's theorem and the fundamental theorem of calculus,

$$\int_{\mathbb{R}}\widetilde{R}_c(x)\,dx = \int_{\mathbb{R}}\left(\int_x^{x+1}\partial_y\widetilde{Q}_c(y)\,dy\right) = \left[\widetilde{Q}_c(x)\right]_{x=-\infty}^{x=\infty} = -2\kappa.$$

Combining these results, we obtain $H(\mathbf{U}_c(t)) = \sinh 2\kappa - 2\kappa$. □

Now, let us linearize (16) around $\mathbf{Q}(t) = \mathbf{Q}_c(t)$ and $\mathbf{Q}'(t) = \mathbf{0}$. This yields

$$\mathbf{p}(t) + e^{\mathbf{Q}_c(t)}(\mathbf{q}(t)-\mathbf{q}'(t)) - e^{-\mathbf{Q}_c(t)}(\mathbf{q}(t)-e^{-\partial}\mathbf{q}'(t)) = 0, \tag{19}$$
$$\mathbf{p}'(t) + e^{\mathbf{Q}_c(t)}(\mathbf{q}(t)-\mathbf{q}'(t)) - e^{\partial}e^{-\mathbf{Q}_c(t)}(\mathbf{q}(t)-e^{-\partial}\mathbf{q}'(t)) = 0. \tag{20}$$

Our aim is to show that this linearized Bäcklund transformation defines a uniformly bounded mapping with respect to t which pulls back every solution of (6) satisfying (8) to a solution of (13). To begin with, we note that linearized Toda equations are well-posed in $l^2_a \times l^2_a$.

Lemma 5. *Let $a \in \mathbb{R}$ and $(\mathbf{q_0}, \mathbf{p_0}) \in l^2_a \times l^2_a$. Then the initial value problems*

$$\begin{cases} \ddot{\mathbf{q}} = (1-e^{-\partial})\{e^{-\mathbf{R}_c}(e^{\partial}-1)\mathbf{q}\}, \\ \mathbf{q}(0) = \mathbf{q_0}, \quad \mathbf{p}(0) = \mathbf{p_0}, \end{cases} \tag{21}$$

and

$$\begin{cases} \ddot{\mathbf{q}}' = (e^{\partial}-2+e^{-\partial})\mathbf{q}' \\ \mathbf{q}'(0) = \mathbf{q}'_\mathbf{0}, \quad \mathbf{p}'(0) = \mathbf{p}'_\mathbf{0}, \end{cases} \tag{22}$$

have a unique solution in the class $C^2(\mathbb{R}; l^2_a \times l^2_a)$, respectively.

Proof. The shift operators e^{∂} and $e^{-\partial}$ and the multiplication operator $e^{-\mathbf{R}_c}$ are bounded on l^2_a and smooth in time. Existence and uniqueness for these linear equations is standard. □

Next we show that the flows generated by (21) and (22) leave the linearized Bäcklund transformation invariant.

Lemma 6. *Let $t_0 \in \mathbb{R}$ and let $\mathbf{q}(t)$ and $\mathbf{q}'(t)$ be solutions to* (21) *and* (22)*, respectively. Let $\mathbf{p} = \dot{\mathbf{q}}$ and $\mathbf{p}' = \dot{\mathbf{q}}'$. Suppose that the linearized Bäcklund transformation* (19) *and* (20) *holds at $t = t_0$. Then* (19) *and* (20) *hold for every $t \in \mathbb{R}$.*

Proof. Let

$$\mathbf{F_1}(t) = \quad \mathbf{p}(t) + e^{\mathbf{Q}_c(t)}(\mathbf{q}(t)-\mathbf{q}'(t)) - e^{-\mathbf{Q}_c(t)}(\mathbf{q}(t)-e^{-\partial}\mathbf{q}'(t)),$$
$$\mathbf{F_2}(t) = \quad \mathbf{p}'(t) + e^{\mathbf{Q}_c(t)}(\mathbf{q}(t)-\mathbf{q}'(t)) - e^{\partial}e^{-\mathbf{Q}_c(t)}(\mathbf{q}(t)-e^{-\partial}\mathbf{q}'(t)).$$

By (17) and (18), we have

$$\dot{\mathbf{Q}}_c = e^{\partial} e^{-\mathbf{Q}_c} - e^{-\mathbf{Q}_c} = e^{-\partial} e^{\mathbf{Q}_c} - e^{\mathbf{Q}_c}. \tag{23}$$

Differentiating $\mathbf{F_1}$ with respect to t and substituting (21) and (23), we find

$$\frac{d\mathbf{F_1}}{dt} = e^{\mathbf{Q}_c}(\mathbf{F_1} - \mathbf{F_2}) - e^{-\mathbf{Q}_c}(\mathbf{F_1} - e^{-\partial}\mathbf{F_2}). \tag{24}$$

Similarly, we find

$$\frac{d\mathbf{F_2}}{dt} = e^{\mathbf{Q}_c}(\mathbf{F_1} - \mathbf{F_2}) - e^{\partial} e^{-\mathbf{Q}_c}(\mathbf{F_1} - e^{-\partial}\mathbf{F_2}). \tag{25}$$

Applying Gronwall's inequality to (24) and (25), we have that for some $C > 0$,

$$\|\mathbf{F_1}(t)\|_{l^2_a} + \|\mathbf{F_2}(t)\|_{l^2_a} \lesssim e^{C|t-t_0|}(\|\mathbf{F_1}(t_0)\|_{l^2_a} + \|\mathbf{F_2}(t_0)\|_{l^2_a}) = 0.$$

This proves Lemma 6. □

Let $\Lambda := \operatorname{diag}(e^{\partial} - 1, 1)$. Given $t \in \mathbb{R}$, we let

$$X_t = \{(\mathbf{q}, \mathbf{p}) \in l^2_a \times l^2_a : \mathbf{u} = \Lambda(\mathbf{q}, \mathbf{p}) \text{ satisfies (8)}\}. \tag{26}$$

This is a subspace corresponding to states $\mathbf{u}$ satisfying the secular term condition (8). Now we proceed to show that for each fixed t, the linearized Bäcklund transformation defines an isomorphism between X_t and $l^2_a \times l^2_a$, provided $0 < a < 2\kappa$.

Proposition 7. *Suppose $0 < a < 2\kappa$. Let $t \in \mathbb{R}$. For every $(\mathbf{q}, \mathbf{p}) \in X_t$, there exists a unique $(\mathbf{q}', \mathbf{p}') \in l^2_a \times l^2_a$ satisfying*

$$\mathbf{p} + e^{\mathbf{Q}_c(t)}(\mathbf{q} - \mathbf{q}') - e^{-\mathbf{Q}_c(t)}(\mathbf{q} - e^{-\partial}\mathbf{q}') = 0, \tag{27}$$

$$\mathbf{p}' + e^{\mathbf{Q}_c(t)}(\mathbf{q} - \mathbf{q}') - e^{\partial} e^{-\mathbf{Q}_c(t)}(\mathbf{q} - e^{-\partial}\mathbf{q}') = 0. \tag{28}$$

Furthermore, the map $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}', \mathbf{p}')$ defines an isomorphism $\Phi_c(t) \colon X_t \to l^2_a \times l^2_a$.

An easy consequence of the fact that $\mathbf{Q}_c(t + c^{-1}) = e^{-\partial}\mathbf{Q}_c(t)$ is the following.

Corollary 8. *It holds that $\Phi_c(t + c^{-1}) = e^{-\partial}\Phi_c(t)e^{\partial}$ for every $t \in \mathbb{R}$.*

To prove Proposition 7, we need the following.

Lemma 9. *Let $-2\kappa < a < 2\kappa$ and $t \in \mathbb{R}$. Let $\mathbf{C}(t) = e^{\mathbf{Q}_c(t)} - e^{-\mathbf{Q}_c(t)}e^{-\partial}$ be an operator on l^2_a, with adjoint $\mathbf{C}^*(t) = e^{\mathbf{Q}_c(t)} - e^{\partial}e^{-\mathbf{Q}_c(t)}$ acting on $l^2_{-a} = (l^2_a)^*$. Then $\mathbf{C}(t)$ is Fredholm,* $\ker \mathbf{C}(t) = \{\mathbf{0}\}$, $\ker \mathbf{C}^*(t) = \operatorname{span}\{\mathbf{P}_c(t)\}$ *and* $\operatorname{Range} \mathbf{C}(t)^* = l^2_{-a}$.

Proof of Lemma 9. Suppose that $\mathbf{r} = (r_n)_{n\in\mathbb{Z}} \in l^2_a$ satisfies $\mathbf{C}(t)\mathbf{r} = 0$. Then it holds

$$r_n = e^{-2\widetilde{Q}_c(n-ct)} r_{n-1} \quad \text{for every } n \in \mathbb{Z}.$$

By (4), we have $e^{-2\widetilde{Q}_c(n-ct)} \sim e^{2\kappa} > 1$ as $n \to \infty$. Since $\mathbf{r} \in l^2_a$, it follows that $\mathbf{r} = \mathbf{0}$. This proves $\ker \mathbf{C}(t) = \{\mathbf{0}\}$.

Now suppose $\mathbf{C}(t)^*\mathbf{r} = 0$. Then

$$r_{n+1} = e^{\widetilde{Q}_c(n+1-ct)+\widetilde{Q}_c(n-ct)} r_n \quad \text{for every } n \in \mathbb{Z}.$$

Because $e^{\widetilde{Q}_c(n+1-ct)+\widetilde{Q}_c(n-ct)} \sim e^{\mp 2\kappa}$ as $n \to \pm\infty$, and $e^{-a+2\kappa} > 1 > e^{-a-2\kappa}$, we see that $\ker \mathbf{C}(t)^*$ is a 1-dimensional subspace of l^2_{-a}. Differentiating (18) with respect to t, we have

$$-e^{\partial} e^{-\mathbf{Q}_c(t)} \mathbf{P}_c(t) + e^{\mathbf{Q}_c(t)} \mathbf{P}_c(t) = 0.$$

Thus we have $\ker \mathbf{C}(t)^* = \operatorname{span}\{\mathbf{P}_c(t)\}$.

Finally, given any $\mathbf{f} \in l^2_{-a}$ it is easy to determine $\mathbf{r} = (r_n)_{n\in\mathbb{Z}}$ so that $\mathbf{C}(t)^*\mathbf{r} = \mathbf{f}$, by fixing $r_0 = 0$ for example. Then, for any $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$|r_{n+1}| \le e^{\epsilon}(e^{-2\kappa}|r_n| + e^{-\kappa}|f_n|) \quad \text{for } n \ge n_0,$$
$$|r_n| \le e^{\epsilon}(e^{-2\kappa}|r_{n+1}| + e^{-\kappa}|f_n|) \quad \text{for } n \le -n_0.$$

It is then not difficult to show that $\mathbf{r} \in l^2_{-a}$. Now $\mathbf{C}(t)^*$ is Fredholm, so $\mathbf{C}(t)$ is Fredholm. □

Proof of Proposition 7. Let $(\mathbf{q}, \mathbf{p}) \in X_t$. Equations (27) and (28) can be rewritten as

$$\begin{cases} \mathbf{C}(t)\mathbf{q}' = \mathbf{p} + (e^{\mathbf{Q}_c(t)} - e^{-\mathbf{Q}_c(t)})\mathbf{q}, \\ \mathbf{p}' = (e^{\mathbf{Q}_c(t)} - e^{\partial} e^{-\mathbf{Q}_c(t)} e^{-\partial})\mathbf{q}' - \hat{\mathbf{C}}(t)\mathbf{q}, \end{cases} \tag{29}$$

where $\hat{\mathbf{C}}(t) = e^{\mathbf{Q}_c(t)} - e^{\partial} e^{-\mathbf{Q}_c(t)}$ (formally $\hat{\mathbf{C}}(t) = \mathbf{C}(t)^*$). Since $\mathbf{C}(t)$ is Fredholm and $\ker \mathbf{C}(t)^* = \operatorname{span}\{\mathbf{P}_c(t)\}$, (29) has a unique solution $(\mathbf{q}', \mathbf{p}') \in l^2_a \times l^2_a$ if and only if

$$\mathbf{p} + (e^{\mathbf{Q}_c(t)} - e^{-\mathbf{Q}_c(t)})\mathbf{q} \;\perp\; \ker \mathbf{C}(t)^*. \tag{30}$$

Differentiating (17) at $t = 0$, we have

$$\dot{\mathbf{P}}_c(t) + (e^{\mathbf{Q}_c(t)} - e^{-\mathbf{Q}_c(t)})\mathbf{P}_c(t) = 0.$$

Thus we have

$$\begin{aligned} \left\langle \mathbf{p} + (e^{\mathbf{Q}_c(t)} - e^{-\mathbf{Q}_c(t)})\mathbf{q}, \mathbf{P}_c(t) \right\rangle &= \langle \mathbf{p}, \mathbf{P}_c(t)\rangle + \left\langle \mathbf{q}, (e^{\mathbf{Q}_c(t)} - e^{-\mathbf{Q}_c(t)})\mathbf{P}_c(t) \right\rangle \\ &= \langle \mathbf{p}, \mathbf{P}_c(t)\rangle - \left\langle \mathbf{q}, \dot{\mathbf{P}}_c(t) \right\rangle. \end{aligned} \tag{31}$$

On the other hand, with $\mathbf{u} = (\mathbf{r}, \mathbf{p}) = ((e^{\partial} - 1)\mathbf{q}, \mathbf{p})$, we find

$$\begin{aligned} \langle \mathbf{u}, J^{-1}\dot{\mathbf{U}}_c(t)\rangle &= \langle H'(\mathbf{U}_c(t)), \mathbf{u}\rangle \\ &= \langle 1 - e^{-\mathbf{R}_c(t)}, \mathbf{r}\rangle + \langle \mathbf{P}_c(t), \mathbf{p}\rangle \\ &= \langle (1 - e^{-\partial})e^{-\mathbf{R}_c(t)}, \mathbf{q}\rangle + \langle \mathbf{P}_c(t), \mathbf{p}\rangle \\ &= -\langle \dot{\mathbf{P}}_c(t), \mathbf{q}\rangle + \langle \mathbf{P}_c(t), \mathbf{p}\rangle, \end{aligned} \tag{32}$$

since $\mathbf{U}_c$ is a solution to (3). Combining the above with $\mathbf{u} \in \Lambda X_t$, we have (30). Thus we see that (29) is solvable and the map $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}', \mathbf{p}')$ defines a bounded linear mapping $\Phi_c(t)\colon X_t \to l^2_a \times l^2_a$.

Because $\ker \mathbf{C}(t) = \{0\}$, we have $\Phi_c(t)(\mathbf{q}, \mathbf{p}) = (\mathbf{0}, \mathbf{0})$ if and only if

$$\mathbf{p} + (e^{\mathbf{Q}_c(t)} - e^{-\mathbf{Q}_c(t)})\mathbf{q} = (e^{\mathbf{Q}_c(t)} - e^{\partial} e^{-\mathbf{Q}_c(t)})\mathbf{q} = \mathbf{0},$$

which implies $\mathbf{u} = \Lambda(\mathbf{q}, \mathbf{p}) = \alpha \dot{\mathbf{U}}_c$ for some $\alpha \in \mathbb{C}$. By Lemma 4, we have $\langle J^{-1}\partial_c \mathbf{U}_c, \dot{\mathbf{U}}_c\rangle = -\frac{d}{dc}H(\mathbf{U}_c) \neq 0$ and hence $\alpha = 0$ follows from the fact that $\mathbf{u} \in \Lambda X_t$. This proves that $\Phi_c(t)$ is injective.

To see that $\Phi_c(t)$ is surjective, let $(\mathbf{q}', \mathbf{p}') \in l^2_a \times l^2_a$. Applying Lemma 9 with a replaced by $-a$, we see that the range of $\hat{\mathbf{C}}(t)$ is l^2_a, so a solution $(\mathbf{q}, \mathbf{p})$ to (29) exists in $l^2_a \times l^2_a$. Automatically $\mathbf{u} = \Lambda(\mathbf{q}, \mathbf{p}) \perp J^{-1}\dot{\mathbf{U}}_c(t)$ follows due to (31), (32) and since the range of $\mathbf{C}(t)$ is $\{\mathbf{P}_c(t)\}^\perp$. Adjusting $(\mathbf{q}, \mathbf{p})$ by a multiple of $\Lambda^{-1}\dot{\mathbf{U}}_c(t)$ if necessary, we get $(\mathbf{q}, \mathbf{p}) \in X_t$. □

Corollary 10. *The map $\Phi_c(t)$ and its inverse are uniformly bounded for $t \in \mathbb{R}$:*

$$\sup_{t\in\mathbb{R}} \left(\|\Phi_c(t)\|_{B(X_t, l^2_a \times l^2_a)} + \|\Phi_c(t)^{-1}\|_{B(l^2_a \times l^2_a, X_t)} \right) < \infty.$$

Proof. Let $U(t, s)$ denote the evolution operator associated with the evolution equation (6), $U_0(t)$ be the C_0-group generated by (13), and let

$$\hat{U}(t, s) = \Lambda^{-1}U(t, s)\Lambda, \qquad \hat{U}_0(t) = \Lambda^{-1}U_0(t)\Lambda. \tag{33}$$

These are the evolution operators associated with the equations in (21) and (22) respectively. Since $JH''(\mathbf{0})$ and $JH''(\mathbf{u}(t))$ are bounded and $JH''(\mathbf{u}(t))$ is continuous in t, we see that $U(t, s)$ and $U_0(t)$ are locally uniformly continuous in t and s on $l^2_a \times l^2_a$ (see [12]).

Because $\hat{U}(t, s)X_s = X_t$ and by Lemma 6, we have that for all $t, s \in \mathbb{R}$,

$$\Phi_c(t)\hat{U}(t, s)|_{X_s} = \hat{U}_0(t - s)\Phi_c(s). \tag{34}$$

Using $s = 0$ in particular, we get

$$\Phi_c(t) = \hat{U}_0(t)\Phi_c(0)\hat{U}(0, t)|_{X_t}.$$

¿From this it follows $\|\Phi_c(t)\|_{B(X_t, l^2_a \times l^2_a)} + \|\Phi_c(t)^{-1}\|_{B(l^2_a \times l^2_a, X_t)}$ is uniformly bounded for t in bounded sets.

Let $\tau \in [0, c^{-1}]$ and $k \in \mathbb{Z}$ and let $t = \tau + kc^{-1}$ and $\mathbf{v} = (\mathbf{q}, \mathbf{p}) \in X_t$. By Corollary 8, $\Phi_c(t) = e^{-k\partial}\Phi_c(\tau)e^{k\partial}$, so $e^{k\partial}\mathbf{v} \in X_\tau$. Note that since e^∂ is an isometry of l^2, for any $\mathbf{x} \in l^2_a$ we have

$$\|e^{k\partial}\mathbf{x}\|_{l^2_a} = \|(e^{an}x_{n+k})_{n\in\mathbb{Z}}\| = e^{-ak}\|\mathbf{x}\|_{l^2_a}.$$

Now we calculate using the local uniform bound established above that

$$\begin{aligned} \|\Phi_c(t)\mathbf{v}\|_{l^2_a} &= \|e^{-k\partial}\Phi_c(\tau)e^{k\partial}\mathbf{v}\|_{l^2_a} = e^{ak}\|\Phi_c(\tau)e^{k\partial}\mathbf{v}\|_{l^2_a} \\ &\leq Ke^{ak}\|e^{k\partial}\mathbf{v}\|_{l^2_a} = K\|\mathbf{v}\|_{l^2_a}. \end{aligned}$$

Similarly we get a uniform bound for $\Phi_c(t)^{-1}$. □

Now, we are in position to prove (L).

Theorem 11. *Let $c > 1$, $a \in (0, 2\kappa(c))$ and $\beta = ca - 2\sinh(a/2)$. Let $U(t, s)$ be the evolution operator associated with* (6). *Then (L) holds: there exists a constant $K > 0$ such that for any $\mathbf{u}_0 \in X_s$,*

$$\|e^{a(\mathbf{n}-ct)}U(t, s)\mathbf{u_0}\|_{l^2} \leq Ke^{-\beta(t-s)}\|e^{a(\mathbf{n}-cs)}\mathbf{u_0}\|_{l^2} \quad \textit{for every } t \geq s \in \mathbb{R}.$$

Proof. This follows directly using the formulas (33) and (34), Corollary 10 and the boundedness of Λ and Λ^{-1}, and the bound on $U_0(t - s)$ coming from Lemma 3. □

3 Characterization of spectrum

The key to determining the point spectrum $\sigma_p(L_c)$ is to relate eigenfunctions to traveling Floquet modes and utilize the linearization of the Bäcklund transformation (16) around $\mathbf{Q}(t) = \mathbf{Q}_c(t)$ and $\mathbf{Q}'(t) = 0$ to make a correspondence between the linearization of equation (1) around $\mathbf{Q}(t) = \mathbf{Q}_c(t)$

$$\ddot{\mathbf{q}} = (1 - e^{-\partial})\{e^{-\mathbf{R}_c}(e^{\partial} - 1)\mathbf{q}\}, \tag{35}$$

and the linearization of equation (1) around $\mathbf{Q}'(t) = 0$

$$\ddot{\mathbf{q}}' = (e^{\partial} - 2 + e^{-\partial})\mathbf{q}'. \tag{36}$$

In view of Lemma 6, it suffices to generate this correspondence between $(\mathbf{q_0}, \mathbf{p_0})$ and $(\mathbf{q_0'}, \mathbf{p_0'})$ at $t = 0$.

Lemma 12. *Let $0 < a < 2\kappa$, $\lambda \in \mathbb{C}$ and let $w \in H_a^2(\mathbb{R})$. Suppose that $\mathbf{q}(t) = \big(e^{\lambda t}\tilde{w}(n - ct)\big)_{n\in\mathbb{Z}}$ is a solution to (35) that satisfies (8) and that $\mathbf{q}(t)$ and $\mathbf{q}'(t)$ satisfy (19) and (20). Then there exists $\tilde{w}' \in H_a^2(\mathbb{R})$ such that $\mathbf{q}'(t) = \big(e^{\lambda t}\tilde{w}'(n - ct)\big)_{n\in\mathbb{Z}}$.*

Proof. Proposition 7 implies that there exists a unique $(\mathbf{q}'(t), \mathbf{p}'(t))$ satisfying (19) and (20) for every $(\mathbf{q}(t), \mathbf{p}(t))$. By the assumption and the definition of $\mathbf{Q}_c(t)$, we have

$$\mathbf{Q}_c(c^{-1}) = e^{-\partial}\mathbf{Q}_c(0), \quad \mathbf{q}(c^{-1}) = e^{\lambda/c}e^{-\partial}\mathbf{q}(0), \quad \dot{\mathbf{q}}(c^{-1}) = e^{\lambda/c}e^{-\partial}\dot{\mathbf{q}}(0).$$

Combining the above with (19) and (20), we obtain

$$\begin{aligned}\mathbf{p}(0) + e^{\mathbf{Q}_c(0)}(\mathbf{q}(0) - e^{-\lambda/c}e^{\partial}\mathbf{q}'(c^{-1})) - e^{-\mathbf{Q}_c(0)}(\mathbf{q}(0) - e^{-\lambda/c}\mathbf{q}'(c^{-1})) = 0,\\ \mathbf{p}'(0) + e^{\mathbf{Q}_c(0)}(\mathbf{q}(0) - e^{-\lambda/c}e^{\partial}\mathbf{q}'(c^{-1})) - e^{\partial}e^{-\mathbf{Q}_c(0)}(\mathbf{q}(0) - e^{-\lambda/c}\mathbf{q}'(c^{-1})) = 0.\end{aligned}$$

Thus by Proposition 7, we have

$$\mathbf{q}'(c^{-1}) = e^{\lambda/c}e^{-\partial}\mathbf{q}'(0), \quad \dot{\mathbf{q}}'(c^{-1}) = e^{\lambda/c}e^{-\partial}\dot{\mathbf{q}}'(0).$$

Since (36) is autonomous, it follows from Lemma 5 that

$$\mathbf{q}'(t + c^{-1}) = e^{\lambda/c}e^{-\partial}\mathbf{q}'(t)$$

for every $t \in \mathbb{R}$ and that $\mathbf{q}'(t) = \big(e^{\lambda t}\tilde{w}'(n - ct)\big)_{n\in\mathbb{Z}}$ for some $\tilde{w}' \in H_a^2(\mathbb{R})$. This completes the proof of Lemma 12. □

Now, we are in position to prove Theorem 2.

Proof of Theorem 2. The characterization of the essential spectrum follows from [7, Lemma 4.2]. Suppose that λ is an eigenvalue of L_c and $\widetilde{W}$ is an eigenfunction belonging to λ. Suppose (a) $\lambda \notin 2\pi ic\mathbb{Z}$ or (b) $\lambda = 2\pi imc$ ($m \in \mathbb{Z}$) and $\widetilde{W}$ is linearly independent of the eigenfunction $e^{2\pi imx}\partial_x\widetilde{U}_c$. In the case (b), we can choose $\widetilde{W}(x)$ so that

$$\int_{\mathbb{R}} \widetilde{W}(x) \cdot e^{-2\pi imx}J^{-1}\partial_x\widetilde{U}_c(x)dx = \int_{\mathbb{R}} \widetilde{W}(x) \cdot e^{-2\pi imx}J^{-1}\partial_c\widetilde{U}_c(x)dx = 0.$$

Then $\mathbf{u}(t) = (e^{\lambda t}\widetilde{W}(n-ct)))_{n\in\mathbb{Z}}$ is a solution to (6) that satisfies (8). Indeed, noting that $\widetilde{W} \perp \ker_g(L_c^* + 2\pi inc)$ for every $n \in \mathbb{Z}$ ($n \neq m$ in case (b)), we have that for every $n \in \mathbb{Z}$,

$$\begin{aligned} &\textstyle\int_{\mathbb{R}} \widetilde{W}(x) \cdot e^{-2\pi inx} J^{-1}\partial_x \widetilde{U}_c(x)dx = 0, \\ &\textstyle\int_{\mathbb{R}} \widetilde{W}(x) \cdot e^{-2\pi inx} J^{-1}\partial_c \widetilde{U}_c(x)dx = 0. \end{aligned}$$

This yields (8), by (2.17) of [7].

Let $(\mathbf{q}(t), \mathbf{p}(\mathbf{t})) = \Lambda^{-1}\mathbf{u}(t)$. Then $\mathbf{q}(t)$ is a solution to (21) such that $(\mathbf{q_0}, \mathbf{p_0}) = \Lambda^{-1}\mathbf{u}(0) \in X_0 \subset l_a^2 \times l_a^2$, and $\mathbf{q}(t) = \big(e^{\lambda t} w(n-ct)\big)_{n\in\mathbb{Z}}$ where $w \in H_a^2(\mathbb{R})$.

By Proposition 7 and Lemma 12, there exists a solution $\mathbf{q}'(t)$ to (22) and a $\tilde{w}' \in H_a^2(\mathbb{R})$ such that $(\mathbf{q_0'}, \mathbf{p_0'}) = \Phi_c(0)(\mathbf{q_0}, \mathbf{p_0})$ and $\mathbf{q}'(t) = \big(e^{\lambda t}\tilde{w}'(n-ct)\big)_{n\in\mathbb{Z}}$. Put $\varphi(x) = e^{ax}\tilde{w}'(x)$. Then $\varphi \in L^2(\mathbb{R})$ and

$$(c(\partial - a) - \lambda)^2 \varphi = (e^{\partial - a} - 2 + e^{-\partial + a})\varphi.$$

Hence it follows that $K(\xi,\lambda)\hat{\varphi}(\xi) = 0$, where $\hat{\varphi}$ denotes Fourier transform and

$$\begin{aligned} K(\xi,\lambda) &:= (c(i\xi - a) - \lambda)^2 - 4\sinh^2\left(\tfrac{i\xi - a}{2}\right) \\ &= (\lambda - k_+(\xi))(\lambda - k_-(\xi)), \end{aligned}$$

with

$$k_\pm(\xi) = c(i\xi - a) \pm 2\sinh\left(\frac{i\xi - a}{2}\right) = ic\xi + \mu_\pm(\xi).$$

For each $\lambda \in \mathbb{C}$ we have $K(\xi,\lambda) \neq 0$ for a.e. $\xi \in \mathbb{R}$. In fact, we have $k_+(\xi) = \lambda$ or $k_-(\xi) = \lambda$ if $K(\xi,\lambda) = 0$. By (15),

$$\Im k_\pm(\xi) = c\xi \pm 2\cosh\frac{a}{2}\sin\frac{\xi}{2},$$

and $\Im k_\pm(\xi) = \Im\lambda$ has at most a finite number of solutions. This is a contradiction. This proves $\sigma_p(L_c) = 2\pi ic\mathbb{Z}$ and that each eigenvalue is geometrically simple.

Since $L_c(e^{2\pi inx}W(x)) = e^{2\pi inx}(L_c + 2\pi inc)W(x)$ for any $W : \mathbb{R} \to \mathbb{C}^2$, every generalized eigenspace belonging to $\lambda \in 2\pi ic\mathbb{Z}$ has the same structure. Now we will show that the algebraic multiplicity of $\lambda = 0$ is two. In fact, we have $\operatorname{Range} L_c \subset \ker(L_c^*)^\perp$. Because $J^{-1}\partial_x\widetilde{U}_c \in \ker(L_c^*)$ and

$$\int_{\mathbb{R}} \partial_c\widetilde{U}_c(x) \cdot J^{-1}\partial_x\widetilde{U}_c(x)dx = -\frac{1}{c}\frac{d}{dc}H(\mathbf{U}_c) \neq 0,$$

it follows from the Fredholm alternative that $L_cW = \partial_c\widetilde{U}_c$ has no solution in $L^2(\mathbb{R}; e^{2ax}dx)$ and that the algebraic multiplicity of $\lambda = 0$ is two. This completes the proof. □

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References

[1] Alexander J C, Pego R L and Sachs R L 1997 On the transverse instability of solitary waves in the Kadomtsev-Petviashvili equation, *Phys. Lett. A* **226** 187–192

[2] Bona J L and Sachs R L 1988 Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation *Comm. Math. Phys.* **118** 15–29

[3] Chen H H and Liu C S 1975 Backlund transformation solutions of the Toda lattice equation *J. Math. Phys.* **16** 1428–1430

[4] Flaschka H 1974 On the Toda lattice. II. Inverse-scattering solution *Progr. Theoret. Phys.* **51** 703–716

[5] Friesecke G and Pego R L 1999 Solitary waves on FPU lattices I, Qualitative properties, renormalization and continuum limit *Nonlinearity* **12** 1601–1627

[6] Friesecke G and Pego R L 2002 Solitary waves on FPU lattices. II, Linear implies nonlinear stability *Nonlinearity* **15** 1343–1359

[7] Friesecke G and Pego R L 2004 Solitary waves on Fermi-Pasta-Ulam lattices III, Howland-type Floquet theory *Nonlinearity* **17** 207–227

[8] Friesecke G and Pego R L 2004 Solitary waves on Fermi-Pasta-Ulam lattices IV, Proof of stability at low energy *Nonlinearity* **17** 229–251

[9] Grillakis M, Shatah J and Strauss W A 1987 Stability Theory of solitary waves in the presence of symmetry I *J. Diff. Eq.* **74** 160–197

[10] Henry D 1981 *Daniel Geometric Theory of Semilinear Parabolic Equations (Lecture Notes in Mathematics* vol 840) (Berlin-New York: Springer-Verlag)

[11] Molinet L, Saut J C and Tzvetkov N 2007 Global well-posedness for the KP-I equation on the background of a non-localized solution *Comm. Math. Phys.* **272** 775–810

[12] Pazy A 1983 *Semigroups of Linear Operators and Applications to Partial Differential Equations (Applied Mathematical Sciences* vol 44) (New York: Springer-Verlag)

[13] Fibich G, Sivan Y and Weinstein M I 2006 Waves in Nonlinear Lattices: Ultrashort Optical Pulses and Bose-Einstein Condensates, *Phys. Rev. Lett.* **97** 193902

[14] Toda M 1989 *Nonlinear Waves and Solitons (Mathematics and its Applications (Japanese Series*) vol 5) (Tokyo: Kluwer Academic Publishers Group)

[15] Wadati M and Toda M 1975 Backlund transformation for the exponential lattice *J. Phys. Soc. Japan* **39** 1196–1203

[16] Wadati M and Toda M 1975 A canonical transformation for the exponential lattice *J. Phys. Soc. Japan* **39** 1204–1211

[17] Zakharov V E 1975 Instability and nonlinear oscillations of solitons *JETP Lett.* **22** 172–173