

Asymptotic Stability of Traveling Waves for Scalar Viscous Conservation Laws with Non-convex Nonlinearity

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Abstract: The asymptotic stability of traveling wave solutions with shock profile is considered for scalar viscous conservation laws $u_t + f(u)_x = \mu u_{xx}$ with the initial data u_0 which tend to the constant states u_{\pm} as $x \rightarrow \pm \infty$. Stability theorems are obtained in the absence of the convexity of f and in the allowance of s (shock speed) $= f'(u_{\pm})$. Moreover, the rate of asymptotics in time is investigated. For the case $f'(u_+) < s < f'(u_-)$, if the integral of the initial disturbance over $(-\infty, x)$ is small and decays at the algebraic rate as $|x| \rightarrow \infty$, then the solution approaches the traveling wave at the corresponding rate as $t \rightarrow \infty$. This rate seems to be almost optimal compared with the rate in the case $f = u^2/2$ for which an explicit form of the solution exists. The rate is also obtained in the case $f'(u_{\pm}) = s$ under some additional conditions. Proofs are given by applying an elementary weighted energy method to the integrated equation of the original one. The selection of the weight plays a crucial role in those procedures.

1. Introduction

We consider the Cauchy problem for scalar viscous conservation laws:

$$u_t + f(u)_x = \mu u_{xx}, \quad x \in \mathbf{R}, t > 0, \tag{1.1}$$

$$u(0, x) = u_0(x), \quad x \in \mathbf{R}, \tag{1.2}$$

where $f \in C^2$ under consideration, μ is a positive constant and the initial data $u_0(x)$ is asymptotically constant as $x \rightarrow \pm \infty$:

$$u_0(x) \rightarrow u_{\pm} \quad \text{as } x \rightarrow \pm \infty. \tag{1.3}$$

Let Eq. (1.1) admit traveling wave solutions with shock profile such that

$$u = U(x - st) \equiv U(\xi), \quad U(\xi) \rightarrow u_{\pm} \quad \text{as } \xi \rightarrow \pm \infty, \tag{1.4}$$

where the constants u_{\pm} and s (shock speed) satisfy the Rankine–Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0, \tag{1.5}$$

and the generalized shock condition

$$h(u) \equiv -s(u - u_{\pm}) + f(u) - f(u_{\pm}) \begin{cases} < 0 & \text{if } u_+ < u < u_- \\ > 0 & \text{if } u_- < u < u_+ . \end{cases} \tag{1.6}$$

It is noted that the condition (1.6) implies

$$f'(u_+) \leq s \leq f'(u_-) \tag{1.7}$$

and that, especially when $f'' > 0$, the condition (1.6) is equivalent to

$$f'(u_+) < s < f'(u_-) ,$$

which is well-known as Lax’s shock condition (Lax [5]).

When $f'' > 0$ and so $f'(u_+) < s < f'(u_-)$, the stability theorems of the traveling wave solutions have been investigated by Il’in and Oleinik [1], Kawashima and Matsumura [3], Nishihara [7] and etc. Recently, in the absence of $f'' > 0$ the stability theorems have been also investigated by Kawashima and Matsumura [4], Jones, Gardner and Kapitula [2], Mei [6]. See also Osher and Ralston [8], Weinberger [9].

Especially, for $f \in C^2$ and $f'(u_+) < s < f'(u_-)$ Jones et al. [2] have obtained the stability of traveling wave solutions and the speed of asymptotics, which are based on the spectral analysis. But its rate of asymptotic speed is less sufficient than that of Kawashima and Matsumura [3] for $f'' > 0$. When f has only one inflection point, the stability theorem has been obtained by Kawashima and Matsumura [4] including the system case and the rate of speed by Mei [6], both of which are due to the energy method. Mei [6] also has treated the case of $s = f'(u_{\pm})$ for the first time.

Our purpose is to show the stability and the rate of asymptotic speed for any $f \in C^2$ which satisfy the generalized shock condition (1.6) only. Even when $f'(u_+) < s < f'(u_-)$, our rate is sharper than that of Kawashima and Matsumura [3], and seems to be almost optimal from the viewpoint of the optimality in Nishihara [7] for $f = u^2/2$. Our results are given by a rather elementary weighted energy method, but the weight function will be suitably taken.

Our plan of this paper is as follows. After stating the notations, the main theorems will be given in Sect. 2. In Sect. 3 we reformulate our problem and state theorems for the reformulated one. Those proofs will be given in Sects. 4–6.

Notations. We denote several positive constants depending on a, b, \dots by $C_{a,b,\dots}$ or only by C without confusion. We also denote $f(x) \sim g(x)$ as $x \rightarrow a$ when $C^{-1}g < f < Cg$ in a neighborhood of a . For function spaces, L^2 denotes the space of square integrable functions on \mathbf{R} with the norm

$$\|f\| = \left(\int_{\mathbf{R}} |f(x)|^2 dx \right)^{1/2} .$$

Here and after the integrand \mathbf{R} is abbreviated. $H^l (l \geq 0)$ denotes the usual l^{th} order Sobolev space with the norm

$$\|f\|_l = \left(\sum_{j=0}^l \|\partial_x^j f\|^2 \right)^{1/2} .$$

For the weight function w , L_w^2 denotes the space of measurable functions f satisfying $\sqrt{w}f \in L^2$ with the norm

$$|f|_w = \left(\int w(x) |f(x)|^2 dx \right)^{1/2}.$$

When $w(x) = \langle x \rangle^\alpha = (1+x^2)^{\alpha/2}$, we write $L_w^2 = L_\alpha^2$ and $|\cdot|_w = |\cdot|_\alpha$ without confusion. Moreover when w is replaced by $\langle x \rangle^\alpha w$, we denote that space by $L_{\alpha,w}^2$ with the norm

$$|f|_{\alpha,w} = \left(\int \langle x \rangle^\alpha w(x) |f(x)|^2 dx \right)^{1/2}.$$

When $C^{-1} \leq w(x) \leq C$, we note that $L^2 = H^0 = L_0^2 = L_w^2$ with $\|\cdot\| = \|\cdot\|_0 = |\cdot|_0 \sim |\cdot|_w$ and that $L_{\alpha,w}^2 = L_\alpha^2$ with $|\cdot|_{\alpha,w} \sim |\cdot|_\alpha$.

2. Preliminaries and Theorems

We first state the existence of traveling wave solutions with shock profile. Substituting (1.4) into (1.1) we have

$$\mu U_{\xi\xi} = -sU_\xi + f(U)_\xi = h'(U)U_\xi. \quad (2.1)$$

Integrating (2.1) over $(\pm\infty, \xi)$ and noting the Rankine–Hugoniot condition (1.5) we also have

$$\mu U_\xi = -s(U - u_\pm) + f(U) - f(u_\pm) = h(U), \quad (2.2)$$

which has a unique smooth solution U up to a shift satisfying $U(\pm\infty) = u_\pm$ provided that $h(U) \rightarrow 0$ at finite order as $U \rightarrow u_\pm$. Moreover, if $f'(u_+) < s < f'(u_-)$, then $|h(U)| \sim |U - u_\pm|$ as $U \rightarrow u_\pm$ because of $h'(u_\pm) \neq 0$. Hence $|U(\xi) - u_\pm| \sim \exp(-c|\xi|)$ as $\xi \rightarrow \pm\infty$ for some constant c . When $s = f'(u_+)$ or $f'(u_-)$, $|U(\xi) - u_\pm| \sim |\xi|^{-1/k_\pm}$ as $\xi \rightarrow \pm\infty$ provided $|h(U)| \sim |U - u_\pm|^{1+k_\pm}$ for $k_\pm > 0$. We note $k_\pm = n$ if $h'(u_\pm) = \dots = h^{(n)}(u_\pm) = 0$ and $h^{(n+1)}(u_\pm) \neq 0$. Thus we have

Lemma 1. Assume (1.5), (1.6) and

$$|h(U)| \sim |U - u_\pm|^{1+k_\pm}, \quad U \rightarrow u_\pm \quad (2.3)$$

with $k_\pm \geq 0$. Then there exists a traveling wave solution $U(\xi)$ of (1.1) with $U(\pm\infty) = u_\pm$, unique up to a shift, which is determined by the ordinary differential equation (2.1) or (2.2). Moreover, it holds as $\xi \rightarrow \pm\infty$,

$$|U(\xi) - u_\pm| \sim \exp(-c|\xi|) \quad \text{if } f'(u_+) < s < f'(u_-), \quad (2.4)$$

$$|U(\xi) - u_+| \sim |\xi|^{-1/k_+} \quad \text{if } s = f'(u_+), \quad (2.5)$$

and

$$|U(\xi) - u_-| \sim |\xi|^{-1/k_-} \quad \text{if } s = f'(u_-). \quad (2.6)$$

To investigate the stability of traveling wave solution U , we assume $u_0 - U$ is integrable. First we determine a unique shift of U as

$$\int_{-\infty}^{\infty} (u_0(x) - U(x)) dx = 0, \quad (2.7)$$

and define

$$\psi_0(x) = \int_{-\infty}^x (u_0(y) - U(y)) dy. \tag{2.8}$$

Then our three theorems are given as follows.

Theorem 1 (Stability). *Assume (1.5), (1.6) and (2.3) and let U be a traveling wave solution uniquely determined by (2.7). Then the following hold:*

(i) *When $f'(u_+) < s < f'(u_-)$, suppose $u_0 - U$ is integrable and $\psi_0 \in H^2$. Then there exists a positive constant ε_1 such that if $\|\psi_0\|_2 < \varepsilon_1$, then the Cauchy problem (1.1), (1.2) has a unique global solution $u(t, x)$ satisfying*

$$u - U \in C^0([0, \infty); H^1) \cap L^2(0, \infty; H^2)$$

and moreover

$$\sup_{\mathbf{R}} |u(t, x) - U(x - st)| \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{2.9}$$

(ii) *When $s = f'(u_+) < f'(u_-)$, there exists a positive constant ε_1 such that if $\|\psi_0\|_2 + |\psi_0|_{\langle \xi \rangle_+} < \varepsilon_1$ then the Cauchy problem (1.1), (1.2) has a unique global solution $u(t, x)$ satisfying*

$$u - U \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^2 \cap L^2_{\langle \xi \rangle_+}),$$

and moreover

$$\sup_{\mathbf{R}} |u(t, x) - U(x - st)| \rightarrow 0 \quad \text{as } t \rightarrow \infty, \tag{2.10}$$

where $\langle \xi \rangle_+ = \begin{cases} \sqrt{1 + \xi^2} & \xi \geq 0 \\ 1 & \xi < 0. \end{cases}$

(iii) *When $f'(u_+) < s = f'(u_-)$ or $s = f'(u_+) = f'(u_-)$, then $L^2_{\langle \xi \rangle_+}$ in (ii) should be replaced by $L^2_{\langle \xi \rangle_-}$ or $L^2_{\langle \xi \rangle} = L^2$, respectively. Here $\langle \xi \rangle_- = \begin{cases} \sqrt{1 + \xi^2} & \xi < 0 \\ 1 & \xi \geq 0. \end{cases}$*

Remark 1. When $s = f'(u_+)$ or $f'(u_-)$, we need a weight of order $\langle \xi \rangle = \sqrt{1 + \xi^2}$ as $\xi \rightarrow +\infty$ or $-\infty$ for a stability theorem.

Theorem 2 (Rate of asymptotic speed for $f'(u_+) < s < f'(u_-)$). *Let u be a solution obtained in Theorem 1(i) and let ψ_0 lie in L^2_α for some $\alpha > 0$. If α is an integer, then it holds*

$$\sup_{\mathbf{R}} |u(t, x) - U(x - st)| \leq C(1 + t)^{-\alpha/2} (\|u_0 - U\|_1 + |\psi_0|_\alpha), \tag{2.11}$$

while if α is not an integer, then

$$\sup_{\mathbf{R}} |u(t, x) - U(x - st)| \leq C_\varepsilon(1 + t)^{-\alpha/2 + \varepsilon} (\|u_0 - U\|_1 + |\psi_0|_\alpha) \tag{2.12}$$

for any constant $\varepsilon > 0$ and some constant C_ε such that $C_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Remark 2. For $f \in C^2$ Jones et al. [2] have shown $\sup_{\mathbf{R}} |u - U| \leq Ct^{-[a]/4}$ and, for $f'' > 0$, Kawashima and Matsumura [3] have shown $\sup_{\mathbf{R}} |u - U| \leq Ct^{-[a]/2}$. So, Theorems 1 and 2 improve both results. Further, for $f = u^2/2$ Nishihara [7] showed, using an explicit formula, that if $|\psi_0(x)| \leq C|x|^{-\alpha/2}$ as $|x| \rightarrow \infty$, then

$\sup_{\mathbf{R}}|u-U| \leq Ct^{-\alpha/2}$ and this decay rate is optimal in general. Hence the decay rate in Theorem 2 seems to be almost optimal in the L^2 -space version.

Next we state the result for $f'(u_+) = s < f'(u_-)$. When $f'(u_+) < s = f'(u_-)$ or $s = f'(u_+) = f'(u_-)$, the similar result is obtained as in Theorem 1(iii).

Theorem 3 (Rate of asymptotic speed for $f'(u_+) = s < f'(u_-)$). *Let u be a solution obtained in Theorem 1(ii) and $f''(u_+) = \dots = f^{(n)}(u_+) = 0$ and $f^{(n+1)}(u_+) \neq 0$ for $n \geq 1$. Then, if $\psi_0 \in L^2_{\alpha, \langle \xi \rangle_+}$ ($0 < \alpha < 2/n$), it holds for any $\varepsilon > 0$,*

$$\sup_{\mathbf{R}}|u(t, x) - U(x - st)| \leq C_\varepsilon(1+t)^{-\alpha/4+\varepsilon}(\|u_0 - U\|_1 + |\psi_0|_{\alpha, \langle \xi \rangle_+}). \quad (2.13)$$

Remark 3. Our rate seems to be not so sharp. But, the rate in this case is not known as far as we know. More contributions will be expected.

3. Reformulation of the Problem

Letting $U(\xi)$ be the traveling wave solution in Theorem 1, we put

$$u(t, x) = U(\xi) + \psi_\xi(t, \xi), \quad \xi = x - st. \quad (3.1)$$

Then the problem (1.1), (1.2) is reduced to

$$\psi_t - s\psi_\xi + f(U + \psi_\xi) - f(U) = \mu\psi_{\xi\xi}, \quad (3.2)$$

$$\psi(0, \xi) = \psi_0(\xi) \equiv \int_{-\infty}^{\xi} (u_0 - U)(\eta) d\eta. \quad (3.3)$$

Equation (3.2) is rewritten as

$$\psi_t + h'(U)\psi_\xi - \mu\psi_{\xi\xi} = F, \quad (3.4)$$

$$F \equiv -(f(U + \psi_\xi) - f(U) - f'(U)\psi_\xi). \quad (3.5)$$

Now we select the weight as

$$w = w(U) = \left| \frac{(U - u_+)(U - u_-)}{h(U)} \right|. \quad (3.6)$$

Since $w(U) \sim \text{const.}$ for the case $f'(u_+) < s < f'(u_-)$, $L^2_{\alpha, w(U)} = L^2_\alpha$. While if $s = f'(u_+) < f'(u_-)$, then $w(U) \sim |U - u|^{-k_+}$ as $U \rightarrow u_+$ and $w(U(\xi)) \sim \langle \xi \rangle$ as $\xi \rightarrow +\infty$, and hence $L^2_{w(U)} = L^2_{\langle \xi \rangle_+}$. Also, $L^2_{w(U)} = L^2_{\langle \xi \rangle_-}$ if $f'(u_+) < f'(u_-) = s$ and $L^2_{w(U)} = L^2_{\langle \xi \rangle} = L^2_1$ if $f'(u_+) = f'(u_-) = s$. Noting these we define the solution space of (3.2) and (3.3),

$$X(0, T) = \{ \psi \in C^0([0, T]; H^2 \cap L^2_{w(U)}), \psi_\xi \in L^2(0, T; H^2 \cap L^2_{w(U)}) \},$$

with $0 < T \leq +\infty$. Then the problem (3.2), (3.3) can be solved globally in time as follows.

Theorem 3.1. *Suppose $\psi_0 \in H^2 \cap L^2_{w(U)}$. Then there exists a positive constant ε_2 such that if $\|\psi_0\|_2 + |\psi_0|_{w(U)} < \varepsilon_2$, the problem (3.2), (3.3) has a unique global solution $\psi \in X(0, \infty)$ satisfying*

$$\|\psi(t)\|_2^2 + |\psi(t)|_{w(U)}^2 + \int_0^t (\|\psi_\xi(\tau)\|_2^2 + |\psi_\xi(\tau)|_{w(U)}^2) d\tau \leq C(\|\psi_0\|_2^2 + |\psi_0|_{w(U)}^2) \quad (3.7)$$

for any $t \geq 0$. Moreover, ψ_ξ tends to 0 in the maximum norm as $t \rightarrow \infty$, that is,

$$\sup_{\mathbf{R}} |\psi_\xi(t, \xi)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

For the decay rate we have the following.

Theorem 3.2 ($f'(u_+) < s < f'(u_-)$). Suppose $f'(u_+) < s < f'(u_-)$. Then the solution $\psi(t)$ obtained in Theorem 3.1 satisfies

$$(1+t)^\gamma \|\psi(t)\|_2^2 + \int_0^t (1+\tau)^\gamma \|\psi_\xi(\tau)\|_2^2 d\tau \leq C(|\psi_0|_\alpha^2 + \|\psi_0\|_2^2) \quad (3.8)$$

for any γ such that $0 \leq \gamma \leq \alpha$ if α is an integer and that $0 \leq \gamma < \alpha$ if α is not an integer.

Theorem 3.3 ($s = f'(u_+) < f'(u_-)$). Suppose $s = f'(u_+) < f'(u_-)$ and $f''(u_+) = \dots = f^{(n)}(u_+) = 0$ and $f^{(n+1)}(u_+) \neq 0$ for $n \geq 1$. If $0 < \alpha < 2/n$, then the solution $\psi(t, x)$ obtained in Theorem 3.1 satisfies

$$(1+t)^\gamma \|\psi(t)\|_2^2 + \int_0^t (1+\tau)^\gamma \|\psi_\xi(\tau)\|_2^2 d\tau \leq C(\|\psi_0\|_2^2 + |\psi_0|_{\alpha, w(U)}^2) \quad (3.9)$$

for γ such that $0 \leq \gamma < \alpha/2$.

All assertions (i)–(iii) in Theorem 1 are direct consequences of Theorem 3.1. Theorem 2 and Theorem 3 are, respectively, consequences of Theorem 3.2 and Theorem 3.3. Theorems 3.1–3.3 are all proved by a weighted energy method combining the local existence with a priori estimates. These are on the same line in Kawashima and Matsumura [3] etc.

Proposition 3.1 (Local existence). For any $\varepsilon_0 > 0$, there exists a positive constant T_0 depending on ε_0 such that if $\psi_0 \in H^2 \cap L_{w(U)}^2$ and $\|\psi_0\|_2 \leq \varepsilon_0$, then the problem (3.2), (3.3) has a unique solution $\psi \in X(0, T_0)$ satisfying $\|\psi(t)\|_2 < 2\varepsilon_0$ for $0 \leq t \leq T_0$.

Proposition 3.2 (A priori estimate). Let ψ be a solution in $X(0, T)$ for a positive constant T . Then there exists a positive constant ε_3 such that if $\sup_{0 \leq t \leq T} \|\psi(t)\|_2 < \varepsilon_3$, then $\psi(t)$ satisfies (3.7) for $0 \leq t \leq T$.

Proposition 3.1 can be proved in the standard way. So we omit its proof. Proposition 3.2 will be proved in the next section. For the proofs of Theorems 3.2 and 3.3 more estimates are necessary.

In later sections we only show the case $u_+ < u_-$ and $h(U) \leq 0$ for $U \in [u_+, u_-]$. The other case is shown in the same way.

4. Basic Estimate and Stability Theorem

Assuming $u_+ < u_-$ and $h(U) < 0$ for $U \in (u_+, u_-)$, we first derive the basic estimate in all our proofs.

Lemma 4.1. Let $\psi(t) \in X(0, T)$ be a solution of (3.2), (3.3). Then it holds

$$\frac{1}{2} |\psi(t)|_{w(U)}^2 + \int_0^t (\|\sqrt{-U_\xi} \psi(\tau)\|^2 + \mu |\psi_\xi(\tau)|_{w(U)}^2) d\tau \leq \frac{1}{2} |\psi_0|_{w(U)}^2 + \int_0^t \int w(U) \psi F dx d\tau. \quad (4.1)$$

Proof. Multiplying (3.4) by $w(U(\xi))\psi(t, \xi)$ we have

$$\begin{aligned} & \left(\frac{1}{2} w(U) \psi^2 \right)_t + \left(\frac{1}{2} (wh)'(U) \psi^2 - \mu w(U) \psi_\xi \psi \right)_\xi + \mu w(U) \psi_\xi^2 \\ & - \frac{1}{2} (wh)''(U) U_\xi \psi^2 = w(U) \psi F. \end{aligned} \quad (4.2)$$

Here we have used $\mu U_\xi = h(U)$. Since we have taken the weight w as (3.6), we obtain (4.1) by integrating (4.2) over $(0, t) \times \mathbf{R}$ and noting $U_\xi < 0$. Q.E.D.

We now put

$$N(t) = \sup_{0 \leq \tau \leq t} \|\psi(\tau)\|_2,$$

and assume $N(t) \leq \varepsilon_0$. Since $|\psi| \leq N(t)$, $|F| \leq C\psi_\xi^2$. Hence, if $N(t) < \varepsilon_3$ for sufficiently small $\varepsilon_3 > 0$, then we have

$$|\psi(t)|_{w(U)}^2 + \int_0^t |\psi_\xi(\tau)|_{w(U)}^2 d\tau \leq C |\psi_0|_{w(U)}^2. \quad (4.3)$$

Moreover, we apply ∂_ξ to (3.4), multiply it by $\partial_\xi \psi$ and $\partial_\xi^3 \psi$ and integrate the resulting equations over $(0, t) \times \mathbf{R}$. Noting $|F_\xi| \leq o(1)|\psi_\xi| + C|\psi_\xi \psi_{\xi\xi}|$ as $\sup_{\mathbf{R}} |\psi_\xi| \rightarrow 0$ we can get the next lemma. We omit the details.

Lemma 4.2. *There is a positive constant $\varepsilon_4 (\leq \varepsilon_0)$ such that if $N(t) \leq \varepsilon_4$, the estimate holds:*

$$\|\psi_\xi(t)\|_1^2 + \int_0^t \|\psi_{\xi\xi}(\tau)\|_1^2 d\tau \leq C (|\psi_0|_{w(U)}^2 + \|\psi_{0\xi}\|_1^2).$$

Combining Lemma 4.2 with (4.3) gives the proof of Proposition 3.2 and so Theorem 3.1.

5. Decay Rate for the Case $f'(u_+) < s < f'(u_-)$

We proceed to more a priori estimates of the solution ψ of the problem (3.2), (3.3). Since $h(U) < 0$, $U \in (u_+, u_-)$, there exists a unique number $\xi_* \in \mathbf{R}$ such that

$$U(\xi_*) = \bar{u} \equiv \frac{u_+ + u_-}{2}. \quad (5.1)$$

Putting $\langle \xi - \xi_* \rangle = \sqrt{1 + (\xi - \xi_*)^2}$ and multiplying (3.2) by $2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U) \psi$, we get

$$\begin{aligned} & ((1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U) \psi^2)_t + (\dots)_\xi + 2\mu(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U) \psi_\xi^2 \\ & - \gamma(1+t)^{\gamma-1} \langle \xi - \xi_* \rangle^\beta w(U) \psi^2 + (1+t)^\gamma \langle \xi - \xi_* \rangle^{\beta-1} A_\beta \psi^2 \\ & + 2\mu\beta(1+t)^\gamma \langle \xi - \xi_* \rangle^{\beta-2} (\xi - \xi_*) w(U) \psi \psi_\xi \\ & = 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U) \psi F, \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} A_\beta(\xi) &= -\langle \xi - \xi_* \rangle U_\xi (wh)''(U) - \beta \frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle} (wh)'(U) \\ &= -2\langle \xi - \xi_* \rangle U_\xi - 2\beta \frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle} (U - \bar{u}) \end{aligned}$$

by virtue of (3.6).

Lemma 5.1. *Let α be a given positive number. For $\beta \in [0, \alpha]$, there is a positive number c_0 independent of β such that*

$$A_\beta(\xi) \geq c_0 \beta \quad \text{for any } \xi \in \mathbf{R}. \quad (5.3)$$

Proof. Let $g(\xi) = -(wh)'(U(\xi)) = -2(U(\xi) - \bar{u})$, then $g(\xi_*) = 0$ by (5.1) and $g'(\xi) = -2U'(\xi) > 0$, so $g(\xi) \rightarrow u_\mp - u_\pm$ as $\xi \rightarrow \pm\infty$, respectively. Hence

$$-\frac{\xi - \xi_*}{\langle \xi - \xi_* \rangle} (wh)'(U(\xi)) \geq c(\delta), \quad |\xi - \xi_*| \geq \delta \quad (5.4)$$

for any $\delta > 0$. On the other hand,

$$\begin{aligned} -\langle \xi - \xi_* \rangle U_\xi (wh)''(U) &= -2\langle \xi - \xi_* \rangle U_\xi = 2\langle \xi - \xi_* \rangle \frac{-h(U(\xi))}{\mu} \\ &\geq \frac{-h(\bar{u})}{\mu}, \quad |\xi - \xi_*| \leq \delta_0 \end{aligned} \quad (5.5)$$

for some constant δ_0 . Combining (5.4) with (5.5) we obtain (5.3), where $c_0 = \min \left\{ c(\delta_0), \frac{-h(\bar{u})}{\mu\alpha} \right\}$. Q.E.D.

Integrating (5.2) over $[0, t] \times \mathbf{R}$ and noting $C^{-1} \leq w(U) \leq C$, we have

$$\begin{aligned} &(1+t)^\gamma |\psi(t)|_\beta^2 + \beta \int_0^t (1+\tau)^\gamma |\psi(\tau)|_{\beta-1}^2 d\tau + \int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|_\beta^2 d\tau \\ &\leq C \left\{ |\psi_0|_\beta^2 + \gamma \int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_\beta^2 d\tau \right. \\ &\quad \left. + \beta \int_0^t (1+\tau)^\gamma \int \langle \xi - \xi_* \rangle^{\beta-1} |\psi \psi_\xi| d\xi d\tau \right. \\ &\quad \left. + \int_0^t \int (1+\tau)^\gamma \langle \xi - \xi_* \rangle^\beta |\psi| |F| d\xi d\tau \right\}. \end{aligned} \quad (5.6)$$

Since

$$C\beta \langle \xi - \xi_* \rangle^{\beta-1} |\psi \psi_\xi| \leq \frac{\beta}{2} \langle \xi - \xi_* \rangle^{\beta-1} \psi^2 + \frac{C^2\beta}{2} \langle \xi - \xi_* \rangle^{\beta-1} \psi_\xi^2$$

and

$$\begin{aligned} & \int \frac{C^2\beta}{2} \langle \xi - \xi_* \rangle^{\beta-1} \psi_\xi^2 d\xi \\ &= \int_{|\xi - \xi_*| > R} \frac{C^2\beta}{2 \langle \xi - \xi_* \rangle} \langle \xi - \xi_* \rangle^\beta \psi_\xi^2 d\xi + \int_{|\xi - \xi_*| \leq R} \frac{C^2\beta}{2} \langle \xi - \xi_* \rangle^{\beta-1} \psi_\xi^2 d\xi \\ &\leq \frac{1}{2} |\psi_\xi|_\beta^2 + \beta C_R \|\psi_\xi\|^2. \end{aligned}$$

For some fixed $R > 0$, we have

$$\begin{aligned} & (1+t)^\gamma |\psi(t)|_\beta^2 + \int_0^t \left\{ \frac{\beta}{2} (1+\tau)^\gamma |\psi(\tau)|_{\beta-1}^2 + \left(\frac{1}{2} - CN(\tau) \right) (1+\tau)^\gamma |\psi_\xi(\tau)|_\beta^2 \right\} d\tau \\ &\leq C \{ |\psi_0|_\beta^2 + \gamma \int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_\beta^2 d\tau + \beta \int_0^t (1+\tau)^\gamma \|\psi_\xi(\tau)\|^2 d\tau \}. \end{aligned}$$

Thus we get the following.

Lemma 5.2. *There is a positive constant ε_5 such that if $N(T) < \varepsilon_5$, then it holds for $t \in [0, T]$,*

$$\begin{aligned} & (1+t)^\gamma |\psi(t)|_\beta^2 + \int_0^t \{ \beta (1+\tau)^\gamma |\psi(\tau)|_{\beta-1}^2 + (1+\tau)^\gamma |\psi_\xi(\tau)|_\beta^2 \} d\tau \\ &\leq C \{ |\psi_0|_\beta^2 + \gamma \int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_\beta^2 d\tau + \beta \int_0^t (1+\tau)^\gamma \|\psi_\xi(\tau)\|^2 d\tau \}. \end{aligned} \quad (5.7)$$

Applying the induction to (5.7) we have

Lemma 5.3. *It holds for $k = 0, 1, \dots, [\alpha]$,*

$$\begin{aligned} & (1+t)^k |\psi(t)|_{\alpha-k}^2 + \int_0^t \{ (\alpha-k)(1+\tau)^k |\psi(\tau)|_{\alpha-k-1}^2 \\ & \quad + (1+\tau)^k |\psi_\xi(\tau)|_{\alpha-k}^2 \} d\tau \leq C |\psi_0|_\alpha^2. \end{aligned} \quad (5.8)_k$$

Consequently, if α is an integer, then the following estimate holds for $0 \leq \gamma \leq \alpha$:

$$(1+t)^\gamma \|\psi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\psi_\xi(\tau)\|^2 d\tau \leq C |\psi_0|_\alpha^2. \quad (5.9)$$

Proof. First, letting $\gamma = 0$ and $\beta = \alpha$ in (5.7) we have (5.8)₀, which shows the lemma for $\alpha < 1$. Here we have used (4.3). Next we take $1 \leq \alpha < 2$. Letting $\beta = 0$ and $\gamma = 1$ and letting $\beta = \alpha - 1$ and $\gamma = 1$ in (5.7) show (5.8)₁. Hence the proof for $\alpha < 2$ is completed. Repeating the same procedure we can get the desired estimate (5.8)_k for any $\alpha \geq 0$. Q.E.D.

Methods used in this section till now are almost same as in Kawashima and Matsumura [3]. Further we show a sharper estimate. Let α be not an integer and γ be $[\alpha] < \gamma < \alpha$. Taking $\beta = 0$ in (5.7) we have

$$(1+t)^\gamma |\psi(t)|_0^2 + \int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|_0^2 d\tau \leq C (|\psi_0|_0^2 + \gamma \int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_0^2 d\tau). \quad (5.10)$$

Using (5.8)_k with $k = [\alpha]$,

$$(1+t)^{[\alpha]} |\psi(t)|_{\alpha-[\alpha]}^2 + \int_0^t \{(\alpha-[\alpha])(1+\tau)^{[\alpha]} |\psi(\tau)|_{\alpha-[\alpha]-1}^2 + (1+\tau)^{[\alpha]} |\psi_\xi(\tau)|_\alpha^2\} d\tau \leq C |\psi_0|_\alpha^2, \quad (5.8)_{[\alpha]}$$

we estimate the final term in (5.10):

$$\begin{aligned} & \int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_0^2 d\tau \\ &= \int_0^t (1+\tau)^{\gamma-1} \int \langle \xi - \xi_* \rangle^{(\alpha-[\alpha])([\alpha]+1-\alpha) - (\alpha-[\alpha])([\alpha]+1-\alpha)} (\psi^2)^{([\alpha]+1-\alpha) + (\alpha-[\alpha])} d\xi d\tau \\ &\leq \int_0^t (1+\tau)^{\gamma-1} \left(\int \langle \xi - \xi_* \rangle^{\alpha-[\alpha]} \psi^2 d\xi \right)^{[\alpha]+1-\alpha} \left(\int \langle \xi - \xi_* \rangle^{-([\alpha]+1-\alpha)} \psi^2 d\xi \right)^{\alpha-[\alpha]} d\tau \\ &= \int_0^t (1+\tau)^{-([\alpha]+1-\gamma)} \left((1+\tau)^{[\alpha]} |\psi|_{\alpha-[\alpha]}^2 \right)^{[\alpha]+1-\alpha} \left((1+\tau)^{[\alpha]} |\psi|_{\alpha-[\alpha]-1}^2 \right)^{\alpha-[\alpha]} d\tau \\ &\leq C |\psi_0|_\alpha^{2([\alpha]+1-\alpha)} \int_0^t (1+\tau)^{-([\alpha]+1-\gamma)} \left((1+\tau)^{[\alpha]} |\psi|_{\alpha-[\alpha]-1}^2 \right)^{\alpha-[\alpha]} d\tau \\ &\leq C |\psi_0|_\alpha^{2([\alpha]+1-\alpha)} \left(\int_0^t (1+\tau)^{-\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}} d\tau \right)^{[\alpha]+1-\alpha} \left(\int_0^t (1+\tau)^{[\alpha]} |\psi|_{\alpha-[\alpha]-1}^2 d\tau \right)^{\alpha-[\alpha]} \\ &\leq C |\psi_0|_\alpha^2, \end{aligned}$$

because of $\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha} > 1$. Thus we have the following from (5.10).

Lemma 5.4. *If α is not an integer, then it holds for any given $\gamma < \alpha$,*

$$(1+t)^\gamma \|\psi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\psi_\xi(\tau)\|^2 d\tau \leq C |\psi_0|_\alpha^2. \quad (5.11)$$

Similar estimates to Lemma 4.2 give the same decay rate for derivatives of the solution. The details are omitted.

Lemma 5.5. *Let $l=1$ and 2. For any $\gamma \leq \alpha$ (α : integer) or $\gamma < \alpha$ (α : not integer), there exists a positive constant ε_6 such that if $N(T) \leq \varepsilon_6$, then the decay estimate*

$$(1+t)^\gamma \|\partial_\xi^l \psi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\partial_\xi^{l+1} \psi(\tau)\|^2 d\tau \leq C (\|\psi_0\|_2^2 + |\psi_0|_\alpha^2). \quad (5.12)$$

Combining the latter part of Lemma 5.3, Lemma 5.4 with Lemma 5.5 we complete the proof of Theorem 3.2.

6. Decay Rate for the Case $s = f'(u_+) < f'(u_-)$

First we show the following estimate for the solution ψ obtained in Theorem 3.1.

Lemma 6.1. For $0 < \beta \leq \alpha < 2/n (n \geq 1)$, there exists a positive constant ε_7 such that if $N(T) \leq \varepsilon_7$, then the estimate

$$\begin{aligned} & \int w(U)^{1+\beta} \psi(t, \xi)^2 d\xi + \int_0^t \int_{\xi > 0} w(U)^{\beta-1} \psi(\tau, \xi)^2 d\xi d\tau \\ & + \int_0^t \int w(U)^{1+\beta} \psi_\xi(\tau, \xi)^2 d\xi d\tau \leq C \int w(U)^{1+\beta} \psi_0(\xi)^2 d\xi. \end{aligned} \quad (6.1)_\beta$$

Proof. Letting $z(\xi)$ be a positive function and multiplying (3.4) by $2zw(U)\psi$, we have

$$\begin{aligned} & (zw(U)\psi^2)_t + (\cdots)_\xi + 2\mu zw(U)\psi_\xi^2 - 2(z(hw)'(U))_\xi \psi^2 \\ & + 2\mu z_\xi w(U)\psi\psi_\xi = 2zw(U)\psi F. \end{aligned} \quad (6.2)$$

Since

$$-(z(hw)'(U))_\xi = 2(\tilde{u} - U)z_\xi - 2zU_\xi$$

and

$$|2\mu z_\xi w(U)\psi\psi_\xi| \leq 2\varepsilon \mu zw(U)\psi_\xi^2 + \frac{\mu w(U)z_\xi^2}{2\varepsilon z} \psi^2$$

for $\varepsilon \in (0, 1)$, Eq. (6.2) yields

$$\begin{aligned} & (zw(U)\psi^2)_t + (\cdots)_\xi + 2(1-\varepsilon)\mu zw(U)\psi_\xi^2 + 2 \left\{ -2zU_\xi + \left(2(\tilde{u} - U) - \frac{\mu w(U)z_\xi}{2\varepsilon z} \right) z_\xi \right\} \psi^2 \\ & \leq 2zw(U)|\psi F|. \end{aligned} \quad (6.3)$$

Taking $z = w(U)^\beta$, we have

$$I \equiv \left(2(\tilde{u} - U) - \frac{\mu w(U)z_\xi}{2\varepsilon z} \right) z_\xi = \frac{\beta}{\mu} w(U)^{\beta-1} w'(U) h(U) \left(2(\tilde{u} - U) - \frac{\beta w'(U)h(U)}{2\varepsilon} \right).$$

If we put $\delta = U - u_+ > 0$ and $\tilde{u} = u_- - u_+$, then we have near u_+ or $\xi = +\infty$,

$$\begin{aligned} h(U)w'(U) &= 2\delta - \tilde{u} - \frac{\delta(\delta - \tilde{u})h'(U)}{h(U)} \\ &= 2\delta - \tilde{u} - \frac{\delta(\delta - \tilde{u})(h''(u_+)\delta + \cdots + \frac{h^{(n+1)}(u_+)}{n!}\delta^n + o(\delta^n))}{\frac{h''(u_+)}{2!}\delta^2 + \cdots + \frac{h^{(n+1)}(u_+)}{(n+1)!}\delta^{n+1} + o(\delta^{n+1})} \\ &= 2\delta - \tilde{u} - (\delta - \tilde{u})((n+1) + O(\delta)) \\ &= \tilde{u}n + O(\delta), \end{aligned}$$

and hence

$$I = \frac{\beta w(U)^{\beta-1}}{\mu} (\tilde{u}n + O(\delta)) \left(\tilde{u} \left(1 - \frac{\beta n}{2\varepsilon} \right) + O(\delta) \right).$$

Since $\beta \leq \alpha < \frac{2}{n}$, if we set $\varepsilon < 1$ as $1 - \frac{\beta n}{2\varepsilon} > 0$, then there are positive constants c_1 and R_1 such that

$$I \geq c_1 \quad \text{for} \quad \xi \geq R_1. \tag{6.4}$$

Noting $C^{-1} \leq w(U) \leq C$, $C^{-1} \leq w'(U) \leq C$ as $\xi \rightarrow -\infty$ and using Lemma 4.1, we have

$$\int_0^t \int_{\xi \leq R_1} 2I \cdot \psi^2 d\xi d\tau \leq C |\psi_0|_{w(U)}^2. \tag{6.5}$$

Because of (6.4) and (6.5) the integration of (6.3) over $(0, t) \times \mathbf{R}$ gives the estimate (6.1). Q.E.D.

Again multiplying (3.2) by $2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(U) \psi$ and integrating its equation (= (5.2)) over $(0, t) \times \mathbf{R}$, we have for $0 \leq \beta \leq \alpha$,

$$\begin{aligned} & (1+t)^\gamma |\psi(t)|_{\beta, w(U)}^2 + (1-CN(T)) \int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|_{\beta, w(U)}^2 d\tau \\ & + \beta \int_0^t (1+\tau)^\gamma |\psi(\tau)|_{\beta-1}^2 d\tau \\ & \leq C \{ |\psi_0|_{\beta, w(U)}^2 + \gamma \int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_{\beta, w(U)}^2 d\tau \\ & + \beta \int_0^t (1+\tau)^\gamma \int \langle \xi - \xi_* \rangle^{\beta-1} w(U) |\psi \psi_\xi| d\xi d\tau \}. \end{aligned} \tag{6.6}_{\gamma, \beta}$$

For (6.6) _{γ, β} with $\gamma=0$ and $\beta \leq \alpha$,

$$\begin{aligned} |\text{last term in (6.6)}_{0, \beta}| & \leq \int_0^t \int \frac{\beta}{2} \langle \xi - \xi_* \rangle^{\beta-1} \psi^2 \\ & + \frac{C^2 \beta}{2} \langle \xi - \xi_* \rangle^{\beta-1} w(U)^2 \psi_\xi^2 d\xi d\tau, \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} & \frac{C^2 \beta}{2} \int_0^t \int \langle \xi - \xi_* \rangle^{\beta-1} w(U)^2 \psi_\xi^2 d\xi d\tau \\ & \leq C \int_0^t \int_{\xi > R_2} w(U)^{\beta+1} \psi_\xi^2 d\xi d\tau + \frac{1}{2} \int_0^t \int_{\xi < -R_3} \langle \xi - \xi_* \rangle^\beta \psi_\xi^2 d\xi d\tau \\ & + C \int_0^t \int_{-R_3 \leq \xi \leq R_2} \psi_\xi^2 d\xi d\tau \end{aligned} \tag{6.8}$$

for some constants $R_2, R_3 > 0$, because $w(U(\xi)) \sim \xi$ as $\xi \rightarrow \infty$ and $w(U(\xi)) \sim \text{const.}$ as $\xi \rightarrow -\infty$. Applying Lemma 6.1 and Lemma 4.1 to (6.6) _{$0, \beta$} , (6.7) and (6.8), and taking $\beta = \alpha$, we get the following.

Lemma 6.2. *There is a positive constant ε_7 such that if $N(T) \leq \varepsilon_7$, then the estimate*

$$|\psi(t)|_{\alpha, w(U)}^2 + \int_0^t (|\psi(\tau)|_{\alpha-1}^2 + |\psi_\xi(\tau)|_{\alpha, w(U)}^2) d\tau \leq C |\psi_0|_{\alpha, w(U)}^2 \quad (6.9)$$

holds for $\alpha < \frac{2}{n}$ ($n \geq 1$).

Next we consider $(6.6)_{\gamma, \beta}$ with $\gamma < \alpha/2$ and $\beta = 0$:

$$\begin{aligned} & (1+t)^\gamma |\psi(t)|_{0, w(U)}^2 + (1-N(T)) \int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|_{0, w(U)}^2 d\tau \\ & \leq C (|\psi_0|_{0, w(U)}^2 + \gamma \int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_{0, w(U)}^2 d\tau). \end{aligned} \quad (6.6)_{\gamma, 0}$$

We estimate the final term in a similar fashion to the proof of Lemma 5.4. We divide the integrand as

$$\begin{aligned} & C\gamma \int_0^t (1+\tau)^{\gamma-1} |\psi(\tau)|_{0, w(U)}^2 d\tau \\ & \leq C \int_0^t (1+\tau)^{\gamma-1} \left(\int_{\xi>0} + \int_{\xi<0} \right) w(U(\xi)) \psi(\tau, \xi)^2 d\xi d\tau \\ & \leq C \int_0^t (1+\tau)^{\gamma-1} \int_{\xi>0} w(U) \psi(\tau, \xi)^2 d\xi d\tau + C \int_0^t (1+\tau)^{\gamma-1} \int_{\xi<0} \psi(\tau, \xi)^2 d\xi d\tau \\ & \equiv I_1 + I_2. \end{aligned}$$

Noting $\int_{\xi>0} w(U)^{1+\alpha} \psi^2 d\xi \sim \int_{\xi>0} \langle \xi - \xi_* \rangle^\alpha w(U) \psi^2 d\xi$, we have

$$\begin{aligned} I_1 & \leq C \int_0^t (1+\tau)^{\gamma-1} \left(\int_{\xi>0} w(U)^{1+\alpha} \psi^2 d\xi \right)^{\frac{2-\alpha}{2}} \left(\int_{\xi>0} w(U)^{\alpha-1} \psi^2 d\xi \right)^{\frac{\alpha}{2}} d\tau \\ & \leq C |\psi_0|_{\alpha, w(U)}^{2-\alpha} \int_0^t (1+\tau)^{\gamma-1} \left(\int_{\xi>0} w(U)^{\alpha-1} \psi^2 d\xi \right)^{\alpha/2} d\tau \\ & \leq C |\psi_0|_{\alpha, w(U)}^{2-\alpha} \left(\int_0^t (1+\tau)^{-\frac{2(1-\gamma)}{2-\alpha}} d\tau \right)^{\frac{2-\alpha}{2}} \left(\int_0^t \int_{\xi>0} w(U)^{\alpha-1} \psi^2 d\xi d\tau \right)^{\frac{\alpha}{2}} \\ & \leq C |\psi_0|_{\alpha, w(U)}^2 \end{aligned}$$

by $(6.1)_\alpha$ and (6.9) and $\frac{2(1-\gamma)}{2-\alpha} > 1$. While, since $\gamma < \frac{\alpha}{2} < \frac{1}{n} \leq 1$ by $n \geq 1$,

$$I_2 \leq C \int_0^t \int_{\xi<0} \psi(\tau, \xi)^2 d\xi d\tau \leq C |\psi_0|_{w(U)}^2$$

by virtue of (6.9) provided $\alpha \geq 1$ or $n = 1$. Let $n \geq 2$ and so $\alpha < 1$. Then, by virtue of (6.9)

$$\begin{aligned} I_2 &\leq C \int_0^t (1+\tau)^{\gamma-1} \int_{\xi < 0} \langle \xi \rangle^{\alpha(1-\alpha)+\alpha(\alpha-1)} \psi^2 d\xi d\tau \\ &\leq C \int_0^t (1+\tau)^{\gamma-1} \left(\int_{\xi < 0} \langle \xi \rangle^\alpha \psi^2 d\xi \right)^{1-\alpha} \left(\int_{\xi < 0} \langle \xi \rangle^{\alpha-1} \psi^2 d\xi \right)^\alpha d\tau \\ &\leq C |\psi_0|_{\alpha, w(\bar{v})}^2 \int_0^t (1+\tau)^{-(1-\gamma)} |\psi(\tau)|_{\alpha-1}^{2\alpha} d\tau \\ &\leq C |\psi_0|_{\alpha, w(\bar{v})}^2 \left(\int_0^t (1+\tau)^{-\frac{1-\gamma}{1-\alpha}} d\tau \right)^{1-\alpha} \left(\int_0^t |\psi(\tau)|_{\alpha-1}^2 d\tau \right)^\alpha \\ &\leq C |\psi_0|_{\alpha, w(v)}^2 \end{aligned}$$

because of $\frac{1-\gamma}{1-\alpha} > 1$ by $\gamma < \alpha/2$. Thus we have had a desired estimate.

Lemma 6.3. For $N(T) \leq \varepsilon_7$, it holds for $\gamma < \alpha/2 < 1/n$,

$$(1+t)^\gamma |\psi(t)|_{w(v)}^2 + \int_0^t (1+\tau)^\gamma |\psi_\xi(\tau)|_{w(v)}^2 d\tau \leq C |\psi_0|_{\alpha, w(v)}^2.$$

We complete the proof of Theorem 3.3 by obtaining the decay rate for derivatives of the solution, similar to Lemma 5.5. The details are omitted too.

Lemma 6.4. Let $l=1$ and $l=2$. There exists a positive constant ε_8 such that if $N(T) \leq \varepsilon_8$, then the decay estimate

$$(1+t)^\gamma \|\partial_\xi^l \psi(t)\|^2 + \int_0^t (1+\tau)^\gamma \|\partial_\xi^{l+1} \psi(\tau)\|^2 d\tau \leq C (\|\psi_0\|_2^2 + |\psi_0|_{\alpha, w(v)}^2)$$

holds for any $\gamma, 0 \leq \gamma < \alpha/2 < 1/n$.

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