Proceedings of the Royal Society of Edinburgh, **134A**, 579–594, *2004*

Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion

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(MS received 8 April 2003; accepted 13 January 2004)

This paper considers the nonlinear stability of travelling wavefronts of a time-delayed diffusive Nicholson blowflies equation. We prove that, under a weighted L^2 norm, if a solution is sufficiently close to a travelling wave front initially, it converges exponentially to the wavefront as $t \to \infty$. The rate of convergence is also estimated.

1. Introduction and main results

Blowflies are an important parasite of the sheep industry in countries like Australia. For the purposes of prevention, control and elimination, it is of interest to investigate both temporal and spatial variations of the blowflies population using mathematical models. Based on the experimental data of Nicholson [14, 15], Gurney *et al*. [5] established a dynamical model, the Nicholson blowflies equation,

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} + \mathrm{d}N(t) = pf(N(t-r)),$$

where $N(t)$ denotes the total mature population of the blowflies at time t, $d > 0$ is the death rate of the mature population, $r > 0$ is the maturation delay, the time required for a newborn to become matured, $p > 0$ is the impact of the death on the immature population and

$$f(N(t-r)) = N(t-r)\mathrm{e}^{-aN(t-r)}$$

is Nicholson's birth function, where $a > 0$ is a constant. One can approximate the spatial variability of blowflies by considering a nonlinear time-delayed reaction–diffusion equation,

$$\frac{\partial N(t,x)}{\partial t} - \frac{\partial^2 N(t,x)}{\partial x^2} + \mathrm{d}N(t,x) = pf(N(t-r,x)). \tag{1.1}$$

So and Yang [23, 31] investigated (1.1) and established Hopf bifurcations for the Neumann problem, and the stability of the steady-state solutions for the Dirichlet problem. So *et al.* [25] investigated Hopf bifurcations for the Dirichlet problem of (1.1). So and Zou [24] proved the existence of travelling waves for (1.1).

Equation (1.1) admits two constant equilibria, $N_- = 0$ and $N_+ = (1/a)\ln(p/d)$. If $p > d$, then $N_- < N_+$. We will always assume $p > d$ for the rest of the paper. A travelling wavefront connecting N_- and N_+ is a solution to (1.1) of the form $N(t,x) = \phi(x+ct)$ with speed $c > 0$, and the front profile $\phi(\xi)$ satisfies $\phi'(\xi) > 0$ and

$$c\phi'(\xi) - \phi''(\xi) + d\phi(\xi) = pf(\phi(\xi - cr)), \qquad \phi(\pm\infty) = N_\pm, \tag{1.2}$$

where $\xi = x + ct \in (-\infty, +\infty)$ and a prime indicates differentiation with respect to ξ. So and Zou [24] proved the existence of travelling wave solutions to (1.1) with monotone wave profile $\phi(\xi)$.

PROPOSITION 1.1 (cf. [24]). *Suppose* $1 < p/d < e$. *Then there exists*

$$0 < c^* < 2\sqrt{p-d}$$

such that, for any $c > c^*$, *there exists a monotone front travelling wave* $\phi(x+ct)$ *for equation (1.1) connecting* $N_\pm$, *with* $\phi'(\xi) > 0$ *and* $0 = N_- < \phi(\xi) < N_+$ *for* $\xi = x + ct \in (-\infty, \infty)$.

In the present paper, we consider the Cauchy problem to (1.1) with initial conditions

$$N(s,x) = N_0(s,x), \quad s \in [-r,0], \quad x \in \boldsymbol{R}, \tag{1.3}$$

that satisfies

$$N_0(s,x) \to N_\pm \quad \text{for } s \in [-r,0] \quad \text{as } x \to \pm\infty.$$

We provide a stability analysis of travelling wave solutions to (1.1). More specifically, we prove that there exists a unique global solution $N(t,x)$ to the Cauchy problem (1.1) and (1.3), and $N(t,x) \to \phi(x+ct)$ as $t \to \infty$ provided $N_0(s,x) - \phi(x+cs)$ is sufficiently small in a weighted norm, for each $s \in [-r,0]$.

For reaction–diffusion equations without delay, stability of travelling waves has been extensively studied in the literature (see, for example, [1,2,4,8,18–20,28,29] and the references therein). There is also a survey paper of Xin [30] and a textbook of Volpert *et al.* [26]. Sattinger [20] used the spectrum-analysis method to prove the wave stability for the Fisher–KPP nonlinearity, when the initial perturbation has an exponential decay. Stability of wavefronts with critical speeds was studied in Kirchgässner [8] and Gallay [4]. For reaction–diffusion equations with time delays, few results exist on the stability of travelling waves (see the interesting papers by Schaaf [21], Ogiwara and Matano [16] and Smith and Zhao [22]). Schaaf [21] proved linearized stability for Fisher–KPP nonlinearity by a spectral method. Smith and Zhao [22] considered a 'bi-stable' nonlinearity of the form $f(N) = N(1-N)(N-b)$ ($b \in (0,1)$), and proved a global stability result for the travelling wave solution. The methods used in [21,22] do not apply to the nonlinearity in (1.1), because our $N_- = 0$ is an 'unstable node'. Here we adopt a weighted energy method in our stability analysis of wave front solutions to (1.1).

For a travelling wavefront $\phi(x+ct)$ with speed $c > 2\sqrt{p-d} > c^*$, we define a weight function

$$w(x) = \begin{cases} e^{-\alpha(x - x_* - cr)} & \text{for } x \leqslant x_* + cr, \\ 1 & \text{for } x > x_* + cr, \end{cases} \tag{1.4}$$

where

$$\alpha = \eta_0 c, \qquad \eta_0 = \frac{c^2 + 4(p-d)}{2c^2} > \frac{1}{2}, \tag{1.5}$$

and x_* is determined by

$$\frac{d}{p} = [1 - a\phi(x_*)]\mathrm{e}^{-a\phi(x_*)}. \tag{1.6}$$

The existence and uniqueness of such a number x_* is given in lemma 3.5 below. It is also easy to get $0 < \eta_0 < 1$, since $c > 2\sqrt{p-d}$.

For an interval $I \subset \boldsymbol{R}$, let $L^2(I)$ denote the space of square-integrable functions on I, and $H^k(I)$ ($k \geqslant 0$) the Sobolev space of L^2 functions $f(x)$ defined on I whose derivatives $\partial_x^i f$, $i = 1, \dots, k$, also belong to $L^2(I)$. Let $L^2_w(I)$ be the weighted L^2 space with weight $w(x) > 0$ and norm

$$\|f\|_{L^2_w} = \left(\int_I w(x) f(x)^2 \, \mathrm{d}x\right)^{1/2},$$

and $H^k_w(I)$ be the weighted Sobolev space with norm

$$\|f\|_{H^k_w} = \left(\sum_{j=0}^{k} \int_I w(x) |\partial_x^j f(x)|^2 \, \mathrm{d}x\right)^{1/2}.$$

For $T > 0$ and a Banach space $\mathcal{B}$, we denote by $C^0([0,T];\mathcal{B})$ the space of $\mathcal{B}$-valued continuous functions on $[0,T]$, and by $L^2([0,T];\mathcal{B})$ the space of $\mathcal{B}$-valued L^2 functions on $[0,T]$. The corresponding spaces of $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

Our main results are stated in the following.

THEOREM 1.2 (global existence and uniqueness). *Suppose that $N_0(s,x) \geqslant 0$ and is continuous for $(s,x) \in [-r,0] \times \boldsymbol{R}$. For a given travelling wave solution $\phi(x+ct)$, if $N_0(s,x) - \phi(x+cs) \in C^0([-r,0]; H^1(\boldsymbol{R}))$, then there exists a unique global solution $N(t,x)$ of the Cauchy problem (1.1) and (1.3) such that $N(t,x) - \phi(x+ct) \in C^0([0,+\infty); H^1(\boldsymbol{R}))$ and $N(t,x) \geqslant 0$ in $(0,\infty) \times \boldsymbol{R}$.*

THEOREM 1.3 (stability). *For a given travelling wave solution $\phi(x+ct)$ with speed c satisfying*

$$c > 2\sqrt{p-d}, \tag{1.7}$$

if $N_0(s,x) - \phi(x+cs) \in C^0([-r,0]; H^1_w(\boldsymbol{R}))$, where $w(x)$ is the weight function given in (1.4), then there exist positive constants $\delta_0 = \delta_0(d,p,\phi(x_* + cr))$ and $\mu = \mu(d,p,\phi(x_*+cr))$ such that, when $\|N_0(s,\cdot) - \phi(\cdot + cs)\|_{H^1_w} \leqslant \delta_0$ for $s \in [-r,0]$, the unique solution $N(t,x)$ of the Cauchy problem (1.1) and (1.3) satisfies

$$N(t,x) - \phi(x+ct) \in C^0([0,\infty); H^1_w(\boldsymbol{R})) \cap L^2([0,\infty); H^2_w(\boldsymbol{R}))$$

and

$$\sup_{x \in \boldsymbol{R}} |N(t,x) - \phi(x+ct)| \leqslant C\mathrm{e}^{-\mu t}, \quad 0 \leqslant t \leqslant \infty. \tag{1.8}$$

Noting the weight $w(x)$ given in (1.4) and (1.5), we recognize from theorem 1.3 that, as the sufficient condition, the initial perturbation must converge to 0 in the form

$$|u_0(x,s) - \phi(x - cr)| \sim \mathrm{e}^{-\eta_0 c|x|/2}, \quad \eta_0 > \tfrac{1}{2}, \quad \text{as } x \to -\infty.$$

Comparing this with the sufficient condition for the initial data by Sattinger in [20] for the Fisher–KPP equation

$$|u_0(x) - \phi(x)| \sim \mathrm{e}^{-c|x|/2} \quad \text{as } x \to -\infty,$$

our condition is much weaker.

Furthermore, we can improve the stability in theorem 1.3 by allowing the initial perturbation to satisfy

$$|u_0(x,s) - \phi(x - cr)| \sim \mathrm{e}^{-c|x|/4} \quad \text{as } x \to -\infty,$$

if the coefficients p, d and r satisfy

$$4p[1 - \mathrm{e}^{-r(p-d)}] > 5(p - d). \tag{1.9}$$

THEOREM 1.4 (improved stability). *Let the weight function be as follows,*

$$w_2(x) = \begin{cases} \mathrm{e}^{-c(x - x_* - cr)/2} & \textit{for } x \leqslant x_* + cr, \\ 1 & \textit{for } x > x_* + cr, \end{cases} \tag{1.10}$$

and suppose that (1.9) holds. For a given travelling wave solution $\phi(x + ct)$ with speed c satisfying (1.7), if $N_0(s,x) - \phi(x + cs) \in C^0([-r,0]; H^1_{w_2}(\boldsymbol{R}))$, then there exist positive constants $\delta_2 = \delta_2(d, p, \phi(x_* + cr))$ and $\mu_2 = \mu_2(d, p, \phi(x_* + cr))$ such that, when $\|N_0(s,\cdot) - \phi(\cdot + cs)\|_{H^1_{w_2}} \leqslant \delta_2$ for $s \in [-r,0]$, the unique solution $N(t,x)$ of the Cauchy problem (1.1) and (1.3) satisfies

$$N(t,x) - \phi(x + ct) \in C^0([0,\infty); H^1_{w_2}(\boldsymbol{R})) \cap L^2([0,\infty); H^2_{w_2}(\boldsymbol{R}))$$

and

$$\sup_{x \in \boldsymbol{R}} |N(t,x) - \phi(x + ct)| \leqslant C\mathrm{e}^{-\mu_2 t}, \quad 0 \leqslant t \leqslant \infty. \tag{1.11}$$

In § 2, we prove theorem 1.2, the global existence and uniqueness of solutions to the Cauchy problem (1.1) and (1.3). In § 3, we prove theorems 1.3 and 1.4, the stability of travelling wave solutions.

2. Global existence and uniqueness

The method used in the proof of theorem 1.2 is standard (see, for example, [27]). We only outline the important steps. First we establish the non-negativity of all global solutions.

THEOREM 2.1. *Let $N(t,x)$ be the solution of (1.1) and (1.3) in $(0,\infty) \times \boldsymbol{R}$. If $N_0(s,x) \geqslant 0$ holds in $[-r,0] \times \boldsymbol{R}$, then $N(t,x) \geqslant 0$ holds in $(0,\infty) \times \boldsymbol{R}$.*

Proof. For $t \in [0, r]$, we have

$$f(N(t-r,x)) = N(t-r,x)\mathrm{e}^{-aN(t-r,x)} = N_0(t-r,x)\mathrm{e}^{-aN_0(t-r,x)} \geqslant 0.$$

Thus $N(t,x)$ satisfies the differential inequality

$$N_t - N_{xx} + \mathrm{d}N = pN(t-r,x)\mathrm{e}^{-aN(t-r,x)} \geqslant 0, \quad t \in [0,r].$$

Applying the standard comparison principle for linear parabolic equations, we have $N(t,x) \geqslant 0$ on $[0,r]$. The proof is completed by repeating this procedure to each of the intervals $[nr,(n+1)r]$, $n = 1,2,\dots$. □

Let

$$u(t,x) = N(t,x) - \phi(x+ct),$$

where $\phi(x+ct)$ is a given travelling wave solution. Then the Cauchy problem (1.1) and (1.3) can be rewritten as

$$\left.\begin{aligned} u_t(t,x) - u_{xx}(t,x) + \mathrm{d}u(t,x) &= pG(t-r,x), \quad (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}, \\ u(s,x) = N_0(s,x) - \phi(x+cs) &=: u_0(s,x), \quad (s,x) \in [-r,0] \times \boldsymbol{R}, \end{aligned}\right\} \tag{2.1}$$

where

$$G(t-r,x) = f(u(t-r,x) + \phi(x+ct-cr)) - f(\phi(x+ct-cr)).$$

We have the following result.

THEOREM 2.2. *Under the assumptions of theorem 1.2, there exists a unique global solution $u(t,x)$ of the Cauchy problem (2.1) such that $u(t,x) \in C^0([0,\infty); H^1(\boldsymbol{R}))$.*

Theorem 1.2 follows immediately from theorems 2.1 and 2.2. The rest of this section is devoted to the proof of theorem 2.2. The following result on the local existence, uniqueness and extension of solutions is standard. It can be proved using the standard iteration method (cf. [3, 6, 9, 17]). The proof is omitted.

PROPOSITION 2.3 (local existence and uniqueness). *Given $u_0(s,x) \in C^0([-r,0]; H^1(\boldsymbol{R}))$, there exists $t_0 > 0$ such that problem (2.1) has a unique solution $u(t,x) \in C^0([0,t_0); H^1(\boldsymbol{R}))$. Furthermore, let $[0,T)$ be its maximal interval of existence and $u(t,x) \in C^0([0,T_0); H^1(\boldsymbol{R}))$. Then either $T_0 = +\infty$ or $T_0 < +\infty$, and in the latter case $\lim_{t\to T_0^-} \|u(t,\cdot)\|_{H^1(\boldsymbol{R})} = \infty$.*

PROPOSITION 2.4 (boundedness). *Let $u(t,x)$ be a solution in $C^0([0,T); H^1(\boldsymbol{R}))$ for $0 < T < \infty$. Then there exists positive constant C_0, independent of T, such that*

$$\|u(t)\|_{H^1}^2 \leqslant C_0\biggl(\|u_0(0)\|_{H^1}^2 + \int_{-r}^0 \|u_0(s)\|_{H^1}^2\,\mathrm{d}s\biggr)\mathrm{e}^{(p^2/2d)t}, \quad 0 \leqslant t < T. \tag{2.2}$$

Proof. Multiplying (2.1) by $2u(t,x)$ and integrating over $[0,t] \times \boldsymbol{R}$, $t \in [0,T)$, we have

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + 2\int_0^t \|u_x(s)\|_{L^2}^2\,\mathrm{d}s + 2d\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s \\ &\qquad = \|u_0(0)\|_{L^2}^2 + 2p\int_0^t\int_{-\infty}^{\infty} G(s-r,x)u(s,x)\,\mathrm{d}x\mathrm{d}s, \quad t \in [0,T). \end{aligned} \tag{2.3}$$

By the mean-value theorem, there exists a function, $\bar{u}(t,x)$, between $\phi(x+ct)$ and $N(t,x)=\phi(x+ct)+u(t,x)$, such that

$$|G| = |f(u+\phi)-f(\phi)| = |f(N)-f(\phi)| = |f'(\bar{u})(N-\phi)| = |f'(\bar{u})u|. \tag{2.4}$$

Since $\phi \geqslant 0$ (by proposition 1.1) and $N \geqslant 0$ (by theorem 2.1), we have $\bar{u} \geqslant 0$. Thus

$$|f'(\bar{u})| = |(1-a\bar{u})\mathrm{e}^{-a\bar{u}}| \leqslant 1 \quad \text{for all } \bar{u} \in (0,\infty).$$

Combining this with (2.4) leads to

$$|G(s,x)| \leqslant |u(s,x)|. \tag{2.5}$$

We can use (2.5) and the Cauchy–Schwarz inequality, $ab \leqslant \varepsilon a^2 + (1/4\varepsilon)b^2$ for $\varepsilon > 0$, to estimate the last term of (2.3) as follows:

$$\begin{aligned} 2p\int_0^t\int_{-\infty}^{\infty} G(s-r,x)u(s,x)\,\mathrm{d}x\mathrm{d}s &\leqslant 2p\int_0^t\int_{-\infty}^{\infty} |u(s-r,x)u(s,x)|\,\mathrm{d}x\mathrm{d}s \\ &\leqslant 2p\int_0^t\int_{-\infty}^{\infty}\left[\varepsilon u(s,x)^2 + \frac{1}{4\varepsilon}u(s-r,x)^2\right]\mathrm{d}x\mathrm{d}s \\ &= 2p\varepsilon\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s + \frac{p}{2\varepsilon}\int_0^t \|u(s-r)\|_{L^2}^2\,\mathrm{d}s. \end{aligned} \tag{2.6}$$

The last term of (2.6) can be estimated as follows:

$$\begin{aligned} \frac{p}{2\varepsilon}\int_0^t \|u(s-r)\|_{L^2}^2\,\mathrm{d}s &= \frac{p}{2\varepsilon}\int_{-r}^{t-r} \|u(s)\|_{L^2}^2\,\mathrm{d}s \\ &\leqslant \frac{p}{2\varepsilon}\int_{-r}^{0} \|u_0(s)\|_{L^2}^2\,\mathrm{d}s + \frac{p}{2\varepsilon}\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s. \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6) and letting $\varepsilon = d/p$ yield

$$\begin{aligned} &2p\int_0^t\int_{-\infty}^{\infty} G(s-r,x)u(s,x)\,\mathrm{d}x\mathrm{d}s \\ &\qquad \leqslant 2d\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s + \frac{p^2}{2d}\int_{-r}^{0} \|u_0(s)\|_{L^2}^2\,\mathrm{d}s + \frac{p^2}{2d}\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.3), we obtain

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + 2\int_0^t \|u_x(s)\|_{L^2}^2\,\mathrm{d}s \\ &\qquad \leqslant \|u_0(0)\|_{L^2}^2 + \frac{p^2}{2d}\int_{-r}^{0} \|u_0(s)\|_{L^2}^2\,\mathrm{d}s + \frac{p^2}{2d}\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s, \quad t \in [0,T). \end{aligned} \tag{2.9}$$

Applying Gronwall's inequality to (2.9), we have

$$\|u(t)\|_{L^2}^2 \leqslant \left(\|u_0(0)\|_{L^2}^2 + \frac{p^2}{2d}\int_{-r}^{0} \|u_0(s)\|_{L^2}^2\,\mathrm{d}s\right)\mathrm{e}^{(p^2/2d)t}, \quad t \in [0,T). \tag{2.10}$$

Similarly, we can prove that

$$\|u_x(t)\|_{L^2}^2 \leqslant C_1\biggl(\|u_0(0)\|_{H^1}^2 + \int_{-r}^0 \|u_0(s)\|_{H^1}^2\,\mathrm{d}s\biggr)\mathrm{e}^{(p^2/2d)t}, \quad t \in [0, T), \tag{2.11}$$

for some positive constant $C_1 > 0$. Relations (2.10) and (2.11) lead to (2.2). □

Theorem 2.2 now follows from propositions 2.3 and 2.2.

3. Stability of travelling waves

In this section, we prove theorem 1.3 by using a weighted energy method. The proof of theorem 1.4 is similar to that of theorem 1.3, we shall omit the details, and only give the key lemma 3.6. Let $N(t, x)$ be the solution of the Cauchy problem (1.1) and (1.3), and let $\phi(x + ct)$ be a travelling wave solution to (1.1). Set

$$v(t, \xi) = N(t, x) - \phi(\xi), \quad \xi = x + ct.$$

The original problem (1.1) and (1.3) can be reformulated as

$$\left.\begin{aligned} &v_t(t, \xi) + cv_\xi(t, \xi) - v_{\xi\xi}(t, \xi) + \mathrm{d}v(t, \xi) \\ &\qquad - pf'(\phi(\xi - cr))v(t - r, \xi - cr) = pQ(t - r, \xi - cr), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (t, \xi) \in \boldsymbol{R}_+ \times \boldsymbol{R}, \\ &v(s, \xi) = N_0(s, \xi) - \phi(\xi - cs) =: v_0(s, \xi), \quad (s, \xi) \in [-r, 0] \times \boldsymbol{R}. \end{aligned}\right\} \tag{3.1}$$

The nonlinear term $Q(t - r, \xi - cr)$ is

$$Q(t - r, \xi - cr) = f(\phi + v) - f(\phi) - f'(\phi)v, \tag{3.2}$$

where $\phi = \phi(\xi - cr)$ and $v = v(t - r, \xi - cr)$. Theorem 1.3 is equivalent to the following result.

THEOREM 3.1. *For a given travelling wave $\phi(\xi)$ ($\xi = x + ct$) with speed c satisfying (1.7), if $v_0(s, \xi) \in C^0([-r, 0]; H_w^1(\boldsymbol{R}))$, where $w(\xi)$ is the weight function defined in (1.4), then there exist positive constants $\delta_0 = \delta_0(d, p, \phi(x_* + cr))$ and $\mu = \mu(d, p, \phi(x_* + cr))$ such that, when $\sup_{s\in[-r,0]} \|v_0(s)\|_{H_w^1} \leqslant \delta_0$, the solution $v(t, \xi)$ of the Cauchy problem (3.1) satisfies*

$$v(t, \xi) \in C^0([0, \infty); H_w^1(\boldsymbol{R})) \cap L^2([0, \infty); H_w^2(\boldsymbol{R}))$$

and

$$\sup_{\xi\in\boldsymbol{R}} |v(t, \xi)| \leqslant C\mathrm{e}^{-\mu t}, \quad 0 \leqslant t \leqslant \infty. \tag{3.3}$$

For $\tau \geqslant 0$ and $T > 0$, define

$$X(\tau - r, T + \tau) = \{v \mid v(t, \xi) \in C^0([\tau - r, T + \tau]; H_w^1(\boldsymbol{R})) \cap L^2([\tau - r, T + \tau]; H_w^2(\boldsymbol{R}))\}$$

and

$$M_\tau(T) := \sup_{t\in[\tau-r,T+\tau]} \|v(t)\|_{H_w^1}.$$

When $\tau = 0$, we write $M(T) = M_0(T)$. The following local estimate can be derived by an elementary energy method. We omit the proof.

PROPOSITION 3.2 (local estimate). *Consider the Cauchy problem with the initial time* $\tau \geqslant 0$,

$$\left.\begin{aligned} &v_t(t,\xi) + cv_\xi(t,\xi) - v_{\xi\xi}(t,\xi) + \mathrm{d}v(t,\xi) \\ &\qquad - pf'(\phi(\xi - cr))v(t-r,\xi-cr) = pQ(t-r,\xi-cr), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (t,\xi) \in [\tau,\infty) \times \boldsymbol{R}, \\ &v(s,\xi) = N_0(s,\xi) - \phi(\xi - cs) =: v_\tau(s,\xi), \quad (s,\xi) \in [\tau - r, \tau] \times \boldsymbol{R}. \end{aligned}\right\} \tag{3.4}$$

If $v_\tau(s,\xi) \in H^1_w$ *and* $M_\tau(0) \leqslant \delta_1$ *for* $\delta_1 > 0$, *then there exists* $t_0 = t_0(\delta_1) > 0$ *such that* $v(t,\xi) \in X(\tau - r, \tau + t_0)$ *and* $M_\tau(t_0) \leqslant \sqrt{2(1+r)}M_\tau(0)$.

Next, we state a result on *a priori* estimate.

PROPOSITION 3.3 (*a priori* estimate). *Let* $v(t,\xi) \in X(-r,T)$ *be a local solution of (3.1) for a given constant* $T > 0$. *Then there exist positive constants* δ_2, μ *and* $C_2 > 1$, *independent of* T, *such that* $M(T) \leqslant \delta_2$ *implies*

$$\|v(t)\|^2_{H^1_w} + 2(1-\eta_0)\int_0^t \|v(s)\|^2_{H^2_w}\,\mathrm{d}s + 2\mu\int_0^t \|v(s)\|^2_{H^1_w}\,\mathrm{d}s \leqslant C_2\left(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v(s)\|^2_{H^1_w}\,\mathrm{d}s\right) \quad \textit{for } 0 \leqslant t \leqslant T \tag{3.5}$$

and

$$\|v(t)\|^2_{H^1_w} \leqslant C_2\left(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w}\,\mathrm{d}s\right)\mathrm{e}^{-2\mu t}, \quad 0 \leqslant t \leqslant T, \tag{3.6}$$

where $0 < \eta_0 < 1$ *is given in (1.5).*

The proof of proposition 3.3 will be given in the last part of this section. Based on propositions 3.2 and 3.3, we can prove theorem 3.1 using a continuation argument (cf. [7, 10–13]).

Proof of theorem 3.1. Let δ_2, μ and C_2 be constants in proposition 3.3, independent of T. Set

$$\delta_1 = \delta_2, \qquad \delta_0 = \frac{\delta_2}{\sqrt{2(1+r)}} \tag{3.7}$$

and

$$M(0) \leqslant \delta_0. \tag{3.8}$$

By proposition 3.2, there exists $t_0 = t_0(\delta_1) > 0$ such that $v(t,x) \in X(-r,t_0)$ and

$$M(t_0) \leqslant \sqrt{2(1+r)}M(0) \leqslant \sqrt{2(1+r)}\delta_0 \leqslant \delta_2.$$

On the interval $[0,t_0]$, applying proposition 3.3, we obtain (3.6) for $t \in [0,t_0]$, and

$$\sup_{t\in[0,t_0]} \|v(t)\|_{H^1_w} \leqslant \sup_{t\in[0,t_0]} \left\{C_2\left(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w}\,\mathrm{d}s\right)\right\}^{1/2}\mathrm{e}^{-\mu t} \leqslant \sqrt{C_2(1+r)}M(0) \leqslant \sqrt{C_2(1+r)}\delta_0 \leqslant \frac{\delta_2}{\sqrt{2(1+r)}}. \tag{3.9}$$

Now consider the Cauchy problem (3.4) at the initial time $\tau = t_0$. Using (3.8), (3.9) and (3.7), we obtain

$$\begin{aligned} M_{t_0}(0) &= \sup_{s\in[t_0-r,t_0]} \|v(s)\|_{H^1_w} \\ &\leqslant \max\Big\{\sup_{s\in[-r,0]} \|v(s)\|_{H^1_w}, \sup_{s\in[0,t_0]} \|v(s)\|_{H^1_w}\Big\} \\ &\leqslant \max\Big\{M(0), \frac{\delta_2}{\sqrt{2(1+r)}}\Big\} \\ &\leqslant \delta_1. \end{aligned} \tag{3.10}$$

Applying proposition 3.2 once more, we can show that $v(t,x) \in X(-r, 2t_0)$ and $M_{t_0}(t_0) \leqslant \sqrt{2(1+r)}M_{t_0}(0)$. On the other hand,

$$\begin{aligned} M_{t_0}(0) &= \sup_{t\in[t_0-r,t_0]} \|v(s)\|_{H^1_w} \\ &\leqslant \max\Big\{\sup_{s\in[-r,0]} \|v(s)\|_{H^1_w}, \sup_{s\in[0,t_0]} \|v(s)\|_{H^1_w}\Big\} \\ &\leqslant \max\Big\{\delta_0, \frac{\delta_2}{\sqrt{2(1+r)}}\Big\} \\ &\leqslant \frac{\delta_2}{\sqrt{2(1+r)}}, \end{aligned} \tag{3.11}$$

we have

$$M_{t_0}(t_0) \leqslant \sqrt{2(1+r)}M_{t_0}(0) \leqslant \delta_2.$$

Therefore,

$$\begin{aligned} M(2t_0) &= \sup_{s\in[-r,2t_0]} \|v(s)\|_{H^1_w} \\ &\leqslant \max\Big\{\sup_{s\in[-r,0]} \|v(s)\|_{H^1_w}, \sup_{s\in[0,t_0-r]} \|v(s)\|_{H^1_w}, \sup_{s\in[t_0-r,2t_0]} \|v(s)\|_{H^1_w}\Big\} \\ &\leqslant \max\Big\{\delta_0, \frac{\delta_2}{\sqrt{2(1+r)}}, \delta_2\Big\} \\ &\leqslant \delta_2. \end{aligned} \tag{3.12}$$

We can apply proposition 3.3 to obtain (3.6) for $0 \leqslant t \leqslant 2t_0$ and

$$\begin{aligned} \sup_{t\in[0,2t_0]} \|v(t)\|_{H^1_w} &\leqslant \sup_{t\in[0,2t_0]} \Big\{C_2\Big(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w}\,\mathrm{d}s\Big)\Big\}^{1/2} \mathrm{e}^{-\mu t} \\ &\leqslant \sqrt{C_2(1+r)}M(0) \leqslant \sqrt{C_2(1+r)}\delta_0 \leqslant \frac{\delta_2}{\sqrt{2(1+r)}}. \end{aligned} \tag{3.13}$$

Repeating the preceding procedure, we can prove $v(t,x) \in X(-r,\infty)$ and the relation (3.6) for all $0 \leqslant t < \infty$. Also (3.3) follows immediately from (3.6). This completes the proof of theorem 3.1. □

To prove proposition 3.3, we first give two lemmas.

LEMMA 3.4. *For the travelling wave solution $\phi(x+ct)$ with speed $c > 2\sqrt{p-d}$, there exists a unique number $x_* \in (-\infty, \infty)$ such that (1.6) holds, namely,*

$$\frac{d}{p} = f'(\phi(x_*)). \tag{3.14}$$

Furthermore,

$$\mu_0 := \min\{\tfrac{1}{4}(c^2 - 4(p-d)), d - pf'(\phi(x_* + cr))\} > 0. \tag{3.15}$$

Proof. Since $f'(z) = (1 - az)\mathrm{e}^{-az} \geqslant 0$ is strictly decreasing on $[0, 1/a]$ and

$$0 = N_- < N_+ = \frac{1}{a} \ln \frac{p}{d} < \frac{1}{a}$$

due to $1 < p/d < \mathrm{e}$ (see the assumptions in proposition 1.1), we know that $f'(z) > 0$ is strictly decreasing on $[N_-, N_+]$. On the other hand, $f'(N_-) = 1 > d/p$ and

$$0 < f'(N_+) = \left(1 - \ln \frac{p}{d}\right)\frac{d}{p} < \frac{d}{p}.$$

Therefore, $d/p \in (f'(N_+), f'(N_-))$ and there exists a unique $\phi_* \in (N_-, N_+)$ such that $f'(\phi_*) = d/p$. By the strict monotonicity of $\phi(\xi)$, there exists a unique $x_* \in (-\infty, \infty)$ such that $\phi(x_*) = \phi_*$ and $f'(\phi(x_*)) = d/p$.

To prove equation (3.15), we note that the assumption $c > 2\sqrt{p-d}$ implies $\frac{1}{4}(c^2 - 4(p-d)) > 0$. Relation (3.14), together with the fact that $f'(\phi(\xi))$ is strictly decreasing, implies that $d - pf'(\phi(x_* + cr)) > d - pf'(\phi(x_*)) = 0$. □

Now we are going to prove the following lemma, which plays a key role in the proof of the *a priori* estimates.

LEMMA 3.5. *Let $w(\xi)$ be the weight function as defined in (1.4)–(1.6) and let*

$$B_{\eta_0}(\xi) := -c\frac{w'(\xi)}{w(\xi)} - \frac{1}{2\eta_0}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 + 2d - pf'(\phi(\xi - cr)) - p\frac{w(\xi + cr)}{w(\xi)} f'(\phi(\xi)). \tag{3.16}$$

Then

$$B_{\eta_0}(\xi) \geqslant \mu_0 \quad \text{for all } \xi \in \boldsymbol{R}, \tag{3.17}$$

where μ_0 is given in (3.15).

Proof. We consider the following cases.

CASE 1 ($\xi \leqslant x_*$). In this case,

$$w(\xi) = \mathrm{e}^{-\alpha(\xi - x_* - cr)} \quad \text{and} \quad w(\xi + cr) = \mathrm{e}^{-\alpha(\xi - x_*)}.$$

Note that $f'(\phi(\xi))$ is strictly decreasing for $\xi \in (-\infty, \infty)$, so that

$$0 < \frac{p}{d}\left(1 - \ln \frac{p}{d}\right) = f'(N_+) < f'(\phi(\xi)) < f'(\phi(\xi - cr)) < f'(N_-) = 1.$$

From (1.5), (1.6) and (3.15) for the definitions of α, η_0 and μ_0, we obtain

$$\begin{aligned}B_{\eta_0}(\xi) &= c\alpha - \frac{1}{2\eta_0}\alpha^2 + 2d - pf'(\phi(\xi - cr)) - p\mathrm{e}^{-\alpha cr} f'(\phi(\xi)) \\ &> c\alpha - \frac{1}{2\eta_0}\alpha^2 + 2d - p - p\mathrm{e}^{-\alpha cr} > c\alpha - \frac{1}{2\eta_0}\alpha^2 + 2(d-p) \\ &= \tfrac{1}{4}(c^2 - 4(p-d)) \geqslant \mu_0. \end{aligned} \tag{3.18}$$

Case 2 ($x_* < \xi \leqslant x_* + cr$). In this case, $w(\xi) = \mathrm{e}^{-\alpha(\xi - x_* - cr)}$ and $w(\xi + cr) = 1$. Thus

$$\begin{aligned}B_{\eta_0}(\xi) &= c\alpha - \frac{1}{2\eta_0}\alpha^2 + 2d - pf'(\phi(\xi - cr)) - p\mathrm{e}^{\alpha(\xi - x_* - cr)} f'(\phi(\xi)) \\ &> c\alpha - \frac{1}{2\eta_0}\alpha^2 + 2d - p - p\mathrm{e}^{\alpha(\xi - x_* - cr)} \\ &\geqslant c\alpha - \frac{1}{2\eta_0}\alpha^2 + 2(d-p) \geqslant \mu_0. \end{aligned} \tag{3.19}$$

Case 3 ($\xi > x_* + cr$). In this case, $w(\xi) = w(\xi + cr) = 1$. Since $0 < f'(\phi(\xi)) < f'(\phi(x_* + cr))$ and $0 < f'(\phi(\xi - cr)) < f'(\phi(x_*)) = d/p$, we have

$$\begin{aligned}B_{\eta_0}(\xi) &= 2d - pf'(\phi(\xi - cr)) - pf'(\phi(\xi)) \\ &> 2d - pf'(\phi(x_*)) - pf'(\phi(x_* + cr)) \\ &= d - pf'(\phi(x_* + cr)) \geqslant \mu_0. \end{aligned} \tag{3.20}$$

Relation (3.17) follows from (3.18)–(3.20), and the proof is complete. □

The next key lemma is for the proof of theorem 1.4.

Lemma 3.6. *Let (1.9) hold and $w_2(\xi)$ be the weight function as defined in (1.10), and let*

$$B_{\eta_2}(\xi) := -c\frac{w_2{}'(\xi)}{w_2(\xi)} - \frac{1}{2\eta_0}\left(\frac{w_2{}'(\xi)}{w_2(\xi)}\right)^2 + 2d - pf'(\phi(\xi - cr)) - p\frac{w_2(\xi + cr)}{w_2(\xi)} f'(\phi(\xi)), \tag{3.21}$$

where η_2 is a constant satisfying

$$0 < \frac{c^2}{4c^2 - 16(p-d) + 8p(1 - e^{-c^2 r/2})} < \eta_2 < 1. \tag{3.22}$$

Then

$$B_{\eta_2}(\xi) \geqslant \mu_3 \quad \textit{for all } \xi \in \boldsymbol{R}, \tag{3.23}$$

where

$$\mu_3 := \min\left\{\tfrac{1}{2}c^2 - \frac{1}{8\eta_2}c^2 - 2(p-d) + p(1 - \mathrm{e}^{-c^2 r/2}), d - pf'(\phi(x_* + cr))\right\} > 0. \tag{3.24}$$

The proof of this lemma is omitted, because it can be obtained in the similar way of lemma 3.5.

Proof of proposition 3.3. Let $w(\xi)$ be a weight function to be specified later. Multiplying (3.1) by $w(\xi)v(t,\xi)$, we have

$$\begin{aligned}\{\tfrac{1}{2}wv^2\}_t + \{\tfrac{1}{2}cwv^2 - wvv_\xi\}_\xi + wv_\xi^2 + w'v_\xi v \\ + \left\{-\tfrac{1}{2}c\frac{w'}{w} + d\right\}wv^2 - pwvf'(\phi(\xi - cr))v(t-r,\xi-cr) \\ = pwvQ(t-r,\xi-cr), \qquad (3.25)\end{aligned}$$

where $w = w(\xi)$, $v = v(t,\xi)$. Using the Cauchy–Schwarz inequality, we obtain

$$|w'(\xi)v_\xi(t,\xi)v(t,\xi)| \leqslant \eta wv_\xi^2 + \frac{1}{4\eta}\left(\frac{w'}{w}\right)^2 wv^2$$

for any $\eta > 0$. Substituting it into (3.25) and integrating the resulting inequality over $[0,t]\times \boldsymbol{R}$, we obtain

$$\begin{aligned}&\|v(t)\|_{L_w^2}^2 + 2(1-\eta)\int_0^t \|v_\xi(s)\|_{L_w^2}^2\,\mathrm{d}s \\ &\quad + \int_0^t\int_R \left\{-c\frac{w'(\xi)}{w(\xi)} + 2d - \frac{1}{2\eta}\left(\frac{w'(\xi)}{w(\xi)}\right)^2\right\}w(\xi)v(s,\xi)^2\,\mathrm{d}\xi\mathrm{d}s \\ &\quad - 2p\int_0^t\int_{-\infty}^{\infty} f'(\phi(\xi-cr))w(\xi)v(s,\xi)v(s-r,\xi-cr)\,\mathrm{d}\xi\mathrm{d}s \\ &\qquad \leqslant \|v_0(0)\|_{L_w^2}^2 + 2p\int_0^t\int_{-\infty}^{\infty} w(\xi)v(s,\xi)Q(s-r,\xi-cr)\,\mathrm{d}\xi\mathrm{d}s.\end{aligned} \tag{3.26}$$

Using the Cauchy–Schwarz inequality and the fact $f'(\phi(\xi - cr)) > 0$ for all $\xi \in \boldsymbol{R}$ (see the proof of lemma 3.5), and making the change of variables $\xi - cr \to \xi$, $s - r \to s$, we can bound the delay term on the left-hand side of (3.26) by

$$\begin{aligned}&2p\left|\int_0^t\int_{-\infty}^{\infty} w(\xi)f'(\phi(\xi-cr))v(s-r,\xi-cr)v(s,\xi)\,\mathrm{d}\xi\mathrm{d}s\right| \\ &\quad\leqslant p\int_0^t\int_{-\infty}^{\infty} w(\xi)f'(\phi(\xi-cr))[v^2(s,\xi)+v^2(s-r,\xi-cr)]\,\mathrm{d}\xi\mathrm{d}s \\ &\quad= p\int_0^t\int_{-\infty}^{\infty} w(\xi)f'(\phi(\xi-cr))v^2(s,\xi)\,\mathrm{d}\xi\mathrm{d}s \\ &\qquad + p\int_{-r}^{t-r}\int_{-\infty}^{\infty} w(\xi+cr)f'(\phi(\xi))v^2(s,\xi)\,\mathrm{d}\xi\mathrm{d}s \\ &\quad\leqslant p\int_0^t\int_{-\infty}^{\infty} w(\xi)f'(\phi(\xi-cr))v^2(s,\xi)\,\mathrm{d}\xi\mathrm{d}s \\ &\qquad + p\int_0^t\int_{-\infty}^{\infty} w(\xi+cr)f'(\phi(\xi))v^2(s,\xi)\,\mathrm{d}\xi\mathrm{d}s \\ &\qquad + p\int_{-r}^{0}\int_{-\infty}^{\infty} w(\xi+cr)f'(\phi(\xi))v_0^2(s,\xi)\,\mathrm{d}\xi\mathrm{d}s.\end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.26) yields

$$\|v(t)\|_{L_w^2}^2 + 2(1-\eta)\int_0^t \|v_\xi(s)\|_{L_w^2}^2\,\mathrm{d}s + \int_0^t\int_{-\infty}^{\infty} B_\eta(\xi)w(\xi)v^2(s,\xi)\,\mathrm{d}\xi\mathrm{d}s$$
$$\leqslant \|v_0(0)\|_{L_w^2}^2 + p\int_{-r}^0\int_{-\infty}^{\infty} w(\xi+cr)f'(\phi(\xi))v_0^2(s,\xi)\,\mathrm{d}\xi\mathrm{d}s$$
$$+ 2\int_0^t\int_{-\infty}^{\infty} w(\xi)v(s,\xi)Q(s-r,\xi-cr)\,\mathrm{d}\xi\mathrm{d}s, \quad (3.28)$$

where

$$B_\eta(\xi) := -c\frac{w'(\xi)}{w(\xi)} - \frac{1}{2\eta}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 + 2d - pf'(\phi(\xi-cr)) - p\frac{w(\xi+cr)}{w(\xi)}f'(\phi(\xi)). \quad (3.29)$$

We select a suitable weight function $w(\xi)$ for a given $0<\eta<1$ so that $B_\eta(\xi)>0$ for all $\xi \in R$. Set

$$\eta = \eta_0 = \frac{c^2+4(p-d)}{2c^2}$$

and

$$w(\xi) = \begin{cases} \mathrm{e}^{-\alpha(\xi-x_*-cr)}, & x \leqslant x_*+cr, \\ 1, & x > x_*+cr. \end{cases}$$

Then, according to lemma 3.5,

$$B_{\eta_0}(\xi) \geqslant \mu_0 > 0 \quad (3.30)$$

for the positive constant μ_0 defined in (3.15).

Next, we estimate the nonlinear term on the right-hand side of (3.28). By using the standard Sobolev embedding inequality $H^1(\boldsymbol{R}) \hookrightarrow C^0(\boldsymbol{R})$, and the modified inequality $H_w^1(\boldsymbol{R}) \hookrightarrow H^1(\boldsymbol{R})$ for $w(\xi)$ given in (1.4) (the proof can be similarly given as in [12]), we first have

$$|v(t,\xi)| \leqslant \sup_{\xi\in R}|v(t,\xi)| \leqslant C\|v(t,\cdot)\|_{H^1}$$
$$\leqslant C\|v(t,\cdot)\|_{H_w^1} \leqslant CM(t). \quad (3.31)$$

Then, applying Taylor's formula to (3.2) to get

$$|Q(t-r,\xi-cr)| \sim C|v(t-r,\xi-cr)|^2,$$

and noting $w(\xi+cr)/w(\xi) \leqslant C$ for all $\xi \in R$, as in the above estimate for the linear delay term, we finally have

$$\int_0^t\int_{-\infty}^{\infty} w(\xi)v(s,\xi)Q(s-r,\xi-cr)\,\mathrm{d}\xi\mathrm{d}s$$
$$\leqslant CM(t)\int_0^t\int_{-\infty}^{\infty} w(\xi)|v(s-r,\xi-cr)|^2\,\mathrm{d}\xi\mathrm{d}s$$
$$= CM(t)\int_{-r}^{t-r}\int_{-\infty}^{\infty} w(\xi+cr)|v(s,\xi)|^2\,\mathrm{d}\xi\mathrm{d}s$$

$$\leqslant CM(t)\left\{\int_0^t\int_{-\infty}^{\infty}\frac{w(\xi+cr)}{w(\xi)}w(\xi)|v(s,\xi)|^2\,\mathrm{d}\xi\mathrm{d}s + \int_{-r}^0\int_{-\infty}^{\infty}\frac{w(\xi+cr)}{w(\xi)}w(\xi)|v_0(s,\xi)|^2\,\mathrm{d}\xi\mathrm{d}s\right\}$$

$$\leqslant CM(t)\left\{\int_0^t \|v(s)\|^2_{L^2_w}\,\mathrm{d}s + \int_{-r}^0 \|v_0(s)\|^2_{L^2_w}\,\mathrm{d}s\right\}. \tag{3.32}$$

Substituting (3.30) and (3.32) into (3.28), we have

$$\|v(t)\|^2_{L^2_w} + 2(1-\eta_0)\int_0^t \|v_\xi(s)\|^2_{L^2_w}\,\mathrm{d}s + [\mu_0 - C_3M(t)]\int_0^t \|v(s)\|^2_{L^2_w}\,\mathrm{d}s \leqslant \|v_0(0)\|^2_{L^2_w} + C_4[1+M(t)]\int_{-r}^0 \|v_0(s)\|^2_{L^2_w}\,\mathrm{d}s \tag{3.33}$$

for some constants $C_3 > 0$ and $C_4 > 0$.

Let δ_2 be such that

$$\mu_0 - C_3\delta_2 > 0, \quad \text{i.e. } \delta_2 < \mu_0/C_3. \tag{3.34}$$

Clearly, δ_2 can be chosen so that it depends only on c, p, d and $\phi(x_* + cr)$, since

$$\mu_0 = \mu_0(c,d,p,\phi(x_*+cr))$$

(see (3.15)). Define

$$\mu := \tfrac{1}{2}(\mu_0 - C_3\delta_2). \tag{3.35}$$

When $M(T) \leqslant \delta_2$, we have

$$\mu_0 - C_3M(T) \geqslant \mu_0 - C_3\delta_2 = 2\mu$$

and

$$\|v(t)\|^2_{L^2_w} + 2(1-\eta_0)\int_0^t \|v_\xi(s)\|^2_{L^2_w}\,\mathrm{d}s + 2\mu\int_0^t \|v(s)\|^2_{L^2_w}\,\mathrm{d}s \leqslant \|v_0(0)\|^2_{L^2_w} + C\int_{-r}^0 \|v_0(s)\|^2_{L^2_w}\,\mathrm{d}s. \tag{3.36}$$

Similarly, by differentiating equation (3.1) with respect to ξ, multiplying the result by $w(\xi)v_\xi(t,\xi)$ and then integrating over $[0,t]\times\boldsymbol{R}$ for $t \leqslant T$, we obtain (using the basic energy estimate (3.36))

$$\|v_\xi(t)\|^2_{L^2_w} + 2(1-\eta_0)\int_0^t \|v_{\xi\xi}(s)\|^2_{L^2_w}\,\mathrm{d}s + 2\mu\int_0^t \|v_\xi(s)\|^2_{L^2_w}\,\mathrm{d}s \leqslant C\left(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w}\,\mathrm{d}s\right), \tag{3.37}$$

provided $M(T) \leqslant \delta_2$. The details are omitted. Combining (3.36) and (3.37), we have

$$\|v(t)\|^2_{H^1_w} + 2(1-\eta_0)\int_0^t \|v_{\xi\xi}(s)\|^2_{L^2_w}\,\mathrm{d}s + 2\mu\int_0^t \|v(s)\|^2_{H^1_w}\,\mathrm{d}s \leqslant C_2\biggl(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w}\,\mathrm{d}s\biggr), \quad 0 \leqslant t \leqslant T, \tag{3.38}$$

for some absolute constant $C_2 > 0$, independent of T and $v(t,x)$. By Gronwall's inequality, we have

$$\|v(t)\|^2_{H^1_w} \leqslant C_2\biggl(\|v_0(0)\|^2_{H^1_w} + \int_{-r}^0 \|v_0(s)\|^2_{H^1_w}\,\mathrm{d}s\biggr)\mathrm{e}^{-2\mu t}, \quad 0 \leqslant t \leqslant T.$$

The proof is complete. □

Acknowledgments

M.M. is grateful to Professor Martin Gander for helpful discussions and suggestions. Research is partly supported by NSERC. M.Y.L. is also supported by NSF grant DMS0078250.

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(*Issued 29 June 2004*)