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Asymptotic Sufficiency of Maximum Likelihood Estimator in a Truncated Location Family

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Introduction

Let $f(x)$ be a probability density function on real line which vanishes on $(-\infty, 0]$ and twice continuously differentiable in $(0, \infty)$. We consider the case that for $\alpha \geqq 2$, $f(x) \sim Ax^{\alpha-1}$ as $x \to +0$ and $f'(x) \sim Bx^{\alpha-2}$ as $x \to +0$ $(0<A, B<\infty)$. Let $X_1, \cdots, X_n$ be an independent identically distributed random sample of size n $(n=1, 2, \cdots)$ according to a distribution P_θ with density $f(x-\theta)$, and let $\{\hat{\theta}_n\} = \{\hat{\theta}_n(X_1, \cdots, X_n)\}$ be the maximum likelihood estimator (or MLE) of θ. In this paper we prove that under some assumptions (See Section 1.), $\{\hat{\theta}_n\}$ is asymptotically sufficient statistic for $\{P_\theta: \theta \in \Theta\}$ in the sense of LeCam [5]. Our theory of asymptotic sufficiency of MLE is based on the asymptotic properties of MLE and likelihood function, which have been studied in non-regular cases by Akahira [1], Takeuchi [6], Takeuchi and Akahira [7] and Woodroofe [9]. Asymptotic sufficiency of MLE has been discussed under the regular conditions by Kaufman [4] and LeCam [5]. In Akahira [2], asymptotic sufficiency has been discussed in a non-regular case when the density function, with a location parameter, has a compact support on R^1 and positive values at the end points.

In Section 1 notations and assumptions are stated, and in Section 2 we state some known results concerning order of consistency of MLE and $\min(X_1, \cdots, X_n)$ (cf. [1], [6], [7], [9]). In Section 3 we will show that MLE is asymptotically sufficient for $\{P_\theta: \theta \in \Theta\}$ in our non-regular case.

§1. Notations and assumptions.

Let X be a sample space whose generic point is denoted by x, $\mathscr{B}$ a σ-field of subset of X and $\{P_\theta: \theta \in \Theta\}$ a set of probability measures on

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$\mathscr{B}$, where Θ is called a parameter space. In this paper it will be assumed that $X=\Theta=R^1$. For each $n=1, 2, \cdots$, let $(X^n, \mathscr{B}^n)$ be the cartesian product of n copies of $(X, \mathscr{B})$ and $P_{n\theta}$ corresponding product measure of P_θ. The point of X^n will be denoted by $\tilde{x}_n=(x_1, \cdots, x_n)$ and the corresponding random sample by $\tilde{X}_n=(X_1, \cdots, X_n)$. We suppose that $P_\theta(\theta\in\Theta)$ is absolutely continuous with respect to the Lebesgue measure μ on R^1. Then we denote the density $dP_\theta/d\mu$ by $f(x, \theta)$.

We suppose that θ is a location parameter (i.e., $f(x, \theta)=f(x-\theta)$) and consider following assumptions (I), (II), (III), (IV) and (V).

(I) $f(x)>0$ if $x>0$

$f(x)=0$ if $x\leqq 0$

(II) $f(x)$ is twice continuously differentiable in $(0, \infty)$, and for $\alpha\geqq 2$

$$\lim_{x\to+0} x^{1-\alpha}f(x)=A \qquad 0<A<\infty ,$$
$$\lim_{x\to+0} x^{2-\alpha}f'(x)=B \qquad 0<B<\infty ,$$

and

$$\lim_{x\to+\infty} f(x)=0$$

and $f''(x)$ is a bounded function.

Let $g(x)=\log f(x)$ $(0<x<\infty)$.

(III) $\int_0^\infty |g(x)|f(x)dx<\infty$,

(IV) for every $\delta>0$, $\int_\delta^\infty g'(x)^2f(x)dx<\infty$,

(V) for every $a>0$, there exists a δ $(0<\delta<a)$ for which

$$\int_a^\infty \sup_{|t|\leqq\delta} |g''(x-t)|f(x)dx<\infty .$$

These assumptions are much the same as Woodroofe's conditions in [9] except for the assumption (III), but the assumption (II) is slightly different from his condition. The assumption (III) will be needed to prove the consistency of MLE (cf. Wald [8]).

§2. Order of consistency of MLE and minimum statistic.

Under the assumption (II), if $X_1, \cdots, X_n$ is an independent identically distributed random sample from the population with density $f(x-\theta)$, then maximum likelihood estimators exist in the interval $(-\infty, M_n)$, where $M_n=\min(X_1, \cdots, X_n)$. We denote it by $\{\hat{\theta}_n\}=\{\hat{\theta}_n(X_1, \cdots, X_n)\}$.

We have the following lemma by the similar method as in Akahira [1], Takeuchi and Akahira [7] and Cramér [3].

LEMMA 2.1. *Suppose that the assumptions* (I), (II), (IV) *and* (V) *are satisfied with* $\alpha>2$. *Then* $I<\infty$ *and* $I=-\int_0^\infty g''(x)f(x)dx$, *where* $I=\int_0^\infty g'(x)^2 f(x)dx$ *denotes Fisher information number.*

The first part in the following theorem is obtained in [1], [6], [7],, [8] and [9], and the second part is obtained in [1], [3], [6], [7] and [8].

THEOREM 2.1. *Suppose that the assumptions* (I)$\sim$(V) *are satisfied.*

(i) *If* $\alpha=2$, *then for any compact subset* K *of* Θ, $\sqrt{c_1 n\log n}\,(\hat{\theta}_n-\theta)$ *converges in law to the standard normal distribution* $N(0,1)$ *as* $n\to\infty$ *uniformly in* $\theta\in K$, *where* $c_1=B^2/2A$.

(ii) *If* $\alpha>2$, *then for any compact subset* K *of* Θ, $\sqrt{nI}\,(\hat{\theta}_n-\theta)$ *converges in law to the standard normal distribution* $N(0,1)$ *as* $n\to\infty$ *uniformly in* $\theta\in K$.

The following definition is due to Akahira [1] or Takeuchi [6].

DEFINITION 2.1. For an increasing sequence of positive numbers $\{c_n\}$ (c_n tending to infinity) an estimator $\{T_n\}$ ($n=1, 2, \cdots$) is called *consistent with order* $\{c_n\}$, if for every $\varepsilon>0$ and every $\theta'\in\Theta$, there exist a sufficiently small number δ and a sufficiently large number L such that

$$\overline{\lim_{n\to\infty}}\ \sup_{\theta:|\theta-\theta'|<\delta} P_{n\theta}(\{c_n|T_n-\theta|\geqq L\})<\varepsilon\ .$$

By Definition 2.1 and Theorem 2.1 we can state that if $\alpha=2$ then MLE is consistent with order $\{(n\log n)^{1/2}\}$, and if $\alpha>2$ then MLE is consistent with order $\{n^{1/2}\}$.

Next we state a result concerning M_n.

THEOREM 2.2 (Woodroofe [9]). *Suppose that the assumptions* (I), (II) *are satisfied. If* $\alpha\geqq 2$, *then* M_n *is consistent with order* $\{n^{1/\alpha}\}$.

More precisely, it will be obtained that if $\alpha\geqq 2$, then for all $t>0$, $P_{n\theta}(n^{1/\alpha}(M_n-\theta)>t)\to\exp(-At^\alpha/\alpha)$ as $n\to\infty$ uniformly in $\theta\in\Theta$. However we will not require the exact limit distribution for M_n in the sequel.

§3. Asymptotic sufficiency of MLE.

In the beginning we state some lemmas.

LEMMA 3.1 (Woodroofe [9]). *Suppose that the assumptions* (I) *and* (II) *are satisfied with* $\alpha=2$. *Let* $0<\delta<\infty$ *and define* $Z_i=X_i^{-1}$ *if* $0<X_i<\delta$

and $Z_i=0$ *if* $X_i \geqq \delta$ $i=1, 2, \cdots, n$. *Then*

$$(c_2 n \log n)^{-1} \sum_{i=1}^{n} Z_i^2 \longrightarrow 1 \quad \textit{in probability as } n \longrightarrow \infty ,$$

where $c_2 = A/2$.

We define

$$p_n(\tilde{x}_n, \theta) = \prod_{i=1}^{n} f(x_i - \theta)$$
$$\lambda_n(\tilde{x}_n, \theta) = \log p_n(\tilde{x}_n, \theta) \quad \text{if } M_n > \theta$$
$$G_n''(t) = [\partial^2 \lambda_n(\tilde{x}_n, \theta)/\partial\theta^2]_{\theta=t} .$$

Next we state a result concerning likelihood function. The following lemma is a slight modification of Lemma 3.4 in [9] and it will be shown by similar method.

LEMMA 3.2. *Suppose that the assumptions* (I), (II) *and* (V) *are satisfied.*

(i) *If* $\alpha=2$, *then for positive* β_n *satisfying* $\beta_n^{-1} = o(n^{-1/2})$,

$$\sup_{|t| \leqq 1} |(c_1 n \log n)^{-1} G_n''(\theta + t\beta_n^{-1}) + 1| \longrightarrow 0$$

in $P_{n\theta}$*-probability as* $n \to \infty$ *uniformly in* $\theta \in \Theta$.

(ii) *If* $\alpha > 2$, *then for positive* β_n *satisfying* $\beta_n^{-1} = o(n^{-1/\alpha})$,

$$\sup_{|t| \leqq 1} |n^{-1} G_n''(\theta + t\beta_n^{-1}) + I| \longrightarrow 0$$

in $P_{n\theta}$*-probability as* $n \to \infty$ *uniformly in* $\theta \in \Theta$.

PROOF. Since θ is a location parameter, we can restrict our attention to the case $\theta=0$.

At first we prove the part (i). From the assumption (II), we have $g''(x) \sim -B^2/(A^2x^2)$ as $x \to +0$. For arbitrarily given $0<\varepsilon<1$, let $a>0$ be so small that $|(A^2x^2g''(x))/B^2+1| \leqq \varepsilon$ for $0<x\leqq 2a$. For $0<c<d\leqq\infty$, let $\sum_c^d$ denote the summation over all $i=1, 2, \cdots, n$ for which $c<X_i<d$. If $M_n \geqq \beta_n^{-1}/\varepsilon$, which holds with probability approaching to one, then for t and β_n satisfying $|t|\leqq 1$ and $\beta_n^{-1}<a$ respectively,

$$\begin{aligned}
(3.1) \qquad & (c_1 n \log n)^{-1} G_n''(t\beta_n^{-1}) \\
&= (c_1 n \log n)^{-1} \Big(\sum_0^a g''(X_i - t\beta_n^{-1}) + \sum_a^\infty g''(X_i - t\beta_n^{-1}) \Big) \\
&\leqq -\frac{B^2(1-\varepsilon)}{A^2} (c_1 n \log n)^{-1} \sum_0^a (X_i - t\beta_n^{-1})^{-2}
\end{aligned}$$

$$+(c_1 n \log n)^{-1} \sum_a^\infty \sup_{|t| \leq \beta_n^{-1}} |g''(X_i - t)|$$

$$\leqq -(1-\varepsilon)(1+\varepsilon)^{-2}(c_2 n \log n)^{-1} \sum_0^a X_i^{-2} + o_p(1)$$

$$\longrightarrow -(1-\varepsilon)(1+\varepsilon)^{-2} \text{ in } P_{n0}\text{-probability as } n \longrightarrow \infty .$$

We have used Lemma 3.1 and assumption (V) in the final steps in (3.1). Similarly we obtain

$$\text{(3.2)} \qquad \varliminf_{n\to\infty} (c_1 n \log n)^{-1} G_n''(t\beta_n^{-1}) \geqq -(1+\varepsilon)(1-\varepsilon)^{-2}$$

in P_{n0}-probability. Since $\varepsilon > 0$ is arbitrary, from (3.1) and (3.2), we have completed the proof of part (i).

Next we prove the part (ii). For arbitrarily given $0 < \varepsilon < 1$, probability of the event $M_n \geqq \beta_n^{-1}/\varepsilon$ approaches to one as $n \to \infty$. From the assumption (II), if $2 < \alpha < 3$ then $g''(x) \sim -B^2/A^2x^2$ as $x \to +0$ and if $\alpha \geqq 3$ then $x^{\alpha-1}g''(x) = O(1)$ as $x \to +0$. Therefore, we divide the proof into two cases. At first we prove the lemma in the case $2 < \alpha < 3$. Let a so small that

$$\left| \frac{A^2x^2}{B^2} g''(x) + 1 \right| \leqq \varepsilon \quad \text{for } 0 < x \leqq 2a .$$

If $M_n \geqq \beta_n^{-1}/\varepsilon$, then for t and β_n satisfying $|t| \leqq 1$ and $\beta_n^{-1} < a$ respectively and for a suitable $\delta > 0$ and $b > a$ we have

$$\begin{aligned}
&n^{-1}G_n''(t\beta_n^{-1}) \\
&= n^{-1}\Big(\sum_0^a g''(X_i - t\beta_n^{-1}) + \sum_a^\infty g''(X_i - t\beta_n^{-1}) \Big) \\
&\leqq -\frac{B^2(1-\varepsilon)}{nA^2} \sum_0^a (X_i - t\beta_n^{-1})^{-2} + \frac{1}{n} \sum_a^\infty \sup_{|t| \leq \beta_n^{-1}} g''(X_i - t) \\
&\leqq -\frac{B^2(1-\varepsilon)(1+\varepsilon)^{-2}}{nA^2} \sum_0^a X_i^{-2} + \frac{1}{n} \sum_a^\infty \sup_{|t| \leq \beta_n^{-1}} g''(X_i - t) \\
&\leqq \frac{(1-\varepsilon)(1+\varepsilon)^{-3}}{n} \sum_0^a g''(X_i) + \frac{1}{n} \sum_a^b \sup_{|t| \leq \delta} g''(X_i - t) \\
&\quad + \frac{1}{n} \sum_b^\infty \sup_{|t| \leq \delta} g''(X_i - t) \\
&= (1-\varepsilon)(1+\varepsilon)^{-3} J_{1n}(a) + J_{2n}(a, b, \delta) + J_{3n}(b, \delta) ,
\end{aligned}$$

where

$$J_{1n}(a) = \frac{1}{n} \sum_0^a g''(X_i) ,$$

$$J_{2n}(a, b, \delta)=\frac{1}{n}\sum_{a}^{b}\sup_{|t|\leqq\delta} g''(X_t-t) ,$$

$$J_{3n}(b, \delta)=\frac{1}{n}\sum_{b}^{\infty}\sup_{|t|\leqq\delta} g''(X_t-t) .$$

By Lemma 2.1 and assumption (V),

(3.3) $J_{1n}(a)\longrightarrow J_1(a)$ in P_{n0}-probability as $n\longrightarrow\infty$,

(3.4) $J_{2n}(a, b, \delta)\longrightarrow J_2(a, b, \delta)$ in P_{n0}-probability as $n\longrightarrow\infty$,

(3.5) $J_{3n}(b, \delta)\longrightarrow J_3(b, \delta)$ in P_{n0}-probability as $n\longrightarrow\infty$,

where

$$J_1(a)=\int_0^a g''(x)f(x)dx ,$$

$$J_2(a, b, \delta)=\int_a^b \sup_{|t|\leqq\delta} g''(x-t)f(x)dx ,$$

$$J_3(b, \delta)=\int_b^\infty \sup_{|t|\leqq\delta} g''(x-t)f(x)dx .$$

From Lemma 2.1, assumption (V) and the continuousness of $g''(x)$, we obtain that for sufficiently small a, sufficiently large b and suitable δ,

(3.6) $J_1(a)<\varepsilon$,

(3.7) $J_2(a, b, \delta)<\int_a^b g''(x)f(x)dx+\varepsilon<-I+2\varepsilon$,

(3.8) $J_3(b, \delta)<\varepsilon$.

By (3.3)~(3.8), we have

(3.9)
$$\begin{aligned}&(1-\varepsilon)(1+\varepsilon)^{-3}J_{1n}(a)+J_{2n}(a, b, \delta)+J_{3n}(b, \delta)\\ &\longrightarrow(1-\varepsilon)(1+\varepsilon)^{-3}J_1(a)+J_2(a, b, \delta)+J_3(b, \delta)\\ &\qquad(\text{in } P_{n0}\text{-probability as } n\longrightarrow\infty)\\ &\leqq\varepsilon(1-\varepsilon)(1+\varepsilon)^{-3}+3\varepsilon-I .\end{aligned}$$

Similarly we obtain

(3.10) $\varliminf_{n\to 0} n^{-1}G_n''(t\beta_n^{-1})\geqq-\varepsilon(1+\varepsilon)(1-\varepsilon)^{-3}-3\varepsilon-I$ in P_{n0}-probability .

Since $\varepsilon>0$ is arbitrary, from (3.9) and (3.10) we have completed the proof in the case $2<\alpha<3$.

Next we prove the lemma in the case $3\leqq\alpha$. For some constant

$M > 0$, let a be so small that

$$|x^{\alpha-1}g''(x)| \leqq M \quad \text{for } 0<x\leqq 2a .$$

If $M_n \geqq \beta_n^{-1}/\varepsilon$, then for t and β_n satisfying $|t|\leqq 1$ and $\beta_n^{-1}<a$ respectively and for a suitable $\delta>0$ and $b>a$ we have

$$\begin{aligned}
&\frac{1}{n}G_n''(t\beta_n^{-1}) \\
&\leqq \frac{M}{n}\sum_0^a (X_i-t\beta_n^{-1})^{1-\alpha}+\frac{1}{n}\sum_a^\infty g''(X_i-t\beta_n^{-1}) \\
&\leqq \frac{M(1+\varepsilon)^{1-\alpha}}{n}\sum_0^a X_i^{1-\alpha}+\frac{1}{n}\sum_a^b \sup_{|t|\leqq\delta} g''(X_i-t) \\
&\quad +\frac{1}{n}\sum_b^\infty \sup_{|t|\leqq\delta} g''(X_i-t) \\
&= M(1+\varepsilon)^{1-\alpha}J_{1n}'(a)+J_{2n}'(a, b, \delta)+J_{3n}'(b, \delta) ,
\end{aligned}$$

where

$$\begin{aligned}
J_{1n}'(a) &= \frac{1}{n}\sum_0^a X_i^{1-\alpha} , \\
J_{2n}'(a, b, \delta) &= \frac{1}{n}\sum_a^b \sup_{|t|\leqq\delta} g''(X_i-t) , \\
J_{3n}'(b, \delta) &= \frac{1}{n}\sum_b^\infty \sup_{|t|\leqq\delta} g''(X_i-t) .
\end{aligned}$$

By the assumptions (II) and (V),

(3.11) $J_{1n}'(a) \longrightarrow M'a$ in P_{n0}-probability as $n\to\infty$, where $M'>0$ is some constant,

(3.12) $J_{2n}'(a, b, \delta) \longrightarrow J_2'(a, b, \delta)$ in P_{n0}-probability as $n\to\infty$,

(3.13) $J_{3n}'(b, \delta) \longrightarrow J_3'(b, \delta)$ in P_{n0}-probability as $n\to\infty$,

where

$$\begin{aligned}
J_2'(a, b, \delta) &= \int_a^b \sup_{|t|\leqq\delta} g''(x-t)f(x)dx , \\
J_3'(b, \delta) &= \int_b^\infty \sup_{|t|\leqq\delta} g''(x-t)f(x)dx .
\end{aligned}$$

According to the similar method treated in the case $2<\alpha<3$, we have for a sufficiently small $a>0$ and a sufficiently large b, (3.14) $$M'a<\varepsilon\ ,$$

(3.15) $$J_2'(a,\ b,\ \delta)<-I+2\varepsilon\ ,$$

(3.16) $$J_3'(b,\ \delta)<\varepsilon\ .$$

From (3.11)~(3.16),

(3.17) $$\begin{aligned} &M(1+\varepsilon)^{1-\alpha}J_{1n}'(a)+J_{2n}'(a,\ b,\ \delta)+J_{3n}'(b,\ \delta) \\ &\longrightarrow MM'(1+\varepsilon)^{1-\alpha}a+J_2'(a,\ b,\ \delta)+J_3'(b,\ \delta)\ \text{(in } P_{n0}\text{-probability as } n\longrightarrow\infty) \\ &\leqq M\varepsilon(1+\varepsilon)^{1-\alpha}+3\varepsilon-I\ . \end{aligned}$$

Similarly we obtain

(3.18) $$\varliminf_{n\to\infty} n^{-1}G_n''(t\beta_n^{-1})\geqq -M\varepsilon(1+\varepsilon)^{1-\alpha}-3\varepsilon-I\ \text{in } P_{n0}\text{-probability as } n\longrightarrow\infty\ .$$

By (3.17) and (3.18), we have completed the proof in the case $3\leqq\alpha$, since $\varepsilon>0$ is arbitrary.

In the following we make use of next notations.

$$\begin{aligned} A_{n\theta}^{(1)}(\delta)&=\{\tilde{x}_n:\ \sqrt{n\log n}\ |\hat{\theta}_n-\theta|<\delta\} \\ A_{n\theta}^{(2)}(\delta)&=\{\tilde{x}_n:\ \sqrt{n}\ |\hat{\theta}_n-\theta|<\delta\} \\ B_n^{(1)}(\varepsilon)&=\{\tilde{x}_n:\ |(c_1 n\log n)^{-1}G_n''(\hat{\theta}_n)+1|<\varepsilon\} \\ B_n^{(2)}(\varepsilon)&=\{\tilde{x}_n:\ |n^{-1}G_n''(\hat{\theta}_n)+I|<\varepsilon\} \\ C_n^{(1)}&=\{\tilde{x}_n:\ \hat{\theta}_n+(n\log n)^{-1/2}<M_n\} \\ C_n^{(2)}&=\{\tilde{x}_n:\ \hat{\theta}_n+n^{-1/2}<M_n\}\ . \end{aligned}$$

LEMMA 3.3. *Suppose that the assumptions* (I), (II), (III), (IV) *and* (V) *are satisfied. If* $\alpha=2$ ($\alpha>2$), *then for any compact subset* K *of* Θ, $P_{n\theta}(C_n^{(1)})$ ($P_{n\theta}(C_n^{(2)})$)$\to 1$ *as* $n\to\infty$ *uniformly in* $\theta\in K$, *and there exists a positive null sequence* $\{\varepsilon_n\}$ *and a positive divergent sequence* $\{\delta_n\}$ *such that* $P_{n\theta}(A_{n\theta}^{(1)}(\delta_n))$ ($P_{n\theta}(A_{n\theta}^{(2)}(\delta_n))$)$\to 1$ *and* $P_{n\theta}(B_n^{(1)}(\varepsilon_n))$($P_{n\theta}(B_n^{(2)}(\varepsilon_n))$)$\to 1$ *as* $n\to\infty$ *both uniformly on any compact subset of* Θ *and that* $\delta_n^2\varepsilon_n\to 0$ *as* $n\to\infty$.

PROOF. Let K be any compact subset of Θ throughout this proof. By Theorem 2.1 we obtain that for any positive divergent sequence $\{\delta_n\}$, $P_{n\theta}(A_{n\theta}^{(1)}(\delta_n))\to 1$ as $n\to\infty$ uniformly in $\theta\in K$ when $\alpha=2$, and $P_{n\theta}(A_{n\theta}^{(2)}(\delta_n))\to 1$ as $n\to\infty$ uniformly in $\theta\in K$ when $\alpha>2$. By Theorems 2.1 and 2.2, we have

$$
\begin{aligned}
&P_{n\theta}(C_n^{(1)}) \\
&\quad = P_{n\theta}(\{\tilde{x}_n : \sqrt{n \log n}\,(\hat{\theta}_n - \theta) + 1 < \sqrt{n \log n}\,(M_n - \theta)\}) \\
&\quad \longrightarrow 1 \text{ as } n \longrightarrow \infty \text{ uniformly in } \theta \in K \text{ when } \alpha = 2\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&P_{n\theta}(C_n^{(2)}) \\
&\quad = P_{n\theta}(\{\tilde{x}_n : \sqrt{n}\,(\hat{\theta}_n - \theta) + 1 < \sqrt{n}\,(M_n - \theta)\}) \\
&\quad \longrightarrow 1 \text{ as } n \longrightarrow \infty \text{ uniformly in } \theta \in K \text{ when } \alpha > 2\,.
\end{aligned}
$$

By the first part in Lemma 3.2, there exists a positive null sequence $\{\varepsilon_n\}$ such that

$$
\begin{aligned}
(3.19)\qquad & P_{n\theta}(\{\tilde{x}_n : \sup_{|t| \leq 1} |(c_1 n \log n)^{-1} G_n''(\theta + t(n \log \log n)^{-1/2}) + 1| > \varepsilon_n\}) \longrightarrow 0 \\
& \text{as } n \longrightarrow \infty \text{ uniformly in } \theta \in K\,.
\end{aligned}
$$

Moreover, for the sequence $\{\varepsilon_n\}$ satisfying (3.19) we can choose a positive divergent sequence $\{\delta_n\}$ such that

$$
\begin{aligned}
(3.20)\qquad & \delta_n^2 \varepsilon_n \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ and } \delta_n \sqrt{n \log \log n} / \sqrt{n \log n} \longrightarrow 0 \\
& \text{as } n \longrightarrow \infty\,.
\end{aligned}
$$

From (3.19) and (3.20), we can choose a positive null sequence $\{\varepsilon_n\}$ and a positive divergent sequence $\{\delta_n\}$ such that

$$
\begin{aligned}
(3.21)\qquad & P_{n\theta}(\{\tilde{x}_n : \sup_{|t| \leq 1} |(c_1 n \log n)^{-1} G_n''(\theta + t\delta_n (n \log n)^{-1/2}) + 1| > \varepsilon_n\}) \longrightarrow 0 \\
& \text{as } n \longrightarrow \infty \text{ uniformly in } \theta \in K\,, \\
(3.22)\qquad & \delta_n^2 \varepsilon_n \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ and } \delta_n \sqrt{n \log \log n} / \sqrt{n \log n} \longrightarrow 0 \\
& \text{as } n \longrightarrow \infty\,.
\end{aligned}
$$

By (3.21), (3.22) and the result which was shown in the beginning,

$$
\begin{aligned}
&P_{n\theta}(B_n^{(1)}(\varepsilon_n)) \\
&\quad = P_{n\theta}(\{\tilde{x}_n : |(c_1 n \log n)^{-1} G_n''(\hat{\theta}_n) + 1| < \varepsilon_n\}) \\
&\quad \geqq P_{n\theta}(\{\tilde{x}_n : |(c_1 n \log n)^{-1} G_n''(\hat{\theta}_n) + 1| < \varepsilon_n\} \cap A_{n\theta}^{(1)}(\delta_n)) \\
&\quad = P_{n\theta}(\{\tilde{x}_n : |(c_1 n \log n)^{-1} G_n''(\hat{\theta}_n) + 1| < \varepsilon_n\} \mid A_{n\theta}^{(1)}(\delta_n)) \\
&\qquad \times P_{n\theta}(A_{n\theta}^{(1)}(\delta_n)) \\
&\quad \geqq P_{n\theta}(\{\tilde{x}_n : \sup_{|t| \leq 1} |(c_1 n \log n)^{-1} G_n''(\theta + t\delta_n (n \log n)^{-1/2}) + 1| < \varepsilon_n\}) \\
&\qquad \times P_{n\theta}(A_{n\theta}^{(1)}(\delta_n)) \\
&\quad \longrightarrow 1 \text{ as } n \longrightarrow \infty \text{ uniformly in } \theta \in K\,.
\end{aligned}
$$

Next, by the second part in Lemma 3.2 there exists a positive null sequence $\{\varepsilon_n\}$ such that

(3.23) $$P_{n\theta}(\{\tilde{x}_n: \sup_{|t|\leq 1} |n^{-1}G_n''(\theta+tn^{-2/\alpha})+I|>\varepsilon_n\}) \longrightarrow 0$$
$$\text{as } n \longrightarrow \infty \text{ uniformly in } \theta \in K .$$

For the sequence $\{\varepsilon_n\}$ satisfying (3.23), we can choose a positive divergent sequence $\{\delta_n\}$ such that

(3.24) $$\delta_n^2\varepsilon_n \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ and } \delta_n n^{2/\alpha}/n^{1/2} \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Thus we can choose a positive null sequence $\{\varepsilon_n\}$ and a positive divergent sequence $\{\delta_n\}$ such that

$$P_{n\theta}(\{\tilde{x}_n: \sup_{|t|\leq 1} |n^{-1}G_n''(\theta+t\delta_n n^{-1/2})+I|>\varepsilon_n\}) \longrightarrow 0$$
$$\text{as } n \longrightarrow \infty \text{ uniformly in } \theta \in K ,$$

$$\delta_n^2\varepsilon_n \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ and } \delta_n n^{2/\alpha}/n^{1/2} \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

By the similar method as that of previous argument, we have

$$P_{n\theta}(B_n^{(2)}(\varepsilon_n)) \longrightarrow 1 \text{ as } n \longrightarrow \infty \text{ uniformly in } \theta \in K .$$

Thus the proof has been completed.

The following definition is due to LeCam [5].

DEFINITION 3.1. A statistic $\{T_n\}=\{T_n(\tilde{X}_n)\}$ is called *asymptotically sufficient for* $\{P_\theta: \theta \in \Theta\}$ if there exist non-negative functions $q_n(\tilde{x}_n, \theta)$ such that for each $n=1, 2, \cdots$, $q_n(\tilde{x}_n, \theta)$ is the product of a function of $\tilde{x}_n$ only by a function of T_n and θ only and

$$\lim_{n\to\infty} \sup_{\theta \in K} \int_{X^n} \left| \prod_{i=1}^{n} f(x_i, \theta) - q_n(\tilde{x}_n, \theta) \right| \prod_{i=1}^{n} dx_i = 0$$

for any compact subset K of Θ.

Now we prove the asymptotic sufficiency of MLE.

THEOREM 3.1. *If the assumptions* (I)～(V) *are satisfied, then MLE is asymptotically sufficient for* $\{P_\theta: \theta \in \Theta\}$.

PROOF. At first we prove the theorem in the case $\alpha=2$. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be the sequences which were given in the previous lemma, and let

$$q_n(\tilde{x}_n, \theta) = p_n(\tilde{x}_n, \theta) \exp\left[-\frac{c_1}{2}(\sqrt{n \log n}\,(\hat{\theta}_n - \theta))^2\right]$$

$$\times I_{A_{n\theta}^{(1)}(\delta_n)\cap B_n^{(1)}(\varepsilon_n)\cap C_n^{(1)}}(\tilde{x}_n) ,$$

where $I_E(\cdot)$ denotes the indicator function of a set E.

$$\begin{aligned}\sup_{\theta\in K}\int_{X^n}|p_n(\tilde{x}_n,\theta)-q_n(\tilde{x}_n,\theta)|\prod_{i=1}^n dx_i \\ \leqq \sup_{\theta\in K}\int_{A_{n\theta}^{(1)}(\delta_n)\cap B_n^{(1)}(\varepsilon_n)\cap C_n^{(1)}}\left|1-\frac{q_n(\tilde{x}_n,\theta)}{p_n(\tilde{x}_n,\theta)}\right|p_n(\tilde{x}_n,\theta)\prod_{i=1}^n dx_i \\ +\sup_{\theta\in K}P_{n\theta}(\{A_{n\theta}^{(1)}(\delta_n)\cap B_n^{(1)}(\varepsilon_n)\cap C_n^{(1)}\}^c) .\end{aligned}$$

By Lemma 3.3, the second term in the right-hand side converges to zero as $n\to\infty$ for any compact subset K of Θ. We prove that the first term converges to zero as $n\to\infty$. If $\tilde{x}_n\in C_n^{(1)}$, then $\lambda_n(\tilde{x}_n,\theta)$ is twice continuously differentiable with respect to θ in $(n\log n)^{-1/2}$-neighborhood of $\hat{\theta}_n$. Thus, for each $\tilde{x}_n\in C_n^{(1)}$ we can expand $\lambda_n(\tilde{x}_n,\theta)$ with respect to θ around $\hat{\theta}_n$ by Taylor's theorem. We have

$$\begin{aligned}\lambda_n(\tilde{x}_n,\theta)=\lambda_n(\tilde{x}_n,\hat{\theta}_n)+(\theta-\hat{\theta}_n)\left[\frac{\partial}{\partial\theta}\lambda_n(\tilde{x}_n,\theta)\right]_{\theta=\hat{\theta}_n} \\ +\frac{1}{2}(\theta-\hat{\theta}_n)^2\left[\frac{\partial^2}{\partial\theta^2}\lambda_n(\tilde{x}_n,\theta)\right]_{\theta=\theta_n^*} ,\end{aligned}$$

where $|\theta_n^*-\theta|<|\hat{\theta}_n-\theta|$. Since $\hat{\theta}_n$ is MLE for each n, the second term in the right-hand side vanishes. Therefore, from Lemma 3.3, $\tilde{x}_n\in A_{n\theta}^{(1)}(\delta_n)\cap B_n^{(1)}(\varepsilon_n)\cap C_n^{(1)}$ implies that

$$\begin{aligned}&\left|1-\frac{q_n(\tilde{x}_n,\theta)}{p_n(\tilde{x}_n,\theta)}\right| \\ &=\left|1-\exp\left[-\frac{c_1}{2}(\sqrt{n\log n}(\hat{\theta}_n-\theta))^2\left(\frac{1}{c_1 n\log n}\left[\frac{\partial^2}{\partial\theta^2}\lambda_n(\tilde{x}_n,\theta)\right]_{\theta=\theta_n^*}+1\right)\right]\right| \\ &\leqq\exp\left[\frac{c_1}{2}(\sqrt{n\log n}(\hat{\theta}_n-\theta))^2\left|\frac{1}{c_1 n\log n}\left[\frac{\partial^2}{\partial\theta^2}\lambda_n(\tilde{x}_n,\theta)\right]_{\theta=\theta_n^*}+1\right|\right]-1 \\ &\leqq\exp\left(\frac{c_1}{2}\delta_n^2\varepsilon_n\right)-1 \\ &\longrightarrow 0 \text{ as } n\longrightarrow\infty .\end{aligned}$$

Thus we have

$$\sup_{\theta\in K}\int_{A_{n\theta}^{(1)}(\delta_n)\cap B_n^{(1)}(\varepsilon_n)\cap C_n^{(1)}}\left|1-\frac{q_n(\tilde{x}_n,\theta)}{p_n(\tilde{x}_n,\theta)}\right|p_n(\tilde{x}_n,\theta)\prod_{i=1}^n dx_i\longrightarrow 0 \text{ as } n\longrightarrow\infty .$$

The proof has been completed in the case $\alpha=2$.

Next we prove the theorem in the case $\alpha>2$. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be the sequences which were given in Lemma 3.3, and let

$$q_n(\tilde{x}_n, \theta)=p_n(\tilde{x}_n, \theta)\exp\left[-\frac{I}{2}(\sqrt{n}(\hat{\theta}_n-\theta))^2\right]\mathrm{I}_{A_{n\theta}^{(2)}(\delta_n)\cap B_n^{(2)}(\varepsilon_n)\cap C_n^{(2)}}(\tilde{x}_n)\ .$$

Then

$$\begin{aligned}&\sup_{\theta\in K}\int_{X^n}|p_n(\tilde{x}_n, \theta)-q_n(\tilde{x}_n, \theta)|\prod_{i=1}^n dx_i\\&\leqq\sup_{\theta\in K}\int_{A_{n\theta}^{(2)}(\delta_n)\cap B_n^{(2)}(\varepsilon_n)\cap C_n^{(2)}}\left|1-\frac{q_n(\tilde{x}_n, \theta)}{p_n(\tilde{x}_n, \theta)}\right|p_n(\tilde{x}_n, \theta)\prod_{i=1}^n dx_i\\&\quad+\sup_{\theta\in K}P_{n\theta}(\{A_{n\theta}^{(2)}(\delta_n)\cap B_n^{(2)}(\varepsilon_n)\cap C_n^{(2)}\}^c)\ .\end{aligned}$$

By Lemma 3.3, the second term in the right-hand side converges to zero as $n\to\infty$. By the similar method as that of previous argument, for each $\tilde{x}_n\in C_n^{(2)}$ we have

$$\lambda_n(\tilde{x}_n, \theta)=\lambda_n(\tilde{x}_n, \hat{\theta}_n)+\frac{1}{2}(\hat{\theta}_n-\theta)^2\left[\frac{\partial^2}{\partial\theta^2}\lambda_n(\tilde{x}_n, \theta)\right]_{\theta=\theta_n^*}\ ,$$

where $|\theta_n^*-\theta|<|\hat{\theta}_n-\theta|$.

From Lemma 3.3, $\tilde{x}_n\in A_{n\theta}^{(2)}(\delta_n)\cap B_n^{(2)}(\varepsilon_n)\cap C_n^{(2)}$ implies that

$$\begin{aligned}&\left|1-\frac{q_n(\tilde{x}_n, \theta)}{p_n(\tilde{x}_n, \theta)}\right|\\&=\left|1-\exp\left[-\frac{1}{2}(\sqrt{n}(\hat{\theta}_n-\theta))^2\left(\frac{1}{n}\left[\frac{\partial^2}{\partial\theta^2}\lambda_n(\tilde{x}_n, \theta)\right]_{\theta=\theta_n^*}+I\right)\right]\right|\\&\leqq\exp\left[\frac{1}{2}(\sqrt{n}(\hat{\theta}_n-\theta))^2\left|\frac{1}{n}\left[\frac{\partial^2}{\partial\theta^2}\lambda_n(\tilde{x}_n, \theta)\right]_{\theta=\theta_n^*}+I\right|\right]-1\\&\leqq\exp\left(\frac{1}{2}\delta_n^2\varepsilon_n\right)-1\\&\longrightarrow 0 \text{ as } n\longrightarrow\infty\ .\end{aligned}$$

Thus we have

$$\sup_{\theta\in K}\int_{A_{n\theta}^{(2)}(\delta_n)\cap B_n^{(2)}(\varepsilon_n)\cap C_n^{(2)}}\left|1-\frac{q_n(\tilde{x}_n, \theta)}{p_n(\tilde{x}_n, \theta)}\right|p_n(\tilde{x}_n, \theta)\prod_{i=1}^n dx_i\longrightarrow 0 \text{ as } n\longrightarrow\infty\ .$$

The proof has been completed in the case $\alpha>2$.

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