SLAC-PUB-510
September 1968
(TH)

ASYMPTOTIC SUM RULES AT INFINITE MOMENTUM*

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ABSTRACT

By combining the $q_0 \to i\infty$ method for asymptotic sum rules with the $P \to \infty$ method of Fubini and Furlan, we relate the structure functions W_2 and W_1 in inelastic lepton-nucleon scattering to matrix elements of commutators of currents at almost-equal times at infinite momentum. We argue that the infinite momentum limit for these commutators does not diverge but may vanish. If the limit is nonvanishing we predict

$$\nu W_2(\nu, q^2) \longrightarrow f_2\left(\frac{\nu}{q^2}\right)$$

$$W_1(\nu, q^2) \longrightarrow f_1\left(\frac{\nu}{q^2}\right)$$

as ν and q^2 tend to ∞. From a similar analysis for neutrino processes, we conclude that at high energies the total neutrino-nucleon cross sections rise linearly with neutrino laboratory energy until nonlocality of the weak current-current coupling sets in. The sum of νp and $\bar{\nu} p$ cross sections is determined by the equal-time commutator of Cabibbo-current with its time derivative, taken between proton states at infinite momentum.

(submitted to Phys. Rev.)

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*Work supported by the U.S. Atomic Energy Commission.

I. INTRODUCTION

Inelastic lepton-nucleon scattering at high momentum transfer is a very direct means of probing small-distance nucleon structure. Reflecting this fact is the profound state of theoretical ignorance on what, even qualitatively, can be expected in this process.[1] Some small inroads have been recently made using the techniques of current algebra.[2] In particular, Cornwall and Norton[3] have written down a large class of asymptotic sum rules, valid at large q^2, for inelastic electron scattering. Of these, the sum rule of Callan and Gross[4] relating an asymptotic integral over electron scattering cross sections to a piece of the commutator of electromagnetic current with its time derivative is of special interest. The purpose of this paper is to discuss such sum rules in a slightly different language — that of the infinite momentum method. We show that the electron scattering data is related in a direct way to matrix elements of electromagnetic current commutators at infinite nucleon momentum.[5,6] In particular, we find that the structure functions $W_2(q^2, \nu)$ and $W_1(q^2, \nu)$ describing inelastic scattering[7] tend to simple limits for large q^2:

$$\lim_{q^2\to\infty,\ \frac{\nu}{q^2}\ \text{fixed}} \nu W_2(q^2, \nu) = M F_2\left(\frac{-q^2}{\nu}\right) \tag{I.1}$$

$$\lim_{q^2\to\infty,\ \frac{\nu}{q^2}\ \text{fixed}} W_1(q^2, \nu) = \frac{F}{M}\left(\frac{-q^2}{\nu}\right) \tag{I.2}$$

with

$$F_t(\omega) \equiv F_1(\omega) = \frac{-i}{\pi} \lim_{P_z\to\infty} \int_0^\infty d\tau\, \sin\omega\tau \int d^3x \left\langle P_z \middle| \left[J_x\left(\underset{\sim}{x}, \frac{\tau}{P_0}\right), J_x(0)\right] \middle| P_z \right\rangle \tag{I.3}$$

and

$$F_\ell(\omega) \equiv \frac{F_2(\omega)}{\omega} - F_1(\omega) = \frac{i}{\pi} \lim_{P_z \to \infty} \int_0^\infty d\tau \sin \omega\tau \int d^3x \left\langle P_z \left| \left[J_z\left(\underset{\sim}{x}, \frac{\tau}{P_0}\right), J_z(0) \right] \right| P_z \right\rangle \tag{I.4}$$

where $\omega = -\frac{q^2}{\nu}$. The existence of these limits (aside from the Brandt-Sucher disease[8]) is guaranteed by a finite value of the integral appearing in the Callan-Gross sum rule. Although the present data[9] appears to indicate that F_2 is non-vanishing at $q^2 \sim 1\text{-}2\ \mathrm{BeV}^2$, it is still possible that $F_2 \to 0$ and the infinite-momentum commutators in (I.3) and (I.4) vanish in the limit. In such a case the content of this paper is empty.

Sum rules such as Cornwall and Norton have written down[3] may be obtained by taking the sine-transform of (I.3) and (I.4) and expanding both sides in a power series in τ. For example, for n = 1,3,5...

$$\int_0^2 d\omega \omega^n F_t(\omega) = \lim_{q^2 \to -\infty} \left| q^2 \right|^{n+1} \int_0^\infty \frac{d\nu}{\nu^{n+2}} W_1(q^2, \nu)$$

$$= \lim_{P_z \to \infty} \frac{(-i)^n}{2} \int \frac{d^3x}{P_0^n} \left\langle P_z \left| \left[\frac{\partial^n J_x(x,t)}{\partial t^n}, J_x(0) \right] \right| P_z \right\rangle_{t=0}$$

$$n = 1,3,5... \tag{I.5}$$

with a similar expression for F_ℓ or W_2. However, the content of these results is more succinctly discussed in terms of (I.1) - (I.4).

Although a straightforward generalization of these relations to different currents and momentum states is not difficult, what is not straightforward is the interpretation of the almost-equal-time commutators appearing in (I.3) and (I.4). In particular, the spectrum of intermediate "frequencies" $\omega = \frac{-q^2}{\nu}$ is bounded above, corresponding to at most the intermediate energy appropriate to the single-nucleon Z-diagram (see Fig. 1). Assuming the limit (I.1) and (I.2) is nontrivial (nonvanishing), it will be most interesting to construct models with the kind of asymptotic behavior expressed in (I.1) - (I.4). This, however, is beyond the scope of this paper.

In Section II, a simple derivation of the result is given. Section III remedies the swindle perpetrated on the reader in Section II, by providing a more honest derivation. In Section IV, we attempt a generalization to an arbitrary kinematical situation. In Section V, we apply the same method to neutrino-reactions and find that νp and $\bar{\nu} p$ total cross sections should rise linearly with energy. The sum of νp and $\bar{\nu} p$ cross sections is determined by the equal-time commutator of the Cabibbo current with its first-time derivative. Section VI summarizes our conclusions.

II. SIMPLE DERIVATION OF THE ASYMPTOTIC LIMIT

The inelastic scattering cross section from an unpolarized nucleon may be written[7] as

$$\frac{d\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \left[W_2(q^2, \nu) \cos^2 \frac{\theta}{2} + 2W_1(q^2, \nu) \sin^2 \frac{\theta}{2}\right] \tag{II.1}$$

with

E, E'	=	energy of incident and scattered electron
θ	=	scattering angle of electron
q^2	=	$-4EE' \sin^2 \frac{\theta}{2}$
ν	=	$q \cdot P = (E - E') M$
P	=	momentum of target nucleon
q	=	momentum of virtual photon

and

$$\frac{1}{M^2}\left(P_\mu - \frac{P\cdot q q_\mu}{q^2}\right)\left(P_\nu - \frac{P\cdot q q_\nu}{q^2}\right) W_2(q^2,\nu) - \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(q^2,\nu)$$

$$= \frac{P_0}{M} \overline{\overline{\sum_n}} \langle P|J_\mu(0)|n\rangle \langle n|J_\nu(0)|P\rangle (2\pi)^3 \ \delta^4 (P_n - P - q) \qquad \text{(II.2)}$$

$$= \frac{P_0}{M} \int \frac{d^4x}{2\pi} e^{iq\cdot x} \langle P\left|\left[J_\mu(x),\ J_\nu(0)\right]\right|P\rangle$$

In all matrix elements $\langle P|\ldots|P\rangle$, an average over nucleon spin is implied. Now consider the limit of (II.2) as $P_0 \longrightarrow \infty$, $q_0 \longrightarrow \infty$, $\frac{q_0}{P_0} \longrightarrow -\omega$ fixed, $\underset{\sim}{q}$ fixed $\left(\omega = \frac{-q^2}{\nu}\right)$. Notice that $q^2 \longrightarrow +\infty$ (timelike) in this limit.

Choosing $\mu \neq 0$, $\nu \neq 0$, we find

$$\frac{P_iP_j}{M^2} W_2 + \delta_{ij} W_1 \xrightarrow[P_z \longrightarrow \infty]{} \frac{P_0}{M} \int d^3x \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i\omega P_0 t - i\underset{\sim}{q}\cdot \underset{\sim}{x}} \langle P\left|\left[J_i\left(\underset{\sim}{x}, \frac{P_0 t}{P_0}\right),\ J_j(0)\right]\right|P\rangle \qquad \text{(II.3)}$$

or using (I.1) and (I.2) and the assumption that the commutator vanishes outside the light cone

$$-\ \delta_{i3}\delta_{j3} \frac{F_2(\omega)}{\omega} + \delta_{ij}F_1(\omega) = \lim_{P_z \longrightarrow \infty} \int d^3x \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{-i\omega\tau} \langle P\left|\left[J_i\left(\underset{\sim}{x}, \frac{\tau}{P_0}\right),\ J_j(0)\right]\right|P\rangle$$

$$= \frac{-i}{\pi} \lim_{P_z \longrightarrow \infty} \int d^3x \int_0^{\infty} d\tau\ \sin\omega\tau\ \langle P\left|\left[J_i\left(\underset{\sim}{x}, \frac{\tau}{P_0}\right),\ J_j(0)\right]\right|P\rangle$$

$$(\omega < 0) \qquad \text{(II.4)}$$

Now (II.4) defines F_1 and F_2 for positive ω as well as negative; therefore we let $\omega = \frac{+|q^2|}{\nu} > 0$, as appropriate for inelastic scattering. Then we get

$$F_t(\omega) = F_1(\omega) = \frac{-i}{\pi} \lim_{P_z \longrightarrow \infty} \int d^3x \int_0^{\infty} d\tau\ \sin|\omega|\tau\ \langle P\left|\left[J_x\left(\underset{\sim}{x}, \frac{\tau}{P_0}\right),\ J_x(0)\right]\right|P\rangle$$

$$(\omega > 0) \qquad \text{(II.5)}$$

and

$$F_\ell(\omega) \equiv \frac{F_2(\omega)}{|\omega|} - F_1(\omega) = +\frac{i}{\pi} \lim_{P_z \to \infty} \int d^3x \int_0^\infty d\tau \, \sin|\omega|\tau \, \langle P \left| \left[J_z\left(\underset{\sim}{x}, \frac{\tau}{P_0}\right), J_z(0)\right] \right| P \rangle$$

$$(\omega > 0) \qquad \text{(II. 6)}$$

This is the desired result given in (I.3) and (I.4). Both F_t and F_ℓ are positive; notice the curious sign change between the transverse and longitudinal commutators.

The reader should have noticed the swindle that has been perpetrated in letting $\omega \to -\omega$. There has been no justification that $\omega < 0$ can be extrapolated from $\omega > 0$. The next section is devoted to providing such a justification.

III. JUSTIFICATION OF THE RESULTS

In order to provide a better derivation of the preceding results, we consider the covariant current correlation function

$$T^*_{\mu\nu} = \frac{1}{M^2}\left(P_\mu - \frac{P \cdot q \, q_\mu}{q^2}\right)\left(P_\nu - \frac{P \cdot q \, q_\nu}{q^2}\right) T_2(q^2, \nu) - \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) T_1(q^2, \nu)$$

$$= + \frac{iP_0}{M} \int d^4x \, e^{iq\cdot x} \, \theta(x_0) \, \langle P \left| \left[J_\mu(x), J_\nu(0) \right] \right| P \rangle + \text{Polynomials in } q$$

$$\text{(III. 1)}$$

where, as always in this paper, an average over nucleon spin is implied. According to the dictum of Harari,[10] T_2 satisfies an unsubtracted dispersion relation in ν, while T_1 requires one subtraction, provided $q^2 < 0$, i.e., spacelike.

$$T_2 = \frac{1}{\pi} \int_0^\infty \frac{d\nu'^2 \, \text{Im}\, T_2(\nu', q^2)}{\nu'^2 - \nu^2 - i\epsilon} \qquad \text{(III. 2)}$$

$$T_1(\nu, q^2) = T_1(0, q^2) + \frac{\nu^2}{\pi} \int_0^\infty \frac{d\nu'^2 \, \text{Im}\, T_1(\nu, q^2)}{\nu'^2 (\nu'^2 - \nu^2 - i\epsilon)}$$

$$(q^2 < 0) \qquad \text{(III. 3)}$$

We choose to express the T_i in terms of the variables q^2 and $\omega = (-q^2/\nu)$. Using the fact that

$$\frac{1}{\pi} \mathrm{Im}\, T_i(\nu, q^2) = W_i(\nu, q^2) \tag{III.4}$$

we obtain

$$T_1(\omega, q^2) = T_1(\infty, q^2) - \int_0^4 \frac{d\omega'^2\, W_1(\omega', q^2)}{(\omega'^2 - \omega^2 + i\epsilon)} \tag{III.5}$$

$$T_2(\omega, q^2) = -\omega^2 \int_0^4 \frac{d\omega'^2\, W_2(\omega', q^2)}{\omega'^2(\omega'^2 - \omega^2 + i\epsilon)} \tag{III.6}$$

We now take the limit $q_0 \to i\infty$, $\underset{\sim}{q}$ fixed and P_z temporarily fixed. In this limit

$$\omega = \frac{-q^2}{q_0 P_0} \longrightarrow -\frac{iq_0}{P_Z} \longrightarrow -i\infty$$

$$T_1 \longrightarrow T_1(\infty, q^2) - \frac{P_0^2}{|q_0^2|} \int_0^4 d\omega'^2\, W_1(\omega', q^2)$$

$$T_2 \longrightarrow + \int_0^4 \frac{d\omega'^2}{\omega'^2} W_2(\omega', q^2)$$

On the other hand, from (III.1)

$$T_{\mu\nu} \longrightarrow \frac{iP_0}{M} \int d^3x \int_0^\infty dt\, e^{-|q_0|t} \langle P | [J_\mu(x), J_\nu(0)] | P \rangle + \text{Polynomial}$$

$$\longrightarrow \frac{iP_0}{M|q_0|} \int d^3x \langle P | [J_\mu(\underset{\sim}{x}, 0), J_\nu(0)] | P \rangle + \frac{iP_0}{M|q_0|^2} \int d^3x \langle P | \left[\frac{\partial J_\mu(\underset{\sim}{x}, t)}{\partial t}, J_\nu(0) \right] | P \rangle_{t=0}$$

$$+ \ldots \tag{III.7}$$

Specializing to μ and $\nu \neq 0$, and $\underset{\sim}{P}$ in the z-direction, we find

$$\frac{P_0}{M}\int d^3x \left\langle P\left|\left[\frac{\partial J_x(\underset{\sim}{x}, t)}{\partial t}, J_x(0)\right]\right|P\right\rangle_{t=0} = \lim_{q^2\to -\infty} iq^2\, T_1(\infty, q^2) + iP_0^2 \int_0^4 d\omega'^2\, W_1(\omega', q^2) \tag{III.8}$$

$$\frac{P_0}{M}\int d^3x \left\langle P\left|\left[\frac{\partial J_z(\underset{\sim}{x}, t)}{\partial t}, J_z(0)\right]\right|P\right\rangle_{t=0} - \frac{P_0}{M}\int d^3x \left\langle P\left|\left[\frac{\partial J_x(\underset{\sim}{x}, t)}{\partial t}, J_x(0)\right]\right|P\right\rangle_{t=0}$$

$$= \lim_{q^2\to -\infty} iq^2\, \frac{P_z^2}{M^2}\int_0^4 \frac{d\omega'^2}{\omega'^2}\, W_2(\omega', q^2) \tag{III.9}$$

Hereafter, we shall assume that

$$\lim_{q^2\to -\infty} |q^2| \int_0^4 \frac{d\omega'^2}{\omega'^2}\, W_2(\omega', q^2) = \lim_{q^2\to -\infty} 2|q^2| \int_0^\infty \frac{d\nu'}{\nu'}\, W_2(\nu', q^2) < \infty \tag{III.10}$$

For fixed q^2, the integration in ν converges, if the Harari dictum[10] is correct. The $q^2 \to \infty$ limit is that taken by Callan and Gross[4]; in particular (III.10) is an integral involved in their sum rules. It is unlikely the inelastic scattering is so large that this Callan-Gross integral does not exist; such a circumstance would over-saturate the sum rules for $\int d\nu\, W_2$ from current algebra.[1] More likely is the vanishing of (III.10). If the Callan-Gross integral exists, i.e., (III.10) holds, we can show that almost-equal-time commutators (I.3) and (I.4) necessarily exist as well. To show this, we observe from their kinematical definitions in terms of transverse and longitudinal cross sections,[11] that in our limit

$$\frac{W_1}{W_2} = \left(1 + \frac{\nu^2}{|q^2|}\right)\left(\frac{\sigma_t}{\sigma_t + \sigma_\ell}\right) \lesssim \frac{\nu^2}{|q|^2} \tag{III.11}$$

which implies

$$q^4\int\frac{d\nu'}{\nu'^3}W_1 \le |q^2|\int\frac{d\nu'}{\nu'}W_2 < \infty \tag{III.12}$$

$$\lim_{q^2\to-\infty}\int_0^4 d\omega'^2\, W_1(\omega',q^2) < \infty \tag{III.13}$$

$$\lim_{q^2\to-\infty}|q^2|\int\frac{d\omega'^2}{\omega'^2}W_2(\omega',q^2) < \infty \tag{III.14}$$

and because W_1 and W_2 are positive semidefinite, it follows that for $|\omega| > 2$,

$$\Phi_1(\omega) = \lim_{q^2\to-\infty}\int_0^4\frac{d\omega'^2\, W_1(\omega',q^2)}{\omega^2-\omega'^2} < \infty \tag{III.15}$$

and Φ therefore by analytic continuation exists throughout the cut ω plane barring extreme pathology in the behavior of W_1 in the limit. Similarly, we have

$$\Phi_2(\omega) = \lim_{q^2\to-\infty}|q^2|\omega^2\int_0^4\frac{d\omega'^2\, W_2(\omega',q^2)}{\omega'^2(\omega^2-\omega'^2)} < \infty \tag{III.16}$$

throughout the cut ω plane. These results are then sufficient to guarantee the existence of F_1, F_2, and the almost-equal-time commutators (I.3) and (I.4).

We go back to $T^*_{\mu\nu}$ as defined in (III.1), (III.5) and (III.6) and let $P_z\to\infty$, $q_0\to i\infty$, $\underset{\sim}{q}$ fixed, $\omega = \frac{-q_0}{P_0} = -i\left|\frac{q_0}{P_0}\right|$ fixed.

$$T^*_{ij}\to+\frac{P_iP_j}{M^2}|\omega|^2\int_0^4\frac{d\omega'^2\, W_2(\omega',q^2)}{\omega'^2(\omega'^2+|\omega^2|)}+\delta_{ij}\left[T_1(\infty,q^2)-\int_0^4\frac{d\omega'^2\, W_1(\omega',q^2)}{(\omega'^2+|\omega^2|)}\right] \tag{III.17}$$

In terms of F_1 and F_2, we find from (III. 1)

$$T^*_{ij} \to \frac{P_i P_j}{|q^2|} \frac{|\omega^2|}{M} \int_0^4 \frac{d\omega'^2 \, F_2(\omega')}{|\omega'| \, (\omega'^2 + |\omega^2|)} + \delta_{ij} \left[T_1(\infty, q^2) - \frac{1}{M} \int_0^4 \frac{d\omega'^2 \, F_1(\omega')}{\omega'^2 + |\omega^2|} \right]$$

$$= \lim_{P_z \to \infty} \frac{i}{M} \int d^3x \int_0^\infty d\tau \, e^{-|\omega|\tau} \left\langle P \middle| \left[J_i\left(\underset{\sim}{x}, \frac{\tau}{E}\right), J_j(0) \right] \middle| P \right\rangle + \text{Polynomial} \tag{III. 18}$$

In the limit,

$$\frac{P_i P_j}{|q^2|} \longrightarrow \delta_{i3} \, \delta_{j3} \left| \frac{P_0^2}{q_0^2} \right| \longrightarrow \frac{\delta_{i3} \, \delta_{j3}}{|\omega^2|}$$

and

$$T_1(\infty, q^2) \longrightarrow \text{Polynomial} \, . \tag{III. 19}$$

The existence of the commutator in (III. 18) is guaranteed by the existence of an inverse Laplace-transform of (III. 18). Having taken the limit $q^2 \to \infty$, etc., we may continue (III. 18) into the cut -plane, and obtain

$$\delta_{i3} \, \delta_{j3} \int_0^4 \frac{d\omega'^2 \, F_2(\omega')}{\omega'(\omega'^2 - \omega^2 - i\epsilon)} - \delta_{ij} \int_0^4 \frac{d\omega'^2 \, F_1(\omega')}{\omega'^2 - \omega^2 - i\epsilon}$$

$$= \lim_{P_z \to \infty} i \int d^3x \int_0^\infty d\tau \, e^{i\omega\tau} \left\langle P \middle| \left[J_i\left(\underset{\sim}{x}, \frac{\tau}{P_0}\right), J_j(0) \right] \middle| P \right\rangle \tag{III. 20}$$

Upon taking the imaginary part of this relation, we reproduce (II. 5) and (II. 6). This justifies the short derivation given in Section II.

IV. A GENERALIZATION

The preceding analysis can be generalized to arbitrary currents and momenta of the states. As an example, we consider the case of two different SU(3) × SU(3) currents (q_1, q_2) sandwiched between spin zero hadron states (p_1, p_2) of the same parity. We use the notation of Bander and Bjorken[12]

$$p_1 + q_1 \longrightarrow p_2 + q_2$$

$$P = p_1 + p_2$$

$$\Delta = p_2 - p_1 = q_1 - q_2$$

$$Q = q_1 + q_2$$

$$\nu = P \cdot Q \qquad t = \Delta^2 \qquad \delta = \Delta \cdot Q = q_1^2 - q_2^2 \tag{IV.1}$$

and take the limit

$$E \simeq p_{1z} = p_{2z} \longrightarrow \infty , \qquad Q_0 \longrightarrow i\infty , \qquad \underset{\sim}{Q} = 0 \tag{IV.2}$$

such that

$$\omega = - \frac{Q_0}{P_0} \cong - \frac{Q^2}{\nu} \tag{IV.3}$$

remains finite. Encouraged by the reasonableness of this limit in the special case of Sections I - III, we <u>assume</u> it exists in this case as well.

Under these circumstances, the new general invariants, δ and t, tend to a finite limit:

$$\delta = Q \cdot \Delta \longrightarrow - \omega \left[\left(p_{2\perp}^2 + m_2^2 \right) - \left(p_{1\perp}^2 + m_1^2 \right) \right]$$

$$t = \Delta^2 \longrightarrow - (\underset{\sim}{p}_{2\perp} - \underset{\sim}{p}_{1\perp})^2 \tag{IV.4}$$

The covariant amplitude $M_{\mu\nu}^{*\alpha\beta}$ tends, in the limit to

$$M_{\mu\nu}^{*\alpha\beta} = - (2\pi)^3 \, i \, (4\omega_1 \omega_2)^{1/2} \int d^4x \, e^{iq_1 \cdot x} \, \theta(x_0) \, \langle P_2 \left| \left[j_\mu^\alpha(x), \, j_\nu^\beta(0) \right] \right| P_1 \rangle$$

$$\longrightarrow - (2\pi)^3 \, i \, (2E) \int d^3x \int_0^\infty dt \, e^{-\frac{|Q_0|}{2} t} \lim_{P_z \to \infty} \langle P_z , P_{2\perp} \left| \left[j_\mu^\alpha(\underset{\sim}{x}, t), \, j_\nu^\beta(0) \right] \right| P_z , P_{1\perp} \rangle$$

$$\longrightarrow - 2i \, (2\pi)^3 \int d^3x \int_0^\infty d\tau \, e^{-|\omega| \tau} \lim_{P_z \to \infty} \langle P_z P_{2\perp} \left| \left[j_\mu^\alpha \left(\underset{\sim}{x}, \frac{\tau}{E} \right), \, j_\nu^\beta(0) \right] \right| P_z P_{1\perp} \rangle$$

$$(+ \text{ Polynomials}) \tag{IV.5}$$

This latter expression is a function of $\underset{\sim}{p}_{21}$, $\underset{\sim}{p}_{11}$, and ω alone. Upon writing out $M_{\mu\nu}^{\alpha\beta}$ in invariants (suppressing indices $\alpha\beta$)

$$M_{\mu\nu} = P_\mu P_\nu A_1 + (P_\mu Q_\nu + P_\nu Q_\mu) A_2 + (P_\mu Q_\nu - P_\nu Q_\mu) A_3$$
$$+ (P_\mu \Delta_\nu + P_\nu \Delta_\mu) A_4 + (P_\mu \Delta_\nu - P_\nu \Delta_\mu) A_5 + (Q_\mu \Delta_\nu + Q_\nu \Delta_\mu) A_6$$
$$+ (Q_\mu \Delta_\nu - Q_\nu \Delta_\mu) A_7 + Q_\mu Q_\nu A_8 + \Delta_\mu \Delta_\nu A_9 + g_{\mu\nu} A_{10} \tag{IV.6}$$

we see that A_4, A_5, A_6, and A_7 would have to tend to $(Q^2)^{-1/2}$ in order that the limit be nonvanishing and finite. We consider this unlikely, but cannot exclude it. Here we <u>assume</u> that in the limit these A_i do not contribute.

We write

$$A_i^{\alpha\beta} \longrightarrow \frac{1}{Q^2} F_i^{\alpha\beta}(\omega, t, \epsilon) \qquad i = 1,2,3,8$$

$$Q_0 A_i^{\alpha\beta} \longrightarrow 0 \qquad i = 4,5,6,7 \tag{IV.7}$$

$$A_i^{\alpha\beta} \longrightarrow F_i^{\alpha\beta}(\omega, t, \epsilon) \qquad i = 9,10$$

where we introduce the variable

$$\epsilon = \frac{-\delta}{\omega} = \left(p_{21}^2 + m_2^2\right) - \left(p_{11}^2 + m_1^2\right) \tag{IV.8}$$

When these limits (IV.7) are inserted into (IV.6), we find in the asymptotic infinite-momentum limit

$$M_{\mu\nu}^{\alpha\beta} \longrightarrow \frac{\theta_\mu \theta_\nu}{\omega^2} F_1 - \frac{(\theta_\mu \eta_\nu + \theta_\nu \eta_\mu)}{\omega} F_2 - \frac{(\theta_\mu \eta_\nu - \theta_\nu \eta_\mu)}{\omega} F_3$$
$$+ \eta_\mu \eta_\nu F_8 + \Delta_\mu \Delta_\nu F_9 + g_{\mu\nu} F_{10} \tag{IV.9}$$

where $\theta_\mu = (1,1,0,0)$ $\eta_\mu = (1,0,0,0)$.

We are now free to identify various combinations of these form factors in terms of the almost-equal-time current commutators at infinite momentum. Using i or j to indicate transverse components,

$$M_{00}^{\alpha\beta} \longrightarrow \frac{F_1^{\alpha\beta}}{\omega^2} - \frac{2F_2^{\alpha\beta}}{\omega} + F_8^{\alpha\beta} + F_{10}^{\alpha\beta} \qquad M_{zz}^{\alpha\beta} \longrightarrow \frac{F_1^{\alpha\beta}}{\omega^2} - F_{10}^{\alpha\beta}$$

$$M_{0z}^{\alpha\beta} \longrightarrow \frac{-F_2^{\alpha\beta} + F_3^{\alpha\beta}}{\omega} \qquad M_{iz}^{\alpha\beta} \longrightarrow 0$$

$$M_{z0}^{\alpha\beta} \longrightarrow \frac{-F_2^{\alpha\beta} - F_3^{\alpha\beta}}{\omega} \qquad M_{ij}^{\alpha\beta} \longrightarrow \left(\Delta_i\Delta_j\, F_9^{\alpha\beta} - \delta_{ij}\, F_{10}^{\alpha\beta}\right)$$

$$M_{0i}^{\alpha\beta} \longrightarrow 0 \qquad M_{i0}^{\alpha\beta} \longrightarrow 0 \tag{IV.10}$$

Recall

$$M_{\mu\nu}^{\alpha\beta} \longrightarrow -2i\,(2\pi)^3 \int d^3x \int_0^\infty d\tau\; e^{-|\omega|\tau} \left\langle P_{2\perp}\left|\left[j_\mu^\alpha\left(\underset{\sim}{x}, \frac{\tau}{E}\right),\, j_\nu^\beta(0)\right]\right| P_{1\perp}\right\rangle_{P_z \to \infty} \tag{IV.11}$$

Thus all invariant functions F_i can be determined in terms of the various current correlation-functions, which then play the central role. Similarly, an infinite set of convergent sum rules, whose right hand side involves commutators of J_ν with $\partial_0^n\, J_\mu$, can be obtained by expanding (IV.10) and (IV.11) in inverse powers of ω, and comparing coefficients as $\omega \to \infty$. These are independent of the asymptotic sum rules of Bander and Bjorken,[12] because in this case δ does not tend to zero, but rather to ∞. We do not know what to do with these results, and shall not pursue them further here.

V. NEUTRINO PROCESSES

If we write the analogue of (II.1) for the process $\bar{\nu}_\mu + P \rightarrow \mu^+ +$ hadrons as[13,14,1]

$$\frac{\pi}{EE'}\frac{d\sigma}{d\Omega dE'} = \frac{Md\sigma}{d|q^2|d\nu} = \frac{E'}{E}\frac{G^2}{2\pi}\left[W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2} + \frac{(E+E')}{M}W_3\sin^2\frac{\theta}{2}\right] \tag{V.1}$$

with kinematics as in Section II, and

$$\frac{P_0}{M}\int\frac{d^4x}{2\pi}e^{iq\cdot x}\langle P\left|\left[j_\mu(x),\ j_\nu^\dagger(0)\right]\right|P\rangle = \frac{P_\mu P_\nu}{M^2}W_2 - g_{\mu\nu}W_1 - i\frac{\epsilon_{\mu\nu\alpha\beta}}{2M^2}P^\alpha q^\beta W_3 + \ldots \tag{V.2}$$

where j_μ is the Cabibbo current,[15] it follows from the arguments in the preceding section that under our assumptions

$$W_2 \rightarrow \frac{M}{\nu}F_2\left(\frac{-q^2}{\nu}\right)$$
$$MW_1 \rightarrow F_1\left(\frac{-q^2}{\nu}\right) \tag{V.3}$$
$$W_3 \rightarrow \frac{M}{\nu}F_3\left(\frac{-q^2}{\nu}\right)$$

as $q^2 \rightarrow -\infty, \nu \rightarrow \infty$.

Introducing the variables $\omega = \frac{-q^2}{\nu}$ and $x = \frac{\nu}{ME}$, the total cross section coming from (V.1) becomes (for $E \gg M$)

$$\sigma_{tot} \rightarrow \int_0^2 d\omega\int_0^1 dx\left(\frac{G^2ME}{2\pi}\right)\left[(1-x)F_2(\omega) + \frac{1}{2}x^2\omega F_1(\omega) + \frac{x}{2}\left(1-\frac{x}{2}\right)\omega F_3(\omega)\right]$$
$$= \frac{G^2ME}{4\pi}\int_0^2 d\omega\left[F_2(\omega) + \frac{1}{3}\omega F_1(\omega) + \frac{1}{3}\omega F_3(\omega)\right] \tag{V.4}$$

Therefore the cross section is predicted to rise linearly with laboratory neutrino energy. The coefficient is controlled again by the behavior of the current commutators at almost equal time and at infinite momentum. To determine this, we take various components of (V.2) in the q_0, $P_z \longrightarrow \infty$ limit, in parallel with the discussion leading to (II.5)

$$\begin{aligned}
&\lim_{P_z\to\infty} \int d^3x \int_{-\infty}^{\infty} \frac{d\tau}{2\pi}\, e^{-i\omega\tau} \langle P\left|\left[j_x\left(\underset{\sim}{x}, \tfrac{\tau}{E}\right), j_x^\dagger(0)\right]\right| P\rangle = F_1(\omega) \\
&\lim_{P_z\to\infty} \int d^3x \int_{-\infty}^{\infty} \frac{d\tau}{2\pi}\, e^{-i\omega\tau} \langle P\left|\left[j_z\left(\underset{\sim}{x}, \tfrac{\tau}{E}\right), j_z^\dagger(0)\right]\right| P\rangle = \left[F_1(\omega) - \frac{F_2(\omega)}{\omega}\right] \qquad \text{(V.5)} \\
&\lim_{P_z\to\infty} \int d^3x \int_{-\infty}^{\infty} \frac{d\tau}{2\pi}\, e^{-i\omega\tau} \langle P\left|\left[j_x\left(\underset{\sim}{x}, \tfrac{\tau}{E}\right), j_y^\dagger(0)\right]\right| P\rangle = -\frac{i}{2} F_3(\omega)
\end{aligned}$$

Substituting (V.5) into (V.4) we find, upon extending the ω-integration to ∞

$$\sigma_{tot} \longrightarrow \frac{G^2ME}{4\pi} \int_0^{\infty} d\omega\, \omega \int_{-\infty}^{\infty} \frac{d\tau}{2\pi}\, e^{-i\omega\tau}\, C(\tau) \qquad \text{(V.6)}$$

where

$$C(\tau) = \lim_{P_z\to\infty} \int d^3x \,\langle P_z\left|\frac{4}{3}\left[j_x\left(\underset{\sim}{x}, \tfrac{\tau}{E}\right), j_x^\dagger(0)\right] - \left[j_z\left(\underset{\sim}{x}, \tfrac{\tau}{E}\right), j_z^\dagger(0)\right] + \frac{2i}{3}\left[j_x\left(\underset{\sim}{x}, \tfrac{\tau}{E}\right), j_y^\dagger(0)\right]\right|P_z\rangle \qquad \text{(V.7)}$$

An interesting result is obtained upon taking the sum of antineutrino and neutrino cross sections. By crossing symmetry,

$$\sigma_{tot}^{\bar{\nu}p} + \sigma_{tot}^{\nu p} = \frac{G^2ME}{4\pi} \int_{-\infty}^{\infty} \omega\, d\omega \int_{-\infty}^{\infty} \frac{d\tau}{2\pi}\, e^{-i\omega\tau}\, C(\tau) = \frac{G^2ME}{4\pi}(-i)\left.\frac{\partial C(\tau)}{\partial\tau}\right|_{\tau=0}$$

$$\begin{aligned}
&= \frac{G^2ME}{4\pi} \lim_{P_z\to\infty} (-i) \int \frac{d^3x}{P_0} \langle P_z\left|\frac{4}{3}\left[\frac{\partial j_x(\underset{\sim}{x},t)}{\partial t}, j_x^\dagger(0)\right] - \left[\frac{\partial j_z(\underset{\sim}{x},t)}{\partial t}, j_z^\dagger(0)\right]\right. \\
&\quad \left. + \frac{2i}{3}\left[\frac{\partial j_x(\underset{\sim}{x},t)}{\partial t}, j_y^\dagger(0)\right]\right|P_z\rangle_{t=0} \qquad \text{(V.8)}
\end{aligned}$$

Therefore, we predict not only that $\bar{\nu}p$ and νp total cross sections depend linearly on energy, but that the sum of the total cross sections is determined by the equal-time commutator of the Cabibbo current with its time-derivative at infinite momentum.

The linear rise of cross sections predicted here would be cut off, were there an intermediate boson W exchanged, with the cutoff at $E \sim \frac{M_W^2}{M_p}$. Data from the deep-mine cosmic-ray neutrino experiments[16,17] are as yet inconclusive; however, a linear rise of neutrino cross sections up to 10-100 BeV is not inconsistent with the data.[18]

VI. CONCLUSIONS

By combining the $q_0 \to i\infty$ asymptotic limit with the infinite-momentum method, we have shown that in a certain limit, inelastic electron scattering structure functions

$$\lim_{\substack{q^2 \to \infty \\ \frac{-q^2}{\nu} = \omega \text{ fixed}}} MW_1(\nu, q^2) \equiv F_t\left(\frac{-q^2}{\nu}\right) \tag{VI.1}$$

$$\lim_{\substack{q^2 \to \infty \\ \omega \text{ fixed}}} (-q^2)\, W_2(\nu, q^2) - W_1(\nu, q^2) \equiv \frac{F_\ell\left(\frac{-q^2}{\nu}\right)}{M} \tag{VI.2}$$

are directly related to Fourier transforms of almost-equal-time commutators at infinite nucleon momentum, given in (II.5) and (II.6). Provided the Callan-Gross[4] integral is finite:

$$\lim_{|q^2| \to \infty} |q^2| \int_0^\infty \frac{d\nu}{\nu} W_2(\nu, q^2) < \infty \tag{VI.3}$$

we have shown that these commutators are not infinite, but may be zero (or ambiguous). The hypothesis that these commutators are indeed finite is equivalent to the prediction

$$MW_1 \xrightarrow[q^2 \to \infty]{} F_1\left(\frac{-q^2}{\nu}\right) \tag{VI.4}$$

$$\frac{\nu W_2}{M} \xrightarrow[q^2 \to \infty]{} F_2\left(\frac{-q^2}{\nu}\right) \tag{VI.5}$$

Under similar assumptions, total $\bar{\nu}p$ and νp cross sections are predicted to rise linearly with laboratory neutrino energy. Of particular interest is the behavior of the sum of cross sections, dependent, according to (V.8), only on the equal-time commutator of Cabibbo current with its time derivative, evaluated between nucleon states at infinite momentum.

An extension of this technique to more general kinematical conditions, presented in Section IV, is possible, but by itself does not seem to lead further insight into the nature of this limit. A more physical approach into what is going on is, without question, needed.

ACKNOWLEDGMENTS

We thank R. Norton, R. P. Feynman, members of the SLAC theory group, and the SLAC electron-scattering group for useful and enjoyable conversations.

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$$W_1 = \frac{\left(\nu - \frac{|q2|}{2}\right)\sigma_t}{4\pi^2 \alpha M} \qquad W_2 = \frac{\left(\nu - \frac{|q|^2}{2}\right)\left(\sigma_t + \sigma_\ell\right)}{4\pi^2 \alpha M\left(1 + \frac{\nu^2}{M^2 |q^2|}\right)}$$

Notice that F_t and F_ℓ Eqs. (I.3) and (I.4) are proportional to σ_t and σ_ℓ in the limit:

$$\frac{4\pi^2 \alpha F_t}{\nu} \rightarrow \left(1-\frac{\omega}{2}\right)\sigma_t \qquad \frac{4\pi^2 \alpha F_\ell}{\nu} \rightarrow \left(1-\frac{\omega}{2}\right)\sigma_\ell$$

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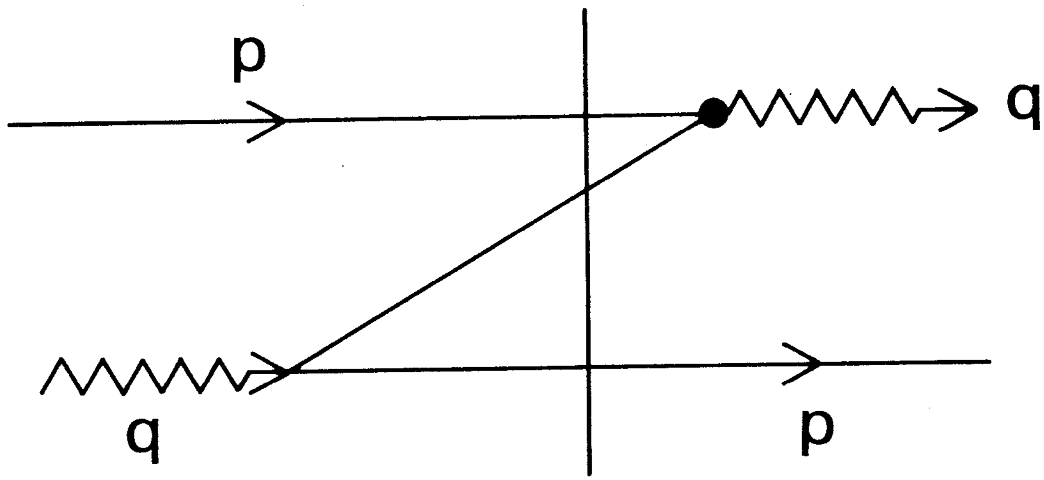

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