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ASYMPTOTIC THEORY OF LIKELIHOOD RATIO AND RANK ORDER TESTS IN SOME MULTIVARIATE LINEAR MODELS[1]

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1. Summary and introduction. The purpose of this paper is two-fold: (i) to develop the asymptotic distribution theory of the normal theory likelihood ratio test statistic for the (multivariate) general linear hypothesis problem when the parent distribution is not necessarily normal and (ii) to develop the theory of the multivariate analysis of covariance based on general rank scores. The problem (i) extends the distribution theory of the likelihood ratio statistic developed by the authors in [9] for the multivariate general linear hypothesis problem (for a class of simple alternatives) to the more general case where one has also to deal with a set of concomittant variables, and the problem (ii) extends the results of the authors' earlier paper [8] on the rank order theory of the univariate analysis of covariance to the corresponding multivariate case.

Let $\mathbf{Z}_{k\alpha} = (\mathbf{Y}_{k\alpha}, \mathbf{X}_{k\alpha})$; [where $\mathbf{Y}_{k\alpha} = (Y_{k\alpha}^{(1)}, \cdots, Y_{k\alpha}^{(p)})$ and $\mathbf{X}_{k\alpha} = (X_{k\alpha}^{(1)}, \cdots, X_{k\alpha}^{(q)})$, p, $q \geqq 1$], $\alpha = 1, \cdots, n_k$ be n_k independent and identically distributed random vectors (i.i.d.r.v.) having a $(p+q)$-variate continuous cumulative distribution function (cdf) $G_k(\mathbf{z})$, $\mathbf{z} \in R^{p+q}$, for $k = 1, \cdots, c$. It is assumed that $\mathbf{Z}_{11}, \cdots, \mathbf{Z}_{cn_c}$ are mutually independent. Let us denote by $F_k^{(1)}(\mathbf{x})$ the (marginal) joint cdf of $\mathbf{X}_{k\alpha}$, and let $F_k^{(2)}(\mathbf{y} \mid \mathbf{x})$ be the conditional cdf of $\mathbf{Y}_{k\alpha}$, given $\mathbf{X}_{k\alpha} = \mathbf{x}$, $k = 1, \cdots, c$. As in the univariate theory (cf. [8, 10]), we assume that

$$F_1^{(1)}(\mathbf{x}) = \cdots = F_c^{(1)}(\mathbf{x}), \qquad \mathbf{x} \in R^q \tag{1.1}$$

and frame the null hypothesis as

$$H_0: F_1^{(2)}(\mathbf{y} \mid \mathbf{x}) = \cdots = F_c^{(2)}(\mathbf{y} \mid \mathbf{x}). \tag{1.2}$$

We may note that under the usual additive model, viz.,

$$F_k^{(2)}(\mathbf{y} \mid \mathbf{x}) = F^{(2)}(\mathbf{y} - \boldsymbol{\tau}_k \mid \mathbf{x}), \qquad \boldsymbol{\tau}_k = (\tau_k^{(1)}, \cdots, \tau_k^{(p)}), \tag{1.3}$$

$k = 1, \cdots, c$, the null hypothesis H_0 in (1.2) implies that $\boldsymbol{\tau}_1 = \cdots = \boldsymbol{\tau}_c$. We are interested in the set of alternatives that (1.2) does not hold, which under the model (1.3) implies that not all $\boldsymbol{\tau}_k$, $k = 1, \cdots, c$ are identical.

The problem of multivariate analysis of covariance (MANOCA) can be viewed as a special case of the general linear hypothesis problem, considered in Anderson (1958, chapter 8). Two problems arise in this context: (i) how the likelihood ratio (l.r.) test behaves when the parent distribution is not necessarily normal, and (ii) how the multivariate generalizations of the tests considered in [8, 10] compare with

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the normal theory l.r. test. The purpose of the present investigation is to study these problems thoroughly.

2. The normal theory l.r. test for the general linear hypotheses. In this section we shall study the asymptotic distribution theory of the normal theory l.r. test for the general linear hypothesis problem when the parent cdf's are not necessarily normal. The MANOCA problem will be studied subsequently as a special case of the general linear hypothesis problem. For simplicity of presentation, we first consider the case of non-stochastic regression variables. Later on, the results will be generalized to stochastic regression variables.

2.1. *Non-stochastic regression variables.* Consider the following model:

$$\mathbf{Y}_\alpha = \boldsymbol{\beta}\mathbf{X}_\alpha + \mathbf{e}_\alpha, \qquad \alpha = 1, \cdots, N, \tag{2.1}$$

where $\mathbf{e}_1, \cdots, \mathbf{e}_n$ are i.i.d.r.v.'s distributed according to the cdf $F(\mathbf{e})$, $\mathbf{e} \in R^p$; $\boldsymbol{\beta}$ is a $p \times q$ matrix of unknown regression constants, and $\mathbf{X}_\alpha$, $\alpha = 1, \cdots, N$ are known (non-stochastic) q-vectors. We partition $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} = [\boldsymbol{\beta}_1, \boldsymbol{\beta}_2] \tag{2.2}$$

where $\boldsymbol{\beta}_i$ is of the order $p \times q_i$, $i = 1, 2$ and $q_1 + q_2 = q$. The problem is to test the null hypothesis

$$H_0 : \boldsymbol{\beta}_1 = \mathbf{0} \tag{2.3}$$

against the alternative $H_1 : \boldsymbol{\beta}_1 \neq \mathbf{0}$. We introduce the following notations:

$$\mathbf{A}_N = \textstyle\sum_{\alpha=1}^N \mathbf{X}_\alpha \mathbf{X}_\alpha'/N, \qquad \mathbf{C}_N = \textstyle\sum_{\alpha=1}^N \mathbf{Y}_\alpha \mathbf{X}_\alpha'/N, \tag{2.4}$$

$$\mathbf{V}_N = \textstyle\sum_{\alpha=1}^N \mathbf{e}_\alpha \mathbf{X}_\alpha'/N, \qquad \mathbf{W}_N = \textstyle\sum_{\alpha=1}^N \mathbf{e}_\alpha \mathbf{e}_\alpha'/N, \qquad \mathbf{S}_N = \textstyle\sum_{\alpha=1}^N \mathbf{Y}_\alpha \mathbf{Y}_\alpha'/N. \tag{2.5}$$

Note that

$$\mathbf{C}_N = \boldsymbol{\beta}\mathbf{A}_N + \mathbf{V}_N, \qquad \mathbf{S}_N = \boldsymbol{\beta}\mathbf{A}_N\boldsymbol{\beta}' + \boldsymbol{\beta}\mathbf{V}_N' + \mathbf{V}_N\boldsymbol{\beta}' + \mathbf{W}_N. \tag{2.6}$$

We partition $\mathbf{V}_N$, $\mathbf{C}_N$ and $\mathbf{A}_N$ as follows:

$$\mathbf{V}_N = [\mathbf{V}_{N1}, \mathbf{V}_{N2}], \qquad \mathbf{C}_N = [\mathbf{C}_{N1}, \mathbf{C}_{N2}], \tag{2.7}$$

$$\mathbf{A}_N = \begin{pmatrix} \mathbf{A}_{N11} & \mathbf{A}_{N12} \\ \mathbf{A}_{N21} & \mathbf{A}_{N22} \end{pmatrix} \tag{2.8}$$

where $\mathbf{V}_{Ni}$ (C_{Ni}) is of the order $p \times q_i$ and $\mathbf{A}_{Nij}$ is of the order $q_i \times q_j$, for $i, j = 1, 2$.

Then the normal theory l.r. test is based on the statistic [cf. Anderson (1958), page 188)]

$$\lambda_N = \{|\hat{\boldsymbol{\Sigma}}_\Omega| / |\hat{\boldsymbol{\Sigma}}_\omega|\}^{N/2} \tag{2.9}$$

where

$$\hat{\boldsymbol{\Sigma}}_\Omega = \mathbf{S}_N - \hat{\boldsymbol{\beta}}_N \mathbf{A}_N \hat{\boldsymbol{\beta}}_N'; \qquad \hat{\boldsymbol{\beta}}_N = \mathbf{C}_N \mathbf{A}_N^{-1} = [\hat{\boldsymbol{\beta}}_{N1}, \hat{\boldsymbol{\beta}}_{N2}], \tag{2.10}$$

$$\hat{\boldsymbol{\Sigma}}_\omega = \mathbf{S}_N - \hat{\boldsymbol{\beta}}_{N2}^* \mathbf{A}_{N22} \hat{\boldsymbol{\beta}}_{N2}^{*\prime}; \qquad \hat{\boldsymbol{\beta}}_{N2}^* = \mathbf{C}_{N2} \mathbf{A}_{N22}^{-1}. \tag{2.11}$$

Anderson (1958, Chapter 8) has shown that when the underlying cdf $F(\mathbf{e})$ is non-singular multivariate normal, $2\log\lambda_N$ has asymptotically a chi-square distribution with pq_1 degrees of freedom (df). We shall show that the above result holds even when F is not necessarily normal. This will follow as a special case of the more general theorems considered below.

We shall say that $\{\mathbf{X}_1, \cdots, \mathbf{X}_N\}$ satisfies the *generalized Noether condition*, if

$$\max_{1 \le i \le q}[\max_{1 \le \alpha \le N} |X_\alpha^{(i)}| \{\textstyle\sum_{\alpha=1}^N [X_\alpha^{(i)}]^2\}^{-\frac{1}{2}}] \to 0 \quad \text{as} \quad N \to \infty, \tag{2.12}$$

where $\mathbf{X}_\alpha = (X_\alpha^{(1)}, \cdots, X_\alpha^{(q)})$, $\alpha = 1, \cdots, N$. We define the covariance matrix of F by

$$\mathbf{\Sigma} = E(\mathbf{e}_\alpha \mathbf{e}_\alpha') \quad \text{and assume that} \quad 0 < |\mathbf{\Sigma}| < \infty. \tag{2.13}$$

Also, we assume that

$$\lim_{N \to \infty} \mathbf{A}_N = \mathbf{A} \tag{2.14}$$

exists and is positive definite, where $\mathbf{A}_N$ is defined by (2.4) and (2.8).

THEOREM 2.1. *Under the assumptions* (2.12)–(2.14), $N^{\frac{1}{2}}(\hat{\boldsymbol{\beta}}_N - \boldsymbol{\beta})$ *has asymptotically a multinormal distribution with means zero and dispersion matrix* $\mathbf{\Sigma} \otimes \mathbf{A}^{-1}$, *where* $\otimes$ *stands for the Kronecker product of two matrices.*

PROOF. Using (2.4), (2.6) and (2.10), we obtain that

$$\hat{\boldsymbol{\beta}}_N = \mathbf{C}_N \mathbf{A}_N^{-1} = \boldsymbol{\beta} + \mathbf{V}_N \mathbf{A}_N^{-1}. \tag{2.15}$$

We now consider an arbitrary linear combination of the elements of $N^{\frac{1}{2}}(\boldsymbol{\beta}_N - \boldsymbol{\beta})$, say,

$$Z_N = N^{\frac{1}{2}} \textstyle\sum_{i=1}^p \sum_{j=1}^q d_{ij} [\hat{\beta}_{Nij} - \beta_{ij}]. \tag{2.16}$$

We denote by $\mathbf{D} = ((d_{ij}))$. Then, by (2.15) and (2.16), we have

$$Z_N = N^{\frac{1}{2}} \operatorname{tr} [\mathbf{V}_N \mathbf{A}_N^{-1} \mathbf{D}'] \tag{2.17}$$

which, by (2.4), equals

$$N^{-\frac{1}{2}} \textstyle\sum_{\alpha=1}^N \{\operatorname{tr} [\mathbf{e}_\alpha \mathbf{X}_\alpha' \mathbf{A}_N^{-1} \mathbf{D}']\}. \tag{2.18}$$

Let us denote by $\mathbf{g}_{\alpha,N} = N^{-\frac{1}{2}} \mathbf{X}_\alpha' \mathbf{A}_N^{-1} \mathbf{D}'$, $\alpha = 1, \cdots, N$. Then, we have from the preceding two equations

$$Z_N = \textstyle\sum_{\alpha=1}^N \operatorname{tr} [\mathbf{e}_\alpha \mathbf{g}_{\alpha,N}] = \sum_{\alpha=1}^N Z_{N,\alpha}, \quad \text{say}. \tag{2.19}$$

We define $\mathbf{j}_p = (1, \cdots, 1)'$, $\mathbf{J}_p = \mathbf{j}_p \mathbf{j}_p'$. Then, it follows that

$$V(Z_{N,\alpha}) = \operatorname{tr} [\mathbf{\Sigma} \mathbf{g}_{\alpha,N}' \mathbf{g}_{\alpha,N}], \qquad \alpha = 1, \cdots, N. \tag{2.20}$$

By virtue of (2.14), it is easy to check that for any fixed $\mathbf{D}$, $\mathbf{g}_{\alpha,N}$, $\alpha = 1, \cdots, N$, satisfy the generalized Noether condition (2.12). And this implies that

$$\max_{1 \le \alpha \le N} \{V(Z_{N,\alpha}) / V(Z_N)\} \to 0 \quad \text{as} \quad N \to \infty. \tag{2.21}$$

Furthermore, it is easy to check that

$$V(Z_N) = \mathrm{tr}\left[\mathbf{\Sigma}\left(\textstyle\sum_{\alpha=1}^{N} \mathbf{g}'_{\alpha,N}\mathbf{g}_{\alpha,N}\right)\right] = O(1), \tag{2.22}$$

by virtue of (2.12), (2.13) and (2.14). The proof of the asymptotic normality of Z_N then follows as an application of Theorem 2 in Gnedenko and Kolmogorov [1954, page 128], the conditions of which can be shown to hold in the present content. This completes the proof.

Let us now define

$$A^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{2.23}$$

where B_{ij} is of the order $q_i \times q_j$, $i, j = 1, 2$. Then, we have the following.

COROLLARY 2.1. *Under the assumptions of Theorem* 2.1, $N^{\frac{1}{2}}(\hat{\boldsymbol{\beta}}_{N1}-\boldsymbol{\beta}_1)$ *converges in law to a* pq_1*-variate normal distribution with zero means and dispersion matrix* $\mathbf{\Sigma}\otimes\mathbf{B}_{11}$.

Consider now the sequence $\{H_N\}$ of alternative hypotheses

$$H_N: \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{1N} = N^{-\frac{1}{2}}\mathbf{\Lambda}_1; \qquad \mathbf{\Lambda}_1 = (\lambda_{ij}) \tag{2.24}$$

where $\lambda_{ij}(i = 1, \cdots, p, j = 1, \cdots, q_1)$ are all finite. Denote

$$A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21} = ((a_{rs\cdot 2})) \qquad r, s = 1, \cdots q_1, \tag{2.25}$$

where A_{ij} is the limit (as $N\to\infty$) of A_{Nij}; $i, j = 1, 2$ [cf. (2.8) and (2.14)]. Also, let $\mathbf{\Sigma}^{-1} = ((\sigma^{ij}))$, and

$$\Delta_\lambda = \textstyle\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{r=1}^{q_1}\sum_{s=1}^{q_1}\sigma^{ij}a_{rs\cdot 2}\lambda_{ir}\lambda_{js}. \tag{2.26}$$

LEMMA 2.1. *Under* $\{H_N\}$ *in* (2.24) *and the conditions of Theorem* 2.1, $N\hat{\boldsymbol{\beta}}_{N1}\mathbf{A}_{N11.2}\hat{\boldsymbol{\beta}}'_{N1}$ *has elements all bounded in probability, as* $N\to\infty$, *where* $\mathbf{A}_{N11.2} = \mathbf{A}_{N11} - \mathbf{A}_{N12}\mathbf{A}_{N22}^{-1}\mathbf{A}_{N21}$.

PROOF. It follows from Corollary 2.1 that under (2.24),

$$|N^{\frac{1}{2}}\hat{\beta}_{N,ij}-\lambda_{ij}| \text{ is bounded in probability, as } N\to\infty, \tag{2.27}$$

for all $i = 1, \cdots, p, j = 1, \cdots, q_1$. Also, a typical element of $N\hat{\boldsymbol{\beta}}_{N1}\mathbf{A}_{N11.2}\hat{\boldsymbol{\beta}}'_{N1}$ is $\sum_{r=1}^{q}\sum_{s=1}^{q}(N^{\frac{1}{2}}\hat{\beta}_{Nir})a_{rs\cdot 2}(N^{\frac{1}{2}}\hat{\beta}_{Njs})$. Using (2.25) and (2.27), the result follows □.

THEOREM 2.2. *Under* $\{H_N\}$ *in* (2.24) *and the conditions of Theorem* 2.1, $2\log\lambda_N$ [where λ_N is defined by (2.9)] *has asymptotically a non-central* χ^2*-distribution with* pq_1 df *and the non-centrality parameter* Δ_λ, *defined by* (2.26).

PROOF. It is well known [cf. Anderson (1958, page 190)] that

$$\hat{\mathbf{\Sigma}}_\omega = \hat{\mathbf{\Sigma}}_\Omega + \hat{\boldsymbol{\beta}}_{N1}\mathbf{A}_{N11.2}\hat{\boldsymbol{\beta}}'_{N1}, \tag{2.28}$$

where $\hat{\mathbf{\Sigma}}_\Omega$ and $\hat{\mathbf{\Sigma}}_\omega$ are defined in (2.10) and (2.11) respectively. Thus, by Lemma 2.1, we have

$$\hat{\mathbf{\Sigma}}_\omega = \hat{\mathbf{\Sigma}}_\Omega + N^{-1}\mathbf{Q}_N; \qquad \mathbf{Q}_N = ((q_{Nij})) = N\hat{\boldsymbol{\beta}}_{N1}\mathbf{A}_{N11.2'}\ \hat{\boldsymbol{\beta}}_{N1} \tag{2.29}$$

where by Lemma 2.1, $q_{Nij}(i = 1, \cdots, p, j = 1, \cdots, q_1)$ are all bounded, in probability. Also, we may note that by (2.6) and (2.10),

$$\hat{\boldsymbol{\Sigma}}_\Omega = [\hat{\boldsymbol{\beta}}_N - \boldsymbol{\beta}]\mathbf{A}_N[\hat{\boldsymbol{\beta}}_N - \boldsymbol{\beta}] - 2\hat{\boldsymbol{\beta}}_N \mathbf{A}_N[\hat{\boldsymbol{\beta}}_N - \boldsymbol{\beta}] + 2\boldsymbol{\beta}\mathbf{V}_N' + \mathbf{W}_N, \tag{2.30}$$

where $\mathbf{A}_N$, $\mathbf{V}_N$, $\mathbf{W}_N$ and $\hat{\boldsymbol{\beta}}_N$ are defined by (2.4), (2.5) and (2.10). Now, by Theorem 2.1, the first two terms on the right-hand side of (2.30) converge, in probability, to null matrices. Also, by (2.5), $E(\mathbf{V}_N) = \mathbf{0}$ and

$$E(\mathbf{V}_N' \otimes \mathbf{V}_N) = N^{-1}\mathbf{A}_N \otimes \boldsymbol{\Sigma} \to \mathbf{0}^{qp \times pq} \quad \text{as} \quad N \to \infty. \tag{2.31}$$

Thus, $\mathbf{V}_N \to_p \mathbf{0}$. Since $\boldsymbol{\beta}$ is a matrix of finite elements ($\boldsymbol{\beta}_{N1} \to \mathbf{0}$ but $\boldsymbol{\beta}_{N2}$ has finite elements), $\boldsymbol{\beta}\mathbf{V}_N \to_p 0$. Finally, by Khinchin's law of large numbers (on the i.i.d.r.v.'s $\mathbf{e}_\alpha$, $\alpha = 1, \cdots, N$) $\mathbf{W}_N = N^{-1}\sum_{\alpha=1}^N \mathbf{e}_\alpha \mathbf{e}_\alpha' \to_p \boldsymbol{\Sigma}$.
Thus,

$$\hat{\boldsymbol{\Sigma}}_\Omega \to_p \boldsymbol{\Sigma}, \tag{2.32}$$

and hence, by (2.13), $\hat{\boldsymbol{\Sigma}}_\Omega$ is finite and positive definite, in probability. Let $\hat{\Sigma}_{\Omega ij}$ be the cofactor of the (i,j)th element of $\hat{\boldsymbol{\Sigma}}_\Omega$ for $i, j = 1, \cdots, p$. Then by (2.29), we have by standard determinant expansion

$$\begin{aligned} |\hat{\boldsymbol{\Sigma}}_\omega| &= |\hat{\boldsymbol{\Sigma}}_\Omega + N^{-1}\mathbf{Q}_N| = |\hat{\boldsymbol{\Sigma}}_\Omega| + N^{-1}\textstyle\sum_{i=1}^p \sum_{j=1}^p \hat{\Sigma}_{\Omega ij} q_{Nij} + O_p(N^{-2}) \\ &= |\hat{\boldsymbol{\Sigma}}_\Omega| \{1 + N^{-1}\textstyle\sum_{i=1}^p \sum_{j=1}^p \hat{\sigma}_\Omega^{ij} q_{Nij} + O_p(N^{-2})\} \end{aligned} \tag{2.33}$$

where

$$\hat{\boldsymbol{\Sigma}}_\Omega^{-1} = ((\hat{\sigma}_\Omega^{ij})), \qquad \hat{\sigma}_\Omega^{ij} = \hat{\Sigma}_{\Omega ij}/|\hat{\boldsymbol{\Sigma}}_\Omega|, \qquad i,j = 1, \cdots, p$$

Using (2.9) and (2.33), we get after simplications that

$$\begin{aligned} 2\log\lambda_N &= \textstyle\sum_{i=1}^p \sum_{j=1}^p \hat{\sigma}_\Omega^{ij} q_{Nij} + O_p(N^{-1}) \\ &= N\textstyle\sum_{i=1}^p \sum_{j=1}^p \sigma^{ij} \sum_{r=1}^{q_1} \sum_{s=1}^{q_1} {}_N\hat{\beta}_{ir}\hat{\beta}_{Njs} a_{Nrs\cdot 2} + o_p(1), \end{aligned} \tag{2.34}$$

as by (2.13) and (2.32) $\hat{\boldsymbol{\Sigma}}_\Omega^{-1} \to_p \boldsymbol{\Sigma}^{-1}$. Since the matrix $A_{N11.2} \otimes \boldsymbol{\Sigma}^{-1}$ converges to $[B_{11} \otimes \boldsymbol{\Sigma}]^{-1}$, [cf. (2.14) and (2.25)], as $N \to \infty$ and $N^{\frac{1}{2}}E\{\hat{\boldsymbol{\beta}}_{N1} \mid H_N\} \to \boldsymbol{\Lambda}_1$, the rest of the proof follows from (2.34), Corollary 2.1 and the well-known results on the asymptotic distribution of quadratic forms associated with multinormal distributions. □

2.2. *Stochastic regression variables.* We now extend the model (2.1) to the case of stochastic regression variables. We assume that $\mathbf{Z}_\alpha = (\mathbf{Y}_\alpha, \mathbf{X}_\alpha)$, $\alpha = 1, \cdots, N$ are i.i.d.r.v.'s distributed according to a $(p+q)$-variate cdf $G(\mathbf{z})$ having the mean vector $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ and the dispersion matrix

$$\boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} \\ \boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{pmatrix}.$$

We denote the marginal cdf of $\mathbf{X}_\alpha$ by $F^{(1)}(\mathbf{x})$, $\mathbf{x} \in R^q$ and the conditional cdf of $\mathbf{Y}_\alpha$, given $\mathbf{X}_\alpha = \mathbf{x}$, by

$$F^{(2)}(\mathbf{y} \mid \mathbf{x}) = F^{(2)}(\mathbf{y} - \boldsymbol{\beta}\mathbf{x}). \tag{2.35}$$

Then

$$\boldsymbol{\Gamma}_{11.2} = \boldsymbol{\Gamma}_{11} - \boldsymbol{\Gamma}_{12}\boldsymbol{\Gamma}_{22}^{-1}\boldsymbol{\Gamma}_{21} \tag{2.36}$$

is the conditional dispersion matrix of $\mathbf{Y}_\alpha$ given $\mathbf{X}_\alpha = \mathbf{x}$. Denote

$$\boldsymbol{\Gamma}_{ij} + \boldsymbol{\mu}_i \boldsymbol{\mu}_j = \boldsymbol{\Sigma}_{ij}, \qquad i,j = 1,2. \tag{2.37}$$

It follows from (2.9), (2.5) and the Khinchin law of large numbers that as $N \to \infty$,

$$\mathbf{A}_N \to_p \boldsymbol{\Sigma}_{22}, \qquad \mathbf{C}_N \to_p \boldsymbol{\Sigma}_{12}, \quad \text{and} \quad \mathbf{S}_N \to_p \boldsymbol{\Sigma}_{11}. \tag{2.38}$$

Let now

$$\boldsymbol{\Sigma}_{22} = \begin{pmatrix} \boldsymbol{\Sigma}_{11.22} & \boldsymbol{\Sigma}_{12.22} \\ \boldsymbol{\Sigma}_{21.22} & \boldsymbol{\Sigma}_{22.22} \end{pmatrix}, \tag{2.39}$$

where $\boldsymbol{\Sigma}_{ij.22}$ is of the order $q_i \times q_j$; $i, j = 1, 2$, and define

$$\boldsymbol{\nu}_{11.2} = \boldsymbol{\Sigma}_{11.22} - \boldsymbol{\Sigma}_{12.22}\boldsymbol{\Sigma}_{22.22}^{-1}\boldsymbol{\Sigma}_{21.22}. \tag{2.40}$$

Then, using (2.38), it follows that as $N \to \infty$

$$\mathbf{A}_{N11.2} \to_p \boldsymbol{\nu}_{11.2} = ((\nu_{rs\cdot 2})); \tag{2.41}$$

where $\mathbf{A}_{N11.2}$ is defined in Lemma 2.1. Finally, let us define

$$\boldsymbol{\Gamma}_{11.2}^{-1} = ((\gamma_2{}^{ij})). \tag{2.42}$$

Then, proceeding as in the proof of Theorem 2.1, we arrive at the following result.

THEOREM 2.3. *Under* (2.36)–(2.42), $N^{\frac{1}{2}}(\hat{\boldsymbol{\beta}}_N - \boldsymbol{\beta})$ *has asymptotically a multinormal distribution with means zero and dispersion matrix* $\boldsymbol{\Gamma}_{11.2} \otimes \boldsymbol{\Sigma}_{22}^{-1}$.

Let us now define

$$\Delta_\lambda{}^* = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{r=1}^{q_1} \sum_{s=1}^{q_1} \gamma_2{}^{ij} \gamma_{rs\cdot 2} \lambda_{ir} \lambda_{js} \tag{2.43}$$

where λ_{ij}, $\nu_2{}^{ij}$ and $\nu_{rs\cdot 2}$ are defined by (2.24), (2.42) and (2.40) respectively.

Modifying Corollary (2.1) and Lemma 2.1 in the light of Theorem 2.3, we arrive at the following:

THEOREM 2.4. *Under* $\{H_N\}$ *in* (2.24) *and the conditions of Theorem* 2.3, $2 \log \lambda_N$ *has asymptotically the non-central chi-square distribution with* pq_1 df *and the non-centrality parameter* $\Delta_\lambda{}^*$ *defined in* (2.43).

2.3. *Mixed model.* We now consider the model (2.1) when some of the regression variables are stochastic and the remaining non-stochastic. In this case we have to proceed as in Section 2.2 with the necessary modifications from Section 2.1. For brevity of presentation, the details are omitted.

3. Asymptotic theory of the normal theory l.r. test for MANOCA. We define $\mathbf{Z}_{k\alpha}$, $\alpha = 1, \cdots, n_k$, $k = 1, \cdots, c$ as in Section 1, and the conditional cdf of $\mathbf{Y}_{k\alpha}$, given $\mathbf{X}_{k\alpha} = \mathbf{x}$, as in (1.3). Then, under (2.35), we have the model

$$E(\mathbf{Y}_{k\alpha} \mid \mathbf{X}_{k\alpha} = \mathbf{x}) = \boldsymbol{\tau}_k + \boldsymbol{\beta}\mathbf{x}, \qquad k = 1, \cdots, c, \tag{3.1}$$

where $\tau_1, \cdots, \tau_c$ are non-stochastic. Thus, (3.1) corresponds to the mixed model in Section 2.3. We now consider $\boldsymbol{\beta}$ to be a nuisance parameter (matrix). We desire to test the null hypothesis

(3.2) $$H_0: \boldsymbol{\tau}_1 = \cdots = \boldsymbol{\tau}_c,$$

against the sequence of alternatives $\{H_N\}$ where

(3.3) $$H_N: \boldsymbol{\tau}_k = \boldsymbol{\tau}_{kN} = N^{-\frac{1}{2}}\boldsymbol{\theta}_k, \qquad k = 1, \cdots, c,$$

and $\boldsymbol{\theta}_k$, $k = 1, \cdots, c$ are vectors with real and finite elements. We introduce the following notations:

(3.4) $\bar{\mathbf{Y}}_k = n_k^{-1}\sum_{\alpha=1}^{n_k}\mathbf{Y}_{k\alpha}, \quad k = 1, \cdots, c; \qquad \bar{\mathbf{Y}}_N = N^{-1}\sum_{k=1}^{c} n_k \bar{\mathbf{Y}}_k;$

(3.5) $\mathbf{S}_N = N^{-1}\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k}(\mathbf{Y}_{k\alpha}-\bar{\mathbf{Y}}_k)(\mathbf{Y}_{k\alpha}-\bar{\mathbf{Y}}_k)'$,

(3.6) $\mathbf{S}_N^* = N^{-1}\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k}(\mathbf{Y}_{k\alpha}-\bar{\bar{\mathbf{Y}}})(\mathbf{Y}_{k\alpha}-\bar{\bar{Y}})'$,

(3.7) $A_N = N^{-1}\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k}\mathbf{X}_{k\alpha}\mathbf{X}_{k\alpha}'$,

(3.8) $\mathbf{C}_N = N^{-1}\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k}(\mathbf{Y}_{k\alpha}-\bar{\mathbf{Y}}_k)\mathbf{X}_{k\alpha}', \qquad C_N^* = N^{-1}\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k}(\mathbf{Y}_{k\alpha}-\bar{\bar{\mathbf{Y}}})\mathbf{X}_{k\alpha}'.$

Finally, let

(3.9) $$\hat{\boldsymbol{\beta}}_N = \mathbf{C}_N\mathbf{A}_N^{-1} \quad \text{and} \quad \boldsymbol{\beta}_N^* = C_N^* A_N^{-1}.$$

Then, the normal-theory l.r. test statistic for the MANOCA problem is based on the statistic

(3.10) $$\lambda_N = \{|\mathbf{S}_N - \hat{\boldsymbol{\beta}}_N\mathbf{A}_N\hat{\boldsymbol{\beta}}_N'|/|\mathbf{S}_N^* - \boldsymbol{\beta}_N^*\mathbf{A}_N\boldsymbol{\beta}_N^{*\prime}|\}^{N/2};$$

the null hypothesis in (3.2) is rejected when λ_N exceeds a critical value $\lambda_{N,\varepsilon}$.

From the results of Section 2, it follows that under H_0 in (3.2), $2\log\lambda_N$ has asymptotically the χ^2-distribution with $p(c-1)$ df. Hence,

(3.11) $$\lim_{N\to\infty} 2\log\lambda_{N,\varepsilon} = \chi^2_{p(c-1),\varepsilon},$$

where $\chi^2_{t,\varepsilon}$ is the upper $100\varepsilon\%$ point of the χ^2-distribution with t df.

We define the dispersion matrix of $\mathbf{Z}_{k\alpha}$ by $\boldsymbol{\Sigma}^*$ (of the order $(p+q) \times (p+q)$). We write

(3.12) $$\boldsymbol{\Sigma}^* = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$

where $\boldsymbol{\Sigma}_{11}(\boldsymbol{\Sigma}_{22})$ is the dispersion matrix of $\mathbf{Y}_{k\alpha}(\mathbf{X}_{k\alpha})$ and $\boldsymbol{\Sigma}_{12}$ is the covariance matrix of $(\mathbf{Y}_{k\alpha}, \mathbf{X}_{k\alpha})$. Let then

(3.13) $$\boldsymbol{\Sigma}_{11.2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} = ((\sigma_{ij\cdot 2})),$$

and let

(3.14) $$\boldsymbol{\Sigma}_{11.2}^{-1} = ((\sigma_{11.2}^{ij})).$$

From the theorems of Section 2, we then arrive at the following.

THEOREM 3.1. *Assume* (i) $\lim_{N\to\infty} n_k/N = \rho_k : 0 < \rho_k < 1, k = 1, \cdots, c$, *and* (ii) $\mathbf{\Sigma}^*$. *defined in* (3.12), *is positive definite. Then, under* (3.3), $2\log\lambda_N$ [*where* λ_N *is defined by* (3.10)] *has asymptotically the non-central* χ^2*-distribution with* $p(c-1)$ df *and the non-centrality parameter*

$$\Delta_\lambda = \textstyle\sum_{k=1}^c \rho_k(\boldsymbol{\theta}_k - \bar{\boldsymbol{\theta}})\mathbf{\Sigma}_{11.2}^{-1}(\boldsymbol{\theta}_k - \bar{\boldsymbol{\theta}})'; \qquad \bar{\boldsymbol{\theta}} = \textstyle\sum_{k=1}^c \rho_k \boldsymbol{\theta}_k. \tag{3.15}$$

4. Rank order tests for MANOCA. In this section we generalize the rank order tests for ANOCA considered in [8, 10] to the multivariate problem under consideration. For convenience of presentation, we shall use, as far as possible, the notations employed in [8].

Let us denote the sample point $\mathbf{Z}_N^* = (\mathbf{Z}_{11}, \cdots, \mathbf{Z}_{cn_c})$ where $\mathbf{Z}_{k\alpha} = (Y_{k\alpha}, X_{k\alpha})\cdot \mathbf{Z}_N^*$ is a matrix of order $(p+q) \times N$. Ranking the N elements in each row of $\mathbf{Z}_N^*$ in ascending order of magnitude, we get a $(p+q) \times N$ rank matrix

$$\mathbf{R}_N = \begin{Bmatrix} S_{11}^{(1)} \dots S_{1n_1}^{(1)} \dots S_{c1}^{(1)} \dots S_{cn_c}^{(1)} \\ \vdots \qquad\qquad\qquad \vdots \\ S_{11}^{(p)} \dots S_{1n_1}^{(p)} \dots S_{c1}^{(p)} \dots S_{cn_c}^{(p)} \\ R_{11}^{(1)} \dots R_{1n_1}^{(1)} \dots R_{c1}^{(1)} \dots R_{cn_c}^{(1)} \\ \vdots \qquad\qquad\qquad \vdots \\ R_{11}^{(q)} \dots R_{1n_1}^{(q)} \dots R_{c1}^{(q)} \dots R_{cn_c}^{(q)} \end{Bmatrix} \tag{4.1}$$

where $S_{k\alpha}^{(i)}(R_{k\alpha}^{(j)})$ is the rank of $Y_{k\alpha}^{(i)}(X_{k\alpha}^{(j)})$ among the N observations on the ith (jth) Y-variate (X-variate); $i = 1, \cdots, p (j = 1, \cdots, q)$. We define the rank order statistics

$$S_{Ni}^{(k)} = \textstyle\sum_{\alpha=1}^{n_k} E_N^{(i)}(S_{k\alpha}^{(i)})/n_k, \qquad i = 1, \cdots, p, \quad k = 1, \cdots, c; \tag{4.2}$$

$$T_{Nj}^{(k)} = \textstyle\sum_{\alpha=1}^{n_k} E_N^{*(j)}(R_{k\alpha}^{(j)})/n_k, \qquad j = 1, \cdots, q, \quad k = 1, \cdots, c \tag{4.3}$$

where the $E_N^{(i)}(\alpha)$ and $E_N^{*(j)}(\alpha)$, $\alpha = 1, \cdots, N$ are the general rank scores satisfying the conditions of Section 4 of [8]. Define also

$$\bar{E}_N^{(j)} = \textstyle\sum_{\alpha=1}^N E_{N(\alpha)}^{(i)}/N, \qquad \bar{E}_N^{*(j)} = \textstyle\sum_{\alpha=1}^N E_{N(\alpha)}^{*(j)}/N, \tag{4.4}$$

for $i = 1, \cdots, p, j = 1, \cdots, q$.

$$v_{ij\cdot 11} = N^{-1}\textstyle\sum_{k=1}^c \sum_{\alpha=1}^{n_k} E_N^{(i)}(S_{k\alpha}^{(i)})E_N^{(j)}(S_{k\alpha}^{(j)}) - \bar{E}_N^{(i)}\bar{E}_n^{(j)} \tag{4.5}$$

for $i, j = 1, \cdots, p$; $\mathbf{V}_{11} = ((v_{ij\cdot 11}))$;

$$v_{ij\cdot 22} = N^{-1}\textstyle\sum_{k=1}^c \sum_{\alpha=1}^{n_k} E_N^{*(i)}(R_{k\alpha}^{(i)})E_N^{*(j)}(R_{k\alpha}^{(j)}) - \bar{E}_N^{*(i)}\bar{E}_N^{*(j)} \tag{4.6}$$

for $i, j = 1, \cdots, q$; $\mathbf{V}_{22} = ((v_{ij\cdot 22}))$;

$$v_{ij\cdot 12} = N^{-1}\textstyle\sum_{k=1}^c \sum_{\alpha=1}^{n_k} E_N^{(i)}(S_{k\alpha}^{(i)})E_N^{*(j)}(R_{k\alpha}^{(j)}) - \bar{E}_N^{(j)}\bar{E}_N^{*(j)} \tag{4.7}$$

for $i = 1, \cdots, p; j = 1, \cdots, q$; $\mathbf{V}_{12} = ((v_{ij.12}))$ and let

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{12}' & \mathbf{V}_{22} \end{pmatrix}. \tag{4.8}$$

Let $\mathscr{P}_N$ be the permutational (conditional) probability measure generated by the

$N!$ conditionally equally likely permutations of the column of $\mathbf{R}_N$; for details, the reader is referred to [6]. Then as in [6], it follows that

$$E(S_{Ni}^{(k)} - \bar{E}_N{}^{(i)} \mid \mathscr{P}_N) = 0, \qquad i = 1, \cdots, p, \quad k = 1, \cdots, c \tag{4.9}$$

$$E(T_{Ni}^{(k)} - \bar{E}_N{}^{*(i)} \mid \mathscr{P}_N) = 0, \qquad i = 1, \cdots, q, \quad k = 1, \cdots, c \tag{4.10}$$

$$\operatorname{Cov}(S_{Ni}^{(k)}, S_{Nj}^{(l)} \mid \mathscr{P}_N) = (\delta_{kl} N - n_k) v_{ij \cdot 11} / n_k(N-1), \tag{4.11}$$

$$\operatorname{Cov}(T_{Ni}^{(k)}, T_{Nj}^{(l)} \mid \mathscr{P}_N) = (\delta_{kl} N - n_k) v_{ij \cdot 22} / n_k(N-1), \tag{4.12}$$

$$\operatorname{Cov}(S_{Ni}^{(k)}, T_{Nj}^{(l)} \mid \mathscr{P}_N) = (\delta_{kl} N - n_k) v_{ij \cdot 12} / n_k(N-1) \tag{4.13}$$

where δ_{kl} is the usual Kronecker delta. Proceeding again as in [8], we obtain the adjusted mean rank scores for the c samples as

$$\mathbf{S}_N{}^* = \mathbf{S}_N - \bar{\mathbf{E}}_N + \mathbf{V}_{12} \mathbf{V}_{22}^{-1} [\mathbf{T}_N - \bar{\mathbf{E}}_N{}^*], \tag{4.14}$$

where $\mathbf{V}_{22}$ and $\mathbf{V}_{12}$ are defined by (4.6) and (4.7) respectively, and

$$\mathbf{S}_N = ((S_{N,i}^{(k)})), \qquad \mathbf{S}_N{}^* = ((S_{N,i}^{*(k)})) \quad \text{are } p \times c \text{ matrices}, \tag{4.15}$$

$$\bar{\mathbf{E}}_N = ((\bar{E}_N{}^{(i)})), \quad \text{i.e. the } i\text{th row has all the } c \text{ elements } \bar{E}_N{}^{(i)}, \tag{4.16}$$

$$\bar{\mathbf{E}}_N{}^* = ((E_N{}^{*(i)})), \quad \text{i.e. the } i\text{th row has all the } c \text{ elements } \bar{E}^{*(i)}, \tag{4.17}$$

$$\mathbf{T}_N = ((T_{N,i}^{(k)})) \quad \text{is of the order} \quad q \times c. \tag{4.18}$$

Further, let

$$\mathbf{V}_{11.22} = \mathbf{V}_{11} - \mathbf{V}_{12} \mathbf{V}_{22}^{-1} \mathbf{V}_{12}'. \tag{4.19}$$

Then, the proposed test statistic is

$$\mathscr{L}'_N = \textstyle\sum_{k=1}^{c} n_k \mathbf{S}_{N,k}^{*\prime} V_{11.22}^{-1} \mathbf{S}_{N,k}^*, \tag{4.20}$$

where $S_{N,k}^*$ is the kth column of $\mathbf{S}_N{}^*$, $k = 1, \cdots, c$. Note that when $p = 1$, the statistic (4.20) reduces to the statistic considered in the corresponding ANOCA problem [8]. As in [8], the permutation distribution of $\mathscr{L}_N$ will not depend on the underlying unknown cdf when the null hypothesis holds, and hence the permutation distribution of $\mathscr{L}_N$ leads to a conditionally distribution-free test of H_0. The small sample test procedure can thus be based on the exact sample permutation distribution of $\mathscr{L}_N$. However, in view of the excessive labor involved in this process, we consider the large sample procedure.

5. Large sample permutation test based on $\mathscr{L}_N$. Let $E_N{}^{(i)}(\alpha) = J_N{}^{(i)}(\alpha/N+1)$, $i = 1, \cdots, p$, and $E_N{}^{*(i)}(\alpha) = J_N{}^{*(i)}(\alpha/N+1)$, $i = 1, \cdots, q$ where the functions $J_N{}^{(i)}$ and $J_N{}^{*(i)}$ satisfy the conditions of Section 4 of [8]. Furthermore, we assume that

$$0 < \rho_0 \leqq \rho_N{}^{(k)} = n_k/N \leqq 1 - \rho_0 < 1; \qquad \text{for all } k = 1, \cdots, c, \tag{5.1}$$

and $\rho_0 \leqq 1/c$. Then, we have the following

THEOREM 5.1. *Under the above assumptions the statistic $\mathscr{L}_N$ defined by* (4.20) *has* *asymptotically, in probability, (under the permutation measure* $\mathscr{P}_N$, *the chi-square distribution with* $p(c-1)$ df.

The proof of this theorem follows along the lines of Theorem 4.1 of [8] with adaptations from Theorems 4.1 and 4.2 of Puri and Sen (1966).

By virtue of this theorem, we have the following large sample test procedure:

If $\mathscr{L}_N > \chi^2_{p(c-1),\varepsilon}$ reject H_0 in (1.2), otherwise accept H_0, where $\chi^2_{t,\varepsilon}$ is defined just after (3.11).

To study the power properties of this test, it is necessary to study the unconditional distribution of $\mathscr{L}_N$ under suitable sequences of alternatives. This in turn requires the study of the joint asymptotic distribution of the statistics $\mathbf{S}_N{}^*$ or equivalently of $\mathbf{S}_N$ and $\mathbf{T}_N$. This has been studied in detail by Puri and Sen (1966) (Section 5) when $G_1, \cdots, G_c$ are arbitrary. Here we shall specifically consider the asymptotic distribution of $\mathbf{S}_N{}^*$ under the following sequence of alternative hypotheses $\{k_N\}$ defined by

$$k_N : \boldsymbol{\tau}_k = \boldsymbol{\tau}_{kN} = N^{-\frac{1}{2}}\boldsymbol{\theta}_k, \qquad k = 1, \cdots, c \tag{5.2}$$

where $\boldsymbol{\theta}_k = (\theta_{k1}, \cdots, \theta_{kp})$, $k = 1, \cdots, c$ are c real p-vectors. Also, we assume that

$$\lim_{N\to\infty} \rho_N{}^{(k)} = \rho^{(k)} = 0 < \rho^{(k)} < 1, \qquad k = 1, \cdots, c. \tag{5.3}$$

Let $G(\mathbf{Z})$ denote the common distribution of $\mathbf{Z}_{k\alpha} = (\mathbf{Y}_{k\alpha}, \mathbf{X}_{k\alpha})$ under the null hypothesis; and let $G_{[1]}, \cdots, G_{[p]}$ be the marginal cdf's of the p Y-variates corresponding to the joint cdf G. We assume that the marginal distributions are absolutely continuous, and the following integrals exist:

$$B^{(i)} = \int_{-\infty}^{\infty} \frac{d}{dx} J^{(i)}(G_{[i]}^{(x)})\, dG_{[i]}^{(x)}, \qquad i = 1, \cdots, p; \tag{5.4}$$

where $J^{(i)}(u) = \lim_{N\to\infty} J_N{}^{(i)}(u)$, $i = 1, \cdots, p$. Also let $J^{*(i)}(u) = \lim_{N\to\infty} J_N{}^{*(i)}(u)$, $i = 1, \cdots, q$.

Define

$$\nu_{ii\cdot 11} = \int_0^1 [J^{(i)}(u)]^2\, du - \mu_i{}^2; \quad \mu_i = \int_0^1 J^{(i)}(u)\, du \tag{5.5}$$

$$\nu_{ij\cdot 11} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J^{(i)}(G_{[i]}^{(x)}) J^{(j)}(G_{[j]}^{(y)})\, dG_{[i,j]}^{(x,y)} - \mu_i \mu_j \tag{5.6}$$

where $G_{[i,j]}$ is the marginal cdf of the i and jth Y-variate, corresponding to the cdf G, $i \neq j = 1, \cdots, p$.

$$\nu_{ii\cdot 22} = \int_0^1 [J^{*(i)}(u)]^2\, du - \mu_i{}^{*2}; \qquad \mu_i{}^* = \int_0^1 J^{*(i)}(u)\, du; \qquad i = 1, \cdots, q \tag{5.7}$$

$$\nu_{ij\cdot 22} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J^{*(i)}(G_{[i]}^{*(y)}) J^{*(j)}(G_{[j]}^{*(y)})\, dG_{[i,j]}^{*(x,y)} - \mu_i{}^*\mu_j{}^*, \tag{5.8}$$

where $G^*_{[i]}$ and $G^*_{[i,j]}$ are the marginal cdf's of the ith and (i,j)th X-variates corresponding to the cdf G; $i \neq j = 1, \cdots, q$. Furthermore, let

$$\nu_{ij\cdot 12} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J^{(i)}(G_{[i]}^{(x)}) J^{*(j)}(G_{[j]}^{*(y)})\, dG_{[i,j]}^{**(x,y)} - \mu_i \mu_j{}^* \tag{5.9}$$

where $G_{[i,j]}^{**(x,y)}$ is the joint distribution of the ith Y-variate and jth X-variate corresponding to the cdf G; $i = 1, \cdots, p, j = 1, \cdots, q$.

Finally, denote

$$(5.10) \qquad \boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_{11} & \boldsymbol{v}_{12} \\ \boldsymbol{v}_{12}' & \boldsymbol{v}_{22} \end{pmatrix}; \qquad \boldsymbol{v}_{rs} = ((v_{ij \cdot rs})); \qquad r, s = 1, 2,$$

and

$$(5.11) \qquad \boldsymbol{v}_{11.2} = \boldsymbol{v}_{11} - \boldsymbol{v}_{12} \boldsymbol{v}_{22}^{-1} \boldsymbol{v}_{12}'.$$

Then, we have the following

THEOREM 5.2. *Under the conditions of Section* 4 of [8], (5.2), (5.3) and (5.4), *the statistic* $\mathscr{L}_N$ *defined by* (4.20) *has asymptotically the non-central chi-square distribution with* $p(c-1)$ df *and non-centrality parameter*

$$(5.12) \qquad \Delta_{\mathscr{L}} = \textstyle\sum_{k=1}^{c} \rho_k (\boldsymbol{\eta}_k - \bar{\boldsymbol{\eta}}) \boldsymbol{v}_{11.2}^{-1} (\boldsymbol{\eta}_k - \bar{\boldsymbol{\eta}})'$$

where

$$(5.13) \qquad \boldsymbol{\eta}_k = (\eta_{k1}, \cdots, \eta_{kp}); \qquad \eta_{ki} = \theta_{ki} B^{(i)}, \qquad i = 1, \cdots, p; \quad k = 1, \cdots, c$$

and $\bar{\boldsymbol{\eta}} = \sum_{k=1}^{c} \rho_k \boldsymbol{\eta}_k$.

The proof is a multivariate generalization of Theorem 5.1 of [8] and can be accomplished along the lines of Theorem 6.1 of Puri and Sen (1966).

REMARK. It is of interest to note that the convergence of the distribution of $\mathscr{L}_N$ is based on the covergence of the distribution of the statistics $\mathbf{S}_N$ and $\mathbf{T}_N$. Actually the above theorem and the theorems considered in [6, 7, 8, 11 and some of the references cited therein] deal exclusively with the convergence of the distributions of the rank statistics to the appropriate normal distributions. It is, however, not known whether such a convergence is accompanied by the convergence of the corresponding moments, say of means and covariances. This result will follow if $N^{\frac{1}{2}}$ times the higher order terms in the Chernoff-Savage theorem, or its generalizations considered in the references cited above, all converge in mean square to zero. This is not true in general. For bounded scores ($J'(u)$ bounded), this will of course be true. But for unbounded scores this result is not generally true. However, if, in addition to the stated conditions on the score functions $J^{(i)}(u)$ and $J^{*(i)}(u)$, we assume that $J^{(i)}(u) = J_1^{(i)}(u) - J_2^{(i)}(u)$, and $J^{*(i)}(u) = J_1^{*(i)}(u) - J_2^{*(i)}(u)$ where $J_k^{(i)}(u)$ as well as $J_k^{*(i)}(u)$, $k = 1, 2$ are nondecreasing in u, then the convergence of the covariances of the rank order statistics to the corresponding covariances of the asymptotic normal distribution can easily be established by proceeding exactly as in Theorems 2.1 and 2.3 of Hájek (1968). However, this does not guarantee the convergence of $N^{\frac{1}{2}}E(T_{Nj}^{(k)})$ or that of $N^{\frac{1}{2}}E(S_{Ni}^{(k)})$. But recently Hoeffding (1968) has shown (under conditions less restrictive than Chernoff and Savage, and more restrictive than Hájek (1968)), that this convergence also holds (even for general alternatives, i.e. not only for (5.2)). For brevity, we omit the details.

6. Asymptotic efficiency of the proposed tests. It follows from Theorem 3.1 and Theorem 5.2 that the asymptotic (Pitman) relative efficiency (A.R.E.) of the test based on $\mathscr{L}_N$ with respect to the normal theory l.r. test is equal to

$$\varepsilon_{\mathscr{L},\lambda} = \Delta_{\mathscr{L}}/\Delta_\lambda, \tag{6.1}$$

where Δ_λ and $\Delta_{\mathscr{L}}$ are defined in (3.15) and (5.12), respectively. Since the A.R.E. (6.1) depends on the matrices $\boldsymbol{v}_{11.2}$, $\boldsymbol{\Sigma}_{11.2}$, ρ_k, $k = 1, \cdots, c$ and the shifts $\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_c$, we do not get a single numerical value, as in the corresponding univariate problem (cf. (6.2) of [6]). This feature is common to other multivariate problems (viz, [5, 8]). Two interesting results are worth mentioning in this content. First, the MANOCA test considered in this paper is asymptotically *at least as efficient* as the *MANOVA test*, considered in Puri and Sen [7]. Second, the MANOCA test based on normal scores statistics is asymptotically as efficient as the normal theory l.r. test for normal alternatives. These we consider below.

If we totally disregard the concomitant variates and consider rank order tests based only on the statistics $S_{N,k}^{(i)}$, $i = 1, \cdots, p$, $k = 1, \cdots, c$, the results of [7] are directly applicable. The corresponding test-statistic in this situation reduces to

$$\mathscr{L}_N^{\,0} = \sum_{k=1}^{c} n_k[\mathbf{S}_{N,k} - \overline{\mathbf{E}}_N]'\mathbf{V}_{11}^{-1}[\mathbf{S}_{N,k} - \overline{\mathbf{E}}_N], \tag{6.2}$$

where $\overline{\mathbf{E}}_N = (\bar{E}_N^{(1)}, \cdots, \bar{E}_N^{(p)\prime})$ and $\mathbf{S}_{N,k}$ is the kth column of $\mathbf{S}_N$, defined by (4.15), $k = 1, \cdots, c$. It follows from Theorem 6.2 of [7] that under the sequence of alternatives $\{H_N\}$ in (5.2), $\mathscr{L}_N^{\,0}$ has asymptotically a non-central χ^2-distribution with $p(c-1)$ df and non-centrality parameter

$$\Delta_{\mathscr{L}}^{\,0} = \sum_{k=1}^{c} \rho_k(\boldsymbol{\eta}_k - \bar{\boldsymbol{\eta}})'\boldsymbol{v}_{11}^{-1}(\boldsymbol{\eta}_k - \bar{\boldsymbol{\eta}}), \tag{6.3}$$

where $\boldsymbol{v}_{11}$ and $\boldsymbol{\eta}_k$ are defined by (5.10) and (5.13) respectively. Now, the A.R.E. of the MANOCA test ($\mathscr{L}_N$) with respect to the MANOVA test ($\mathscr{L}_N^{\,0}$) is given by

$$\varepsilon_{\mathscr{L},\mathscr{L}^0} = \Delta_{\mathscr{L}}/\Delta_{\mathscr{L}^0}; \tag{6.4}$$

which also, like (6.1), depends on $\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_c$ as well as on the parent distribution G. We shall show that (6.4) is bounded below by 1, uniformly in $\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_c$ and G. To prove this, we use the well-known result by Courant [cf. Bodewig (1956)] on the ratio of two quadratic forms. This leads [by virtue of (5.12) and (6.3)] to

$$\inf_{\theta_1,\ldots,\theta_c} \varepsilon_{\mathscr{L},\mathscr{L}^0} = C_{\min}[\boldsymbol{v}_{11.2}^{-1}\boldsymbol{v}_{11}] = C_{\max}[\boldsymbol{v}_{11}^{-1}\boldsymbol{v}_{11.2}], \tag{6.5}$$

where $C_{\min}(C_{\max})$ stands for the minimum (maximum) characteristic root of the matrix under parenthesis. Thus, by (5.11), we have

$$\begin{aligned} C_{\max}[\boldsymbol{v}_{11}^{-1}\boldsymbol{v}_{11.2}] &= C_{\max}[\boldsymbol{v}_{11}^{-1}\{\boldsymbol{v}_{11} - \boldsymbol{v}_{12}\boldsymbol{v}_{22}^{-1}\boldsymbol{v}_{21}\}] \\ &= C_{\max}[\mathbf{I}_{q_1} - \boldsymbol{v}_{11}^{-1}\{\boldsymbol{v}_{12}\boldsymbol{v}_{22}^{-1}\boldsymbol{v}_{21}\}] \leqq 1, \end{aligned} \tag{6.6}$$

as $\boldsymbol{v}_{11}^{-1}\{\boldsymbol{v}_{12}\boldsymbol{v}_{22}^{-1}\boldsymbol{v}_{21}\}$ is positive semidefinite. Hence, from (6.5) and (6.6), we obtain that

$$\inf_{\theta_1,\cdots,\theta_c} \varepsilon_{\mathscr{L},\mathscr{L}^0} \geqq 1, \quad \text{uniformly in} \quad G. \tag{6.7}$$

This clearly establishes the superiority of the MANOCA test over the corresponding MANOVA test.

Now, if the parent cdf G is non-singular multinormal, and we use the normal scores [i.e., $J^{(i)}(u) = \Phi^{-1}(u): 0 < u < 1$, $i = 1, \cdots, p$, $J^{*(i)}(u) = \Phi^{-1}(u): 0 < u < 1$, $i = 1, \cdots, q$, where $\Phi(x)$ is the standard normal cdf], it can be easily shown that $\boldsymbol{\eta}_k = \boldsymbol{\theta}_k$, $k = 1, \cdots, c$, and $\boldsymbol{\nu} = \boldsymbol{\Sigma}^*$, where $\boldsymbol{\nu}$ and $\boldsymbol{\Sigma}^*$, $\boldsymbol{\eta}_k$ and $\boldsymbol{\theta}_k$ are defined by (5.10), (3.12), (5.13) and (5.2) respectively. Hence, from (3.15), (5.12) and (6.1), we have

$$\varepsilon_{\mathscr{L}(\Phi),\lambda} = 1 \quad \text{for all} \quad \boldsymbol{\Sigma}^*, \tag{6.8}$$

where $\mathscr{L}(\Phi)$ stands for the normal scores MANOCA statistic.

Let us now consider the *rank sum test for MANOCA*. Here we select $E_N^{(i)}(\alpha) = \alpha/(N+1)$, $1 \leqq \alpha \leqq N$, $i = 1, \cdots, p$; $E_N^{*(i)}(\alpha) = \alpha/(N+1)$, $1 \leqq \alpha \leqq N$, $i = 1, \cdots, q$. Then the corresponding test $\mathscr{L}_N$ reduces to the rank sum $\mathscr{L}_N(R)$ test. In this case $B_i = \int g_{[i]}^2(x)\,dx$, $i = 1, \cdots, p$; $\nu_{ij,rr} = 1/12$ for $r = 1$, $i = 1, \cdots, p$ or $r = 2$, $i = 1, \cdots, q$

$$\nu_{ij\cdot 11} = \int_{-p}^{\infty}\int_{-p}^{\infty} G_{[i]}(x)G_{[i]}(y)\,dG_{[i,j]}(x, y) - \tfrac{1}{4}$$

$$\nu_{ij\cdot 22} = \int_{-p}^{\infty}\int_{-p}^{\infty} G_{[i]}^*(x)G_{[i]}^*(y)\,dG_{[i,j]}^*(x, y) - \tfrac{1}{4}$$

$$\nu_{ij\cdot 12} = \int_{-p}^{\infty}\int_{-p}^{\infty} G_{[i]}(x)G_{[j]}^*(y)\,dG_{[i,j]}^{**}(x, y) - \tfrac{1}{4}$$

In general, it is difficult to obtain simple bounds for $\nu_{\mathscr{L}(R),\lambda}$ unlike in the univariate case (cf. [8]). However, if the underlying distribution is normal, then $\nu_{ij\cdot rs} = (2\pi)^{-1}\sin^{-1}\frac{1}{2}\rho_{ij\cdot rs}$ where $\rho_{ij\cdot rs}$ are the elements of the product moment correlation matrix of $\mathbf{Z}_{k\alpha}$. The matrix $\boldsymbol{\Sigma}^*$ in (3.12) has the elements $\sigma_{ij\cdot rs}$ which can be written as the product of the corresponding standard deviations and the correlations $\rho_{ij\cdot rs}$. Thus the minimum and the maximum characteristic roots of $\gamma_{11.2}\Sigma_{11\cdot 0}^{-1}$ can be shown to depend on the $\rho_{ij\cdot rs}$ and $\int g_{[i]}^2(x)\,dx$. For some simple special cases, these values may be computed.

REFERENCES

[1] Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[2] Bodewig, E. (1956). *Matrix Calculus.* Interscience, New York.

[3] Gnedenko, B. W. and Kolmogorov, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, Cambridge.

[4] Hájek, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. *Ann. Math. Statist.* **39** 325–346.

[5] Hoeffding, Wassily (1968). On the centering of a simple linear rank statistic. Department of Statistics, University of North Carolina Mimeo Series. No. 585.

[6] Puri, Madan L. (1964). Asymptotic efficiency of a class of c-sample tests. *Ann. Math. Statist.* **35** 102–121.

[7] Puri, Madan L. and Sen, Pranab Kumar (1966). On a class of multivariate multi-sample rank order tests. *Sankhyā Ser. A* **28** 353–376.

[8] Puri, Madan L. and Sen, Pranab Kumar (1969). Analysis of covariance based on general rank scores. *Ann. Math. Statist.* **40** 610–618.

[9] PURI, MADAN LAL and SEN, PRANAB KUMAR (1969). A class of rank order tests for a general linear hypothesis. *Ann. Math. Statist.* **40** 1325–1343.
[10] QUADE, DANA (1967). Rank analysis of covariance. *J. Amer. Statist. Assoc.* **62** 1187–1200.
[11] SEN, PRANAB KUMAR and PURI, MADAN L. (1967). On the theory of rank order tests for location in the multivariate one-sample problem. *Ann. Math. Statist.* **38** 1216–1228.