ASYMPTOTIC THEORY OF SOME TESTS FOR A POSSIBLE CHANGE IN THE REGRESSION SLOPE OCCURRING AT AN UNKNOWN TIME POINT*

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Abstract

Based on least squares estimators and aligned linear rank statistics, some testing procedures for a possible change in the regression slope occurring at an unknown time point are considered. The asymptotic theory of the proposed tests rests on certain invariance principles pertaining to least squares estimators and aligned rank order statistics and these are developed here.

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1. Introduction.

Parametric as well as nonparametric testing procedures for poassible shifts in the location of a distribution function (df) occurring at unknown time points between consecutively taken observations have been proposed and studied by Page (1955), Chernoff and Zacks (1964), Kander and Zacks (1966), Mustafi (1968), Bhattacharyya and Johnson (1968), Sen and Srivastava (1975) and Sen (1977), among others. The object of the present investigation is to consider a related problem of regression where a change in the regression coefficient may occur at a unknown time point and to develop suitable testing procedures.

Given the observations on independent random variables (rv) $Y_i = Y(t_i)$, $i = 1,\ldots,n$, taken at time points $t_1,\ldots,t_n$ (where $t_1 \le \ldots \le t_n$ with at least one strict inequality sign), consider the following regression model:

$$Y(t) = \alpha + I(t<\tau)\beta(t-\tau) + I(t\ge\tau)\gamma(t-\tau) + e(t) \ , \tag{1.1}$$

where α, β, γ and τ are *unknown parameters* $(t_1 \le \tau \le t_n)$, $I(A)$ stands for the indicator function of the set A and $e(t)$ is a *white noise* i.e., for every real e, the df

$$F(e) = P\{e(t) \le e\} \text{ does not depend on } t \ (t_1 \le t \le t_n) \ . \tag{1.2}$$

Note that if $\tau = t_1$ or t_n, then (1.1) reduces to a simple regression model, while for $t_1 < \tau < t_n$ and $\beta \ne \gamma$, it relates to a segmented regression model with a common intercept (α) at time point τ and two different slopes β and γ for $t < \tau$ and $t \ge \tau$, respectively; τ is termed a *transition point*. We assume that

$$t_1 < \tau < t_n \text{ while } \beta \text{ and } \gamma \text{ may or may not be equal} \ . \tag{1.3}$$

Then, under (1.3), (1.1) relates to a simple regression model only when $\beta = \gamma$.

Such a segmented regression model is not very uncommon in practical problems. If we let the t_i stand for the *doses* of a drug and the Y_i for the *responses*, such a segmented *dose-response regression* also arises in some problems, where, at a higher dose, the regression pattern may differ from the one at a lower dose.

We desire to test for

$$(1.4) \qquad H_0: \beta = \gamma \quad \text{vs.} \quad H_1: \beta > \gamma \quad \text{or} \quad H_2: \beta \neq \gamma \,,$$

treating α, β and τ as nuisance parameters. If τ were specified, one could have considered the two samples $\{Y_i : t_i < \tau\}$ and $\{Y_i : t_i \geq \tau\}$ (which are independent) and, bearing in mind the two simple regression models pertaining to these samples, one might have tested for the identity of the two slopes β and γ. Since τ is not specified, our problem is somewhat more complicated. Moreover, we do not assume that F in (1.2) is of a specified form (e.g., normal), and, for this reason, we take recourse to tests based on rank statistics and the classical least squares estimators.

In Section 2, along with the preliminary notions, the proposed test statistics are formulated. Some invariance principles for least squares estimators (LSE) are considered in Section 3 and also these are incorporated there in the study of the asumptotic properties of the tests based on the LSE. Similar invariance principles are developed for (aligned) linear rank statistics (LRS) in Section 4 and these are utilized then in the study of the asymptotic properties of the proposed rank tests. Section 5 deals with the asymptotic comparison of the procedures based on LSE and LRS.

2. Preliminary notions and the proposed tests.

Let us define

(2.1) $$\bar{t}_k = \textstyle\sum k^{-1} {}_{i=1}^{k} t_i \quad \text{and} \quad T_k^2 = \sum_{i=1}^{k}(t_i - \bar{t}_k)^2, \; k \geq 1 .$$

Note that T_k^2 is ↑ in $k(\geq 1)$. Under H_0 in (1.4), based on $Y_1,\ldots,Y_k$, the LSE of β is

(2.2) $$\hat{\beta}_k = T_k^{-2}\sum_{i=1}^{k}(t_i - t_k)Y_k, \quad \text{for} \quad k = 2,\ldots,n; \; \hat{\beta}_1 = 0 .$$

If we assume that F in (1.2) admits of a finite variance σ^2, then under H_0 in (1.4), $\hat{\beta}_2,\ldots,\hat{\beta}_n$ are all unbiased estimators of β with variances $\sigma^2/T_2,\ldots,\sigma^2/T_n$, respectively. On the other hand, if H_0 does not hold and $t_m \leq \tau < t_{m+1}$ for some m: $1 \leq m \leq n-1$, then

(2.3) $$E(\hat{\beta}_k | \beta\ \gamma) = \begin{cases} \beta, & k \leq m, \\ \gamma + (\beta-\gamma)T_k^{-2}\sum_{i=1}^{m}(t_i-\tau)(t_i-\bar{t}_k), & m+1 \leq k \leq n, \end{cases}$$

where the right hand side (rhs) of (2.3), for $k > m$, differs from β (or γ). Thus, the estimators cease to fluctuate around a common β when H_0 does not hold. We consider the residuals

(2.4) $$\hat{Y}_i = Y_i - \hat{\beta}_n t_i, \quad i = 1,\ldots,n$$

and based on the partial set $\{\hat{Y}_1,\ldots,\hat{Y}_k\}$, we compute

(2.5) $$\tilde{\beta}_k = T_k^{-2}\sum_{i=1}^{k}(t_i - \bar{t}_k)\hat{Y}_i = \hat{\beta}_k - \hat{\beta}_n, \; 2 \leq k \leq n; \; \tilde{\beta}_1 = 0 .$$

Under H_0, $\tilde{\beta}_1,\ldots,\tilde{\beta}_n$ al unbaisedly estimate 0, while under H_1 or H_2, they are not so. Our proposed test based on the LSE rests on the statistics

(2.6) $$M_n^+ = \max_{0\leq k\leq n} S_{n,k} \quad \text{and} \quad M_n = \max_{0\leq k\leq n} |S_{n,k}| ,$$

where $S_{n,0} = S_{n,1} = 0$ and

$$S_{n,k} = T_n^{-1}T_k^2\tilde{\beta}_k = T_n^{-1}T_k^2(\hat{\beta}_k - \hat{\beta}_n), \quad 2 \le k \le n , \tag{2.7}$$

so that $S_{n,n} = 0$. The test procedure will be formulated in Section 3.

For the rank tests (to follow), we do not need the existence of the second moment of F. However, to avoid ties among the observations (Y_i), we assume that F is continuous everywhere. Consider the usual LRS

$$L_k = \sum_{i=1}^{k}(t_i - \bar{t}_k)a_k(R_{ki}), \quad k = 1,\ldots,n , \tag{2.8}$$

where, for every $k(\ge 1)$, R_{ki} = *rank* of Y_i among $Y_1,\ldots,Y_k$ for $1 \le i \le k$, the *scores* $a_k(i)$ are defined by

$$a_k(i) = E\phi(U_{ki}), \quad i = 1,\ldots,k , \tag{2.9}$$

$U_{k1} < \cdots < U_{kk}$ are the ordered rv's of a sample of size k from the rectangular $(0,1)$ df and the *score generating function* $\phi = \{\phi(u),\ 0 < u < 1\}$ is assumed to be square integrable inside $(0,1)$. Actually, bearing in mind, the elimination of the nuisance parameters through estimation, we assume that

$$\phi(u) \text{ is } \nearrow \text{ in } u:\ 0 < u < 1 . \tag{2.10}$$

For every real $b:\ -\infty < b < \infty$, let

$$L_k(b) = \sum_{i=1}^{k}(t_i - \bar{t}_k)a_k(R_{ki}(b)), \quad k \ge 1 , \tag{2.11}$$

where $R_{ki}(b)$ = rank of $Y_i - bt_i$ among $Y_1 - bt_1,\ldots,Y_k - bt_k$ for $1 \le i \le k$. Then [cf. Theorem 6.1 of Sen (1969)], under (2.10), for every $k(\ge 1)$,

$$L_k(b) \text{ is } \searrow \text{ in } b:\ -\infty < b < \infty . \tag{2.12}$$

Let then

$$\beta^*_{k,1} = \sup\{b:\ L_k(b) > 0\},\quad \beta^*_{k,2} = \inf\{b:\ L_k(b) < 0\} ; \tag{2.13}$$

(2.14) $$\beta_k^* = \frac{1}{2}(\beta_{k,1}^* + \beta_{k,2}^*),\ k = 1,\dots,n\ .$$

Under H_0 in (2.14), β_k^* is a translation-invariant, robust and consistent estimator of β [viz., Adichie (1967)], for $k \geq 1$. As in (2.4), we consider the residuals

(2.15) $$Y_i^* = Y_i - \beta_n^* t_i \quad \text{for} \quad i = 1,\dots,n$$

and based on the partial set $\{Y_1^*,\dots,Y_k^*\}$, we define

(2.16) $$\begin{aligned} L_{n,k}^* &= L_k(\beta_n^*)/T_n \\ &= T_n^{-1}\textstyle\sum_{i=1}^k (t_i - \bar{t}_k) a_k(R_{ki}^*),\ k = 1,\dots,n\ , \end{aligned}$$

where R_{ki}^* = rank of Y_i^* among $Y_1^*,\dots,Y_k^*$, for $1 \leq i \leq k;\ k \geq 1$. Conventionally, we let $L_{n,0}^* = 0$. Then, parallel to (2.6), our proposed test statistics are

(2.17) $$D_n^+ = A_n^{-1}\left\{\max_{0\leq k\leq n} L_{n,k}^*\right\} \quad \text{and} \quad D_n = A_n^{-1}\left\{\max_{0\leq k\leq n} |L_{n,k}^*|\right\}$$

where for $n \geq 2$,

(2.18) $$A_n^2 = (n-1)^{-1}\textstyle\sum_{i=1}^n [a_n(i) - \bar{a}_n]^2 \quad \text{and} \quad \bar{a}_n = \frac{1}{n}\sum_{i=1}^n a_n(i)\ .$$

The test procedure will be formulated in Section 4.

3. Asymptotic properties of the tests based on M_n^+ and M_n.

For the study of the asymptotic distribution theory of M_n^+ and M_n (under the null as well as local alternative hypothesis), we need to study first some invariance principles relating to the LSE. For this asymptotic study, we consider a triangular array $\{t_{ni},\ 1 \leq i \leq n;\ n \geq 1\}$ of time-variables and $\{Y_i = Y(t_{ni}),\ 1 \leq i \leq n;\ n \geq 1\}$ are defined accordingly as in (1.1). We assume that for every θ: $0 < \theta \leq 1$, (3.1) $$\lim_{n\to\infty} \bar{t}_{n[n\theta]} = \lim_{n\to\infty} [n\theta]^{-1}\sum_{i=1}^{[n\theta]} t_{ni} = \mu(\theta) \quad \text{exists} ,$$

(3.2) $$\lim_{n\to\infty} (T^2_{n[n\theta]}/n) = \lim_{n\to\infty} \left\{ n^{-1}\sum_{i=1}^{[n\theta]} (t_{ni}-\bar{t}_{n[n\theta]})^2 \right\} = \xi^2(\theta) \quad \text{exists}$$

and both $\mu(\theta)$ and $\xi(\theta)$ are continuous inside $[0,1]$. Note that (3.1) and (3.2) insure that

(3.3) $$\max\left\{ T_{nn}^{-1} |t_{ni}-\bar{t}_{nk}| : 1 \le i \le k \le n \right\} \to 0 \quad \text{as} \quad n \to \infty .$$

Let then $U_{n0} = U_{n1} = 0$ and for $k \ge 2$

(3.4) $$U_{nk} = \sum_{i=1}^{k} (t_{ni}-\bar{t}_{nk})Y_i$$

and define a stochastic process $W_n^{(1)} = \{W_n^{(1)}(u),\ 0 \le u \le 1\}$ be letting

(3.5) $$W_n^{(1)}(u) = U_{k_n(u)}/(\sqrt{n}\sigma\xi(1)),\ k_n(u) = \max\{k:\ T^2_{nk} \le uT^2_{nn}\},\ 0 \le u \le 1 .$$

Note that $k_n(u)$ is a nondecreasing, right-continuous and integer-valued function of $u \in [0,1]$ and $W_n^{(1)}$ belongs to the $D[0,1]$ space (having only jump discontinuities) endowed with the Skorokhos J_1-topology. Let $W = \{W(u),\ 0 \le u \le 1\}$ be a standard Wiener process on $[0,1]$. Then, we have the following.

Theorem 3.1. *If* $Ee(t) = 0$, $Ee^2(t) = \sigma^2 < \infty$ *and* (3.1)-(3.2) *hold, then under* H_0^*: $\alpha = \beta = \gamma = 0$,

(3.6) $$W_n^{(1)} \xrightarrow[L]{} W, \quad \textit{in the } J_1\textit{-topology on } D[0,1] .$$

Proof. First, we establish the convergence of the finite-dimensional distributions (f.d.d.) of $\{W_n^{(1)}\}$ to those of W. For every (fixed) $m(\ge 1)$ and $0 \le u_1 < \dots < u_m \le 1$, let $k_j = k_n(u_j)$, $1 \le j \le m$ and for an arbitrary $\underset{\sim}{d} \ne \underset{\sim}{0}$, let

(3.7) $$Z_n = \sum_{j=1}^{m} d_j W_n^{(1)}(u_j) = (\sqrt{n}\sigma\xi(1))^{-1}\sum_{j=1}^{m} d_j \sum_{i=1}^{k_j} (t_{ni}-\bar{t}_{nk_j})Y_i$$

$$= (\sqrt{n}\sigma\xi(1))^{-1}\sum_{i=1}^{n} f_{ni}Y_i ,$$

where

(3.8) $$f_{ni} = \sum_{j=1}^{m} d_j[(t_{ni}-\bar{t}_{nk_j})I(i \le k_j)],\ 1 \le i \le n\ .$$

It follows by some routine steps that $\sum_{i=1}^{n} f_{ni} = 0$ and

(3.9) $$\left(\sum_{i=1}^{n} f_{ni}^2\right)/(n\sigma^2\xi^2(1)) \to \sum_{j=1}^{m}\sum_{\ell=1}^{m} d_j d_\ell (u_j \wedge u_\ell)\ ;$$

(3.10) $$n^{-\frac{1}{2}}[\max\{|f_{ni}|: 1 \le i \le n\}] \to 0 \quad \text{as} \quad n \to \infty\ .$$

Note that the rhs of (3.9) equals the variance of $\sum_{j=1}^{m} d_j W(u_j)$. Further, under H_0^*, the Y_i are independent and identically distributed (i.i.d.) rv's with 0 mean and variance σ^2. Hence, using (3.10) and a special version of the central limit theorem in Hájek and Šidak (1967, p. 153), it follows that Z_n is asymptotically normally distributed. This establishes the convergence of the f.d.d.'s of $\{W_n^{(1)}\}$ to those of W. It remains to show that $\{W_n^{(1)}\}$ is *tight*. Since by definition, $W_n^{(1)}(0) = 0$, with probability 1, it suffices to show that for every $\varepsilon > 0$ and $\eta > 0$, there exist a δ: $0 < \delta < 1$ and an n_0, such that for $n \ge n_0$ and every k: $0 \le k \le n-[n\delta]$, $q = k+[n\delta]$,

(3.11) $$P\left\{\max_{k<m\le q} |U_m - U_k| > \varepsilon\sigma\xi(1)\sqrt{n}\right\} < \eta\delta\ ;$$

see Theorem 8.3 of Billingsley (1968, p. 56). For this note that for every k,q: $0 \le k < q \le n$,

$$\max_{k<m\le q} n^{-\frac{1}{2}}|U_m - U_k| \le |\bar{t}_{nq}-\bar{t}_{nk}|\, n^{-\frac{1}{2}}\left\{\max_{k<m\le q}\left|\sum_{i=k+1}^{m} Y_i\right|\right\}$$

$$+\, n^{-\frac{1}{2}}\left\{\max_{k<m\le q}|\bar{t}_{nk}-\bar{t}_{nm}|\left|\sum_{i=1}^{m} Y_i\right|\right\} + n^{-\frac{1}{2}}\left\{\max_{k<m\le q}\left|\sum_{i=k+1}^{m}(t_{nk}-\bar{t}_{nq})Y_i\right|\right\}$$

(3.12) $$= C_{kq}^{(1)} + C_{kq}^{(2)} + C_{kq}^{(3)}\ , \quad \text{say}\ .$$

By the nondecreasing nature of t_{ni} (in i),

$$(3.13) \qquad \max_{k<m\le q}|\bar{t}_{nm}-\bar{t}_{nk}| = |\bar{t}_{nq}-\bar{t}_{nk}| ,$$

where by (3.1) and the continuity of $\mu(\theta)$, $\theta \in [0,1]$,

$$(3.14) \qquad \lim_{n\to\infty}|\bar{t}_{n[ns]}-\bar{t}_{n[ns+n\delta]}| \to 0 \text{ as } \delta \downarrow 0, \ \forall\, s \in [0,1) .$$

Further, the Y_i are i.i.d.rv's with 0 mean and variance σ^2, so that by the Donsker Theorem [cf. Billingsley (1968)], for every $\varepsilon' > 0$ and $\eta' > 0$, there exist a δ: $0 < \delta < 1$ and an n_0, such that for $n \ge n_0$ and every k: $0 \le k \le n-[n\delta]$, $q = k + [n\delta]$,

$$(3.15) \qquad P\left\{n^{-\frac{1}{2}} \max_{k<m\le q}|\textstyle\sum_{i=k+1}^{m} Y_i| > \varepsilon \right\} < \eta'\delta ,$$

$$(3.16) \qquad P\left\{n^{-\frac{1}{2}} \max_{1\le k\le n}|\textstyle\sum_{i=1}^{k} Y_i| > K_{\eta'\delta}\right\} < \eta'\delta ,$$

where $K_{\eta'\delta}$ is a positive number $(< \infty)$, depending on $\eta'\delta$. Thus, it follows from (3.12) through (3.16) that for every $\varepsilon > 0$ and $\eta > 0$, there exist $\varepsilon' > 0$, $\eta' > 0$ and δ: $0 < \delta < 1$, such that for $q - k = [\delta n]$, $n \ge n_0$,

$$(3.17) \qquad P\left\{C_{kq}^{(1)} + C_{kq}^{(2)} > \frac{1}{2}\varepsilon\right\} < \frac{1}{2}\eta\delta;\ \eta' = \frac{1}{4}\eta,\ \varepsilon' = \varepsilon/4$$

where $\delta(> 0)$ is so small that $\lim_{n\to\infty}|\bar{t}_{nq}-\bar{t}_{nk}|K_{\eta'\delta} < \varepsilon/4$. Hence, to prove (3.11), it suffices to show that for every $\varepsilon' > 0$ and $\eta' > 0$, there exist a δ: $0<\delta<1$ and an n_0, such that for $n \ge n_0$ and every k: $0 \le k \le n-[n\delta]$, $q = k+[n\delta]$,

$$(3.18) \qquad P\left\{C_{kq}^{(3)} > \varepsilon'\right\} < \eta'\delta .$$

If we let $V_{ni} = (t_{ni}-\bar{t}_{nq})Y_i$, $k \le i \le q$, then the V_{ni} are independent, $EV_{ni} = 0$, $\sum_{i=k+1}^{q} EV_{ni}^2 = \sigma^2\sum_{i=k+1}^{q}(t_{ni}-\bar{t}_{nq})^2 = \sigma^2\{T_{nq}^2 - \sum_{i=1}^{k}(t_{ni}-\bar{t}_{nq})^2\} = \sigma^2\{T_{nq}^2-T_{nk}^2-k(\bar{t}_{nk}-\bar{t}_{nq})^2\}$ $\le \sigma^2(T_{nq}^2-T_{nk}^2) \sim n\sigma^2\left[\xi^2(\frac{q}{n}) - \xi^2(\frac{k}{n})\right]$, $\left\{\sum_{s=k+1}^{i} V_{ns},\ i \ge k+1\right\}$ is a martingale and, finally, $(\sqrt{n}\sigma)^{-1}\left[\sum_{s=k+1}^{q} V_{ns}\right]$ is asymptotically normally distributed with 0 mean and variance

$\left(\xi^2(\frac{q}{n}) - \xi^2(\frac{k}{n}) - (k/n)\left[\mu(\frac{a}{n}) - \mu(\frac{k}{n})\right]^2\right)$ $(\le[\xi^2(q/n) - \xi^2(k/n)])$. Hence, (3.18) follows by using Lemma 4 of Brown (1961) and proceeding as in the proof of Theorem 3. Q.E.D.

Let us now consider the process $W_n^{(2)} = \{W_n^{(2)}(u),\ 0 \le u \le 1\}$ where

(3.19) $$W_n^{(2)}(u) = S_{n,k_n(u)}/\sigma,\ 0 \le u \le 1\ ,$$

$k_n(u)$ is defined by (3.5) and the $S_{n,k}$ by (2.7). Note that in (2.7) and, elsewhere, we replace the t_i by t_{ni} and T_k^2 by T_{nk}^2, $1 \le k \le n$. Also, let $W^0 = \{W^0(t) = W(t) - tW(1),\ 0 \le t \le 1\}$ be a standard Brownian bridge on [q1]. Note that by definition in (2.5), the $\tilde{\beta}_k$ are invariant under shift and regression i.e., if we work with $Y_i - a - bt_i$, $1 \le i \le n$, then the resulting $\tilde{\beta}_k$ will be the same as the ones in (2.5) for every real (a,b). Hence, the distribution of $W_n^{(2)}$ under H_0 in (1.4) will be the same as under H_0^*. Further, by definition,

(3.20) $$\begin{aligned} W_n^{(2)}(u) &= (T_{nn}\sigma)^{-1}[U_{nk_n(u)} - T_{nn}^{-2}T_{nk_n(u)}^2 U_{n,n}] \\ &= (\sqrt{n}\ \xi(1)/T_{nn})[W_n^{(1)}(u) - T_{nn}^{-2}T_{nk_n(u)}^2 W_n^{(1)}(1)],\ 0 \le u \le 1\ . \end{aligned}$$

Also, note that $T_{n0}^2 = T_{n1}^2 = 0$, while for $k \ge 1$,

(3.21) $$(T_{nk+1}^2 - T_{nk}^2)/T_{nn}^2 = [(k+1)/k][t_{nk+1} - \bar{t}_{nk+1}]^2/T_n^2 \to 0, \text{ by (3.3)}\ ,$$

while by (3.2), $\sqrt{n}\xi(1)/T_n \to 1$ as $n \to \infty$. Hence,

(3.22) $$W_n^{(2)}(u) = (\sqrt{n}\xi(1)/T_{nn})[W_n^{(1)}(u) - uW_n^{(1)}(1)] + R_n(u)\ ,$$

where $\sup\{|R_n(u)|: 0 \le u \le 1\} \to 0$. From Theorem 3.1 and (3.22), we arrive at the following.

Theorem 3.2. *If* $Ee(t) = 0$, $Ee^2(t) = \sigma^2 < \infty$ *and* (3.1)-(3.2) *hold, then under* H_0 *in* (1.4),

(3.23) $$W_n^{(2)} \underset{L}{\rightarrow} W^0, \text{ in the } J_1\text{-topology on } D[0,1] .$$

For the standard Brownian brigde W^0, it is well known [viz., Billingsley (1968, p.85)] that for every $t \geq 0$

(3.24) $$P\left\{\sup_{0\leq u\leq 1} W^0(u) \leq t\right\} = 1-\exp(-2t^2) ,$$

(3.25) $$P\left\{\sup_{0 u 1} |W^0(u)| \leq t\right\} = 1-2\sum_{k=1}^{\infty}(-1)^k \exp(-2k^2t^2) ,$$

where the rhs of (3.25) is bounded from below by $1-2\exp(-2t^2)$ and is practically equal to this lower bound when t is not very small. On equating the rhs of (3.24) and (3.25) to $1-\varepsilon$, where ε is the desired level of significance $(0 < \varepsilon < 1)$, we denote the solutions by Δ_ε^+ and Δ_ε respectively. Then, by Theorem 3.2, we have on noting that

(3.26) $$M_n^+/\sigma = \sup_{0\leq u\leq 1} W_n^{(2)}(u) \quad \text{and} \quad M_n/\sigma = \sup_{0\leq u\leq 1} |W_n^{(2)}(u)| ,$$

(3.27) $$P\{M_n^+ > \sigma\Delta_\varepsilon^+ | H_0\} \to \varepsilon \quad \text{and} \quad P\{M_n > \sigma\Delta_\varepsilon | H_0\} \to \varepsilon ,$$

and hence, the asymptotic critical values of M_n^+ and M_n (at the desired level of significance ε: $0 < \varepsilon < 1$) are $\sigma\Delta_\varepsilon^+$ and $\sigma\Delta_\varepsilon$, respectively. Thus, if σ is specified, we have the following test procedure:

Compute the $S_{n,k}$, $k \leq n$, defined by (2.7). If, for at least one k:

(3.28) $1 \leq k \leq n-1$, $S_{n,k}$ (or $|S_{n,k}|$) exceeds $\sigma\Delta_\varepsilon^+$ (or $\sigma\Delta_\varepsilon$), reject H_0 in (1.4). If, no such k exists, accept H_0.

In practice, mostly, σ is not specified. We consider the estimator

(3.29) $$\hat{\sigma}_n^2 = n^{-1}\sum_{i=1}^n (Y_i - \bar{Y}_n - \hat{\beta}_n(t_i - \bar{t}_n))^2 \quad \left(\text{where } \bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i\right) .$$

Under H_0 in (1.4), we have a simple regression model, and hence [viz., Sen and Puri (1970)], $\hat{\sigma}_n \xrightarrow{p} \sigma$. As such, we may proceed as in (3.28) where we replace σ by $\hat{\sigma}_n$; the asymptotic level of significance remains equal to ε.

Let us now consider the behavior of these tests when H_0 in (1.4) does not hold. Suppose that (1.1) holds with

(3.30) $$\tau = t_m \text{ for some } m: 1 < m \le n-1 \text{ and } \beta - \gamma = \delta \ (\neq 0) .$$

Let $Y_i^0 = Y_i + \delta(t_i - t_m)I(t \ge t_m)$, $i = 1,\dots,n$ and in (2.2), (2.4), (2.5) and (2.7), we replace the Y_i by Y_i^0 and denote the resulting quantities by $\hat{\beta}_k^0$, $\hat{Y}_i^0$, $\tilde{\beta}_k^0$ and $S_{n,k}^0$, respectively. Then, we have by some direct computations that

(3.31) $$\hat{\beta}_k^0 = \begin{cases} \hat{\beta}_k, & k \le m, \\ \hat{\beta}_k + \delta T_k^{-2}\{T_k^2 - T_m^2 - m(t_m - \bar{t}_m)(\bar{t}_k - t_m)\}, & m < k \le n , \end{cases}$$

where $\bar{t}_k = k^{-1}\sum_{i=1}^k t_i$ is $\nearrow$ in $k (1 \le k \le n)$, and

(3.32) $$S_{n,k}^0 = \begin{cases} S_{n,k} - \delta(T_k^2/T_n)\{1 - T_m^2/T_n^2 - m(t_m - \bar{t}_m)(\bar{t}_n - \bar{t}_m)/T_n^2\}, & k \le m , \\ S_{n,k} - \delta(T_k^2/T_n)\left\{\left(\frac{T_m^2}{T_k^2} - \frac{T_m^2}{T_n^2}\right) - m(t_m - \bar{t}_m)\left[\frac{\bar{t}_n - \bar{t}_m}{T_n^2} - \frac{\bar{t}_k - \bar{t}_m}{T_k^2}\right]\right\}, & m < k \le n . \end{cases}$$

As such, if m/n is bounded away from 0 and 1, then under (3.1), (3.2) and (3.30), $\max\{S_{n,k} - S_{n,k}^0 : 1 \le k \le n\} \to \infty$ as $n \to \infty$ when $\delta > 0$ and $\max\{|S_{n,k} - S_{n,k}| : 1 \le k \le n\} \to \infty$ as $n \to \infty$ when $\delta \neq 0$. On the other hand, for the Y_i^0, the simple regression model holds, so that Theorem 3.2 applies to the $S_{n,k}^0$, and hence,

$\max\{S^0_{n,k}; 1 \le k \le n\}$ or $\max\{|S^0_{n,k}|; 1 \le k \le n\}$ is $O_p(1)$. Thus, $\max\{S_{n,k}: 1 \le k \le n\} \to \infty$, in probability, if $\delta > 0$ and $\max\{|S_{n,k}|: 1 \le k \le n\} \to \infty$, in probability, if $\delta \ne 0$. Hence, the tests based on (3.28) are consistent.

In view of the consistency of the tests, for the study of the asymptotic power properties, we confine ourselves to a suitable sequence $\{H_n\}$ of alter-hypotheses for which the asymptotic power is different from 1. Keeping (3.1)-(3.2) in mind, we assume that

$$\tau = t_m \quad \text{where} \quad m/n \to \nu: 0 < \nu < 1,\ T_m^2/T_n^2 \to \rho: 0 < \rho < 1 . \tag{3.33}$$

Actually, if we let

$$h(u) = \inf\{t: \xi^2(t)/\xi^2(1) \ge u\},\ 0 \le u \le 1 , \tag{3.34}$$

then $\rho = \xi^2(\nu)/\xi^2(1)$ and $\mu(\nu) = \mu(h(\rho))$. We consider then

$$H_n: \beta = \gamma + T_n^{-1}\delta \quad \text{for some real } \delta, \text{ where (3.3) holds .} \tag{3.35}$$

Let us then define $a_\delta = \{a_\delta(t),\ 0 \le t \le 1\}$ by letting

$$a_\delta(t) = \begin{cases} \delta t[1-\rho-\nu(\tau-\mu(\nu))(\mu(1)-\mu(\nu))/\xi^2(1)]/\sigma, & 0 \le t \le \rho , \\ \frac{1}{\sigma}\,\delta[\rho(1-t)-\nu(\tau-\mu(\nu))\{(1-t)\mu(\nu)-\mu(h(t))+t\mu(1)\}/\xi^2(1)], & \rho < t \le 1 . \end{cases} \tag{3.36}$$

In (3.32), replacing δ by δ/T_n and then using Theorem 3.2 (for the $\{S^0_{n,k}\}$) along with (3.5) and (3.33)-(3.36), we arrive at the following.

<u>Theorem 3.3</u>. *If* $Ee(t) = 0$, $Ee^2(t) = \sigma^2 < \infty$ *and* (3.1), (3.2) *and* (3.33) *holds, then under* $\{H_n\}$ *in* (3.35),

$$W_n^{(2)} \xrightarrow{L} W^0 + a_\delta, \text{ in the } J_1\text{-topology on } D[0,1] . \tag{3.37}$$

By virtue of Theorem 3.3, the asymptotic power of the test based on M_n^+, under $\{H_n\}$ in (3.35), is given by

(3.38) $$P\{W^0(t) + a_\delta(t) \geq \Delta_\varepsilon^+, \text{ for some } t \in (0,1)\},$$

and for the test based on M_n, the corresponding expression is

(3.39) $$P\{|W^0(t) + a_\delta(t)| \geq \Delta_\varepsilon, \text{ for some } t \in (0,1)\}.$$

Since a_δ is not a linear (in t) boundary (in general), closed expression for (3.38) and (3.39) are not generally available.

4. Asymptotic properties of the tests based on D_n^+ and D_n.

We need to develop some invariance principles for aligned LRS for the study of the distribution theory of D_n^+ and D_n. Consider first the case of $\{L_k\}$, defined by (2.8), and define $W_n^{(3)} = \{W_n^{(3)}(u), 0 \leq u \leq 1\}$ by letting

(4.1) $$W_n^{(3)}(u) = T_n^{-1}A_n^{-1}L_{k_n^*(u)}, \quad 0 \leq u \leq 1;$$

(4.2) $$k_n^*(u) = \max\{k: T_k^2A_k^2 \leq uT_n^2A_n^2\}, \quad 0 \leq u \leq 1,$$

where T_n^2 and A_n^2 are defined by (2.1) and (2.18), respectively. Here also, $W_n^{(3)}$ belongs to the $D[0,1]$ space. Then, we have the following.

Theorem 4.1. *For scores defined by* (2.9) *with nondecreasing and square integrable* ϕ, *under* (3.3) *and* H_0^*: $\beta = \gamma = 0$ *(refer to* (1.1)),

(4.3) $$W_n^{(3)} \xrightarrow{L} W, \text{ in the } J_1\text{-topology on } D[0,1].$$

Proof: Since under H_0^*, the Y_i are i.i.d.rv with a continuous df $F(x-\alpha)$, (4.3) directly follows from Theorem 2.2 of Sen (1975).

We proceed on to the case where H_0^* may not hold. Here, we assume, that (i) the df F admits of an absolutely continuous probability density function (pdf) f with a finite Fisher information

$$I(f) = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 dF(x) \quad (< \infty) . \tag{4.4}$$

Also, let

$$\psi(u) = -f'(F^{-1}(u)/f(F^{-1}(u)), \; 0 < u < 1 , \tag{4.5}$$

$$A^2 = \int_0^1 \phi^2(u)du - \left(\int_0^1 \phi(u)du\right)^2 \quad \text{and} \quad \lambda(\phi,\psi) = \left(\int_0^1 \phi(u)\psi(u)du\right)/AI^{\frac{1}{2}}(f) . \tag{4.6}$$

We assume that

$$\lambda^* = I^{\frac{1}{2}}(f)\lambda(\phi,\psi) > 0 . \tag{4.7}$$

Further, we assume that (3.1), (3.2) and (3.33) hold and consider a sequence $\{K^*_{n,\underset{\sim}{b}}\}$ of alternative hypotheses where

$$K^*_{n,\underset{\sim}{b}}: \text{(1.1) holds with } \beta = T_n^{-1}b_1, \; \gamma = T_n^{-1}b_2; \; \underset{\sim}{b} = (b_1,b_2) \tag{4.8}$$

and b_1,b_2 are real numbers. Then, we have the following.

<u>Theorem 4.2</u>. *Under* $\{K^*_{n,\underset{\sim}{b}}\}$ *and the assumptions made above,*

$$W_n^{(3)} \underset{L}{\to} W + \omega, \quad \textit{in the } J_1\textit{-topology on } D[0,1] , \tag{4.9}$$

where $\omega = \{\omega(u), \; 0 \le u \le 1\}$ *is given by*

$$\omega(u) = \begin{cases} \lambda^* b_1 u, & 0 \le u \le \rho \\ \lambda^*\{b_2u+(b_1-b_2)[\rho+\nu(\tau-\mu(\nu))(\mu(h(u))-\mu(\nu))/\xi^2(1)]\}, & \rho \le u \le 1 , \end{cases} \tag{4.10}$$

where ν, ρ, $\xi(1)$ *and* $h(u)$ *are defined as in* (3.1), (3.2), (3.33) *and* (3.34).

<u>Proof</u>. Let us denote the joint distribution of $(Y_1,\ldots,Y_n)$ under $K^*_{n,\underset{\sim}{b}}$ by $P_{n,\underset{\sim}{b}}$, so that $P_{n,\underset{\sim}{0}}$ stands for the null hypothesis (H_0^*) case. Then, by an

appeal to the basic results in Hájek and Šidák (1967, Ch. VI), we conclude that under the hypothesis of this theorem, $\{P_{n,\underset{\sim}{b}}\}$ is *contiguous* to $\{P_{n,\underset{\sim}{0}}\}$. Also, (4.3) insures the *tightness* of $W_n^{(3)}$ under $\{P_{n,\underset{\sim}{0}}\}$. Hence, proceeding as in the proof of Theorem 2 of Sen (1976), we conclude that $W_n^{(3)}$ remains tight under $\{K^*_{n,\underset{\sim}{b}}\}$ as well. Hence, to prove (4.9), it suffices to prove the convergence of f.d.d.'s of $\{W_n^{(3)}\}$ to those of $W+\omega$, when $\{K^*_{n,\underset{\sim}{b}}\}$ holds. Towards this, we define

$$S^*_{n,k} = T_{nn}^{-1}\sum_{i=1}^{k}(t_{ni}-\bar{t}_{nk})\phi(F(Y_i-\alpha)),\ 1\le k\le n\ . \tag{4.11}$$

Then, for every (fixed) $m(\ge 1)$ and $0\le u_1<\dots<u_m\le 1$, defining $k_j = k_n^*(u_j)$ by (4.2), $1\le j\le m$, it follows by an appeal to Hájek (1961) that under H_0^*: $\beta=\gamma=0$, $T_{nn}^{-1}L_{k_j} = W_n^{(3)}(u_j)$ is equivalent in quadratic mean to S^*_{n,k_j}, $\forall\ 1\le j\le m$. By contiguity of $\{P_{n,\underset{\sim}{b}}\}$ to $\{P_{n,0}\}$, $|W_n^{(3)}(u_j)-S^*_{n,k_j}|\overset{p}{\to}0$, $\forall\ 1\le j\le m$ under $\{K^*_{n,\underset{\sim}{b}}\}$ as well. Finally, under $\{K^*_{n,\underset{\sim}{b}}\}$, the asymptotic joint normality of $(S^*_{n,k_1},\dots,S^*_{n,k_m})$ can be derived by an appeal to a theorem on page 216 of Hájek and Šidák (1967). In view of the similarity of this proof with that of Theorem 3.1 of Sen (1977), the details are omitted.

For every real d; let us define

$$\bar{L}_{n,k}(d) = L_k(d/T_n),\ 0\le k\le n\ , \tag{4.12}$$

where the $L_k(b)$ are defined by (2.11). Also, in (4.1), replacing the $L_{k_n^*(u)}$ by $\bar{L}_{k_n^*(u)}(d)$, we define the corresponding stochastic process by $\bar{W}^{(3)}_{n,d} = \{\bar{W}^{(3)}_{n,d}(u),\ 0\le u\le 1\}$. Thus, $W_n^{(3)} = \bar{W}^{(3)}_{n,0}$.

<u>Theorem 4.3</u>. *Under* $\{K^*_{n,\underset{\sim}{b}}\}$ *in* (4.8) *and the hypothesis of Theorem* 4.2, *for every (fixed)* d; $-\infty<d<\infty$,

$$\bar{W}^{(3)}_{n,d} \underset{L}{\to} \{W(t)+\omega(t)-d\lambda^* t,\ 0\le t\le 1\} \tag{4.13}$$

where $\omega(t)$ *is defined by* (4.10).

Proof. Let $\Pi^{d}_{n,\underset{\sim}{b}}$ be the distribution of $\overline{W}^{(3)}_{n,d}$ under $\{K^{*}_{n,\underset{\sim}{b}}\}$, defined for Borel subsets $\mathcal{D}$ of $D[0,1]$. Note that by denoting by $R^{*}_{nk}(d)$ = rank of $Y_i - dt_i/T_n$ among $Y_i - dt_1/T_n, \ldots, T_n - dt_n/T_n$, for $i = 1,\ldots,n$, $\underset{\sim}{R}^{*}_{n}(d) = (R^{*}_{n1}(d),\ldots,R^{*}_{nn}(d))'$, it follows that

$$(4.14) \qquad [\underset{\sim}{R}^{*}_{n}(d), \text{ under } K^{*}_{n,\underset{\sim}{b}}] \overset{\mathcal{L}}{=} [\underset{\sim}{R}^{*}_{n}(0), \text{ under } K^{*}_{n,\underset{\sim}{b}-d\underset{\sim}{1}}] ,$$

and hence, for every $D \in \mathcal{D}$,

$$(4.15) \qquad \Pi^{d}_{n,\underset{\sim}{b}}(D) = P\{\overline{W}^{(3)}_{n,\underset{\sim}{d}} \in D | K^{*}_{n,\underset{\sim}{b}}\}$$

$$= P\{\overline{W}^{(3)}_{n,0} \in D | K^{*}_{n,\underset{\sim}{b}-d\underset{\sim}{1}}\}$$

$$= \Pi^{0}_{n,\underset{\sim}{b}-d\underset{\sim}{1}}(D) ,$$

which insures that $\{\overline{W}^{(3)}_{n,d}$, under $K^{*}_{n,\underset{\sim}{b}}\} \overset{\mathcal{L}}{=} \{W^{(3)}_{n}$, under $K^{*}_{n,\underset{\sim}{b}-d\underset{\sim}{1}}\}$, and hence, (4.13) follows from Theorem 4.2. Q.E.D.

Theorem 4.4. *Under* $\{K^{*}_{n,\underset{\sim}{b}}\}$ *in* (4.8) *and the hypothesis of Theorem* 4.2, *for every fixed* $\underset{\sim}{b}$ *and* $K(<\infty)$,

$$(4.16) \qquad \sup\{|\overline{W}^{(3)}_{n,d}(t) - W^{(3)}_{n}(t) + \lambda^{*}dt| : 0 \le t \le 1, |d| \le K\} \overset{p}{\to} 0 .$$

Proof. We may virtually repeat the proof of Theorem 3.3 of Sen (1977). We note that (3.24) and (3.25) of Sen (1977) hold here, by virtue of the basic result of Jurečková (1969) and our Theorem 4.3 here. Hence, the details are omitted.

We are now in a position to study the invariance principles for our aligned LRS$\{L^{*}_{n,k};\ 0 \le k \le n\}$. Consider a sequence $\{K_{n,\underset{\sim}{b}}\}$ of alternative hypotheses, where

$$(4.17) \qquad K_{n,\underset{\sim}{b}}: \text{(1.1) holds with } \beta = \theta + T_n^{-1}b_1,\ \gamma = \theta + T_n^{-1}b_2 ,$$

$\underset{\sim}{b} = (b_1, b_2)$ $(\neq \underset{\sim}{0})$. Further, in (4.1), we replace the $L_{k^{*}_{n}(u)}$ by $L^{*}_{n,k^{*}_{n}(u)}$, $0 \le u \le 1$ and denote the resulting process by $W_n^{(4)} = \{W^{(4)}_{u}(u),\ 0 \le u \le 1\}$.

Note that $\{L_k(\theta+a),\ 1 \le k \le n\}$ under $K^*_{n,\underset{\sim}{b}}$ as the same distribution as $\{L_k(a),\ 1 \le k \le n\}$ under $K_{n,b}$. Hence, for the study of the distribution of $W_n^{(4)}$ under $\{K_{n,\underset{\sim}{b}}\}$, we may, without any loss of generality, set [in (4.17)], $\theta = 0$, i.e., under $\{K^*_{n,\underset{\sim}{b}}\}$ in (4.8). By virtue of (2.13), (2.14), Theorem 4.4 and the above discussion, it follows by some standard steps that under $\{K^*_{n,\underset{\sim}{b}}\}$,

$$\lambda^* T_{nn}\beta^*_n = W_n^{(3)}(1) + o_p(1) , \tag{4.18}$$

so that by Theorem 4.2,

$$|T_{nn}\beta^*_n| = O_p(1), \quad \text{under} \quad \{K^*_{n,\underset{\sim}{b}}\} . \tag{4.19}$$

By Theorem 4.4, (4.18) and (4.19), we obtain that under $\{K^*_{n,b}\}$ (or equivalently, under $\{K_{n,\underset{\sim}{b}}\}$ as $W_n^{(4)}$ remains in variant if the Y_i are replaced by $Y_i - \theta t_i$, $1 \le i \le n$, for any real θ),

$$\sup\{|W_n^{(4)}(u) - W_n^{(3)}(u) + uW_n^{(3)}(1)| :\ u \in [0,1]\} \overset{p}{\to} 0 , \tag{4.20}$$

so that, if we define $W_n^{(5)} = \{W_n^{(5)}(u) = W_n^{(3)}(u) - uW_n^{(3)}(1),\ 0 \le u \le 1\}$, then under $\{K_{n,\underset{\sim}{b}}\}$,

(4.21) $\qquad W_n^{(4)}$ and $W_n^{(5)}$ are convergent equivalent .

Finally, by an appeal to Theorem 4.2, we conclude that under $\{K^*_{n,\underset{\sim}{b}}\}$,

(4.22) $\qquad W_n^{(5)} \underset{L}{\to} W^0 + \omega^0$, *in the* J_1-*topology on* $D[0,1]$,

where W^0 is standard Brownian bridge on $[0,1]$ and

$$\omega^0(t) = \begin{cases} \lambda^*(b_1-b_2)t[1-\rho-\nu(\mu(1)-\mu(\nu))(\tau-\mu(\nu))/\xi^2(1)], & 0 \le t \le \rho , \\ \lambda^*(b_1-b_2)[\rho(1-t)-\nu(\tau-\mu(\nu))\{(1-t)\mu(\nu)-\mu(h(t)) \\ \qquad +t\mu(1)\}/\xi^2(1)], & \rho \le t \le 1 , \end{cases} \tag{4.23}$$

where μ, ξ^2, τ, ρ and $h(t)$ are defined by (3.1), (3.2), (3.33) and (3.34). This leads us to the main theorem of this section.

Theorem 4.5. *Under* $\{K_{n,\underset{\sim}{b}}\}$ *and the hypothesis of Theorem* 4.2,

(4.24) $$W_n^{(4)} \underset{L}{\to} W^0 + \omega^0, \text{ in the } J_1\text{-topology on } D[0,1]$$

where $\omega^0 = \{\omega^0(t),\ 0 \le t \le 1\}$ *is given by* (4.23).

We notice that by definition in (2.17),

(4.25) $$D_n^+ = \sup_{0\le t\le 1} W_n^{(4)}(t) \quad \text{and} \quad D_n = \sup_{0\le t\le 1} |W_n^{(4)}(t)| \ .$$

Hence, by Theorem 4.5 and (4.26), under H_0: $\beta = \gamma$ i.e., $K_{n,\underset{\sim}{0}}$, D_n^+ and D_n have the asymptotic distributions given by (3.24) and (3.25), respectively. This leads us to the following test procedure:

Compute the $L^*_{n,k}$, defined by (2.16). If, for at least one k: $1 \le k \le n-1$,

(4.26) $L^*_{n,k}$ (or $|L^*_{n,k}|$) exceeds $A_n\Delta_\varepsilon^+$ (or $A_n\Delta_\varepsilon$), reject H_0 in (1.4). If, no such k exists, accept H_0 .

As in Section 3, we confine ourselves to local alternatives for the study of the asymptotic power properties of the tests based on (4.26). We assume that the same sequence $\{H_n\}$ of alternative hypotheses in (3.35) holds. Then, by an appeal to Theorem 4.5, we conclude that under the regularity conditions of Theorem 4.5,

(4.27) $$\lim_{n\to\infty} P\{D_n^+ > \Delta_\varepsilon^+ | H_n\} = P\{W^0(t) = \omega_\delta^0(t) > \Delta_\varepsilon^+ \text{ for some } t \in [0,1]\} \ ,$$

(4.28) $$\lim_{n\to\infty} P\{D_n > \Delta_\varepsilon | H_n\} = P\{|W^0(t) + \omega_\delta^0(t)| > \Delta_\varepsilon \text{ for some } t \in [0,1]\} \ ,$$

where

$$(4.29)\qquad \omega_\delta^0(t) = \begin{cases} \lambda^*\delta t[1-\rho-\nu(\mu(1)-\mu(\nu))(\tau-\mu(\nu))/\xi^2(1)], & 0 \le t \le \rho \\ \lambda^*\delta[\rho(1-t)-\nu(\tau-\mu(\nu))\{(1-t)\mu(\nu)-\mu(h(t)) \\ \qquad t\mu(1)\}/\xi^2(1)], & \rho \le t \le 1 . \end{cases}$$

Note that $\omega_\delta^0(t)$ is linear in t for $0 \le t \le \rho$, but, in general, for $\rho \le t \le 1$, $\omega_\delta^0(t)$ is not linear in t [as $\mu(h(t))$ need not be linear in t].

5. Asymptotic comparison of the LSE and LRS procedure.

It may be remarked that both M_n^+/σ and D_n^+ (or M_n/σ and D_n) have the same limiting null distribution (3.24) [or (3.25)]. If, we let

$$(5.1)\qquad b(t) = \begin{cases} (t+1)[1-\rho-\nu(\tau-\mu(\nu))(\mu(1)-\mu(\nu))/\xi^2(1)], & 0 \le t \le \rho/(1-\rho) , \\ \rho-\nu(\tau-\mu(\nu)\{\mu(\nu)+t\mu(1)-(t+1)\mu(h(t/(t+1)))\}/\xi^2(1), & t \ge \rho/(1-\rho) \end{cases}$$

and note that $\left\{(t+1)W^0\left(\frac{t}{t+1}\right),\ 0 \le t < \infty\right\} = \{X(t),\ 0 \le t < \infty\}$, where $\{X(t),\ t \ge 0\}$ is a standard Wiener process on R^+, we obtain then from (3.35), (3.36) and (3.38) that the asymptotic power of M_n^+ is equal to

$$(5.2)\qquad P\{W^0(t) \ge \Delta_\varepsilon^+ - a_\delta(t) \quad \text{for some} \quad t \in [0,1]\}$$

$$= P\left\{(t+1)W^0\left(\frac{t}{t+1}\right) \ge (t+1)\Delta_\varepsilon^+ - (t-1)a_\delta\left(\frac{t}{t+1}\right) \quad \text{for some} \quad t \in R^+\right\}$$

$$= P\left\{X(t) \ge (t+1)\Delta_\varepsilon^+ - \frac{\delta}{\sigma}\, b(t), \quad \text{for some} \quad t \in R^+\right\}$$

Similarly, for M_n, we have the asymptotic power is equal to

$$(5.3)\qquad P\left\{\left|X(t) + \frac{\delta}{\sigma}\, b(t)\right| \ge (t+1)\Delta_\varepsilon \quad \text{for some} \quad t \in R^+\right\} .$$

Likewise, from (4.27)-(4.29) and (5.1), we have the asymptotic power of the D_n^+ test given by

(5.4) $$P\{X(t) \geq (t+1)\Delta_\varepsilon^+ - \lambda^*\delta b(t) \quad \text{for some} \quad t \in R^+\}$$

while for the D_n test, it is equal to

(5.5) $$P\{|X(t) + \lambda^*\delta b(t)| \geq (t+1)\Delta_\varepsilon \quad \text{for some} \quad t \in R^+\} .$$

Note that (5.2) though (5.5) are computed for the common sequence $\{H_n\}$ of alternatives [in (3.35)] when all the statistics are based on the same sample size n. Suppose now that M_n^+ and M_n are based on sample size $\{n\}$ while D_n^+ or D_n are based on $\{N = N(n)\}$, where

(5.6) $$\lim_{n\to\infty} N(n)/n = e^{-1} \quad \text{for some} \quad 0 < e < \infty$$

For D_N^+ and D_N statistics with $N = N(n)$ satisfying (5.6), we may proceed as in Section 4 where we replace $W_n^{(3)}$, $W_n^{(4)}$ and $W_n^{(5)}$ by $W_N^{(3)}$, $W_N^{(4)}$ and $W_N^{(5)}$, respectively, defined for $n = N$, while we stick to the same alternatives $K_{n,\underset{\sim}{b}}$, $K^*_{n,\underset{\sim}{b}}$ and H_n. Since, by (3.2) and (5.6),

(5.7) $$\lim_{n\to\infty} T_N^2/T_n^2 = \lim_{n\to\infty} [\{(N^{-1}T_n^2/(n^{-1}T_n^2)\}\{N/n\}]$$
$$= \lim_{n\to\infty} [N(n)/n] = e^{-1} ,$$

in (4.10), (4.23) and (4.29), $\omega(u)$, $\omega^0(u)$ and $\omega^0(u)$ are to be replaced by $e^{-\frac{1}{2}}\omega(u)$, $e^{-\frac{1}{2}}\omega^0(u)$ and $e^{-\frac{1}{2}}\omega_\delta^0(u)$, respectively. Thus, the asymptotic power of the test based on $\{D_N^+\}$ under $\{H_n\}$ in (3.33), when (5.6) holds, is given by

(5.7) $$P\{X(t) \geq (t+1)\Delta_\varepsilon^+ - e^{-\frac{1}{2}}\lambda^*\delta b(t) \quad \text{for some} \quad t \in R^+\} .$$

Consequently, if we choose e by letting

(5.8) $$e^{-\frac{1}{2}}\lambda^* = \sigma^{-1} \quad \text{i.e.} \quad e = \sigma^2(\lambda^*)^2 ,$$

then (5.2) and (5.7) are equal (for any δ). Hence, θ in the usual Pitman-sense, the asymptotic relative efficiency (A.R.E.) of the LRS procedure with respect to the LSE procedure is

$$(5.9) \qquad e = \lim_{n\to\infty} [n/N(n)] = \sigma^2(\lambda^*)^2 = \sigma^2 I(f)\left\{\int_0^1 \phi(u)\psi(u)du\right\}^2 .$$

The same efficiency result holds for D_n relative to M_n.

Noew (5.9) agrees with the classical Pitman-efficiency of the two-sample rank order test (for location) relative to the Student's t-test. Thus, we may conclude that if $\phi(u) \equiv u$: $0 < u < 1$, then the corresponding rank procedure has an A.R.E. with respect to the LSE procedure equal to $3/\pi$ when F is normal, is bounded from below by 0.864 for all continuous F and is usually ≥ 1 when F has havier (than normal) tails. Also, if $\phi(u) = \Phi^{-1}(u)$, $0 < u < 1$ (i.e., normal scores), then (5.9) is bounded from below by 1 where the lower bound is attained only when F is normal. Thus, from the A.R.E. point of view, the LRS procedures are attractive, they do not require the estimation of σ^2 (as is need for the LSE procedure) and they are expected to be robust.

We conclude this section with the remark that $b(t)$ in (5.1) is linear in t for $t \leq \rho/(1-\rho)$, while for $t > \rho/(1-\rho)$, in general, it is not so. Since $\mu(s)$ is $\nearrow$ in $s \in [0,1]$, we have for $t > \rho/(1-\rho)$

$$(5.10) \qquad \rho - \rho(\tau-\mu(\nu))t(\mu(1)-\mu(\nu)) \leq b(t) \leq \rho + \nu(\tau-\mu(\nu))(\mu(1)-\mu(\nu)) .$$

Hence, for (5.2) and (5.4), bounds for the asymptotic power can be obtained by using the segmented linear boundaries in (5.1) and (5.10) and then using the results in Anderson (1960, Section 6). In general, these are quire complicated to be expressible in closed form.

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