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ASYMPTOTIC THERMOELASTIC BEHAVIOR OF FLAT PLATES*

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Introduction. The derivation of the mechanical behavior of a flat plate as the limit behavior of a three-dimensional flat solid whose thickness tends to zero is a well-established theory since the works of Goldenveizer [**G**] or Ciarlet and Destuynder [**CD1**]. Due attention has been paid to the static response of such plates for various linear or nonlinear mechanical behaviors (Ciarlet–Destuynder [**CD1, CD2**], Ciarlet [**C**], Blanchard–Ciarlet [**BC**]). Much less attention, however, has been devoted to the dynamic response of flat plates from a similar standpoint. The linearly elastic case was considered in Raoult [**R1, R2**], but we are not aware of any further work in that direction.

The present study is devoted to the dynamic behavior of a three-dimensional linearly thermoelastic flat plate. Specifically, a three-dimensional flat plate with small thickness is submitted to an arbitrary system of initial and loading conditions. The limits of the displacement, stress, and temperature fields as the thickness approaches zero are investigated.

Thermoelastic behavior is characterized by a coupling between the mechanical equations of motion and the "energy" equation. The limiting procedure is seriously affected by the presence of the coupling terms. The initial conditions are seen to play an essential role in the analysis. In particular, a change of initial condition generally occurs for the temperature field. A similar phenomenon appears in the homogenization of a thermoelastic composite (Francfort [**F**]). These concurring results seem to indicate that such shifts in initial data are closely linked to any kind of asymptotic problem for coupled systems.

The first section is very short and entirely devoted to notation and basic definitions. In the second section, the problem under investigation is formulated in a mathematical framework. It is then rescaled in the usual manner (Ciarlet–Destuynder [**CD1**]) so as to obtain a family of problems indexed by the thickness of the plate and defined on a fixed domain.

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The third section is concerned with the definition of the limit behavior. It consists of a flexural problem for the component of the displacement field normal to the plate together with a coupled membrane-thermal problem for the components of the displacement fields in the plane of the plate and the temperature field. The membrane problem is quasi-static, whereas the thermal equation is a parabolic evolution equation. The initial membrane displacement field is completely determined by the initial temperature field and by the initial values of the loadings (Theorem 1).

In the fourth section, the thickness of the plate tends to zero. The solution fields of the problems defined on the fixed domains are shown to weakly converge to the solution fields of the limit problems, at least when hypotheses of weak convergence are imposed on the initial conditions and the loadings (Theorem 2). The initial condition on the limit temperature field is seen to generally differ from the limit of the initial condition on that field (Remark 8).

The fifth and last section examines the possibility of strong convergence of the solution fields as the thickness tends to zero. The loadings and initial conditions are assumed to converge strongly. It is then proved in Theorem 3 that strong convergence takes place if and only if a compatibility equation is satisfied by the limits of the initial conditions and initial loadings. An example of initial conditions and loadings that are compatible is given in Remark 9. In that example it is noted that the initial condition on the temperature field remains unchanged in the limiting process. We conjecture that, under mild restrictive assumptions, strong convergence takes place if and only if that initial condition remains unchanged.

1. Notation and basic definitions. As is customary in plate theory, Greek indices range from 1 to 2 and Latin indices from 1 to 3. Any point x of $\mathbf{R}^3$ is decomposed into $y = (x_1, x_2)$ and x_3.

Einstein's summation convention is used throughout the text. An overdot $\dot{}$ denotes differentiation with respect to time, and an overbar $\bar{}$ denotes the integral $\int_{-1}^{1} dx_3$.

Finally, if b_{ij} denotes the ij^{th} component of a second-order tensor b on $\mathbf{R}^3$,

$$\operatorname{Tr} b = b_{ii}, \qquad \operatorname{tr} b = b_{\alpha\alpha}.$$

The three-dimensional flat plate is defined as

$$\Omega(\varepsilon) = \omega \times (-\varepsilon, \varepsilon),$$

where ω is a smooth bounded domain of $\mathbf{R}^2$ and 2ε the thickness of the plate. By definition

$$\Gamma^{\pm}(\varepsilon) = \omega \times \{\pm\varepsilon\}, \qquad \Gamma^{l}(\varepsilon) = \partial\omega \times (-\varepsilon, \varepsilon).$$

The following spaces are defined:

$$H(\varepsilon) = \{v \in H^1(\Omega(\varepsilon));\ v = 0 \text{ on } \Gamma^l(\varepsilon)\},$$

$$\mathbf{H}(\varepsilon) = [H(\varepsilon)]^3,$$

$$Y(\varepsilon) = \{\tau \in [L_2(\Omega(\varepsilon))]^9;\ \tau \text{ is symmetric}\}.$$

$V_{KL}(\varepsilon) = \{ v \in \mathbf{H}(\varepsilon);\ v_3$ is independent of x_3, lies in $H_0^2(\omega)$, and there exists $\mathbf{v}$ in $[H_0^1(\omega)]^2$ such that $v_\alpha = \mathbf{v}_\alpha - x_3\, \partial v_3/\partial x_\alpha$, $\alpha = 1, 2\}$.

We drop the parenthetic (ε) whenever $\varepsilon = 1$; for example,

$$\Omega = \Omega(1).$$

If $u(x)$ is a displacement field, its linearized strain tensor is defined as

$$e_{ij}(u)(x) = e_{ij}(x) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)(x).$$

To each point $x = (y, x_3)$ of Ω we associate the point $x^\varepsilon = (y, \varepsilon x_3)$ of $\Omega(\varepsilon)$. To each vector field $w^\varepsilon(x^\varepsilon)$ we associate the field $W^\varepsilon(x)$ defined as

$$W_\alpha^\varepsilon(x) = w_\alpha^\varepsilon(x^\varepsilon), \qquad W_3^\varepsilon(x) = \varepsilon w_3^\varepsilon(x^\varepsilon).$$

To each scalar field $z^\varepsilon(x^\varepsilon)$ we associate the field $Z^\varepsilon(x)$ defined as

$$Z^\varepsilon(x) = z^\varepsilon(x^\varepsilon).$$

To each tensor field $\tau^\varepsilon(x^\varepsilon)$ we associate the field $T^\varepsilon(x)$ defined as

$$T_{\alpha\beta}^\varepsilon(x) = \tau_{\alpha\beta}^\varepsilon(x^\varepsilon),$$

$$T_{\alpha 3}^\varepsilon(x) = \frac{1}{\varepsilon}\tau_{\alpha 3}^\varepsilon(x^\varepsilon),$$

$$T_{3\alpha}^\varepsilon(x) = \frac{1}{\varepsilon}\tau_{3\alpha}^\varepsilon(x^\varepsilon),$$

$$T_{33}^\varepsilon(x) = \frac{1}{\varepsilon^2}\tau_{33}^\varepsilon(x^\varepsilon).$$

In this manner the spaces $H(\varepsilon)$, $Y(\varepsilon)$, $L_2(\Omega(\varepsilon)), \ldots$ are mapped onto the spaces H, Y, $L_2(\Omega), \ldots$.

From now on, the x dependence (respectively x^ε dependence) of all mathematical expressions will be implicit, unless confusion could arise.

2. Setting of the problem. In this section the evolution problem for the thermoelastic flat plate is formulated on $\Omega(\varepsilon)$. It is then rescaled using the transformations defined in Sec. 1.

The plate $\Omega(\varepsilon)$ is made of an inhomogeneous linearly thermoelastic isotropic material. The Young's modulus $E^\varepsilon(x^\varepsilon)$, Poisson's ratio $\nu^\varepsilon(x^\varepsilon)$, thermal dilation coefficient $\alpha^\varepsilon(x^\varepsilon)$, heat conductivity coefficient $k^\varepsilon(x^\varepsilon)$, specific heat coefficient $\beta^\varepsilon(x^\varepsilon)$, and mass density $\rho^\varepsilon(x^\varepsilon)$ are defined as

$$\begin{aligned}
E^\varepsilon(x^\varepsilon) &= E(x), && \text{with } E(x) > 0,\\
\nu^\varepsilon(x^\varepsilon) &= \nu(x), && \text{with } -1 < \nu(x) < \tfrac{1}{2},\\
\alpha^\varepsilon(x^\varepsilon) &= \alpha(x), &&\\
k^\varepsilon(x^\varepsilon) &= k(x), && \text{with } k(x) > 0,\\
\beta^\varepsilon(x^\varepsilon) &= \beta(x), && \text{with } \beta(x) > 0,\\
\rho^\varepsilon(x^\varepsilon) &= \varepsilon^2\rho(x), && \text{with } \rho(x) > 0,
\end{aligned}$$

where E, ν, α, k, β, ρ are $\mathscr{C}^\infty$ functions on $\overline{\Omega}$ and even functions of x_3.

REMARK 1. The ε^2 dependence of ρ^ε on ε allows for an upward shift in the purely elastic vibration frequencies of the plate as the scaling parameter goes to zero, which in turn renders the limit model sensitive to inertia effects (Raoult [**R1**]). Similar scalings are found in other problems such as the flow of a viscous fluid in a porous medium (Sanchez–Palencia [**S**], Ch. 8).

The constitutive equations for the plate relate the stress tensor σ_{ij}^ε to the linearized strain tensor $e_{ij}(u^\varepsilon)$ and to the temperature increment field θ^ε with respect to a uniform reference temperature T_0. Specifically,

$$e_{ij}(u^\varepsilon) - \alpha^\varepsilon \theta^\varepsilon \delta_{ij} = \frac{1+\nu^\varepsilon}{E^\varepsilon}\sigma_{ij}^\varepsilon - \frac{\nu^\varepsilon}{E^\varepsilon}\operatorname{Tr}\sigma^\varepsilon \delta_{ij}. \tag{1}$$

The hypotheses made on E and ν suffice to ensure the invertibility of the stress-strain relation (1).

The plate $\Omega(\varepsilon)$ is laterally clamped and maintained at the ground temperature T_0. The transient response of $\Omega(\varepsilon)$ under an arbitrary set of initial conditions in displacement, velocity, and temperature $(u_0^\varepsilon, v_0^\varepsilon, \theta_0^\varepsilon)$, body loadings (f_i^ε), and upper (lower) surface loadings $(g_i^{\pm\varepsilon})$ is investigated. The following system of equations governs the evolution of the displacement field $u^\varepsilon(x^\varepsilon)$ and temperature increment field $\theta^\varepsilon(x^\varepsilon)$:

$$\rho^\varepsilon \ddot{u}_i^\varepsilon = \frac{\partial}{\partial x_j^\varepsilon}\sigma_{ij}^\varepsilon + f_i^\varepsilon, \tag{2}$$

$$\beta^\varepsilon \dot{\theta}^\varepsilon = \frac{1}{T_0}\frac{\partial}{\partial x_j^\varepsilon}\left(k^\varepsilon \frac{\partial \theta^\varepsilon}{\partial x_j^\varepsilon}\right) - \frac{E^\varepsilon \alpha^\varepsilon}{1-2\nu^\varepsilon}\operatorname{Tr} e(\dot{u}^\varepsilon), \tag{3}$$

$$\begin{aligned} \sigma_{i3}^\varepsilon &= \pm g_i^{\pm\varepsilon} && \text{on } \Gamma^\pm(\varepsilon),\\ \frac{\partial \theta^\varepsilon}{\partial x_3^\varepsilon} &= 0 && \text{on } \Gamma^\pm(\varepsilon),\\ u^\varepsilon &= 0 && \text{on } \Gamma^l(\varepsilon),\\ \theta^\varepsilon &= 0 && \text{on } \Gamma^l(\varepsilon), \end{aligned} \tag{4}$$

$$\begin{aligned} u^\varepsilon(0) &= u_0^\varepsilon,\\ \dot{u}^\varepsilon(0) &= v_0^\varepsilon,\\ \theta^\varepsilon(0) &= \theta_0^\varepsilon. \end{aligned} \tag{5}$$

In Eqs. (2)–(4), σ_{ij}^ε is the stress tensor associated with u^ε and θ^ε through Eq. (1).

Under the following set of hypotheses:

$$\begin{aligned} f^\varepsilon &\in W^{1,2}\big(0,T;\, [L_2(\Omega(\varepsilon))]^3\big),\\ g^{\pm\varepsilon} &\in W^{2,2}\big(0,T;\, [L_2(\omega)]^3\big),\\ u_0^\varepsilon &\in [H^2(\Omega(\varepsilon))]^3 \cap \mathbf{H}(\varepsilon),\\ v_0^\varepsilon &\in \mathbf{H}(\varepsilon),\\ \theta_0^\varepsilon &\in H^2(\Omega(\varepsilon)) \cap H(\varepsilon), \end{aligned} \tag{6}$$

the solution $(u^\varepsilon, \theta^\varepsilon)$ of the system (1)–(5) can be shown to satisfy

$$u^\varepsilon \in \mathscr{C}^0\big([0,T];\, [H^2(\Omega(\varepsilon))]^3 \cap \mathbf{H}(\varepsilon)\big) \cap \mathscr{C}^1([0,T]; \mathbf{H}(\varepsilon)) \cap \mathscr{C}^2\big([0,T];\, [L_2(\Omega(\varepsilon))]^3\big), \tag{7}$$

$$\theta^\varepsilon \in \mathscr{C}^0\big([0,T];\, H^2(\Omega(\varepsilon)) \cap H(\varepsilon)\big) \cap \mathscr{C}^1([0,T];\, L_2(\Omega(\varepsilon))).$$

REMARK 2. The existence and uniqueness of the solution of (1)–(5) is discussed in Francfort [**F**] or Hughes–Marsden [**HM**] in the framework of semigroup theory. The regularity (7) is a direct application of that theory (see, for example, Brézis [**B**], Ch. 7).

A rescaling of the system (1)–(5) is now performed with the help of the transformations defined in Sec. 1. The images of all the fields entering the system (1)–(5) are denoted by the corresponding capital letters. We obtain

$$e_{\alpha\beta}(U^\varepsilon) - \alpha\Theta^\varepsilon\delta_{\alpha\beta} = \frac{1+\nu}{E}\Sigma^\varepsilon_{\alpha\beta} - \frac{\nu}{E}\left(\operatorname{tr}\Sigma^\varepsilon + \varepsilon^2\Sigma^\varepsilon_{33}\right)\delta_{\alpha\beta},$$

$$e_{\alpha 3}(U^\varepsilon) = \varepsilon^2\frac{1+\nu}{E}\Sigma^\varepsilon_{\alpha 3}, \tag{8}$$

$$e_{33}(U^\varepsilon) - \varepsilon^2\alpha\Theta^\varepsilon = \varepsilon^4\frac{1+\nu}{E}\Sigma^\varepsilon_{33} - \varepsilon^2\frac{\nu}{E}\left(\operatorname{tr}\Sigma^\varepsilon + \varepsilon^2\Sigma^\varepsilon_{33}\right),$$

$$\varepsilon^2\rho\ddot{U}^\varepsilon_\alpha = \frac{\partial}{\partial x_j}\Sigma^\varepsilon_{\alpha j} + F^\varepsilon_\alpha, \qquad \rho\ddot{U}^\varepsilon_3 = \frac{\partial}{\partial x_j}\Sigma^\varepsilon_{3j} + \frac{1}{\varepsilon^2}F^\varepsilon_3, \tag{9}$$

$$\beta\dot{\Theta}^\varepsilon = \frac{1}{T_0}\left(\frac{\partial}{\partial x_\alpha}\left(k\frac{\partial\Theta^\varepsilon}{\partial x_\alpha}\right) + \frac{1}{\varepsilon^2}\frac{\partial}{\partial x_3}\left(k\frac{\partial\Theta^\varepsilon}{\partial x_3}\right)\right) - \frac{E\alpha}{1-2\nu}\left(\operatorname{tr} e(\dot{U}^\varepsilon) + \frac{1}{\varepsilon^2}\frac{\partial\dot{U}^\varepsilon_3}{\partial x_3}\right), \tag{10}$$

$$\begin{aligned}
\Sigma^\varepsilon_{\alpha 3} &= \pm\frac{1}{\varepsilon}G^{\pm\varepsilon}_\alpha && \text{on } \Gamma^\pm,\\
\Sigma^\varepsilon_{33} &= \pm\frac{1}{\varepsilon^3}G^{\pm\varepsilon}_3 && \text{on } \Gamma^\pm,\\
\frac{\partial\Theta^\varepsilon}{\partial x_3} &= 0 && \text{on } \Gamma^\pm,\\
U^\varepsilon &= 0 && \text{on } \Gamma',\\
\Theta^\varepsilon &= 0 && \text{on } \Gamma',
\end{aligned} \tag{11}$$

$$U^\varepsilon(0) = U^\varepsilon_0, \qquad \dot{U}^\varepsilon(0) = V^\varepsilon_0, \qquad \Theta^\varepsilon(0) = \Theta^\varepsilon_0. \tag{12}$$

Introducing the tensor fields

$$\mathbf{e}(U^\varepsilon) = \left(\begin{array}{c|c} e_{\alpha\beta}(U^\varepsilon) & \frac{1}{\varepsilon}e_{\alpha 3}(U^\varepsilon) \\ \hline \frac{1}{\varepsilon}e_{3\beta}(U^\varepsilon) & \frac{1}{\varepsilon^2}e_{33}(U^\varepsilon) \end{array}\right),$$

$$\mathbf{\Sigma}^\varepsilon = \left(\begin{array}{c|c} \Sigma^\varepsilon_{\alpha\beta} & \varepsilon\Sigma^\varepsilon_{\alpha 3} \\ \hline \varepsilon\Sigma^\varepsilon_{3\beta} & \varepsilon^2\Sigma^\varepsilon_{33} \end{array}\right),$$

and the bilinear forms $\mathscr{A}$, $\mathscr{B}$ on Y defined by

$$\mathscr{A}(A, B) = \int_\Omega \left(\frac{1+\nu}{E} A_{ij} B_{ij} - \frac{\nu}{E} (\operatorname{Tr} A)(\operatorname{Tr} B) \right) dx,$$

$$\mathscr{B}(A, B) = \int_\Omega \left(\frac{E}{1+\nu} A_{ij} B_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} (\operatorname{Tr} A)(\operatorname{Tr} B) \right) dx,$$

we obtain the following variational formulation for the system (8)–(12):

$$\mathscr{A}(\boldsymbol{\Sigma}^\varepsilon(t), \Psi) = \int_\Omega \mathbf{e}_{ij}(U^\varepsilon(t)) \Psi_{ij}\, dx - \int_\Omega \alpha \Theta^\varepsilon(t) (\operatorname{Tr} \Psi)\, dx, \tag{13a}$$

or alternatively,

$$\mathscr{B}(\mathbf{e}(U^\varepsilon(t)), \Psi) = \int_\Omega \Sigma^\varepsilon_{ij}(t) \Psi_{ij}\, dx + \int_\Omega \frac{E\alpha}{1-2\nu} \Theta^\varepsilon(t) (\operatorname{Tr} \Psi)\, dx, \tag{13b}$$

for any Ψ in Y,

$$\varepsilon^2 \int_\Omega \rho \ddot{U}^\varepsilon_\alpha(t) W_\alpha\, dx + \int_\Omega \rho \ddot{U}^\varepsilon_3(t) W_3\, dx + \int_\Omega \Sigma^\varepsilon_{ij}(t) e_{ij}(W)\, dx \tag{14}$$
$$= \int_\Omega F^\varepsilon_\alpha(t) W_\alpha\, dx + \frac{1}{\varepsilon^2} \int_\Omega F^\varepsilon_3(t) W_3\, dx + \frac{1}{\varepsilon} \int_{\Gamma^\pm} G^{\pm\varepsilon}_\alpha(t) W_\alpha\, dy + \frac{1}{\varepsilon^3} \int_{\Gamma^\pm} G^{\pm\varepsilon}_3(t) W_3\, dy$$

for any W in $\mathbf{H}$,

$$\int_\Omega \beta \dot{\Theta}^\varepsilon(t) Z\, dx + \frac{1}{T_0} \int_\Omega k \frac{\partial \Theta^\varepsilon}{\partial x_\alpha}(t) \frac{\partial Z}{\partial x_\alpha} + \frac{1}{\varepsilon^2 T_0} \int_\Omega k \frac{\partial \Theta^\varepsilon}{\partial x_3}(t) \frac{\partial Z}{\partial x_3}\, dx \tag{15}$$
$$+ \int_\Omega \frac{E\alpha}{1-2\nu} \operatorname{Tr} \mathbf{e}(\dot{U}^\varepsilon(t)) Z\, dx = 0$$

for any Z in H,

$$U^\varepsilon(0) = U^\varepsilon_0, \qquad \dot{U}^\varepsilon(0) = V^\varepsilon_0, \qquad \Theta^\varepsilon(0) = \Theta^\varepsilon_0. \tag{16}$$

The hypotheses (6) become

$$\begin{aligned} &F^\varepsilon \in W^{1,2}\big(0,T; [L_2(\Omega)]^3\big),\\ &G^{\pm\varepsilon} \in W^{2,2}\big(0,T; [L_2(\omega)]^3\big),\\ &U^\varepsilon_0 \in [H^2(\Omega)]^3 \cap \mathbf{H},\\ &V^\varepsilon_0 \in \mathbf{H},\\ &\Theta^\varepsilon_0 \in H^2(\Omega) \cap H. \end{aligned} \tag{17}$$

The system (13)–(16) has a unique solution $(U^\varepsilon, \Theta^\varepsilon)$ with

$$U^\varepsilon \in \mathscr{C}^0\big([0,T]; [H^2(\Omega)]^3 \cap \mathbf{H}\big) \cap \mathscr{C}^1([0,T]; \mathbf{H}) \cap \mathscr{C}^2\big([0,T]; [L_2(\Omega)]^3\big), \tag{18}$$
$$\Theta^\varepsilon \in \mathscr{C}^0([0,T]; H^2(\Omega) \cap \mathbf{H}) \cap \mathscr{C}^1([0,T]; L_2(\Omega)).$$

Thus the system (13)–(16) holds true for any t in $[0, T]$.

Our goal in the following sections is to examine the behavior of the fields U^ε, Θ^ε, Σ^ε, Σ^ε, $\mathbf{e}(U^\varepsilon)$ as ε tends to zero. The convergences obtained will imply convergence properties for the original fields, i.e., u^ε, θ^ε, σ^ε.

3. The limit behavior. In this section we define *a priori* a limit problem and briefly study its properties. The justification of the model as a valid limit behavior is done in subsequent sections.

We introduce two evolution systems on ω. The first evolution problem reads

$$\bar{\rho}\ddot{u}_3^0 + \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\overline{\left(\frac{E x_3^2}{1+\nu}\right)} \frac{\partial^2 u_3^0}{\partial x_\alpha \partial x_\beta} + \overline{\left(\frac{E \nu x_3^2}{(1-\nu)(1+\nu)}\right)} \Delta u_3^0 \delta_{\alpha\beta} \right) = F_3^0, \tag{19}$$

with

$$u_3^0 = \frac{\partial u_3^0}{\partial n} = 0 \qquad \text{on } \partial\omega \tag{20}$$

as boundary conditions and

$$u_3^0(0) = u_{03}^0, \qquad \dot{u}_3^0(0) = \frac{\overline{\left(\rho v_{03}^0\right)}}{\bar{\rho}} \tag{21}$$

as initial conditions. In (19) F_3^0 is an element of $W^{1,2}(0,T;\, L_2(\omega))$ and in (21) u_{03}^0 and v_{03}^0 are elements of $H_0^2(\omega)$ and $L_2(\Omega)$, respectively. In view of the positivity properties of the coefficients, the system (19)–(21) is classically seen to admit a unique generalized solution u_3^0 in $\mathscr{C}^0([0,T];\, H_0^2(\omega)) \cap \mathscr{C}^1([0,T];\, L_2(\omega))$.

The second evolution problem couples a quasi-static equation with a parabolic equation. Defining

$$\kappa = \overline{\left(\beta + \frac{E\alpha^2(1+\nu)}{(1-\nu)(1-2\nu)}\right)},$$

the problem reads

$$\sigma_{\alpha\beta}^0 = \overline{\left(\frac{E}{1+\nu}\right)} e_{\alpha\beta}(\mathbf{u}^0) + \overline{\left(\frac{E\nu}{(1-\nu)(1+\nu)}\right)} \operatorname{tr} e(\mathbf{u}^0)\delta_{\alpha\beta} - \overline{\left(\frac{E\alpha}{1-\nu}\right)} \theta^0 \delta_{\alpha\beta}, \tag{22}$$

$$\frac{\partial}{\partial x_\beta}\sigma_{\alpha\beta}^0 + \mathbf{F}_\alpha^0 = 0, \tag{23}$$

$$\kappa \frac{\partial \theta^0}{\partial t} = \frac{\partial}{\partial x_\alpha}\left(\bar{k}\frac{\partial \theta^0}{\partial x_\alpha}\right) - \overline{\left(\frac{E\alpha}{1-\nu}\right)} \operatorname{tr} e(\dot{\mathbf{u}}^0), \tag{24}$$

with

$$\mathbf{u}^0 = 0 \quad \text{on } \partial\omega, \qquad \theta^0 = 0 \quad \text{on } \partial\omega, \tag{25}$$

as boundary conditions and

$$\theta^0(0) = \frac{1}{\kappa}\left[\Theta_0^0 - \overline{\left(\frac{E\alpha}{1-\nu}\right)} \operatorname{tr} e(\mathbf{u}^0(0))\right] \tag{26}$$

as initial condition. The following theorem establishes the existence and uniqueness of the solution of the system (22)–(26):

THEOREM 1. If Θ_0^0 is an element of $L_2(\omega)$ and $\mathbf{F}^0$ an element of $W^{1,2}(0,T;[L_2(\omega)]^2)$, the system (22)–(26) has a unique solution $(\mathbf{u}^0, \theta^0)$ in $\mathscr{C}^0([0,T];\ [H_0^1(\omega)]^2 \times L_2(\omega))$. The initial value $\mathbf{u}^0(0)$ is then the unique solution in $[H_0^1(\omega)]^2$ of

$$\frac{\partial}{\partial x_\beta}\left\{\overline{\left(\frac{E}{1+\nu}\right)}\, e_{\alpha\beta}(\mathbf{u}^0(0)) + \left(\overline{\left(\frac{E\nu}{(1-\nu)(1+\nu)}\right)} + \frac{1}{\kappa}\,\overline{\left(\frac{E\alpha}{1-\nu}\right)}^2\right) \operatorname{tr} e(\mathbf{u}^0(0))\,\delta_{\alpha\beta}\right\}$$
$$= \frac{\partial}{\partial x_\alpha}\left(\frac{1}{\kappa}\,\overline{\left(\frac{E\alpha}{1-\nu}\right)}\,\Theta_0^0\right) - \mathbf{F}_\alpha^0(0)$$

Proof of Theorem 1. The positivity properties of the coefficients imply that the mapping S^0 from $[H_0^1(\omega)]^2$ into $[H^{-1}(\omega)]^2$ defined for any $\mathbf{v}$ in $[H_0^1(\omega)]^2$ as

$$(S^0\mathbf{v})_\alpha = \frac{\partial}{\partial x_\beta}\left(\overline{\left(\frac{E}{1+\nu}\right)}\, e_{\alpha\beta}(\mathbf{v}) + \overline{\left(\frac{E\nu}{(1-\nu)(1+\nu)}\right)} \operatorname{tr} e(\mathbf{v})\,\delta_{\alpha\beta}\right)$$

is an isomorphism. It is easily checked that the mapping L_0 from $L_2(\omega)$ into itself defined for any ζ in $L_2(\omega)$ as

$$L_0\zeta = \overline{\left(\frac{E\alpha}{1-\nu}\right)} \operatorname{tr} e\left(S^{0^{-1}}\left(\operatorname{grad}\left\{\overline{\left(\frac{E\alpha}{1-\nu}\right)}\,\zeta\right\}\right)\right)$$

is a bounded positive self-adjoint linear mapping on $L_2(\omega)$. Furthermore, the function

$$r_0(t) = \overline{\left(\frac{E\alpha}{1-\nu}\right)} \operatorname{tr} e\left(S^{0^{-1}}(\mathbf{F}^0(t))\right)$$

is an element of $W^{1,2}(0,T;\ L_2(\omega))$. The system (22)–(26) can then be rewritten as

$$\mathbf{u}^0 = S^{0^{-1}}\left(-\mathbf{F}^0 + \operatorname{grad}\left(\overline{\left(\frac{E\alpha}{1-\nu}\right)}\,\theta^0\right)\right), \tag{27}$$

$$\begin{gathered}(\kappa I + L_0)\frac{\partial\theta^0}{\partial t} = \frac{\partial}{\partial x_\alpha}\left(\bar{k}\frac{\partial\theta^0}{\partial x_\alpha}\right) + \dot{r}_0(t),\\ \theta^0 = 0 \quad \text{on } \partial\omega, \qquad (\kappa I + L_0)\theta^0(0) = \Theta_0^0 + r_0(0),\end{gathered} \tag{28}$$

where I is the identity mapping for $L_2(\omega)$.

The existence of a solution to the system (28) is obtained through application of the following simple lemma.

LEMMA 1. Let H be a Hilbert space and let A be the infinitesimal generator of a strongly continuous contraction semigroup on H. If L is a self-adjoint isomorphism on H and if there exists a strictly positive constant α such that for any u in H,

$$(Lu, u)_H \geqslant \alpha\|u\|_H^2,$$

$L^{-1}A$ generates a strongly continuous semigroup $S(t)$ on H.

Outline of the proof of Lemma 1. The properties of L imply that L defines an inner product on H whose associated norm is equivalent to the norm $\|\ \|_H$. The operator $L^{-1}A$ is easily seen to satisfy the hypotheses of the Lumer–Phillips theorem (Yosida [**Y**], p. 250) for this new inner product. The result of Lemma 1 follows from a direct application of that theorem.

This lemma is applied with $H = L_2(\omega)$, $L = \kappa I + L_0$, $\alpha = \min_{y \in \bar{\omega}}(\kappa(y))$ and it implies the existence and uniqueness of θ^0 in $\mathscr{C}^0([0, T];\ L_2(\omega))$ and, with the help of (27), of $\mathbf{u}^0$ in $\mathscr{C}^0([0, T];\ [H_0^1(\omega)]^2)$. The equation satisfied by $\mathbf{u}^0(0)$ is easily derived from (22) and (23) written at time $t = 0$ and (26).

REMARK 3. In view of the regularity properties of the fields $\mathbf{u}^0$ and u_3^0, the displacement field u^0 defined as $u^0 = (u_\alpha^0, u_3^0)$ with

$$u_\alpha^0 = \mathbf{u}_\alpha^0 - x_3 \frac{\partial u_3^0}{\partial x_\alpha}$$

is an element of $\mathscr{C}^0([0, T];\ V_{KL})$.

4. Weak convergence of the fields. In this section we establish *a priori* estimates on the fields $U^\varepsilon(t)$, $\Theta^\varepsilon(t)$, $\mathbf{e}(U^\varepsilon(t))$, and $\boldsymbol{\Sigma}^\varepsilon(t)$ with the help of Eqs. (13)–(16). These estimates enable us to pass to the weak limit in (13)–(16).

Specifically, we obtain the following.

LEMMA 2. Let us assume that hypotheses (17) hold true and that

$$\begin{aligned}
U_0^\varepsilon &\to u_0^0 && \text{weakly in } \mathbf{H},\\
\varepsilon V_{0\alpha}^\varepsilon &\to v_{0\alpha}^0 && \text{weakly in } L_2(\Omega),\\
V_{03}^\varepsilon &\to v_{03}^0 && \text{weakly in } L_2(\Omega),\\
\mathbf{e}(U_0^\varepsilon) &\to \mathbf{e}_0^0 && \text{weakly in } Y,\\
\Theta_0^\varepsilon &\to \theta_0^0 && \text{weakly in } L_2(\Omega),\\
F_\alpha^\varepsilon &\to f_\alpha^0 && \text{weakly in } W^{1,2}(0, T;\ L_2(\Omega)),\\
\frac{1}{\varepsilon^2} F_3^\varepsilon &\to f_3^0 && \text{weakly in } W^{1,2}(0, T;\ L_2(\Omega)),\\
\frac{1}{\varepsilon} G_\alpha^{\pm\varepsilon} &\to g_\alpha^{\pm 0} && \text{weakly in } W^{1,2}(0, T;\ L_2(\omega)),\\
\frac{1}{\varepsilon^3} G_3^{\pm\varepsilon} &\to g_3^{\pm 0} && \text{weakly in } W^{1,2}(0, T;\ L_2(\omega)),
\end{aligned} \tag{29}$$

as ε tends to zero. Then,

$$\begin{aligned}
U^\varepsilon &\quad \text{is bounded in } L_\infty(0, T;\ \mathbf{H}),\\
\varepsilon \dot{U}_\alpha^\varepsilon &\quad \text{is bounded in } L_\infty(0, T;\ L_2(\Omega)),\\
\dot{U}_3^\varepsilon &\quad \text{is bounded in } L_\infty(0, T;\ L_2(\Omega)),\\
\mathbf{e}(U^\varepsilon) &\quad \text{is bounded in } L_\infty(0, T;\ Y),\\
\Theta^\varepsilon &\quad \text{is bounded in } L_\infty(0, T;\ L_2(\Omega)) \cap L_2(0, T;\ H),\\
\frac{1}{\varepsilon}\frac{\partial \Theta^\varepsilon}{\partial x_3} &\quad \text{is bounded in } L_2(0, T;\ L_2(\Omega)),\\
\boldsymbol{\Sigma}^\varepsilon &\quad \text{is bounded in } L_\infty(0, T;\ Y),
\end{aligned} \tag{30}$$

independently of ε.

Proof of Lemma 2. The regularity properties (18) enable us to use $\mathbf{e}(\dot{U}^\varepsilon)$, $\dot{U}^\varepsilon$, and Θ^ε as trial functions in (13b), (14), and (15), respectively. Observing that, in view of (13b),

$$\int_\Omega \Sigma^\varepsilon_{ij}(t) e_{ij}(\dot{U}^\varepsilon(t))\,dx = \int_\Omega \Sigma^\varepsilon_{ij}(t)\mathbf{e}_{ij}(\dot{U}^\varepsilon(t))\,dx$$

$$= \mathscr{B}(\mathbf{e}(U^\varepsilon(t)), \mathbf{e}(\dot{U}^\varepsilon(t))) - \int_\Omega \frac{E\alpha}{1-2\nu}\Theta^\varepsilon(t)(\operatorname{Tr}\mathbf{e}(\dot{U}^\varepsilon(t)))\,dx,$$

and adding together the expressions resulting from (14) and (15) yields

$$\begin{aligned}
\varepsilon^2 \int_\Omega \rho \ddot{U}^\varepsilon_\alpha(t)\dot{U}^\varepsilon_\alpha(t)\,dx &+ \int_\Omega \rho \ddot{U}^\varepsilon_3(t)\dot{U}^\varepsilon_3(t)\,dx \\
&+ \mathscr{B}(\mathbf{e}(U^\varepsilon(t)), \mathbf{e}(\dot{U}^\varepsilon(t))) + \int_\Omega \beta \Theta^\varepsilon(t)\dot{\Theta}^\varepsilon(t)\,dx \\
&+ \frac{1}{T_0}\int_\Omega k \frac{\partial \Theta^\varepsilon}{\partial x_\alpha}(t)\frac{\partial \Theta^\varepsilon}{\partial x_\alpha}(t)\,dx + \frac{1}{\varepsilon^2 T_0}\int_\Omega k\frac{\partial \Theta^\varepsilon}{\partial x_3}(t)\frac{\partial \Theta^\varepsilon}{\partial x_3}(t)\,dx \\
&= \int_\Omega F^\varepsilon_\alpha(t)\dot{U}^\varepsilon_\alpha(t)\,dx + \frac{1}{\varepsilon^2}\int_\Omega F_3(t)\dot{U}^\varepsilon_3(t)\,dx \\
&+ \frac{1}{\varepsilon}\int_{\Gamma^\pm} G^{\pm\varepsilon}_\alpha(t)\dot{U}^\varepsilon_\alpha(t)\,dy + \frac{1}{\varepsilon^3}\int_{\Gamma^\pm} G^{\pm\varepsilon}_3(t)\dot{U}^\varepsilon_3(t)\,dy.
\end{aligned} \tag{31}$$

Integrating (31) over the time interval $[0, t]$ for $0 \leqslant t \leqslant T$ leads to

$$\begin{aligned}
&\int_\Omega \rho\left\{|\varepsilon \dot{U}^\varepsilon_\alpha(t)|^2 + |\dot{U}^\varepsilon_3(t)|^2\right\}dx + \mathscr{B}(\mathbf{e}(U^\varepsilon(t)), \mathbf{e}(U^\varepsilon(t))) \\
&\quad + \int_\Omega \beta|\Theta^\varepsilon(t)|^2\,dx + \frac{2}{T_0}\int_0^t\int_\Omega k\left\{\sum_{\alpha=1}^2 \left|\frac{\partial \Theta^\varepsilon}{\partial x_\alpha}(s)\right|^2 + \left|\frac{1}{\varepsilon}\frac{\partial \Theta^\varepsilon}{\partial x_3}(s)\right|^2\right\}dx\,ds \\
&= \int_\Omega \rho\left\{|\varepsilon V^\varepsilon_{0\alpha}|^2 + |V^\varepsilon_{03}|^2\right\}dx + \mathscr{B}(\mathbf{e}(U^\varepsilon_0), \mathbf{e}(U^\varepsilon_0)) \\
&\quad + \int_\Omega \beta|\Theta^\varepsilon_0|^2\,dx + 2\left[\int_\Omega \left(F^\varepsilon_\alpha(s)U^\varepsilon_\alpha(s) + \frac{1}{\varepsilon^2}F^\varepsilon_3(s)U^\varepsilon_3(s)\right)dx\right. \\
&\quad \left. + \int_{\Gamma^\pm}\left(\frac{1}{\varepsilon}G^{\pm\varepsilon}_\alpha(s)U^\varepsilon_\alpha(s) + \frac{1}{\varepsilon^3}G^{\pm\varepsilon}_3(s)U^\varepsilon_3(s)\right)dy\right]_0^t \\
&\quad - 2\int_0^t\int_\Omega \left(\dot{F}^\varepsilon_\alpha(s)U^\varepsilon_\alpha(s) + \frac{1}{\varepsilon^2}\dot{F}^\varepsilon_3(s)U^\varepsilon_3(s)\right)dx\,ds \\
&\quad - 2\int_0^t\int_{\Gamma^\pm}\left(\frac{1}{\varepsilon}\dot{G}^{\pm\varepsilon}_\alpha(s)U^\varepsilon_\alpha(s) + \frac{1}{\varepsilon^3}\dot{G}^{\pm\varepsilon}_3(s)U^\varepsilon_3(s)\right)dy\,ds.
\end{aligned} \tag{32}$$

The positivity properties of the coefficients on $\overline{\Omega}$, the equality (32), and the hypotheses (29) imply the existence of a generic strictly positive constant C, independent of ε such that

$$\begin{aligned}\sum_{\alpha=1}^{2} \|\varepsilon\dot{U}_\alpha^\varepsilon\|^2_{L_\infty(0,T;\,L_2(\Omega))} + \|\dot{U}_3^\varepsilon\|^2_{L_\infty(0,T;\,L_2(\Omega))}&\\ + \|\mathbf{e}(U^\varepsilon)\|^2_{L_\infty(0,T;\,Y)} + \|\Theta^\varepsilon\|^2_{L_\infty(0,T;\,L_2(\Omega))}&\\ + \sum_{\alpha=1}^{2}\left\|\frac{\partial\Theta^\varepsilon}{\partial x_\alpha}\right\|^2_{L_2(0,T;\,L_2(\Omega))} + \left\|\frac{1}{\varepsilon}\frac{\partial\Theta^\varepsilon}{\partial x_3}\right\|^2_{L_2(0,T;\,L_2(\Omega))}&\\ \leqslant C\left(1 + \|U^\varepsilon\|_{L_\infty(0,T;[L_2(\Omega)]^3)}\right).&\end{aligned} \tag{33}$$

Poincaré's inequality and Korn's inequality imply that

$$\|U^\varepsilon\|_{L_\infty(0,T;[L_2(\Omega)]^3)} \leqslant C\|\mathbf{e}(U^\varepsilon)\|_{L_\infty(0,T;\,Y)},$$

for any ε less than 1. Inequality (33) then yields the desired estimates on U^ε, $\varepsilon\dot{U}_\alpha^\varepsilon$, $\dot{U}_3^\varepsilon$, $\mathbf{e}(U^\varepsilon)$, Θ^ε, $(1/\varepsilon)\partial\Theta^\varepsilon/\partial x_3$.

Because of the positivity properties of the coefficients, the bilinear form $\mathscr{A}$ defines a norm on Y that is equivalent to the natural $L_2(\Omega)$-norm. The estimate on Σ^ε is then an immediate consequence of equality (13a) with Σ^ε as trial function.

REMARK 4. The hypotheses on U_0^ε and $\mathbf{e}(U_0^\varepsilon)$ imply that u_0^0 belongs to V_{KL}, i.e.,

$$u_{0\alpha}^0 = \mathbf{u}_{0\alpha}^0 - x_3\frac{\partial u_{03}^0}{\partial x_\alpha}, \text{ with } \mathbf{u}_{0\alpha}^0 \text{ in } H_0^1(\omega)$$
$$\text{and } u_{03}^0 \text{ in } H_0^2(\omega).$$

Similarly, the estimate on $(1/\varepsilon)\partial\Theta^\varepsilon/\partial x_3$ implies that the weak-* limits of weakly convergent subsequences of Θ^ε in $L_\infty(0,T;\,L_2(\Omega))$ are independent of x_3, and the estimate on U_α^ε implies that the sequence $\varepsilon\dot{U}_\alpha^\varepsilon$ converges weak-* to 0 in $L_\infty(0,T;\,L_2(\Omega))$.

In view of Lemma 2, we conclude that there exist weakly converging subsequences of all bounded fields appearing in (30). Since we eventually show the uniqueness of the weak limits of these fields, we identify the sequence with its converging subsequences, and denote by u^0, $\mathbf{e}^0$, θ^0, $\boldsymbol{\sigma}^0$, $\mathbf{q}^0$ the weak limits of U^ε, $\mathbf{e}(U^\varepsilon)$, Θ^ε, Σ^ε, and $(1/\varepsilon)\partial\Theta^\varepsilon/\partial x_3$.

With the help of Lemma 2 and of Sec. 3 we are in a position to prove the following theorem of weak convergence of the fields:

THEOREM 2. Let us assume that hypotheses (17) and (29) hold true and that

$$\frac{\partial}{\partial x_\alpha}f_\alpha^0 \in W^{1,2}(0,T;\,L_2(\Omega)), \qquad \frac{\partial}{\partial x_\alpha}g_\alpha^{\pm 0} \in W^{1,2}(0,T;\,L_2(\omega)). \tag{34}$$

Then, as ε tends to zero,

$$
\begin{aligned}
U^\varepsilon &\to u^0 && \text{weak-}* \text{ in } L_\infty(0,T;\mathbf{H}),\\
\varepsilon \dot U^\varepsilon_\alpha &\to 0 && \text{weak-}* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
\dot U^\varepsilon_3 &\to \dot u^0_3 && \text{weak-}* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
\Theta^\varepsilon &\to \theta^0 && \text{weak-}* \text{ in } L_\infty(0,T;L_2(\Omega)) \text{ and weakly in } L_2(0,T;H),\\
\Sigma^\varepsilon_{\alpha\beta} &\to \boldsymbol{\sigma}^0_{\alpha\beta} && \text{weak-}* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
\varepsilon\Sigma^\varepsilon_{\alpha 3} &\to 0 && \text{weak-}* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
\varepsilon^2\Sigma^\varepsilon_{33} &\to 0 && \text{weak-}* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
\frac{1}{\varepsilon}\mathrm{e}_{\alpha 3}(U^\varepsilon) &\to 0 && \text{weak-}* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
\frac{1}{\varepsilon^2}e_{33}(U^\varepsilon) &\to && \alpha\theta^0 - \frac{\nu}{E}\operatorname{tr}\boldsymbol{\sigma}^0 = \alpha\frac{1+\nu}{1-\nu}\theta^0 - \frac{\nu}{1-\nu}\operatorname{tr} e(u^0)\\
& && \qquad\qquad \text{weak-}* \text{ in } L_\infty(0,T;L_2(\Omega)),\\
\frac{1}{\varepsilon}\frac{\partial\Theta^\varepsilon}{\partial x_3} &\to 0 && \text{weakly in } L_2(0,T;L_2(\Omega)),
\end{aligned}
\tag{35}
$$

where

$$
u^0_\alpha = \mathbf{u}^0_\alpha - x_3\frac{\partial u^0_3}{\partial x_\alpha}, \tag{36}
$$

and where u^0_3, $\mathbf{u}^0$, θ^0, $\sigma^0_{\alpha\beta} \overset{\text{def}}{=} \bar{\boldsymbol{\sigma}}^0_{\alpha\beta}$ are the unique solution of (19)–(26) with

$$
\begin{aligned}
F^0_3 &= \bar f^0_3 + \left(g^{+0}_3 + g^{-0}_3\right) + \overline{\left(x_3\frac{\partial}{\partial x_\alpha}f^0_\alpha\right)} + \frac{\partial}{\partial x_\alpha}\left(g^{+0}_\alpha - g^{-0}_\alpha\right),\\
\mathbf{F}^0_\alpha &= \bar f^0_\alpha + \left(g^{+0}_\alpha + g^{-0}_\alpha\right),\\
\Theta^0_0 &= \overline{\left(\beta\theta^0_0\right)} + \overline{\left(\frac{E\alpha}{1-2\nu}\left(\operatorname{tr} e\left(u^0_0\right) + \mathbf{e}^0_{033}\right)\right)}.
\end{aligned}
\tag{37}
$$

REMARK 5. The hypotheses and conclusions of Theorem 2 are easily expressed in terms of the original fields. We assume that (6) and (34) hold true and that the fields

$$
u^\varepsilon_{0\alpha},\ \varepsilon u^\varepsilon_{03},\ \varepsilon v^\varepsilon_0,\ \frac{1}{\varepsilon}\frac{\partial u^\varepsilon_{03}}{\partial x_3},\ \frac{1}{\varepsilon}\frac{\partial u^\varepsilon_{0\alpha}}{\partial x_3} + \frac{\partial u^\varepsilon_{03}}{\partial y_\alpha},\ \theta^\varepsilon_0,\ f^\varepsilon_\alpha,\ \frac{1}{\varepsilon}f^\varepsilon_3,\ \frac{1}{\varepsilon}g^{\pm\varepsilon}_\alpha,\ \frac{1}{\varepsilon^2}g^{\pm\varepsilon}_3
$$

taken at the point $(y,\varepsilon x_3)$ (or $(y,\varepsilon x_3,t)$ for the loadings) converge (in the appropriate weak topologies) to

$$
u^0_{0\alpha},\ u^0_{03},\ v^0_0,\ \mathbf{e}^0_{033},\ 2\mathbf{e}^0_{0\alpha 3},\ \theta^0_0,\ f^0_\alpha,\ f^0_3,\ g^{\pm 0}_\alpha,\ g^{\pm 0}_3
$$

taken at the point (y,x_3) (or (y,x_3,t) for the loadings). We obtain the convergence (in the appropriate weak topologies) of the fields

$$
u^\varepsilon_\alpha,\ \varepsilon u^\varepsilon_3,\ \varepsilon\dot u^\varepsilon_\alpha,\ \varepsilon\dot u^\varepsilon_3,\ \theta^\varepsilon,\ \sigma^\varepsilon_{\alpha\beta},\ \sigma^\varepsilon_{\alpha 3},\ \sigma^\varepsilon_{33},\ \frac{1}{\varepsilon}\frac{\partial u^\varepsilon_\alpha}{\partial x_3} + \frac{\partial u^\varepsilon_3}{\partial y_\alpha},\ \frac{1}{\varepsilon}\frac{\partial u^\varepsilon_3}{\partial x_3},\ \frac{1}{\varepsilon}\frac{\partial\theta^\varepsilon}{\partial x_3}
$$

taken at the point $(y,\varepsilon x_3,t)$ to the fields u^0_α, u^0_3, 0, $\dot u^0_3$, θ^0, $\boldsymbol{\sigma}^0_{\alpha\beta}$, 0, 0, 0, $\alpha\theta^0 - (\nu/E)\operatorname{tr}\boldsymbol{\sigma}^0$, 0 taken at the point (y,x_3,t).

Proof of Theorem 2. Let $W(x)$ be an arbitrary element of $\mathbf{H}$ and let $\varphi(t)$ be an arbitrary element of $\mathscr{C}_0^\infty(0,T)$. Inserting

$$W^\varepsilon(x,t) = \varphi(t)\big(\varepsilon W_\alpha(x),\ \varepsilon^2 W_3(x)\big)$$

as test function in (14), integrating the resulting expression over the time interval $(0,T)$, and letting ε tend to zero, we are left with

$$\int_0^T \int_\Omega \left(\boldsymbol{\sigma}_{\alpha 3}^0 \frac{\partial W_\alpha}{\partial x_3} + \boldsymbol{\sigma}_{33}^0 \frac{\partial W_3}{\partial x_3} \right) \varphi(t)\, dx\, dt = 0.$$

We are at liberty to choose W_α and W_3 of the form $\int_0^{x_3} \chi_\alpha(y, x_3)\, dx_3$ and $\int_0^{x_3} \chi_3(y, x_3)\, dx_3$, where χ is an arbitrary element of $[\mathscr{C}_0^\infty(\Omega)]^3$. Thus

$$\boldsymbol{\sigma}_{\alpha 3}^0 = 0, \qquad \boldsymbol{\sigma}_{33}^0 = 0. \tag{38}$$

A test function of the form $\varphi(t) W(x)$, with $W(x)$ an arbitrary element of V_{KL} (see Sec. 1), is now used in (14). Since

$$e_{\alpha 3}(W) = 0, \qquad e_{33}(W) = 0,$$

a similar procedure yields, for almost any t in $(0,T)$,

$$\int_\Omega \rho \ddot{u}_3^0(t) W_3\, dx + \int_\Omega \boldsymbol{\sigma}_{\alpha\beta}^0(t) e_{\alpha\beta}(W)\, dx$$
$$= \int_\Omega f_\alpha^0(t) W_\alpha\, dx + \int_\Omega f_3^0(t) W_3\, dx + \int_{\Gamma^\pm} g_\alpha^{\pm 0}(t) W_\alpha\, dy + \int_{\Gamma^\pm} g_3^{\pm 0}(t) W_3\, dy. \tag{39}$$

If W_3 is chosen to be equal to zero, (39) yields, for almost any t in $(0,T)$,

$$\int_\Omega \boldsymbol{\sigma}_{\alpha\beta}^0(t) \frac{\partial \mathbf{W}_\alpha}{\partial w_\beta}\, dx = \int_\Omega f_\alpha^0(t) \mathbf{W}_\alpha\, dx + \int_{\Gamma^\pm} g_\alpha^{\pm 0}(t) \mathbf{W}_\alpha\, dy. \tag{40}$$

But $\mathbf{W}_\alpha$ is independent of x_3, thus (40) reads as

$$\frac{\partial}{\partial x_\beta} \bar{\boldsymbol{\sigma}}_{\alpha\beta}^0(t) + \bar{f}_\alpha^0(t) + \big(g_\alpha^{+0}(t) + g_\alpha^{-0}(t)\big) = 0, \tag{41}$$

for almost any t in $(0,T)$.

Similarly if $\mathbf{W}_\alpha$ is chosen to be equal to zero, (39) yields, for almost any t in $(0,T)$,

$$\int_\Omega \rho \ddot{u}_3^0(t) W_3\, dx - \int_\Omega x_3 \boldsymbol{\sigma}_{\alpha\beta}^0(t) \frac{\partial^2 W_3}{\partial x_\alpha \partial x_\beta}\, dx$$
$$= -\int_\Omega x_3 f_\alpha^0(t) \frac{\partial W_3}{\partial x_\alpha}\, dx + \int_\Omega f_3^0(t) W_3\, dx \mp \int_{\Gamma^\pm} g_\alpha^{\pm 0}(t) \frac{\partial W_3}{\partial x_\alpha}\, dy + \int_{\Gamma^\pm} g_3^{\pm 0} W_3\, dy. \tag{42}$$

Since W_3 is independent of x_3, (42) reads as

$$\overline{\ddot{(\rho u_3^0)}}\,(T) - \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \overline{x_3 \boldsymbol{\sigma}_{\alpha\beta}^0}(t) = \bar{f}_3^0(t) + \big(g_3^{+0}(t) + g_3^{-0}(t)\big)$$
$$+ \overline{\left(x_3 \frac{\partial}{\partial x_\alpha} f_\alpha^0\right)}(t) + \left(\frac{\partial}{\partial x_\alpha} g_\alpha^{+0}(t) - \frac{\partial}{\partial x_\alpha} g_\alpha^{-0}(t)\right), \tag{43}$$

for almost any t in $(0, T)$. In view of (43), $\overline{(\rho u_3^0)}^{\cdot\cdot}(t)$ is an element of $L_\infty(0, T; H^{-2}(\omega))$, thus $\overline{(\rho u_3^0)}^{\cdot}(t)$ has a trace at $t = 0$. The computation of the trace requires the use of a test function of the form $\eta(t)W(x)$ in (14), with $W(x)$ an arbitrary element of V_{KL} and $\eta(t)$ an arbitrary element of $\mathscr{C}_0^\infty([0, s))$, $0 < s \leqslant T$. An integration by parts of the term

$$\int_0^s \int_\Omega \rho \ddot{U}_3^\varepsilon(t) W_3 \eta(t)\, dx\, dt$$

is performed; it yields

$$-\int_0^s \int_\Omega \rho \dot{U}_3^\varepsilon(t) W_3 \dot{\eta}(t)\, dx\, dt - \eta(0) \int_\Omega \rho V_{03}^\varepsilon W_3\, dx.$$

Passing to the limit as before, and integrating by parts appropriately, we obtain, with the help of (39),

$$\eta(0)\left(\left\langle\left\langle \overline{(\rho u_3^0)}^{\cdot}(0), W_3 \right\rangle\right\rangle - \int_\Omega \rho v_{03}^0 W_3\, dx\right) = 0$$

where $\langle\langle\, ,\rangle\rangle$ stands for the duality product between $H^{-2}(\omega)$ and $H_0^2(\omega)$. Thus

$$\overline{(\rho u_3^0)}^{\cdot}(0) = \overline{(\rho v_{03}^0)}\,. \tag{44}$$

Of course, since U_3^ε converges weak-$*$ in $W^{1,\infty}(0, T; L_2(\Omega))$ to u_3^0, $U_3^\varepsilon(0)$ converges weakly in $L_2(\Omega)$ to $u_3^0(0)$, *i.e.*,

$$u_3^0(0) = u_{03}^0. \tag{45}$$

We now consider test functions of the form $\varphi(t)\Psi(x)$ in (13a), with Ψ an arbitrary element of Y, and repeat a similar procedure. We obtain that, for almost every t in $(0, T)$,

$$\mathscr{A}(\boldsymbol{\sigma}^0(t), \Psi) = \int_\Omega \mathbf{e}_{ij}^0(t) \Psi_{ij}\, dx - \int_\Omega \alpha \theta^0(t)(\operatorname{Tr} \Psi)\, dx. \tag{46}$$

In view of (38) and of the expression of $\mathscr{A}$, (46) yields

$$\mathbf{e}_{\alpha 3}^0 = 0, \qquad \mathbf{e}_{33}^0 = \alpha\theta^0 - \frac{\nu}{E} \operatorname{tr} \boldsymbol{\sigma}^0, \tag{47}$$

and

$$\mathbf{e}_{\alpha\beta}^0 = e_{\alpha\beta}(u^0) = \alpha\theta^0 \delta_{\alpha\beta} + \frac{1+\nu}{E} \boldsymbol{\sigma}_{\alpha\beta}^0 - \frac{\nu}{E} \operatorname{tr} \boldsymbol{\sigma}^0 \delta_{\alpha\beta},$$

or equivalently

$$\boldsymbol{\sigma}_{\alpha\beta}^0 = \frac{E}{1+\nu} e_{\alpha\beta}(u^0) + \frac{E\nu}{(1-\nu)(1+\nu)} \operatorname{tr} e(u^0) \delta_{\alpha\beta} - \frac{E\alpha}{1-\nu} \theta^0 \delta_{\alpha\beta}. \tag{48}$$

Equalities (47) imply *a fortiori* that

$$e_{\alpha 3}(u^0) = 0, \qquad e_{33}(u^0) = 0,$$

and thus that

$$u^0 \in L_\infty(0, T; V_{KL}), \tag{49}$$

i.e., that

$$u_3^0 \in L_\infty\big(0, T; H_2^0(\omega)\big), \qquad u_\alpha^0 = \mathbf{u}_\alpha^0 - x_3 \frac{\partial u_3^0}{\partial x_\alpha}, \tag{50}$$

where $\mathbf{u}^0$ is an element of $[H_0^1(\omega)]^2$.

We finally consider test functions of the form $\varepsilon\varphi(t)Z(x)$ in (15), with Z an arbitrary element of H, and repeat the familiar procedure. We obtain that, for almost every t in $(0, T)$,

$$\frac{1}{T_0}\int_\Omega \mathbf{q}^0(t)\frac{\partial Z}{\partial x_3}\,dx = 0.$$

An argument similar to that used for $\boldsymbol{\sigma}_{33}^0$ leads to

$$\mathbf{q}^0(t) = 0. \tag{51}$$

A test function of the form $\varphi(t)Z(x)$ with Z an arbitrary element of $H_0^1(\omega)$ is now used in (15) (such a test function is allowed since $H_0^1(\omega) \subset H$). We obtain that, for almost any t in $(0, T)$,

$$\int_\Omega \beta\dot{\theta}^0(t) Z\,dx + \frac{1}{T_0}\int_\Omega k\frac{\partial\theta^0}{\partial x_\alpha}(t)\frac{\partial Z}{\partial x_\alpha}\,dx + \int_\Omega \frac{E\alpha}{1-2\nu}\big(\mathrm{Tr}\,\dot{\mathbf{e}}^0(t)\big)Z\,dx = 0, \tag{52}$$

i.e., since θ^0 is independent of x_3 by virtue of Remark 4,

$$\beta\dot{\theta}^0(t) = \frac{1}{T_0}\frac{\partial}{\partial x_\alpha}\left(\bar{k}\frac{\partial\theta^0}{\partial x_\alpha}\right) - \overline{\left(\frac{E\alpha}{1-2\nu}\mathrm{Tr}\,\dot{\mathbf{e}}^0\right)}. \tag{53}$$

But (47) and (48) imply that

$$\mathrm{Tr}\,\mathbf{e}^0 = \mathrm{tr}\,e(u^0) + \mathbf{e}_{33}^0 = \frac{1-2\nu}{1-\nu}\,\mathrm{tr}\,e(u^0) + \alpha\frac{1+\nu}{1-\nu}\theta^0. \tag{54}$$

With the help of (54), (53) reads as

$$\kappa\dot{\theta}^0(t) = \frac{1}{T_0}\frac{\partial}{\partial x_\alpha}\left(\bar{k}\frac{\partial\theta^0}{\partial x_\alpha}\right) - \overline{\left(\frac{E\alpha}{1-\nu}\,\mathrm{tr}\,e(\dot{u}^0)\right)}, \tag{55}$$

where κ has been defined in Sec. 3. In view of (55),

$$\kappa\theta^0 + \overline{\left(\frac{E\alpha}{1-\nu}\,\mathrm{tr}\,e(u^0)\right)}$$

is an element of $W^{1,2}(0, T; H^{-1}(\omega))$. The computation of its trace at time $t = 0$ requires the use of a test function of the form $\eta(t)Z(x)$ in (15), with $Z(x)$ an arbitrary element of $H_0^1(\omega)$ and $\eta(t)$ an arbitrary element of $\mathscr{C}_0^\infty([0, s))$, $0 < s \leqslant T$. An integration by parts of the term

$$\int_0^s\int_\Omega\left(\beta\dot{\Theta}^\varepsilon(t) + \frac{E\alpha}{1-2\nu}\mathrm{Tr}\,\mathbf{e}\big(\dot{U}^\varepsilon(t)\big)\right)Z\eta(t)\,dx\,dt$$

is performed; it yields

$$-\int_0^s \int_\Omega \left(\beta\Theta^\varepsilon(t) + \frac{E\alpha}{1-2\nu}\operatorname{Tr}\mathbf{e}(U^\varepsilon(t))\right) Z\dot{\eta}(t)\,dx\,dt$$
$$-\eta(0)\int_\Omega \left(\beta\Theta_0^\varepsilon + \frac{E\alpha}{1-2\nu}\operatorname{Tr}\mathbf{e}(U_0^\varepsilon)\right) Z\,dx.$$

Passing to the limit as before, and integrating by parts appropriately, we obtain, with the help of (52),

$$\eta(0)\left(\left\langle\left(\kappa\theta^0 + \overline{\left(\frac{E\alpha}{1-\nu}\operatorname{tr}e(u^0)\right)}\right)(0), Z\right\rangle - \int_\Omega \left(\beta\theta_0^0 + \frac{E\alpha}{1-2\nu}\operatorname{Tr}\mathbf{e}_0^0\right) Z\,dx\right) = 0$$

where $\langle\ ,\ \rangle$ stands for the duality product between $H^{-1}(\omega)$ and $H_0^1(\omega)$. Thus

$$\left(\kappa\theta^0 + \overline{\left(\frac{E\alpha}{1-\nu}\operatorname{tr}e(u^0)\right)}\right)(0) = \overline{(\beta\theta_0^0)} + \overline{\left(\frac{E\alpha}{1-2\nu}\operatorname{Tr}\mathbf{e}_0^0\right)}. \tag{56}$$

To complete the proof of Theorem 2, we merely have to replace u_α^0 by $\mathbf{u}_\alpha^0 - x_3\,\partial u_3^0/\partial x_\alpha$ in (48) and (54).

The expressions obtained for $\sigma_{\alpha\beta}^0$ and $\operatorname{tr}e(u^0)$ are then used to compute $\bar{\sigma}_{\alpha\beta}$, $\overline{-x_3\sigma_{\alpha\beta}^0}$ and $\overline{((E\alpha/(1-\nu))\operatorname{tr}e(u^0))}$. In view of the even character of all coefficients and since u_3^0 and θ^0 are independent of x_3,

$$\sigma_{0\alpha\beta} \overset{\text{def}}{=} \bar{\sigma}_{0\alpha\beta} = \overline{\left(\frac{E}{1+\nu}\right)} e_{\alpha\beta}(\mathbf{u}^0) + \overline{\left(\frac{E\nu}{(1-\nu)(1+\nu)}\right)} \operatorname{tr}e(\mathbf{u}^0)\delta_{\alpha\beta} - \overline{\left(\frac{E\alpha}{1-\nu}\right)}\theta^0\delta_{\alpha\beta},$$

$$\overline{(-x_3\sigma_{0\alpha\beta})} = \overline{\left(\frac{Ex_3^2}{1-\nu}\right)}\frac{\partial^2 u_3^0}{\partial x_\alpha \partial x_\beta} + \overline{\left(\frac{E\nu x_3^2}{(1-\nu)(1+\nu)}\right)}\Delta u_3^0\delta_{\alpha\beta}, \tag{57}$$

$$\overline{\left(\frac{E\alpha}{1-\nu}\operatorname{tr}e(u^0)\right)} = \overline{\left(\frac{E\alpha}{1-\nu}\right)}\operatorname{tr}e(\mathbf{u}^0).$$

Recalling Eqs. (41), (43), (44), (45), (55), (56), and (57), we conclude that u_3^0, $\mathbf{u}^0$, θ^0, $\sigma_{\alpha\beta}^0$ satisfy (19)–(26), which, together with Eqs. (38), (47), and (51) and relation (49), completes the proof of Theorem 2.

REMARK 6. The regularity hypotheses (34) together with the $H_0^2(\omega)$-regularity of u_{03}^0 (cf. Remark 4) are implicitly used to assess the existence and uniqueness of the solution u_3^0 of (19)–(22).

REMARK 7. In view of Remark 3, u^0 lies in $\mathscr{C}^0([0,T]; V_{KL})$. The initial value of u_α^0 is then determined as

$$u_\alpha^0(0) = \mathbf{u}_\alpha^0(0) - x_3\frac{\partial u_{03}^0}{\partial x_\alpha}. \tag{58}$$

REMARK 8. As announced in the introduction, there is in general a change in the initial condition in temperature; let us assume for example that

θ_0^0 is independent of x_3 and $\theta_0^0 \neq 0$ almost everywhere,

$$u_0^0 = 0, \qquad \mathbf{e}_{033}^0 = 0, \qquad f_\alpha^0 = 0, \qquad g_\alpha^{\pm 0} = 0,$$

and also that for every x in $\bar{\Omega}$, $\alpha(x) > 0$. Then (26) and (37) become

$$\Theta_0^0 = \bar{\beta}\theta_0^0,$$

and

$$\theta^0(0) = \frac{1}{\kappa}\left\{\bar{\beta}\theta_0^0 - \overline{\left(\frac{E\alpha}{1-\nu}\right)} \operatorname{tr} e\left(\mathbf{u}^0(0)\right)\right\}, \tag{59}$$

where, according to Theorem 1, $\mathbf{u}^0(0)$ is the solution of

$$\frac{\partial}{\partial x_\beta}\left\{\overline{\left(\frac{E}{1+\nu}\right)} e_{\alpha\beta}\left(\mathbf{u}^0(0)\right) + \left(\overline{\left(\frac{E\nu}{(1-\nu)(1+\nu)}\right)} + \frac{1}{\kappa}\overline{\left(\frac{E\alpha}{(1-\nu)}\right)}^2\right) \operatorname{tr} e\left(\mathbf{u}^0(0)\right)\delta_{\alpha\beta}\right\}$$
$$= \frac{\partial}{\partial x_\beta}\left(\frac{\bar{\beta}}{\kappa}\overline{\left(\frac{E\alpha}{1-\nu}\right)}\theta_0^0\delta_{\alpha\beta}\right). \tag{60}$$

The positivity properties of the coefficients imply that

$$\kappa > \bar{\beta}.$$

If $\theta^0(0) = \theta_0^0$, (59) implies that

$$\theta_0^0 = -\frac{1}{\kappa - \bar{\beta}}\overline{\left(\frac{E\alpha}{1-\nu}\right)} \operatorname{tr} e\left(\mathbf{u}^0(0)\right),$$

and thus (60) becomes

$$\frac{\partial}{\partial x_\beta}\left\{\overline{\left(\frac{E}{1+\nu}\right)} e_{\alpha\beta}\left(\mathbf{u}^0(0)\right) + \left(\overline{\left(\frac{E\nu}{(1-\nu)(1+\nu)}\right)} + \frac{1}{(\kappa - \bar{\beta})}\overline{\left(\frac{E\alpha}{(1-\nu)}\right)}^2\right) \operatorname{tr} e\left(\mathbf{u}^0(0)\right)\delta_{\alpha\beta}\right\} = 0. \tag{61}$$

Then, because of (61),

$$\mathbf{u}^0(0) = 0,$$

which contradicts (59).

This change of the initial datum in the temperature field appears in another asymptotic problem involving thermoelastic behavior, namely a problem of homogenization (Francfort [**F**]). It is shown in that context to be a by-product of rapid oscillations in time of the temperature field. Whether the same phenomenon occurs in the present plate problem is an open question.

5. Strong convergence of the fields. In this section it is shown that strong convergence can occur if stronger hypotheses are satisfied by the loadings and the initial conditions. In particular, the initial conditions must satisfy a compatibility condition that will be specified later.

The method used to prove strong convergence follows that of Raoult [**R1**]. It is based on the convergence of the norms in the Hilbert space $L_2(0, T; L_2(\Omega))$.

Specifically, we prove the following.

THEOREM 3. Let us assume that hypotheses (17) and (34) hold true, and that all the convergences in (29) become strong convergences.

Then, as ε tends to zero,

$$\begin{array}{lll}
U^{\varepsilon} & \to u^0 & \text{strongly in } L_2(0,T;\mathbf{H}),\\
\varepsilon\dot{U}^{\varepsilon}_{\alpha} & \to 0 & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\dot{U}^{\varepsilon}_3 & \to \dot{u}^0_3 & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\Theta^{\varepsilon} & \to \theta^0 & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\Sigma^{\varepsilon}_{\alpha\beta} & \to \sigma^0_{\alpha\beta} & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\varepsilon\Sigma^{\varepsilon}_{\alpha 3} & \to 0 & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\varepsilon^2\Sigma^{3}_{33} & \to 0 & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\frac{1}{\varepsilon}\mathrm{e}_{\alpha 3}(U^{\varepsilon}) & \to 0 & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\frac{1}{\varepsilon^2}\mathrm{e}_{33}(U^{\varepsilon}) & \to \alpha\theta^0 - \frac{\nu}{E}\operatorname{tr}\sigma^0 & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\sqrt{T-t}\,\frac{\partial\Theta^{\varepsilon}}{\partial x_{\alpha}} & \to \sqrt{T-t}\,\frac{\partial\theta^0}{\partial x_{\alpha}} & \text{strongly in } L_2(0,T;L_2(\Omega)),\\
\frac{\sqrt{T-t}}{\varepsilon}\,\frac{\partial\Theta^{\varepsilon}}{\partial x_3} & \to 0 & \text{strongly in } L_2(0,T;L_2(\Omega))
\end{array} \tag{62}$$

if and only if the following compatibility condition is satisfied by the initial conditions and the initial loadings:

$$\int_{\omega}\left[\left(\frac{\overline{(\rho v^0_{03})}^{\,2}}{\bar{\rho}} - \overline{(\rho|v^0_0|^2)}\right) + \left(\kappa\big(\theta^0(0)\big)^2 - \overline{\big(\beta\theta^{0^2}_0\big)}\right) - 2F^0_{\alpha}(0)\big(\mathbf{u}^0_{\alpha}(0) - u^0_{0\alpha}\big)\right.$$
$$\left. + \mathscr{B}_0\big(e(u^0(0)), e(u^0(0))\big) - \mathscr{B}\big(\mathbf{e}^0_0, \mathbf{e}^0_0\big) = 0, \tag{63}$$

where $\mathbf{F}^0_{\alpha}$ is given in (37), $\mathscr{B}$ was defined in Sec. 2, and $\mathscr{B}_0$ is defined as the following bilinear form on Y:

$$\mathscr{B}_0(A,B) = \int_{\Omega}\left(\frac{E}{1+\nu}A_{ij}B_{ij} + \frac{\nu E}{(1-\nu)(1+\nu)}(\operatorname{tr}A)(\operatorname{tr}B)\right)dx. \tag{64}$$

REMARK 9. If

$$v^0_{0\alpha} = 0, \tag{65}$$
$$v^0_{03} \text{ and } \theta^0_0 \text{ are independent of } x_3,$$

the compatibility condition (63) reduces to

$$\int_{\omega}\left[\left(\kappa\big(\theta^0(0)\big)^2 - \bar{\beta}\theta^{0^2}_0\right) - 2\mathbf{F}^0_{\alpha}(0)\big(\mathbf{u}^0_{\alpha}(0) - \mathbf{u}^0_{0\alpha}\big)\right]dy$$
$$+ \mathscr{B}_0\big(e(u^0(0)), e(u^0(0))\big) - \mathscr{B}\big(\mathbf{e}^0_0, \mathbf{e}^0_0\big) = 0. \tag{66}$$

If further

$$\mathbf{u}^0_0,\ \theta^0_0 \text{ satisfy (22) and (23) with } \mathbf{F}^0_{\alpha}(0) \text{ as loading,}$$
$$\mathbf{e}^0_{0\alpha 3} = 0, \qquad \mathbf{e}^0_{033} = \alpha\frac{1+\nu}{1-\nu}\theta^0_0 - \frac{\nu}{1-\nu}\operatorname{tr}e\big(u^0_0\big), \tag{67}$$

then

$$\mathbf{u}^0(0) = \mathbf{u}_0^0, \qquad \theta^0(0) = \theta_0^0,$$

and (66) is satisfied.

Under the hypotheses (65), (67) there is no change in the initial temperature increment field. We are led to state the following conjecture:

$$\text{$\mathbf{u}^0(0) = \mathbf{u}_0^0$ and $\theta^0(0) = \theta_0^0$ are necessary conditions for strong convergence when (65) holds true.} \tag{C}$$

All our attempts to find a counterexample to (C) have failed, and at the present time we do not have any clue about the validity of (C). Of course, it is much more interesting in our opinion to disprove (C) than to prove it.

REMARK 10. The analog of Remark 5 holds true after replacing weak by strong convergences everywhere in that remark.

Proof of Theorem 3. First, let us define the space $\mathscr{L}_2$ as

$$\mathscr{L}_2 = \Big\{\mathscr{G} = (\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4) | \mathscr{G}_1 \in L_2\big(0, T; [L_2(\Omega)]^3\big),$$
$$\mathscr{G}_2 \in Y, \qquad \mathscr{G}_3 \in L_2(0, T; L_2(\Omega)), \qquad \sqrt{T-t}\,\mathscr{G}_4 \in L_2\big(0, T; [L_2(\Omega)]^3\big)\Big\}.$$

Proving the convergences (62) reduces to proving the strong convergence of the field $\mathscr{G}^\varepsilon$ of $\mathscr{L}_2$ defined as

$$\mathscr{G}_1^\varepsilon = \big(\varepsilon \dot{U}_\alpha^\varepsilon, \dot{U}_3^\varepsilon\big),$$
$$\mathscr{G}_2^\varepsilon = \mathbf{e}\big(U^\varepsilon\big),$$
$$\mathscr{G}_3^\varepsilon = \Theta^\varepsilon,$$
$$\mathscr{G}_4^\varepsilon = \left(\frac{\partial \Theta^\varepsilon}{\partial x_\alpha}, \frac{1}{\varepsilon}\frac{\partial \Theta^\varepsilon}{\partial x_3}\right),$$

to the field $\mathscr{G}^0$ of $\mathscr{L}_2$ defined as

$$\mathscr{G}_1^0 = \big(0, \dot{u}_3^0\big),$$
$$\mathscr{G}_2^0 = \mathbf{e}^0 = \left(\begin{array}{c|c} e_{\alpha\beta}(u^0) & 0 \\ \hline 0 & \dfrac{\alpha(1+\nu)\theta^0 - \nu\,\mathrm{tr}\,e(u^0)}{1-\nu} \end{array}\right),$$
$$\mathscr{G}_3^0 = \theta^0,$$
$$\mathscr{G}_4^0 = \left(\frac{\partial \theta^0}{\partial x_\alpha}, 0\right).$$

Indeed, with the help of (8) and of Poincaré's and Korn's inequalities, all fields entering (62) are then found to converge.

For any $\mathscr{G}$ of $\mathscr{L}_2$, we define

$$|||\mathscr{G}||| = \int_0^T \bigg[\int_\Omega \rho|\mathscr{G}_1(t)|^2\,dx + \mathscr{B}(\mathscr{G}_2(t), \mathscr{G}_2(t)) + \int_\Omega \beta|\mathscr{G}_3(t)|^2\,dx + \int_0^t\int_\Omega |\mathscr{G}_4(s)|^2\,dx\,ds\bigg]\,dt,$$

where $\mathscr{B}$ was defined in Sec. 2. In view of the positivity properties of the coefficients, $|||\ |||$ defines a norm on $\mathscr{L}_2$ that is equivalent to the natural L_2-norm on $\mathscr{L}_2$.

Because $L_2(0, T; L_2(\Omega))$ is uniformly convex, and since a direct application of Theorem 2 shows that, as ε tends to zero, $\mathscr{G}^\varepsilon$ weakly converges in $\mathscr{L}_2$ to $\mathscr{G}^0$, the strong convergence of $\mathscr{G}^\varepsilon$ holds true if and only if $|||\mathscr{G}^\varepsilon|||$ converges to $|||\mathscr{G}^0|||$ when ε tends to zero.

Taking $\mathbf{e}(\dot{U}^\varepsilon(t))$, $\dot{U}^\varepsilon(t)$, and $\Theta^\varepsilon(t)$ as test functions in (13b), (14), and (15), adding together the resulting expressions, and integrating the result over the time interval $(0, t)$, then over the time interval $(0, T)$, yields the following expression for $|||\mathscr{G}^\varepsilon|||$:

$$
\begin{aligned}
|||\mathscr{G}^\varepsilon|||^2 = T\Bigg[& \int_\Omega \rho\left(\sum_{\alpha=1}^{2} |\varepsilon V^\varepsilon_{0\alpha}|^2 + |V^\varepsilon_{03}|^2\right) dx + \mathscr{B}(\mathbf{e}(U^\varepsilon_0), \mathbf{e}(U^\varepsilon_0)) \\
& + \int_\Omega \beta |\Theta^\varepsilon_0|^2\, dx - 2\int_\Omega \left(F^\varepsilon_\alpha(0) U^\varepsilon_{0\alpha} + \frac{1}{\varepsilon^2} F^\varepsilon_3(0) U^\varepsilon_{03}\right) dx \\
& - 2\int_{\Gamma^\pm} \left(\frac{1}{\varepsilon} G^{\pm\varepsilon}_\alpha(0) U^\varepsilon_{0\alpha} + \frac{1}{\varepsilon^3} G^{\pm\varepsilon}_3(0) U^\varepsilon_{03}\right) dy \Bigg] \\
& + 2\int_0^T \Bigg\{ \int_\Omega \left(F^\varepsilon_\alpha(t) U^\varepsilon_\alpha(t) + \frac{1}{\varepsilon^2} F^\varepsilon_3(t) U^\varepsilon_3(t)\right) dx \\
& + \int_{\Gamma^\pm} \left(\frac{1}{\varepsilon} G^{\pm\varepsilon}_\alpha(t) U^\varepsilon_\alpha(t) + \frac{1}{\varepsilon^3} G^{\pm\varepsilon}_3(t) U^\varepsilon_3(t)\right) dy \Bigg\}\, dt \\
& - 2\int_0^T \int_0^t \Bigg[\int_\Omega \left(\dot{F}^\varepsilon_\alpha(s) U^\varepsilon_\alpha(s) + \frac{1}{\varepsilon^2} \dot{F}^\varepsilon_3(s) U^\varepsilon_3(s)\right) dx \\
& + \int_{\Gamma^\pm} \left(\frac{1}{\varepsilon} \dot{G}^{\pm\varepsilon}_\alpha(s) U^\varepsilon_\alpha(s) + \frac{1}{\varepsilon^3} \dot{G}^{\pm\varepsilon}_3(s) U^\varepsilon_3(s)\right) dy \Bigg]\, ds\, dt.
\end{aligned}
\tag{68}
$$

The computation of $|||\mathscr{G}^0|||$ necessitates a detailed evaluation of the term $\mathscr{B}(\mathscr{G}^0_2(t), \mathscr{G}^0_2(t))$. Specifically, in view of the even character of all coefficients and of (36),

$$
\begin{aligned}
\mathscr{B}(\mathscr{G}^0_2(t), \mathscr{G}^0_2(t)) = \int_\omega dy \Bigg[& \Bigg\{ \overline{\left(\frac{E}{(1+\nu)}\right)}\, e_{\alpha\beta}(\mathbf{u}^0(t))\, e_{\alpha\beta}(\mathbf{u}^0(t)) \\
& + \overline{\left(\frac{\nu E}{(1-\nu)(1+\nu)}\right)}\, \operatorname{tr} e(\mathbf{u}^0(t))^2 + \overline{\left(\frac{E(1+\nu)\alpha^2}{(1-\nu)(1-2\nu)}\right)}\, (\theta^0(t))^2 \Bigg\} \\
& + \Bigg\{ \overline{\left(\frac{E x_3^2}{1+\nu}\right)}\, \frac{\partial^2 u^0_3}{\partial x_\alpha \partial x_\beta}(t) \frac{\partial^2 u^0_3}{\partial x_\alpha \partial x_\beta}(t) + \overline{\left(\frac{\nu E x_3^2}{(1-\nu)(1+\nu)}\right)}\, (\Delta u^0_3(t))^2 \Bigg\} \Bigg].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|||\mathscr{G}^0||| = \int_0^T \int_\omega \Big\{ & \bar{\rho}\left|\dot{u}_3^0(t)\right|^2 + \overline{\left(\frac{Ex_3^2}{1+\nu}\right)} \frac{\partial^2 u_3^0}{\partial x_\alpha \partial x_\beta}(t) \frac{\partial^2 u_3^0}{\partial x_\alpha \partial x_\beta}(t) \\
& + \overline{\left(\frac{\nu E x_3^2}{(1-\nu)(1+\nu)}\right)} \left(\Delta u_3^0(t)\right)^2 \Big\} \, dy \, dt \\
+ \int_0^T \int_\omega \Big\{ & \overline{\left(\frac{E}{1+\nu}\right)} e_{\alpha\beta}(\mathbf{u}^0(t)) e_{\alpha\beta}(\mathbf{u}^0(t)) \\
& + \overline{\left(\frac{\nu E}{(1-\nu)(1+\nu)}\right)} \operatorname{tr} e(\mathbf{u}^0(t))^2 + \kappa\left|\theta^0(t)\right|^2 \\
& + \int_0^t \bar{k} \frac{\partial \theta^0}{\partial x_\alpha}(s) \frac{\partial \theta^0}{\partial x_\alpha}(s) \, ds \Big\} \, dy \, dt.
\end{aligned}
$$

Multiplication of (19), (23), and (24) by $\dot{u}_3^0(t)$, $\dot{\mathbf{u}}^0(t)$, and $\theta^0(t)$, respectively, appropriate integration by parts, addition of the resulting expressions, and integration over the time interval $(0, t)$, then over the time interval $(0, T)$, lead to the following expression for $|||\mathscr{G}^0|||$:

$$
\begin{aligned}
|||\mathscr{G}^0|||^2 = T \Bigg[& \int_\omega \frac{\overline{\left(\rho v_{03}^0\right)}^2}{\bar{\rho}} \, dy + \mathscr{B}_0\left(e\left(u^0(0)\right), e\left(u^0(0)\right)\right) \\
& + \int_\omega \kappa\left(\theta^0(0)\right)^2 dy - 2\int_\omega \left(\mathbf{F}_\alpha^0(0)\mathbf{u}_\alpha(0) + F_3^0(0) u_{03}^0\right) dy \Bigg] \\
& + 2\int_0^T \int_\omega \left(\mathbf{F}_\alpha^0(t)\mathbf{u}_\alpha^0(t) + F_3^0(t) u_3^0(t)\right) dy \, dt \\
& - 2\int_0^T \int_0^t \int_\omega \left(\dot{\mathbf{F}}_\alpha^0(s)\mathbf{u}_\alpha^0(s) + \dot{F}_3^0(s) u_3^0(s)\right) dy \, ds \, dt,
\end{aligned}
\tag{69}
$$

where $\mathbf{F}_\alpha^0$ and F_3^0 are given in (37) and $\mathscr{B}_0$ is defined by (64).

The study of the convergence of $|||\mathscr{G}^\varepsilon|||$ to $|||\mathscr{G}^0|||$ reduces to the study of the convergence of the right-hand side of equality (68) to the right-hand side of equality (69).

Because all convergences in (29) are assumed to be strong convergences, and U^ε converges weakly in $L_2(0, T; \mathbf{H})$ to u^0, with u^0 in $\mathscr{C}^0([0, T]; V_{KL})$,

$$
\begin{aligned}
\lim_{\varepsilon \to 0} |||\mathscr{G}^\varepsilon|||^2 = T \Bigg[& \int_\omega \overline{\left(\rho|v_0^0|^2\right)} \, dy + \mathscr{B}\left(\mathbf{e}_0^0, \mathbf{e}_0^0\right) \\
& + \int_\omega \Big\{ \overline{\left(\beta\theta_0^{0^2}\right)} - 2\Big(\overline{\left(f_\alpha^0(0) u_{0\alpha}^0\right)} + g_\alpha^{\pm 0}(0) u_{0\alpha}^0(x_3 = \pm 1) \\
& \qquad + \overline{f_3^0}(0) u_{03}^0 + \left(g_3^{\pm 0}(0) + g_3^{-0}(0)\right) u_{03}^0 \Big) \Big\} \, dy \Bigg] \\
& + 2\int_0^T \int_\omega \left(\mathbf{F}_\alpha^0(t)\mathbf{u}_\alpha^0(t) + F_3^0(t) u_3^0(t)\right) dy \, dt \\
& - 2\int_0^T \int_0^t \int_\omega \left(\dot{\mathbf{F}}_\alpha^0(s)\mathbf{u}_\alpha^0(s) + \dot{F}_3^0(s) u_3^0(s)\right) dy \, ds \, dt.
\end{aligned}
\tag{70}
$$

In view of (69) and (70),

$$\lim_{\varepsilon \to 0} |||\mathscr{G}^\varepsilon|||^2 = |||\mathscr{G}^0|||^2$$

if and only if

$$\begin{aligned}\int_\omega \Bigg[& \left(\frac{\overline{(\rho v_{03}^0)}^2}{\bar{\rho}} - \overline{(\rho |v_0^0|^2)} \right) + \left(\kappa(\theta^0(0))^2 - \overline{(\beta \theta_0^{0^2})} \right) \\ & -2\Big(\mathbf{F}_\alpha^0(0)\mathbf{u}_\alpha^0(0) - \Big(\overline{(f_\alpha^0(0) u_{0\alpha}^0)} + g_\alpha^{\pm 0}(0) u_{0\alpha}^0(x_3 = \pm 1) \Big)\Big) \\ & -2\left(\overline{\left(x_3 \frac{\partial}{\partial x_\alpha} f_\alpha^0(0) \right)} + \frac{\partial}{\partial x_\alpha}(g_\alpha^{+0} - g_\alpha^{-0})(0) \right) u_{03}^0 \Bigg] dy \\ & + \mathscr{B}_0\big(e(u^0(0), e(u^0(0)))\big) - \mathscr{B}(\mathbf{e}_0^0, \mathbf{e}_0^0) = 0. \end{aligned} \tag{71}$$

Since u_0^0 lies in V_{KL} (cf. Remark 4), equality (71) is precisely (63).

Concluding remarks. The results obtained in the present study raise two main questions. Can strong convergence take place in the presence of a change of initial data? Does the change of initial condition in the temperature field result from an improper choice of the field variable?

The first question has been commented on in Remark 9. It remains open at the present time. A partial answer to the second question may be given by choosing the entropy field

$$S^\varepsilon = \beta \Theta^\varepsilon + \frac{E\alpha}{1 - 2\nu} \operatorname{tr} \mathbf{e}(U^\varepsilon)$$

as a natural variable in Eq. (10) in place of the temperature field Θ^ε (see Francfort [F] for similiar considerations in the context of homogenization). The field S^ε is easily shown not to undergo any change in initial condition during the asymptotic process. Unfortunately, complete removal of the temperature field in Eqs. (8)–(12) leads to an evolution system with third-order space derivatives whose analysis seems difficult through the classical methods used in the present study.

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