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Asymptotic Waves for a Non Linear System

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Abstract

The term of wave is used in a general way to name any solution of a hyperbolic problem $L(u)=0$, and the method used is a generalization of a geometrical process of optics (W.K.B.) consisting in seeking the solution in the form $u=e^{i\omega\varphi}\sum_{j\geq 0}(i\omega)^{-j}g^j$, in term of a real parameter ω *(frequency)*, unknown functions φ (phase) and g^j (attenuating factors). We thus build such an asymptotic solution in the cases of simple and multiple characteristics, provided that the Cauchy data has an asymptotic expansion, and observe, like other authors, that the phase function is a solution of the classical eikonal equation, and the terms of the series are determined by a recursive system of differential equations. We then deduce a condition for genuine non linearity of the problem which generalizes that of Lax [7] and John [3], and highlight from it the singular behaviour of the first term of the formal solution when the characteristics are distinct. We note that, for sufficiently small frequency, the asymptotic solution is almost global.

Keywords : hyperbolic, asymptotic series, non linearity, blow-up

I. Introduction

We are interested, in this article, with a system of partial derivative equations, hyperbolic and nonlinear, and propose the construction of an asymptotic solution in the direction given by G.K.L [5], G.Boillat [4], C.Bruhat [8]. The process consists in seeking a solution in the form $u \approx \sum_{k=0}^{\infty} \omega^{-k} u_k(x, \omega\varphi)$,

(x a point of R^{n+1}, ω the frequency is a real parameter, φ the phase is a scalar function) and to observe, after replacement in the system, the phase and the term series u_k . Method known as "W.K.B" was used by Lax [6], then Ludwig [2] and G.K.L [5] for the linear systems, generalized then with the nonlinear systems by G.Boillat[4], Y.C.Bruhat [8], and exploited since per many authors [1],,[9],… It is the form suggested by Lax [6] that we use in this article.

II. Statement of the problem

Let the system :

$$\begin{cases} L(u) = A^{\nu}(t,x,u)\partial_{\nu} u = 0 \\ u(0,x) = f(x) \end{cases} \qquad \text{summation in } \nu \qquad (1)$$

Where A^{ν} is a $(k \times k)$ matrix, $(t,x) \in R^{+} \times R^{n}$, $u(t,x) = (u_1,...,u_k)$ the unknown function, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_{\mu} = \frac{\partial}{\partial x_{\mu}}$, $\mu = 1,...,n.$

Under the conditions of regularity of the coefficients and initial data with compact support which ensure the existence and the unicity of a regular local solution u^0 in $[0,T] \times R^n$, one proposes to seek in the polycylinder $\left|u-u^0\right| \le \varepsilon$ the solution of the problem (1) with oscillatory data $f(x) = f^0(x) + \frac{e^{i\omega\psi(x)}}{i\omega} f^1(x)$ (2)

in the form:
$$u(t,x) = u^0(t,x) + \sum_{j=1}^{\infty} \frac{e^{i\omega\varphi(t,x)}}{(i\omega)^j} g^j(t,x) \qquad (3)$$

in term of a real parameter ω (frequency), of a function $\varphi(t,x)$ with real values (phase) which will be to determine as well as the functions $g^j(t,x)$.

By identification, we have :

$$\varphi(0,x) = \psi(x),$$
$$g^1(0,x) = f^1(x), \qquad g^j(0,x) = 0, \forall j \ge 2 \qquad (4)$$

Let us note ∇ the vector $\left(\frac{\partial}{\partial u_1},...,\frac{\partial}{\partial u_k}\right)$ so that :

$$A^{\nu}(u) = A_0^{\nu} + \left(\nabla A^{\nu}\right)_0 \times \left(u - u_0\right) + 0(\varepsilon) \qquad (5)$$

where index zero is allotted to any function of u when $u = u^0$.

While reporting (3) and (5) in (1), one obtains:

$$L(u) = e^{i\omega\varphi} \sum_{j=0}^{\infty} \frac{F_j}{(i\omega)^j} = 0 \quad \text{which involves} \quad F_j = 0, \forall j = 0,1,\dots,\infty. \tag{6}$$

As follows :

$$F_0 = \left(A_0^\nu \varphi_{x_\nu}\right) g^1 = 0 \tag{7}$$

$$\begin{aligned} F_1 &= \left(A_0^\nu \varphi_{x_\nu}\right) g^2 + A_0^\nu \partial_\nu g^1 + \left(\nabla A_0^\nu \times g^1\right) \partial_\nu u^0 \\ &\quad + e^{i\omega\varphi} \left(\nabla A_0^\nu \varphi_{x_\nu} \times g^1\right) g^1 = 0 \end{aligned} \tag{8}$$

$$F_j = \left(A_0^\nu \varphi_{x_\nu}\right) g^{j+1} + G\left(g^j\right) + H\left(g^{j-1}\right) = 0,\ j \ge 2 \tag{9}$$

where :

$$\begin{aligned} G\left(g^j\right) &= A_0^\nu \varphi_{x_\nu} g^j + \left(\nabla A_0^\nu \times g^j\right) \partial_\nu u^0 \\ &\quad + e^{i\omega\varphi} \left[\left(\nabla A_0^\nu \varphi_{x_\nu} g^j\right) g^1 + \left(\nabla A_0^\nu \varphi_{x_\nu} g^1\right) g^1\right] \end{aligned}$$

And :

$$\begin{aligned} H\left(g^{j-1}\right) &= \sum\nolimits_{l=1}^{j-1} e^{i\omega\varphi} \left(\nabla A_0^\nu g^{j-l}\right) \partial_\nu g^l \\ &\quad + \sum\nolimits_{l=1}^{j-2} e^{i\omega\varphi} \left(\nabla A_0^\nu g^{j-l}\right) g^{l+1} \end{aligned}.$$

III. Détermination of the phase φ

According to (7), g^1 is a right eigenvector corresponding to the eigenvalue 0, for the approached characteristic matrix $A = \left(A_0^\nu \varphi_{x_\nu}\right)$ and $d = \det A = 0$; in other words, the sets $\left\{(t,x) \in R^{n+1} / \varphi(t,x) = \text{constante}\right\}$ form a family of characteristic surfaces. Let $\varphi(t,x) = \text{constante}$ to be such a surface, l the displacement of a point of surface, λ its propagation velocity ; one has $\lambda = \frac{dl}{dt}$ and $dx_i = dl \cos(N, x_i) = N_i dl$ where $N_i = \frac{\varphi_{x_i}}{|D_x\varphi|}$, $N = (N_1,\dots,N_n)$ being the unit normal. It is shown that : $d = |D_x\varphi| \det\left(A^i N_i - \lambda A^0\right)$. One standardizes the problem by posing $A^0 = I$ (unit) because A^0 is regular in the neighbourhood of $t = 0, u = u^0$. It is seen that d is a polynomial of k degree in λ. The hyperbolicity induces the existence of two bases of eigenvectors $\left\{R^i\right\}$ and $\left\{L^i\right\}$, respectively rights and lefts, of the matrix $A^i N_i$ and r real eigenvalues, $\lambda_i\,(r \le k)$, with respective multiplicities m_i. While posing $\delta^i = \varphi_t + |D_x\varphi| \lambda^i(t,x,u)$, one shows that: $AR^i = \delta^{(i)} R^{(i)}$, $L^i A = \delta^{(i)} L^{(i)}$ and $d = \prod_{i=1}^{r} \left(\delta^i\right)^{m_i}$;

(one does not summon compared to the index between brackets).

The equation $d = 0$ is thus equivalent to r P.D.E for the function φ :

$$\begin{cases} \varphi_t + |D_x\varphi|\lambda^i\left(t, x, u^0\right) = 0 \\ \varphi(0, y) = \psi(y) \end{cases} \qquad (10)$$

which admit each one a single local solution (the traditional eikonal equation). There are thus r characteristic families of surfaces φ^i , and each one of them defines an asymptotic wave for u. While seeking then u in the form:

$u = u_0 + \sum_{j=1}^{\infty}\sum_{\alpha=1}^{r}\frac{e^{i\omega\varphi^\alpha}}{(i\omega)^j}g^{\alpha j}$, that one reports in $L(u)$, one obtains finally:

$L(u) = \sum_{\alpha=1}^{r}e^{i\omega\varphi^\alpha}\sum_{j=0}^{\infty}\frac{F_j^\alpha}{(i\omega)^j} = 0$, where, while by vanishing F_j^α for each α and j, one finds the equations (7), (8), (9) where φ is replaced by φ^α and g^j by $g^{\alpha j}$.

VI. Calculation of the attenuating factors g^j

Let us write the system bicaracteristic relating to the problem (10):

$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx\mu}{d\tau} = \dfrac{\varphi_{x_\mu}^{(i)}}{\left|D_x\varphi^i\right|}\lambda^{(i)}, x_\mu(0) = y_\mu, \mu = 1,\ldots,n \\ \dfrac{dp_0}{d\tau} = -\left|D_x\varphi^{(i)}\right|\partial_0\lambda^{(i)}, \\ p_0(0) = p_0^i(y) = -|p'|\lambda^i\left(0, y, f^0(y)\right) \\ \dfrac{dp_\mu}{d\tau} = -\left|D_x\varphi^{(i)}\right|\partial_\mu\lambda^{(i)}, p_\mu(0) = \psi_{x_\mu}(y), (p_0, p') = D\varphi \end{cases} \qquad (11)$$

The determination of the factors g^j returns to the resolution of the systems (7), (8), (9) along the bicaracteristics solution of (11). The equation (7) shows that g^1 belongs to subspace generated by the right eigenvectors associated to the null eigenvalue δ_0^i.

Lemma 1: If L^i and R^j are respectively the eigenvectors left and right of A_0 associated with the same eigenvalue $\delta = 0$, then, along the bicaracteristics solution of (11), one has the relation:

$$L^i A_0^\mu R^j = L^i R^j \frac{dx_\mu}{dt}, \mu = 1,\ldots,n \qquad (12).$$

Lemma 2: Standardization :

$L^i R^j = \delta_{ij}$ (symbol of Kronecker)

−**Simple characteristics**: in this case, A has k real eigenvalues distinct $\delta^1,\ldots,\delta^k$ and two bases of eigenvectors $\{L^i\}_1^K$ and $\{R^j\}_1^k$. Let us fix the null eigenvalue $\delta = 0$, with which are associated the eigenvectors L and R. From (7), one deduces that: $g^1 = \sigma_1 R$ (13)

where $\sigma_1(t,x)$ is a function with real values on R^{n+1} which it is necessary to calculate. While reporting (13) in LF_1, (8), using lemmas 1,2 and while placing oneself on a characteristic $(t, x(t,y))$, one obtains the differential equation of Bernoulli:

$$\frac{d\sigma_1}{dt} + \Gamma\sigma_1 + \Theta(\sigma_1)^2 = 0 \qquad (14)$$

where

$$\begin{cases} \Gamma(t,y) = LA_0^\nu \partial_\nu R + L\left(\nabla A^\nu \times R\right)_0 \partial_\nu u^0 \\ \Theta(t,y) = e^{i\omega\varphi} L\left(\nabla A^\nu \varphi_{x_\nu} \times R\right)_0 R \end{cases} \qquad (15)$$

According to (4) and the expression of $f^1(y)$ in the base $\{R^j(0,y)\}$, one has:

$$\sigma_1(0,y) = L(0,y) f^1(y) \qquad (16)$$

The problem (14), (16) defines in a single way $\sigma_1(t,y)$. By reporting its value in the system (8), this one becomes an algebraic linear system compared to g^2 and thus has a particular solution h_2 modulo R, and so $g^2 = h_2 + \sigma_2 R$ where σ_2 is a scalar function which one will determine by the relation $LF_2 = 0$. Let us reason by induction on $j \geq 2$. Let us suppose known the factors $g^1, ..., g^{j-1}$ such as : $g^{j-1} = h_{j-1} + \sigma_{j-1} R$.

The equality (9) : $F_{j-1} = A_0 g^j + G(g^{j-1}) + H(g^{j-2}) = 0$ is a linear system in g^j which the resolution gives us a particular solution h_j modulo R, thus $g^j = h_j + \sigma_j R$ where the unknown function σ_j will be then solution of the equation $LF_j = 0$, which is: $\dfrac{d\sigma_j}{dt} + LG(R)\sigma_j = -LG(h_j) - LH(g^{j-1})$.

It is a linear differential equation for σ_j and the initial value $\sigma_j(0,y) = -L(0,y)h_j(0,y)$ is obtained by expressing the vector h_j in the base $\{R^i\}$. Thus all the factors can be calculated successively.

Multiple characteristics: we treat the case of a multiplicity m with an eigen subspace of dimension m. Let the bases of the eigenvectors $\{R^\beta\}$ and $\{L^\varepsilon\}$, $\beta, \varepsilon = 1,...,m$, respectively on the right and on the left of the matrix A_0, corresponding to the null eigenvalue δ_0. From $F_0 = A_0 g^1 = 0$ one deduces that g^1 is a linear combination of the vectors R^β, so : $g^1 = \sigma_1^\beta R^\beta$ (17)

that one reports in $L^\varepsilon F_1 = 0$, by using lemma 1, to obtain:

$$(L^{\varepsilon} R^{\beta})\frac{d\sigma_1^{\beta}}{dt} + \Gamma_{\varepsilon}^{\beta}\sigma_1^{\beta} + \Theta_{\varepsilon}^{\beta\gamma}\sigma_1^{\beta}\sigma_1^{\gamma} = 0 \qquad (18),$$

with $\beta, \varepsilon, \gamma$ varying from 1 to m ; and

$$\Gamma_{\varepsilon}^{\beta} = L^{\varepsilon} A_0^{\nu} \partial_{\nu} R^{\beta} + L^{\varepsilon}\left(\nabla A^{\nu} \times R^{\beta}\right)_0 \partial_{\nu} u^0,$$

$$\Theta_{\varepsilon}^{\beta\gamma} = e^{i\omega\varphi} L^{\varepsilon}\left(A \times R^{\beta}\right)_0 R^{\gamma} = e^{i\omega\varphi}\left|D_x \varphi\right|\left(A\lambda R^{\beta}\right)_0 L^{\varepsilon} R^{\gamma}.$$

Let us observe that (18) is a system of m differential equations to the m unknown factors $\sigma_1^1,...,\sigma_1^m$ whose initial values are solutions of the system of Cramer:

$$f^1(y) = \sum_{\alpha=1}^{r} g^{1\alpha}(0,y) = \sum_{\alpha=1}^{r} \sigma_1^{\alpha\beta}(0,y) R^{\alpha\beta}(0,y) \qquad (19)$$

where one points out that α indicates the number of roots of the characteristic polynomial, each one of them of multiplicity m_{α}, $\sum_{\alpha=1}^{r} m_{\alpha} = k$, so that the system $\left\{R^{\alpha\beta}\right\}$ is complete. Thus, (18), (19) determine in a single way the m functions $\sigma_1^1,...,\sigma_1^m$ and consequently the factor g^1. The relation $F_1 = 0$ considered as a linear system in g^2 gives us a particular solution h_2 modulo an eigenvector R^{β}, and so : $g^2 = h_2 + \sigma_2^{\beta} R^{\beta}$. Let us reason by induction on $j \geq 2$. Let us suppose known the factors $g^1,...,g^{j-1}$ such as: $g^{j-1} = h_{j-1} + \sigma_{j-1}^{\beta} R^{\beta}$; then the linear system in g^j : $F_{j-1} = A_0 g^j + G\left(g^{j-1}\right) + H\left(g^{j-2}\right) = 0$ have a particular solution h_j modulo R^{β}, then $g^j = h_j + \sigma_j^{\beta} R^{\beta}$, that one reports in $L^{\varepsilon} F_j = 0$, by using lemma 1, to lead finally to the system of m differential equations to the m unknown factors σ_j^{β} : $\left(L^{\varepsilon} R^{\beta}\right)\frac{d\sigma_j^{\beta}}{dt} + L^{\varepsilon} G\left(R^{\beta}\right)\sigma_j^{\beta} = -L^{\varepsilon} G\left(h_j\right) - L^{\varepsilon} H\left(g^{j-1}\right)$.

Initial values $\sigma_j^{\beta}(0,y)$ can be given by the system $\sum_{\alpha=1}^{r} \sigma_j^{\beta}(0,y) R^{\alpha\beta}(0,y) = -\sum_{\alpha=1}^{r} h_j^{\alpha}(0,y)$ whose determinant $\left[R^{\alpha\beta}\right]$ does not vanish. Thus the process is complete.

V. PRINCIPAL RESULTS

V.1. Genuine non linearity

In the linear case, A does not depend on u, $\Theta = 0$, and the equation (14) is reduced to that obtained by Lax [6] and Ludwig [2]. It is obvious that for a truly nonlinear case, Θ does not vanish and it is not enough which A depends on u. The condition of nonlinearity required by Lax [7] when $n = 1, k = 2$, and by F. John [3] when $n = 1$, k unspecified, is : $(\nabla\lambda \times R)_0 \neq 0$.

- Let us show that this condition is the same one here:

$$\Theta = e^{i\omega\varphi} L\left(\nabla A^{\nu}\varphi_{x_\nu} \times R\right)_0 R = e^{i\omega\varphi} L\left(\nabla A \times R\right)_0 R$$

$$\left[\nabla(LA)_0 \times R\right]R = (\nabla L \times R)_0 A_0 R + L(\nabla A \times R)_0 R$$

$$= L(\nabla A \times R)_0 R = e^{i\omega\varphi}\Theta$$

Because $A_0 R = 0$.

In addition:

$$\left[\nabla(LA)_0 \times R\right]R = \left[\nabla(\delta L)_0 \times R\right]R$$

$$= (\nabla\delta \times R)_0 LR + \delta_0 (\nabla L \times R)_0 R = (\nabla\lambda \times R)_0$$

Because $\delta_0 = 0$ and $LR = 1$; and so $\Theta = e^{i\omega\varphi}(\nabla\delta \times R)_0 = e^{i\omega\varphi}|D_x\varphi|(\nabla\delta \times R)_0$

and consequently

$$\underline{\Theta \neq 0 \Leftrightarrow (\nabla\lambda \times R)_0 \neq 0}$$

V.2. Blow up of the solution

It is known that, in the case of this system, the solution exists only in the neighbourhood of the initial data.

- Let us show that one observes the same phenomenon here.

The first term of the asymptotic development of u being $g^1 = \sigma_1 R$ where σ_1 checks : (14) $\frac{d\sigma_1}{dt} + \Gamma(t,y)\sigma_1 + \Theta(t,y)(\sigma_1)^2 = 0$; (16) $\sigma_1(0,y) = L(y) f^1(y)$

along the characteristics $(t, x(t,y))$ where $L(y) = L\left(0, y, f^0(y), D\psi(y)\right)$.

One poses $h(t,y) = \int_0^t \Gamma(s,y)\,ds$ and $z(t,y) = e^{h+i\omega\varphi}\sigma_1(t,y)$.

Let us report the value of σ_1 drawn from (20) in (14), (16), the problem becomes:

$$\begin{cases} \dfrac{dz}{dt} = -e^{-h}|D_x\varphi|(\nabla\lambda \times R)_0 z^2 = a(t,y) z^2 \\ z(0,y) = e^{i\omega\psi(y)} L(y) f^1(y) \end{cases} \qquad (21)$$

where $a(t,y)$ does not vanish since $(\nabla\lambda \times R)_0 \neq 0$. Like $y \in \operatorname{supp} f = B$ compact in R^n , $\exists$ a constant M such as $0 \langle M \leq |a(t,y)|, \forall (t,y) \in [O,T] \times B$.

Let us consider the characteristics C_{2N} and C_{2N+1} with respective initial phases $\psi(y) = \frac{2N\pi}{\omega}$ and $\psi(y) = \frac{(2N+1)\pi}{\omega}, N \in Z$ on which $z(t,y)$ is real and one have:

$$z(t,y) = \frac{L(y) f^1(y)}{1 - L(y) f^1(y) \int_0^t a(s,y)\,ds} \text{ on } C_{2N}\text{, and}$$

$$z(t,y) = \frac{-L(y) f^1(y)}{1 + L(y) f^1(y) \int_0^t a(s,y)\,ds} \text{ on } C_{2N+1}$$

Let $B_1 = \operatorname{supp} f^1$ and let us consider the two parts

$B^{+}=\left\{y\in B_1 / L(y)\, f^1(y)\geq m_1\rangle 0\right\}$ and
$B^{-}=\left\{y\in B_1 / L(y)\, f^1(y)\leq -m_2\langle 0\right\}$.

Let us notice that the continuous functions $z_0(t,y)=\dfrac{m_1}{1-m_1Mt}$ and $z_1(t,y)=\dfrac{m_2}{1-m_2Mt}$ for $t\geq 0$, cannot exist respectively for $t\geq (m_1M)^{-1}$ and $t\geq (m_2M)^{-1}$.

♦ If $(\nabla\lambda\times R)_0\langle 0$, then $a(t,y)\geq M\rangle 0$.

– On the characteristics C_{2N} issued from $(0,y)$ where $y\in B^{+}$, a continues solution $z(t,y)$ of (21) cannot exist for $t\geq (m_1M)^{-1}$, because $z(t,y)\geq z_0(t,y)$.

– On C_{2N+1} issued from $(0,y)$ where $y\in B^{-}$, one have $z(t,y)\geq z_1(t,y)$, then $T\prec (m_2M)^{-1}$.

♦ If $(\nabla\lambda\times R)_0\succ 0$, then $a(t,y)\leq -M\prec 0$.

– On C_{2N} issued from $y\in B^{-}$, one have : $-z(t,y)\geq z_1(t,y)\rangle 0$, then $T\prec (m_2M)^{-1}$.

– On C_{2N+1} issued from $y\in B^{+}$, one have : $-z(t,y)\geq z_0(t,y)\rangle 0$, then $T\prec (m_1M)^{-1}$.

In all the cases, a regular solution of (21) exists only for $\underline{0\prec t\leq T_{\max}=(mM)^{-1}}$

With $m=\min(m_1,m_2)$.

Note: for a sufficiently small frequency ω, y in a compact set, $\psi(y)$ may not take the values $\dfrac{N\pi}{\omega}$, N integer, the solution $z(t,y)$ of (21) with complex values, remains limited, and consequently this last proposal does not exclude the existence of global solutions.

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