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Asymptotic zero distribution of hypergeometric polynomials

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We show that the zeros of the hypergeometric polynomials $F(-n, kn+1; kn+2; z)$, $k, n \in \mathbb{N}$, cluster on the loop of the lemniscate $\{z\colon\ |z^k(1-z)| = k^k/(k+1)^{k+1}, \mathrm{Re}(z) > k/(k+1)\}$ as $n \to \infty$. We also state the equations of the curves on which the zeros of $F(-n, kn+1; (k+\ell)n+2; z)$, $k, \ell, n \in \mathbb{N}$, lie asymptotically as $n \to \infty$. Auxiliary results for the asymptotic zero distribution of other functions related to hypergeometric polynomials are proved, including Jacobi polynomials with varying parameters and associated Legendre functions. Graphical evidence is provided using *Mathematica.*

Keywords: hypergeometric polynomials, asymptotic zero distribution

1. Introduction

The Gauss hypergeometric function is defined by

$$F(a, b; c; z) = 1 + \sum_{m=1}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \quad |z| < 1,$$

where

$$(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1) = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}, \quad \alpha \in \mathbb{C},$$

is Pochhammer's symbol. If, for instance, the parameter a is equal to a negative integer, say $a = -n$, the series terminates and reduces to a polynomial of degree n. Very little is known about the location of the zeros of hypergeometric polynomials except in cases where they are linked to classical orthogonal polynomials. For further discussion on this connection, see, for instance, [1, p. 561].

The purpose of this paper is to determine the equations of the critical curves on which the zeros of some classes of hypergeometric polynomials of degree n cluster as $n \to \infty$ and to provide graphical evidence for the validity of our results using *Mathematica.* Our main result concerns the asymptotic zero distribution of $F(-n, kn+1; kn+2; z)$, $k, n \in \mathbb{N}$, as n tends to infinity. We also state an asymptotic result for the zeros of the more general class $F(-n, kn+1; (k+\ell)n+2; z)$, $k, \ell, n \in \mathbb{N}$, as $n \to \infty$, without giving a detailed proof. Auxiliary results exploit the connection between hypergeometric polynomials and associated functions. In particular, we find the asymptotic zero distribution of a class of Jacobi polynomials with varying parameters, and of associated Legendre functions.

2. Critical curves

Our analysis rests on the Euler integral representation (see, for instance, [5, p. 47])

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t, \tag{1}$$

$\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $|z| < 1$, together with the following theorem of Borwein and Chen (cf. [2, theorem 5.1]).

Theorem A. Let

$$I_n(z) = \int_0^1 \big[Q_z(t)\big]^n\,\mathrm{d}t,$$

where $n = 1, 2, \ldots$, and $Q_z(t) = t^k f_z(t)$ is a polynomial in t and z with exactly one nontrivial critical point. Here, k is a positive integer. Let

$$t^\star = t^\star(z)$$

be the nontrivial zero of $(\mathrm{d}/\mathrm{d}t)Q_z(t)$. Then, as $n \to \infty$, the zeros of $I_n(z)$ will cluster on the critical curve

$$\big\{z\colon\ \big|Q_z\big(t^\star(z)\big)\big| = \big|Q_z(1)\big|\big\}. \tag{2}$$

Moreover, the function $\{I_n(z)\}^{1/n}$ converges either to $Q_z\big(t^\star(z)\big)$ or to $Q_z(1)$ uniformly on compact subsets of each region in the complex plane bounded by the critical curve (2). The zeros of $I_n(z)$ will cluster on those sections of (2) that form a boundary between regions R and S, where $\{I_n(z)\}^{1/n}$ converges to $Q_z\big(t^\star(z)\big)$ on R and to $Q_z(1)$ on S.

3. Main results

Theorem 1. Let k and $n \in \mathbb{N}$. Then, as $n \to \infty$, the zeros of $F(-n, kn+1; kn+2; z)$ cluster on the loop of the lemniscate

$$\left\{z\colon\ \big|z^k(1-z)\big| = \frac{k^k}{(k+1)^{k+1}},\ \mathrm{Re}(z) > \frac{k}{k+1}\right\}. \tag{3}$$

Proof. From the integral representation (1) with $a = -n,\ b = kn + 1,\ c = kn + 2$, we have

$$F(-n, kn+1; kn+2; z) = (kn+1)\int_0^1 \left[t^k(1-zt)\right]^n \mathrm{d}t = (kn+1)\int_0^1 \left[Q_z(t)\right]^n \mathrm{d}t,$$

where $Q_z(t) = t^k(1-zt)$ is a polynomial in t and z. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_z(t) = t^{k-1}\bigl(k - (k+1)zt\bigr),$$

which has the nontrivial zero $t^\star = t^\star(z) = k/((k+1)z)$. It follows immediately from theorem A that the critical curve (2) is given by

$$\left\{z\colon\ \left|z^k(1-z)\right| = \frac{k^k}{(k+1)^{k+1}}\right\}.$$

To ascertain whether $\{\int_0^1 \left[Q_z(t)\right]^n \mathrm{d}t\}^{1/n}$ converges to $Q_z(t^\star(z))$ or to $Q_z(1)$ in the different regions bounded by the lemniscate (3), we observe that $t^\star = t^\star(x) = k/((k+1)x) \in [0,1]$ if and only if $x \geqslant k/(k+1)$. Also, it is straightforward to check that $|Q_x(t^\star(x))| > |Q_x(1)|$ only when $x \in A(x)$, the segment of the positive real axis shown in figure 1. It follows by a saddle point argument (cf. [2, theorem 5.2]) that $\{\int_0^1 [Q_z(t)]^n \,\mathrm{d}t\}^{1/n}$ converges uniformly on compact subsets to $Q_z(1)$ on R_1 and R_2, and to $Q_z(t^\star(z))$ on R_3. We deduce that the zeros of $F(-n, kn+1; kn+2; z)$ cluster on the loop of the lemniscate B shown in figure 1. This completes the proof of theorem 1. □

Figure 2 shows the zeros of $F(-n, kn+1; kn+2; z)$ and the graphs of the lemniscates (3) for various values of $k,\ n \in \mathbb{N}$.

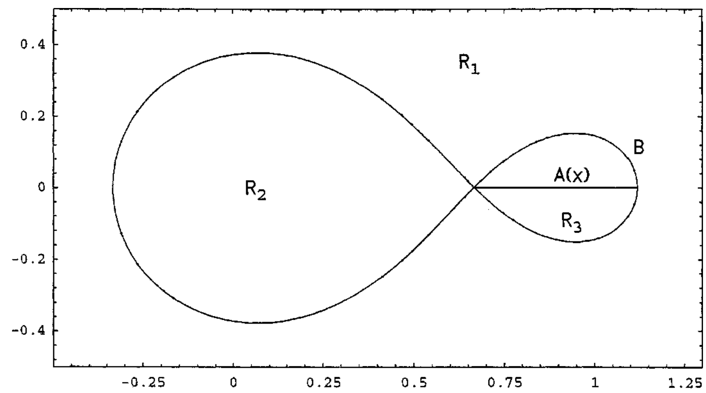


Figure 1. The asymptotic lemniscate.

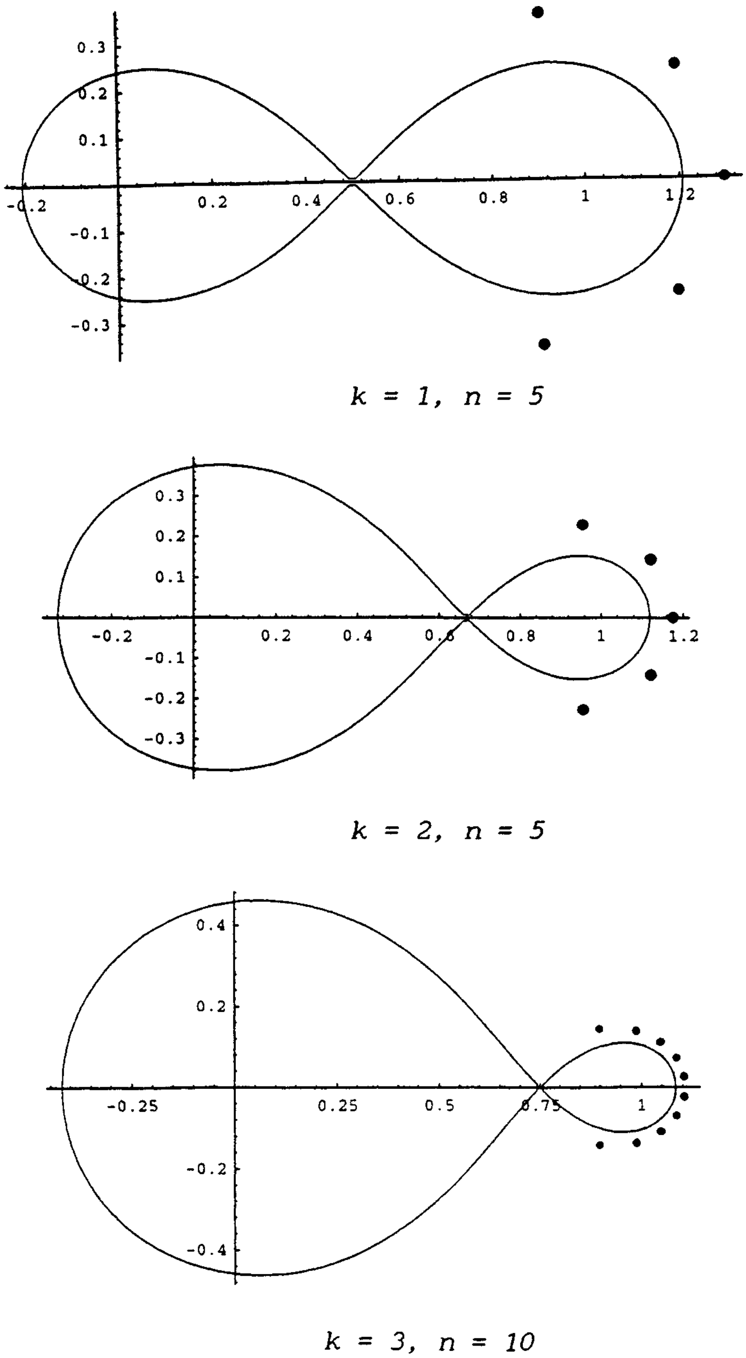
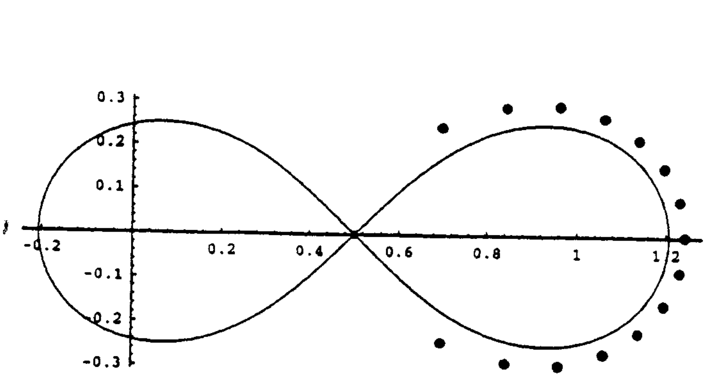
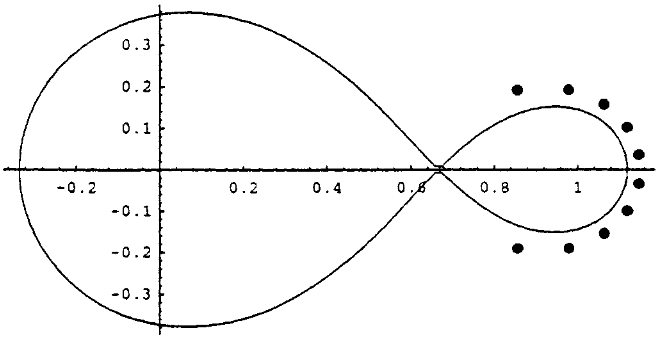


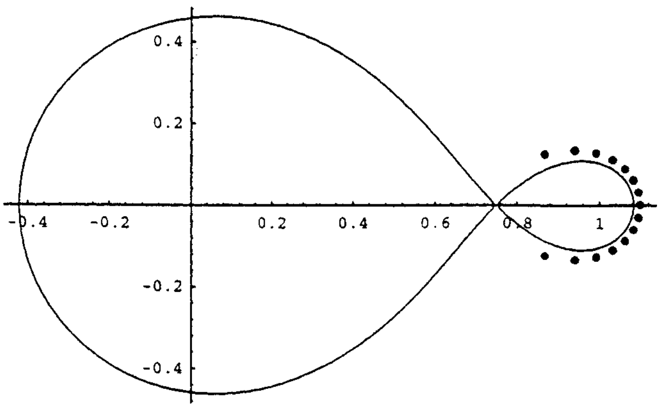
Figure 2. Lemniscate $|z^k(1-z)| = k^k/(k+1)^{k+1}$ and zeros of $F(-n, kn+1; kn+2; z)$.



k = 1, n = 15



k = 2, n = 10



k = 3, n = 15

Figure 2. (Continued.)

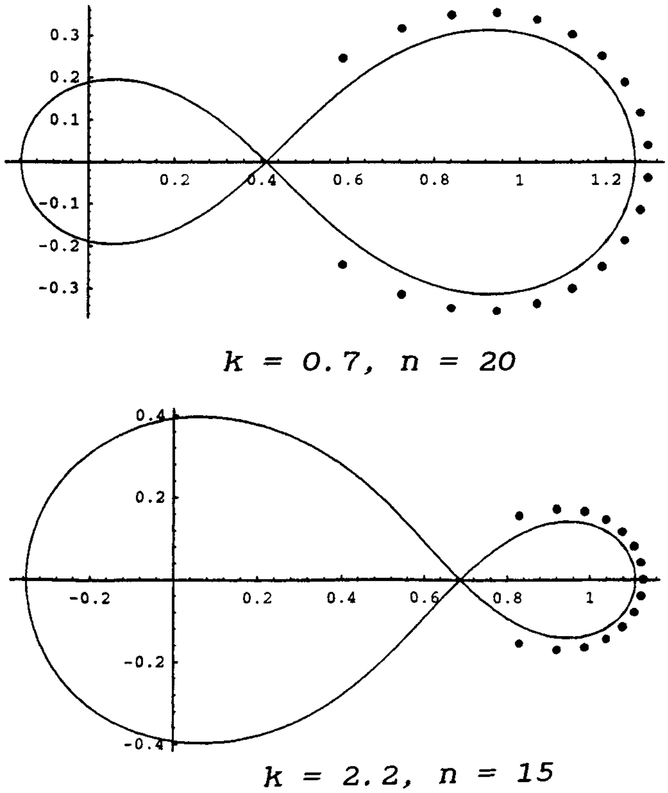


Figure 3. Lemniscate $|z^k(1-z)| = k^k/(k+1)^{k+1}$ and zeros of $F(-n, kn+1; kn+2; z)$, for noninteger values of k.

Remark. Numerical evidence indicates that theorem 1 remains true for all positive k, not necessarily an integer (see figure 3). This more general case is not covered by the Borwein–Chen theorem and we intend to return to this question in a later paper.

Another class of polynomials to which different results of Borwein and Chen from the same paper (cf. [2, theorem 2.5 and corollary 2.6]) can be applied is $F(-n, kn+1; (k+\ell)n+2; z)$, $k, \ell, n \in \mathbb{N}$. We state theorem 2 without proof because the idea is the same as the proof of theorem 1, only the calculations are tedious.

Theorem 2. Let $k, \ell, n \in \mathbb{N}$. Then, as $n \to \infty$, the limit points of the zeros of $F(-n, kn+1;\ (k+\ell)n+2; z)$ lie on the curve

$$\left\{z\colon \left|\frac{(k+1)z+\nu}{(k+1)z+\lambda}\right|^k \left|\frac{(1+k+2\ell)z-\nu}{(1+k+2\ell)z-\lambda}\right|^\ell \left|\frac{(k+1)z-\lambda-2}{(k+1)z-\nu-2}\right| = 1\right\}, \tag{4}$$

where $\mu = [(k+1)^2z^2 - 2(k^2+k\ell+k-\ell)z + (k+\ell)^2]^{1/2}$, $\nu = k+\ell+\mu$, $\lambda = k+\ell-\mu$.

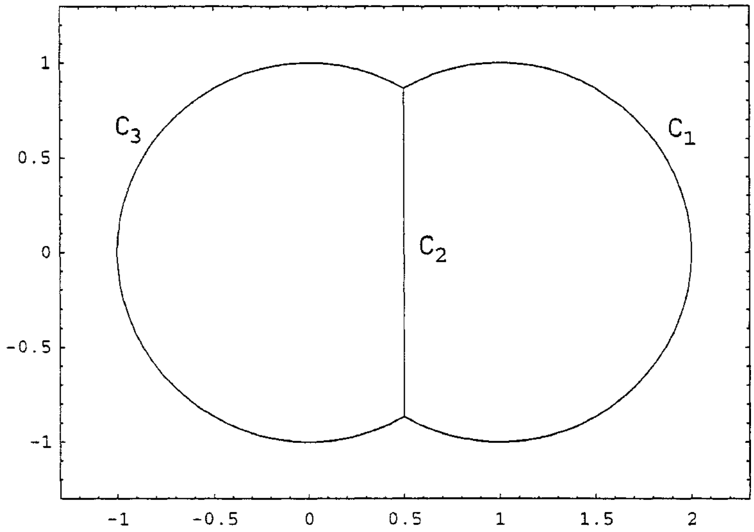


Figure 4. The owl.

Remark. The special case $k = \ell$ in (4) yields the critical curve

$$\left\{z\colon \left|\frac{3\sqrt{3}z(z-1)}{(2z-1)(z+1)(z-2)-2(z^2-z+1)^{3/2}}\right| = 1\right\},$$

which can be shown in an elementary way to be identical to the "owl-shaped" curve $C_1 \cup C_2 \cup C_3$ shown in figure 4, where

$$C_1 = \left\{z\colon\ |z-1| = 1,\ x > \frac{1}{2}\right\}; \qquad C_2 = \left\{z\colon\ z = \frac{1}{2} + \mathrm{i}y,\ |y| < \frac{\sqrt{3}}{2}\right\}$$

and C_3 is the reflection of C_1 about the line $x = \frac{1}{2}$. The theorem then says that the zeros of $F(-n, kn+1; 2kn+2; z)$ tend asymptotically to C_1. However, this statement is contained in a much stronger and more general result. It is shown in the authors' forthcoming paper [3] that for arbitrary $\lambda > -\frac{1}{2}$, and for every $n = 1, 2, \ldots$, the zeros of $F(-n, \lambda; 2\lambda; z)$ actually lie on the circle $|z-1| = 1$ by virtue of their connection with the zeros of ultraspherical polynomials.

4. Auxiliary results

Hypergeometric functions satisfy a variety of linear and quadratic transformations (cf. [1, pp. 559–561]), and hypergeometric polynomials are linked with Jacobi polynomials and associated Legendre functions (cf. [1, pp. 561–562]). These connections lead to the following corollaries of theorem 1.

Corollary 1.

(a) As $n \to \infty$, the zeros of the Jacobi polynomials $\mathcal{P}_n^{(kn+1,-n-1)}(w)$ cluster on the loop of the lemniscate

$$\left\{w\colon\ \left|(w-1)^k(w+1)\right| = \left(\frac{2}{k+1}\right)^{k+1} k^k,\ \operatorname{Re}(w) < 0\right\}. \tag{5}$$

(b) As $n \to \infty$, the zeros of the associated Legendre functions $P_n^{-n-1}(w)$ cluster on the loop of the lemniscate

$$\left\{w\colon\ \left|w^2-1\right| = 1,\ \operatorname{Re}(w) < 0\right\}. \tag{6}$$

Proof. (a) We have (cf. [1, p. 561, equation (15.4.6)])

$$F(-n, kn+1; kn+2; z) = \frac{n!}{(kn+2)_n}\mathcal{P}_n^{(kn+1,-n-1)}(1-2z). \tag{7}$$

Putting $w = 1-2z$ and substituting into the equation of the lemniscate (3), we obtain (5).

(b) Using the identity (cf. [1, p. 562, equation (15.4.17)])

$$F(-n, n+1;\ n+2;\ z) = (n+1)!\left(\frac{1-z}{z}\right)^{(n+1)/2} P_n^{-n-1}(1-2z), \tag{8}$$

putting $w = 1-2z$ and substituting into (3), with $k = 1$, yields (6).

Remark. Note that when $k = 1$, the lemniscate given by (5) is identical to that in (6). In fact, it is clear from (7) with $k = 1$, and (8), that the zero set of the Jacobi polynomial of degree n with parameters $\alpha = n+1$, $\beta = -\alpha$ is identical to that of the Legendre function of degree n with parameter $-n-1$, and, in both cases, the zeros converge to the lemniscate (6) as $n \to \infty$. Figure 5 provides graphical evidence of the assertions in (a).

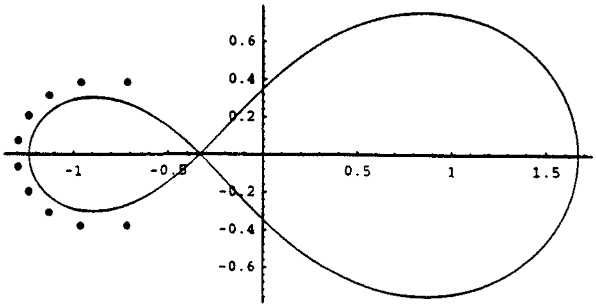
Corollary 2.

(a) The n zeros of the hypergeometric functions $F(\frac{1}{2}, n+1; n+2; w)$ and $F(1, n+\frac{3}{2}; n+2; w)$ cluster on the unit circle $|w| = 1$ as $n \to \infty$.

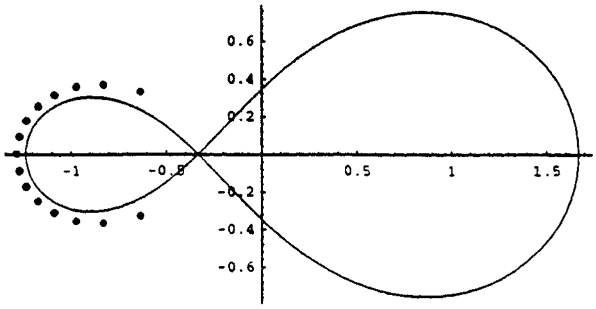
(b) The n zeros of $F(n+\frac{3}{2}, n+1; n+2; w)$ cluster on the vertical line $\operatorname{Re}(w) = \frac{1}{2}$ as $n \to \infty$.

Proof. (a) Invoking the quadratic transformations (cf. [4, p.112, equations (22), (23)]), we have

$$F(-n, n+1;\ n+2;\ z) = (1-z)^{n+1} F\left(\frac{1}{2}, n+1;\ n+2;\ 4z(1-z)\right),$$



k = 2, n = 10



k = 2, n = 15

Figure 5. Zeros of Jacobi polynomial $P_n^{(kn+1,-n-1)}(w)$ and lemniscate $|(w-1)^k(w+1)| = (2/(k+1))^k k^k$.

$$F(-n, n+1; n+2; z) = (1-z)^{n+1}(1-2z)F\left(1, n+\frac{3}{2}; n+2; 4z(1-z)\right).$$

Since the equation of the lemniscate (3) reduces to $\{z\colon \left|4z(1-z)\right| = 1\}$ when $k = 1$, the two identities above immediately yield the stated result.

(b) The transformation (cf. [4, p. 112, equation (24)])

$$\begin{aligned} &F(-n, n+1; n+2; z) \\ &\quad = (1-z)^{n+1}(1-2z)^{-2n-2}F\left(n+\frac{3}{2}, n+1; n+2; 4z(z-1)(1-2z)^{-2}\right), \end{aligned}$$

shows that the zeros of $F(n+\frac{3}{2}, n+1; n+2; w)$ are related to those of $F(-n, n+1; n+2; z)$ by $w = 4z(z-1)(1-2z)^{-2}$. We know from theorem 1 that the zeros of $F(-n, n+1; n+2; z)$ cluster on the curve $|4z(1-z)| = 1$ as $n \to \infty$. But, if $|4z(z-1)| = 1$, then $4z(z-1) = \mathrm{e}^{\mathrm{i}\theta}$ for some θ, and so

$$w = \frac{4z(z-1)}{[1+4z(z-1)]} = \frac{\mathrm{e}^{\mathrm{i}\theta}}{(1+\mathrm{e}^{\mathrm{i}\theta})}.$$

An easy simplification gives $w = \frac{1}{2} + \frac{1}{2}\mathrm{i}\tan(\theta/2)$, and we conclude that the n zeros of $F(n + \frac{3}{2}, n + 1; n + 2; w)$ cluster on the vertical line $\mathrm{Re}(w) = \frac{1}{2}$ as $n \to \infty$.

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