ASYMPTOTICALLY MOST POWERFUL TESTS OF STATISTICAL HYPOTHESES[1]

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1. Introduction. Let $f(x, \theta)$ be the probability density function of a variate x involving an unknown parameter θ. For testing the hypothesis $\theta = \theta_0$ by means of n independent observations $x_1, \cdots, x_n$ on x we have to choose a region of rejection W_n in the n-dimensional sample space. Denote by $P(W_n \mid \theta)$ the probability that the sample point $E = (x_1, \cdots, x_n)$ will fall in W_n under the assumption that θ is the true value of the parameter. For any region U_n of the n-dimensional sample space denote by $g(U_n)$ the greatest lower bound of $P(U_n \mid \theta)$. For any pair of regions U_n and T_n denote by $L(U_n, T_n)$ the least upper bound of

$$P(U_n \mid \theta) - P(T_n \mid \theta).$$

In all that follows we shall denote a region of the n-dimensional sample space by a capital letter with the subscript n.

Definition 1. A sequence $\{W_n\}$, $(n = 1, 2, \cdots$, ad inf.), of regions is said to be an asymptotically most powerful test of the hypothesis $\theta = \theta_0$ on the level of significance α if $P(W_n \mid \theta_0) = \alpha$ and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n \mid \theta_0) = \alpha$, the inequality

$$\limsup_{n \to \infty} L(Z_n, W_n) \leq 0$$

holds.

Definition 2. A sequence $\{W_n\}$, $(n = 1, 2, \cdots$, ad inf.), of regions is said to be an asymptotically most powerful unbiased test of the hypothesis $\theta = \theta_0$ on the level of significance α if $P(W_n \mid \theta_0) = \lim_{n=\infty} g(W_n) = \alpha$, and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n \mid \theta_0) = \lim_{n=\infty} g(Z_n) = \alpha$, the inequality

$$\limsup_{n \to \infty} L(Z_n, W_n) \leq 0$$

holds.

Let $\hat{\theta}_n(x_1, \cdots, x_n)$ be the maximum likelihood estimate of θ in the n-dimensional sample space. That is to say, $\hat{\theta}_n(x_1, \cdots, x_n)$ denotes the value of θ

[1] Presented to the American Mathematical Society at New York, February 24, 1940.

[2] Research under a grant-in-aid from the Carnegie Corporation of New York.

for which the product $\prod_{\nu=1}^{n} f(x_\nu, \theta)$ becomes a maximum. Let W'_n be the region defined by the inequality $\sqrt{n}(\hat{\theta}_n - \theta_0) \geq c'_n$, W''_n defined by the inequality $\sqrt{n}(\hat{\theta}_n - \theta_0) \leq c''_n$, and let W_n consists of all points for which at least one of the inequalities

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \geq a_n, \qquad \sqrt{n}(\hat{\theta}_n - \theta_0) \leq -a_n$$

is satisfied. The constants a_n, c'_n, c''_n are chosen such that

$$P(W'_n \mid \theta_0) = P(W''_n \mid \theta_0) = P(W_n \mid \theta_0) = \alpha.$$

It will be shown in this paper that under certain restrictions on the probability density $f(x, \theta)$ the sequence $\{W'_n\}$ is an asymptotically most powerful test of the hypothesis $\theta = \theta_0$ if θ takes only values $\theta \geq \theta_0$. Similarly $\{W''_n\}$ is an asymptotically most powerful test if θ takes only values $\theta \leq \theta_0$. Finally $\{W_n\}$ is an asymptotically most powerful unbiased test if θ can take any real value.

2. Assumptions on the density function $f(x, \theta)$.

ASSUMPTION 1. *For any positive k*

$$\lim_{n=\infty} P(-k < \hat{\theta}_n - \theta < k \mid \theta) = 1$$

uniformly in θ, where $P(-k < \hat{\theta}_n - \theta < k \mid \theta)$ denotes the probability that $-k \leq \hat{\theta}_n - \theta \leq k$ under the assumption that θ is the true value of the parameter.

Assumption 1 implies somewhat more than consistency of the maximum likelihood estimate $\hat{\theta}_n$. In fact, consistency means only that for any positive k

$$\lim_{n=\infty} P(-k \leq \hat{\theta}_n - \theta \leq k \mid \theta) = 1,$$

without asking that the convergence should be uniform in θ. If $\hat{\theta}_n$ satisfies Assumption 1 we shall say that $\hat{\theta}_n$ is a uniformly consistent estimate of θ. A rigorous proof of the consistency of $\hat{\theta}_n$ (under certain restrictions on $f(x, \theta)$) was given by J. L. Doob.[3] In an appendix to this paper it will be shown that under certain conditions $\hat{\theta}_n$ is uniformly consistent.

Denote by $E_\theta[\psi(x)]$ the expected value of $\psi(x)$ under the assumption that θ is the true value of the parameter. That is to say,

$$E_\theta[\psi(x)] = \int_{-\infty}^{\infty} \psi(x) f(x, \theta)\, dx.$$

For any x, for any positive δ, and for any θ_1, denote by $\varphi_1(x, \theta_1, \delta)$ the greatest lower bound, and by $\varphi_2(x, \theta_1, \delta)$ the least upper bound of $\frac{\partial^2 \log f(x, \theta)}{\partial \theta^2}$ in the interval $\theta_1 - \delta \leq \theta \leq \theta_1 + \delta$.

ASSUMPTION 2. *There exists a positive value k_0 such that the expectations $E_\theta \varphi_1(x, \theta_1, \delta)$ and $E_\theta \varphi_2(x, \theta_1, \delta)$ exist and are continuous functions of θ, θ_1 and δ*

[3] J. L. Doob, "Probability and statistics," *Trans. Am. Math. Soc.*, Vol. 36 (1937).

in the domain D defined by the inequalities: $0 \leq \delta \leq \frac{1}{2}k_0$, $\theta_0 - \frac{1}{2}k_0 \leq \theta_1 \leq \theta_0 + \frac{1}{2}k_0$, $\theta_0 - k_0 \leq \theta \leq \theta_0 + k_0$. *Furthermore the expectations* $E_\theta[\varphi_1(x, \theta_1, \delta)]^2$ *and* $E_\theta[\varphi_2(x, \theta_1, \delta)]^2$ *exist in D and have a finite upper bound in D.*

ASSUMPTION 3. *There exists a positive value k_0 such that*

$$\int_{-\infty}^{\infty} \frac{\partial f(x, \theta)}{\partial \theta} dx = \int_{-\infty}^{\infty} \frac{\partial^2 f(x, \theta)}{\partial \theta^2} dx = 0 \quad \textit{for} \quad \theta_0 - k_0 \leq \theta \leq \theta_0 + k_0.$$

Assumption 3 means simply that we may differentiate with respect to θ under the integral sign. In fact

$$\int_{-\infty}^{\infty} f(x, \theta)\, dx = 1$$

identically in θ. Hence

$$\frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} f(x, \theta)\, dx = \frac{\partial^2}{\partial \theta^2} \int_{-\infty}^{\infty} f(x, \theta)\, dx = 0.$$

Differentiating under the integral sign, we obtain the relations in Assumption 3.

ASSUMPTION 4. *There exists a positive η and a positive k_0 such that*

$$E_\theta \left| \frac{\partial \log f(x, \theta)}{\partial \theta} \right|^{2+\eta}$$

exists and has a finite upper bound in the interval $\theta_0 - k_0 \leq \theta \leq \theta_0 + k_0$.

3. Some propositions. Denote $\sqrt{n}\,(\theta_n - \theta)$ by $z_n(\theta)$ and denote the probability $P[z_n(\theta) < t \mid \theta]$ by $\Phi_n(t, \theta)$.

PROPOSITION I. *Within the θ-interval $[\theta_0 - \frac{1}{2}k_0, \theta_0 + \frac{1}{2}k_0]$ $\Phi_n(t, \theta)$ converges with $n \to \infty$ uniformly in t and θ towards the cumulative normal distribution with zero mean and variance*

$$-1 \Big/ E_\theta \frac{\partial^2 \log f(x, \theta)}{\partial \theta^2}$$

PROOF: In all that follows we assume that θ takes only values in the interval $[\theta_0 - k_0, \theta_0 + k_0]$, except when the contrary is explicitly stated. Furthermore we introduce the variable θ_1 and assume that θ_1 takes only values in the interval $[\theta_0 - \frac{1}{2}k_0, \theta_0 + \frac{1}{2}k_0]$.

Because of Assumption 3 we have

$$E_\theta \frac{\partial \log f(x, \theta)}{\partial \theta} = \int_{-\infty}^{\infty} \frac{\partial f(x, \theta)}{\partial \theta} dx = 0 \tag{1}$$

Since

$$\frac{\partial^2 \log f(x, \theta)}{\partial \theta^2} = \frac{1}{f(x, \theta)} \cdot \frac{\partial^2 f(x, \theta)}{\partial \theta^2} - \frac{1}{[f(x, \theta)]^2} \left[\frac{\partial f(x, \theta)}{\partial \theta}\right]^2$$

we get from Assumption 3

$$E_\theta \left[\frac{\partial \log f(x, \theta)}{\partial \theta}\right]^2 = -E_\theta \frac{\partial^2 \log f(x, \theta)}{\partial \theta^2}. \tag{2}$$

Hence

$$d(\theta) = -E_\theta \frac{\partial^2 \log f(x, \theta)}{\partial \theta^2} > 0. \tag{3}$$

Consider the Taylor expansion

$$\sum_{\alpha=1}^{n} \frac{\partial \log f(x_\alpha, \theta)}{\partial \theta} = \sum_{\alpha=1}^{n} \frac{\partial \log f(x_\alpha, \theta_1)}{\partial \theta} + (\theta - \theta_1) \sum_{\alpha=1}^{n} \frac{\partial^2 \log f(x_\alpha, \theta')}{\partial \theta^2} \tag{4}$$

where θ' lies in the interval $[\theta_1, \theta]$. Denote $\frac{1}{\sqrt{n}} \sum_\alpha \frac{\partial \log f(x_\alpha, \theta_1)}{\partial \theta}$ by $y_n(\theta_1)$. For $\theta = \hat{\theta}_n$ the left hand side of (4) is equal to zero. Hence we have

$$y_n(\theta_1) + [\sqrt{n}(\hat{\theta}_n - \theta_1)] \frac{1}{n} \sum_\alpha \frac{\partial^2 \log f(x_\alpha, \theta')}{\partial \theta^2} = 0, \tag{5}$$

or

$$y_n(\theta_1) + z_n(\theta_1) \frac{1}{n} \sum_\alpha \frac{\partial^2 \log f(x_\alpha, \theta')}{\partial \theta^2} = 0. \tag{6}$$

Let $Q_n(\theta_1)$ be the region defined by the inequality

$$\left| \frac{1}{n} \sum_\alpha \frac{\partial^2 \log f(x_\alpha, \theta')}{\partial \theta^2} + d(\theta_1) \right| < \nu \tag{7}$$

where ν denotes a positive number less than the greatest lower bound of $d(\theta_1)$. We shall prove that

$$\lim_{n=\infty} P[Q_n(\theta_1) \mid \theta_1] = 1 \tag{8}$$

uniformly in θ_1. Let τ_0 be a positive number such that

$$\left| E_{\theta_1} \varphi_i(x, \theta_1, \tau_0) - E_{\theta_1} \frac{\partial^2 \log f(x, \theta_1)}{\partial \theta^2} \right| < \frac{\nu}{2}, \qquad (i = 1, 2) \tag{9}$$

for all values of θ_1. Because of Assumption 2 such a τ_0 certainly exists. Denote by $R_n(\theta_1)$ the region defined by the inequality

$$| \hat{\theta}_n - \theta_1 | \leq \tau_0 . \tag{10}$$

On account of Assumption 1

$$\lim_{n=\infty} P[R_n(\theta_1) \mid \theta_1] = 1 \tag{11}$$

uniformly in θ_1. Since θ' lies in the interval $[\theta_1, \hat{\theta}_n]$, we have

$$| \theta' - \theta_1 | \leq \tau_0 \tag{12}$$

for all points in $R_n(\theta_1)$. Hence at any point in $R_n(\theta_1)$ the inequality

$$\sum_{\alpha=1}^{n} \varphi_1(x_\alpha, \theta_1, \tau_0) \leq \sum_{\alpha=1}^{n} \frac{\partial^2 \log f(x_\alpha, \theta')}{\partial \theta^2} \leq \sum_{\alpha=1}^{n} \varphi_2(x_\alpha, \theta_1, \tau_0) \tag{13}$$

holds.

Let $S_n(\theta_1)$ be defined by the inequality

$$\left| \frac{1}{n} \sum_\alpha \varphi_1(x_\alpha, \theta_1, \tau_0) - E_{\theta_1} \varphi_1(x, \theta_1, \tau_0) \right| < \frac{\nu}{2} \tag{14}$$

and $T_n(\theta_1)$ by the inequality

$$\left| \frac{1}{n} \sum_\alpha \varphi_2(x_\alpha, \theta_1, \tau_0) - E_{\theta_1} \varphi_2(x, \theta_1, \tau_0) \right| < \frac{\nu}{2}. \tag{15}$$

On account of Assumption 2 we have

$$\lim_{n=\infty} P[S_n(\theta_1) \mid \theta_1] = \lim_{n=\infty} P[T_n(\theta_1) \mid \theta_1] = 1 \tag{16}$$

uniformly in θ_1.
Denote by $U_n(\theta_1)$ the common part of the regions $R_n(\theta_1)$, $S_n(\theta_1)$ and $T_n(\theta_1)$. In $U_n(\theta_1)$ we have on account of (9), (14) and (15)

$$\left| \frac{1}{n} \sum_\alpha \varphi_i(x_\alpha, \theta_1, \tau_0) - E_{\theta_1} \frac{\partial^2 \log f(x, \theta_1)}{\partial \theta^2} \right| < \nu \qquad (i = 1, 2). \tag{17}$$

From this we obtain (7) because of (13). That is to say, the inequality (7) is valid everywhere in $U_n(\theta_1)$. Since

$$\lim_{n=\infty} P[U_n(\theta_1) \mid \theta_1] = 1$$

uniformly in θ_1, our statement about $Q_n(\theta_1)$ is proved. From (6) and (7) we get that everywhere in $Q_n(\theta_1)$ the inequalities hold:

$$\frac{y_n(\theta_1)}{d(\theta_1) + \nu} \leq z_n(\theta_1) \leq \frac{y_n(\theta_1)}{d(\theta_1) - \nu} \quad \text{if} \quad y_n(\theta_1) \geq 0; \tag{18}$$

$$\frac{y_n(\theta_1)}{d(\theta_1) + \nu} \geq z_n(\theta_1) \geq \frac{y_n(\theta_1)}{d(\theta_1) - \nu} \quad \text{if} \quad y_n(\theta_1) \leq 0. \tag{19}$$

Let $z_n^*(\theta_1)$ be defined as follows: $z_n^*(\theta_1) = z_n(\theta_1)$ at any point in $Q_n(\theta_1)$, and $z_n^*(\theta_1) = y_n(\theta_1)/d(\theta_1)$ at any point outside $Q_n(\theta_1)$.
On account of (8) we obviously have

$$\lim_{n=\infty} P[z_n^*(\theta_1) < t \mid \theta_1] - P[z_n(\theta_1) < t \mid \theta_1] = 0 \tag{20}$$

uniformly in t and θ_1.
From equation (1) it follows that $E_{\theta_1} y_n(\theta_1) = 0$. From Assumption 4 it follows on account of the general limit theorems

$$\lim_{n=\infty} P[y_n(\theta_1) < t \mid \theta_1] - \frac{1}{\sqrt{2\pi d(\theta_1)}} \int_{-\infty}^t e^{-\frac{1}{2}t^2/d(\theta_1)} \, dt = 0 \tag{21}$$

uniformly in t and θ_1. Hence

$$\lim_{n=\infty} P\left[\frac{y_n(\theta_1)}{d(\theta_1)} < t \mid \theta_1\right] - \sqrt{\frac{d(\theta_1)}{2\pi}} \int_{-\infty}^t e^{-\frac{1}{2}t^2 d(\theta_1)} \, dt = 0$$

uniformly in t and θ_1. Since ν can be chosen arbitrarily small, we get easily from (18), (19), (20) and (21)

$$\lim_{n=\infty} \left| P\left[\frac{y_n(\theta_1)}{d(\theta_1)} < t \mid \theta_1\right] - P[z_n(\theta_1) < t \mid \theta_1] \right| = 0 \tag{22}$$

uniformly in t and θ_1. Proposition 1 follows from (21) and (22).

PROPOSITION 2. *Let $\{W_n\}$ be a sequence of regions of size α, i.e. $P(W_n \mid \theta_0) = \alpha$, and let $V_n(z)$ be the region defined by the inequality*

$$(\hat{\theta}_n - \theta_0)\sqrt{n} < z.$$

Let $U_n(z)$ be the intersection of $V_n(z)$ and W_n, and denote $P[U_n(z) \mid \theta_0]$ by $F_n(z)$. Denote furthermore $P[W_n \mid \theta_0 + \mu/\sqrt{n}]$ by $G(\mu, n)$. If $F_n(z)$ converges to $F(z)$ and if $\lim_{n=\infty} \mu_n = \mu$, then

$$\lim_{n=\infty} G(\mu_n, n) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu z)/c}\, dF(z) \tag{23}$$

where

$$c = -1 \Big/ E_{\theta_0} \frac{\partial^2 \log f(x, \theta_0)}{\partial \theta^2}.$$

PROOF: First we show

$$\int_{-\infty}^{\infty} dF(z) = \alpha. \tag{24}$$

Denote $P[V_n(z) \mid \theta_0]$ by $\Phi_n(z)$. On account of Proposition 1 $\Phi_n(z)$ converges uniformly to the cumulative normal distribution $\psi(z)$ with zero mean and variance c. It is obvious that

$$F_n(z_2) - F_n(z_1) \leq \Phi_n(z_2) - \Phi_n(z_1) \text{ for } z_2 > z_1. \tag{25}$$

Hence

$$F(z_2) - F(z_1) \leq \psi(z_2) - \psi(z_1) \text{ for } z_2 > z_1. \tag{26}$$

From (25) we get

$$\left[\lim_{z=\infty} F_n(z)\right] - F_n(z) = \alpha - F_n(z) \leq 1 - \Phi_n(z). \tag{27}$$

Hence

$$\alpha - F(z) \leq 1 - \psi(z). \tag{28}$$

Since $F_n(z) \leq \alpha$ and therefore also $F(z) \leq \alpha$, we get from (28)

$$0 \leq \alpha - F(z) \leq 1 - \psi(z).$$

Hence

$$\lim_{z=\infty} F(z) = \alpha. \tag{29}$$

Since $F_n(z) \leq \Phi_n(z)$, we have $F(z) \leq \psi(z)$, and therefore

$$\lim_{z=-\infty} F(z) = 0. \tag{30}$$

The equation (24) follows from (29) and (30).

It follows easily from (26) that the integral on the right hand side of the equation (23) exists and is finite.

Let us denote $\theta_0 + \mu_n/\sqrt{n}$ by θ_n. Consider the Taylor expansions

$$\begin{aligned}\sum_\alpha \log f(x_\alpha, \theta_0) = \sum_\alpha \log f(x_\alpha, \hat{\theta}_n) + (\theta_0 - \hat{\theta}_n) \sum_\alpha \frac{\partial}{\partial\theta} \log f(x_\alpha, \hat{\theta}_n) \\ + \tfrac{1}{2}(\theta_0 - \hat{\theta}_n)^2 \sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n')\end{aligned} \tag{31}$$

and

$$\begin{aligned}\sum_\alpha \log f(x_\alpha, \theta_n) = \sum_\alpha \log f(x_\alpha, \hat{\theta}_n) + (\theta_n - \hat{\theta}_n) \sum_\alpha \frac{\partial}{\partial\theta} \log f(x_\alpha, \hat{\theta}_n) \\ + \tfrac{1}{2}(\theta_n - \hat{\theta}_n)^2 \sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n'')\end{aligned} \tag{32}$$

where θ_n' lies in the interval $[\theta_0, \hat{\theta}_n]$ and θ_n'' lies in the interval $[\theta_n, \hat{\theta}_n]$. Since $\hat{\theta}_n$ is the maximum likelihood estimate, we get from (31) and (32)

$$\sum_\alpha \log f(x_\alpha, \theta_0) = \sum_\alpha \log f(x_\alpha, \hat{\theta}_n) + \tfrac{1}{2}(\theta_0 - \hat{\theta}_n)^2 \sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n'), \tag{33}$$

$$\sum_\alpha \log f(x_\alpha, \theta_n) = \sum_\alpha \log f(x_\alpha, \hat{\theta}_n) + \tfrac{1}{2}(\theta_n - \hat{\theta}_n)^2 \sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n''). \tag{34}$$

Denote by β a real variable which can take any value between -2μ and $+2\mu$. Denote by R_n the region defined by the inequality

$$|\hat{\theta}_n - \theta_0| < n^{-\frac{1}{4}}. \tag{35}$$

From Proposition 1 it follows easily that

$$\lim_{n=\infty} P(R_n \mid \theta_0 + \beta/\sqrt{n}) = 1 \tag{36}$$

uniformly in β. Denote $2n^{-\frac{1}{4}}$ by τ_n. Then for almost all n the following inequalities hold at any point in R_n:

$$\sum_\alpha \varphi_1(x_\alpha, \theta_0, \tau_n) \leq \sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n') \leq \sum_\alpha \varphi_2(x_\alpha, \theta_0, \tau_n), \tag{37}$$

$$\sum_\alpha \varphi_1(x_\alpha, \theta_0, \tau_n) \leq \sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n'') \leq \sum_\alpha \varphi_2(x_\alpha, \theta_0, \tau_n). \tag{38}$$

Denote by S_n the region in which (35), (37) and (38) simultaneously hold. It is obvious that

$$\lim_{n=\infty} P(S_n \mid \theta_0 + \beta/\sqrt{n}) = 1$$

uniformly in β. Denote $\theta_0 + \beta/\sqrt{n}$ by $\theta_n(\beta)$. From Assumption 2 it follows easily that

$$\lim_{n=\infty} E_{\theta_n(\beta)} \left\{ \frac{\sum_\alpha \varphi_i(x_\alpha, \theta_0, \tau_n)}{n} \right\} = E_{\theta_0} \frac{\partial^2}{\partial \theta^2} \log f(x, \theta_0) = \frac{-1}{c} \quad (i = 1, 2) \tag{39}$$

uniformly in β. Furthermore the variance of $\sum_\alpha \frac{\varphi_i(x_\alpha, \theta_0, \tau_n)}{n}$, if $\theta_n(\beta)$ is the true value of the parameter θ, converges to zero with $n \to \infty$ uniformly in β. Hence a sequence $\{\lambda_n\}$, $(n = 1, 2, \cdots$, ad inf.), of positive numbers can be given such that

$$\lim_{n=\infty} \lambda_n = 0 \tag{40}$$

and

$$\lim P[T_n \mid \theta_n(\beta)] = 1 \tag{41}$$

uniformly in β, where the region T_n is defined by the inequality

$$\left| \sum_\alpha \frac{\varphi_i(x_\alpha, \theta_0, \tau_n)}{n} + \frac{1}{c} \right| < \lambda_n n^{-\frac{1}{2}} \qquad (i = 1, 2). \tag{42}$$

From (37) and (38) it follows that in the intersection T'_n of T_n and S_n

$$\left| \frac{1}{n} \sum_\alpha \frac{\partial^2}{\partial \theta^2} \log f(x_\alpha, \theta'_n) + \frac{1}{c} \right| < \lambda_n n^{-\frac{1}{2}} \tag{43}$$

and

$$\left| \frac{1}{n} \sum_\alpha \frac{\partial^2}{\partial \theta^2} \log f(x_\alpha, \theta''_n) + \frac{1}{c} \right| < \lambda_n n^{-\frac{1}{2}}. \tag{44}$$

We get from (33), (34), (35), (43) and (44) that at any point in T'_n

$$\sum_\alpha \log f(x_\alpha, \theta_n) - \sum_\alpha \log f(x_\alpha, \theta_0) = \frac{n}{2c} [(\theta_0 - \hat{\theta}_n)^2 - (\theta_n - \hat{\theta}_n)^2] + \lambda'_n, \tag{45}$$

where $|\lambda'_n| \leq \rho\lambda_n$, and ρ denotes a constant not depending on n.
On account of (36) and (41) we have

$$\lim_{n=\infty} P[T'_n \mid \theta_n(\beta)] = 1 \tag{46}$$

uniformly in β.
Denote by $T''_n(z)$ the intersection of $U_n(z)$ (defined in Proposition 2) and T'_n.
Denote furthermore $P[T''_n(z) \mid \theta_0]$ by $F^*_n(z)$.
Since

$$n[(\theta_0 - \hat{\theta}_n)^2 - (\theta_n - \hat{\theta}_n)^2] = n[(\theta_0 - \hat{\theta}_n)^2 - (\theta_0 - \hat{\theta}_n + \mu_n/\sqrt{n})^2]$$
$$= -\mu_n^2 + 2\sqrt{n}\,\mu_n(\hat{\theta}_n - \theta_0),$$

we get from (45) and (46)

$$\lim_{n=\infty} \left\{ P[T''_n(z) \mid \theta_n] - \int_{-\infty}^{z} e^{-\frac{1}{2}(\mu_n^2 - 2\mu_n t)/c} \, dF_n^*(t) \right\} = 0 \tag{47}$$

uniformly in z. It is obvious that

$$\lim_{n=\infty} \{ P[T''_n(z) \mid \theta_n] - P[U_n(z) \mid \theta_n] \} = 0 \tag{48}$$

uniformly in z. Hence we get from (47)

$$\lim_{n=\infty} \left\{ P[U_n(z) \mid \theta_n] - \int_{-\infty}^{z} e^{-\frac{1}{2}(\mu_n^2 - 2\mu_n t)/c} \, dF_n^*(t) \right\} = 0 \tag{49}$$

uniformly in z. It follows from (49) that for any positive L

$$\lim_{n=\infty} \left\{ P[U_n(L) \mid \theta_n] - P[U_n(-L) \mid \theta_n] - \int_{-L}^{L} e^{-\frac{1}{2}(\mu_n^2 - 2\mu_n t)/c} \, dF_n^*(t) \right\} = 0. \tag{50}$$

Since $\lim \mu_n = \mu$, $\lim_{n=\infty} [F_n^*(t) - F_n(t)] = 0$ uniformly in t, and since $\lim_{n=\infty} F_n(t) = F(t)$ uniformly in t, we get from (50)

$$\lim_{n=\infty} \{ P[U_n(L) \mid \theta_n] - P[U_n(-L) \mid \theta_n] \} = \int_{-L}^{L} e^{-\frac{1}{2}(\mu^2 - 2\mu t)/c} \, dF(t). \tag{51}$$

Now let us calculate the limit of $P[V_n(z) \mid \theta_n]$ if $n \to \infty$. The region $V_n(z)$ is defined by the inequality

$$(\hat{\theta}_n - \theta_0)\sqrt{n} < z. \tag{52}$$

This inequality can be written as follows:

$$(\hat{\theta}_n - \theta_n)\sqrt{n} < z - \mu_n. \tag{53}$$

Since $\lim \mu_n = \mu$, we get on account of Proposition 1

$$\begin{aligned} \lim_{n=\infty} P[(\hat{\theta}_n - \theta_n)\sqrt{n} < z - \mu_n \mid \theta_n] &= \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{z-\mu} e^{-\frac{1}{2}t^2/c} \, dt \\ &= \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{z} e^{-\frac{1}{2}(t-\mu)^2/c} \, dt \end{aligned} \tag{54}$$

Hence

$$\lim_{n=\infty} P[V_n(z) \mid \theta_n] = \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{z} e^{-\frac{1}{2}(t-\mu)^2/c} \, dt \tag{55}$$

uniformly in z.

For any positive ϵ let L_ϵ denote the positive number satisfying the condition:

$$\frac{1}{\sqrt{2\pi c}} \left[\int_{-\infty}^{-L_\epsilon} e^{-\frac{1}{2}(t-\mu)^2/c} \, dt + \int_{L_\epsilon}^{\infty} e^{-\frac{1}{2}(t-\mu)^2/c} \, dt \right] = \frac{\epsilon}{2}. \tag{56}$$

From (56) we easily get on account of (26)

$$(57) \qquad 0 \leq \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu t)/c}\, dF(t) - \int_{-L_\epsilon}^{L_\epsilon} e^{-\frac{1}{2}(\mu^2 - 2\mu t)/c}\, dF(t) \leq \frac{\epsilon}{2}.$$

Since the region $U_n(z_2) - U_n(z_1)$ is a subset of $V_n(z_2) - V_n(z_1)$ for $z_2 > z_1$, we have on account of (55) and (56)

$$(58) \quad \limsup_{n \to \infty} | \{P[U_n(\infty) \mid \theta_n] - P[U_n(L_\epsilon) \mid \theta_n] + P[U_n(-L_\epsilon) \mid \theta_n]\} | \leq \frac{\epsilon}{2}.$$

Since

$$P[U_n(\infty) \mid \theta_n] = G(\mu_n, n),$$

we have

$$(59) \qquad \limsup_{n \to \infty} | G(\mu_n, n) - \{P[U_n(L_\epsilon) \mid \theta_n] - P[U_n(-L_\epsilon) \mid \theta_n]\} | \leq \frac{\epsilon}{2}.$$

From (51), (57) and (59) we get

$$(60) \qquad \limsup_{n \to \infty} \left| G(\mu_n, n) - \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu t)/c}\, dF(t) \right| \leq \epsilon.$$

Since ϵ can be chosen arbitrarily small, Proposition 2 is proved.

4. Theorems on asymptotically most powerful tests.

THEOREM 1: *Let M_n be the region defined by the inequality $\sqrt{n}\,(\hat{\theta}_n - \theta_0) \geq A_n$, where A_n is chosen such that $P(M_n \mid \theta_0) = \alpha$. Then $\{M_n\}$ is an asymptotically most powerful test of the hypothesis $\theta = \theta_0$, provided the parameter θ is restricted to values $\geq \theta_0$.*

PROOF: Assume that there exists a test $\{W_n\}$ of size α such that

$$(61) \qquad \limsup_{n \to \infty} L(W_n, M_n) = \delta > 0.$$

Then there exists a subsequence $\{n'\}$ of the sequence $\{n\}$ and a sequence $\{\theta_{n'}\}$ of parameter values $\geq \theta_0$ such that

$$(62) \qquad \lim_{n = \infty} \{P(W_{n'} \mid \theta_{n'}) - P(M_{n'} \mid \theta_{n'})\} = \delta$$

The expression

$$(63) \qquad (\theta_{n'} - \theta_0)\sqrt{n} = \mu_{n'} > 0$$

must be bounded. This can be proved as follows: Since under the assumption $\theta = \theta_0$ the distribution of $\sqrt{n}\,(\hat{\theta}_n - \theta_0)$ converges to a normal distribution with zero mean and finite variance, the sequence $\{A_n\}$ must be bounded. Hence M_n is defined by the inequality

$$(64) \qquad \hat{\theta}_n - \theta_0 \geq A_n/\sqrt{n} = \epsilon_n$$

where

$$\lim_{n=\infty} \epsilon_n = 0. \tag{65}$$

From Assumption 1, (64) and (65) it follows easily that if

$$\lim_{n=\infty} \theta_{n'} = \theta_1 > \theta_0, \qquad \lim_{n=\infty} P(M_{n'} \mid \theta_{n'}) = 1.$$

Hence on account of (62) we must have

$$\lim_{n=\infty} \theta_{n'} = \theta_0. \tag{66}$$

If there would exist a subsequence $\{n^*\}$ of $\{n'\}$ such that $\lim_{n=\infty} \mu_{n^*} = \infty$, then on account of (66) and Proposition 1 we would have $\lim_{n=\infty} P(M_{n^*} \mid \theta_{n^*}) = 1$, which is in contradiction to (62). Hence the expression (63) must be bounded. Let $\{n''\}$ be a subsequence of $\{n'\}$ such that

$$\lim_{n=\infty} \mu_{n''} = \mu > 0. \tag{67}$$

Denote by $F_n(z)$ the probability of the intersection of W_n and the region $(\hat{\theta}_n - \theta_0)\sqrt{n} < z$ under the hypothesis that $\theta = \theta_0$. Consider the subsequence $\{n'''\}$ of the sequence $\{n''\}$ such that $F_{n'''}(z)$ converges with $n \to \infty$ towards a function $F(z)$. The existence of such a subsequence $\{n'''\}$ can be proved as follows: Denote the probability $P[(\hat{\theta}_n - \theta_0)\sqrt{n} < z \mid \theta_0]$ by $\Phi_n(z)$. On account of Proposition 1, $\Phi_n(z)$ converges with $n \to \infty$ uniformly in z towards

$$\psi(z) = \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2/c}\, dt \tag{68}$$

where c has the same value in (23).
We obviously have

$$F_n(z_2) - F_n(z_1) \leq \Phi_n(z_2) - \Phi_n(z_1) \tag{69}$$

for any pair of values z_1, z_2 for which $z_2 > z_1$. Hence

$$\limsup_{n\to\infty} [F_n(z_2) - F_n(z_1)] \leq \psi(z_2) - \psi(z_1). \tag{70}$$

Since $F_n(z)$ is a monotonic function of z, our statement follows easily from (70) and the fact that $\psi(z)$ is uniformly continuous. Hence on account of Proposition 2 we have

$$\lim_{n=\infty} P(W_{n'''} \mid \theta_{n'''}) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu z)/c}\, dF(z) \tag{71}$$

and

$$\lim_{n=\infty} P(M_{n'''} \mid \theta_{n'''}) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu z)/c}\, d\Phi(z) \tag{72}$$

where

$$\Phi(z) = 0 \text{ for } z \leq z_0, \tag{73}$$

$$\Phi(z) = \psi(z) - \psi(z_0) \text{ for } z > z_0, \tag{74}$$

and z_0 is given by

$$1 - \psi(z_0) = \alpha. \tag{75}$$

From (62), (71) and (72) we get

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu z)/c}\, d[F(z) - \Phi(z)] = \delta > 0. \tag{76}$$

Consider a normally distributed variate y with mean ν and variance c. Let B be a critical region of size α for testing the hypothesis $\nu = 0$ by a single observation on y, i.e. B is a subset of the real axis $[-\infty, +\infty]$. Denote by $D(v)$ the intersection of B and the region $C(v)$ defined by the inequality $y < v$. Denote by $H(v)$ the probability of $D(v)$ under the hypothesis $\nu = 0$. Then the power of the test B with respect to the alternative $\nu = \mu$ is given by the following expression

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - \mu v)/c}\, dH(v). \tag{77}$$

If the region B is given by the inequality $y \geq v_0$ where v_0 is chosen such that the size of B is equal to α, then $H(v) = \Phi(v)$ where the function Φ is defined by the equations (73), (74) and (75). Since the latter test is uniformly most powerful[4] with respect to all alternatives $\nu > 0$, for any positive μ the inequality

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - \mu v)/c}\, d[H(v) - \Phi(v)] \leqq 0 \tag{78}$$

holds. Let

$$\psi(v) = \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{v} e^{-\frac{1}{2}t^2/c}\, dt.$$

It is obvious that

$$H(v_2) - H(v_1) \leq \psi(v_2) - \psi(v_1) \text{ for } v_2 > v_1 \tag{79}$$

and

$$\int_{-\infty}^{\infty} dH(v) = \alpha. \tag{80}$$

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[4] See for instance J. Neyman and E. S. Pearson, "Contributions to the theory of testing statistical hypotheses," *Stat. Res. Memoirs*, Vol. 1 (1936).

On the other hand, if $K(v)$ is a monotonically non-decreasing non-negative function of v such that

$$K(v_2) - K(v_1) \leq \psi(v_2) - \psi(v_1) \text{ for } v_2 > v_1 \tag{79'}$$

and

$$\int_{-\infty}^{\infty} dK(v) = \alpha \tag{80'}$$

hold, then there exists a sequence $\{B^{(i)}\}$, ($i = 1, 2, \cdots$, ad inf.), of regions of size α such that

$$\lim_{i=\infty} H^{(i)}(v) = K(v)$$

uniformly in v. Since (78) holds for $H(v) = H^{(i)}(v)$, and since

$$H^{(i)}(v_2) - H^{(i)}(v_1) \leq \psi(v_2) - \psi(v_1) \text{ for } v_2 > v_1,$$

it is easy to see that (78) will hold also for $H(v) = K(v)$. Hence for any monotonically non-decreasing non-negative function $K(v)$ for which (79′) and (80′) are fulfilled, also (78) must hold. Since $F(v)$ is a distribution function which satisfies (79′) and (80′), we have a contradiction to (76). This proves Theorem 1.

THEOREM 2: *Let M_n be the region defined by the inequality $\sqrt{n}\,(\hat{\theta}_n - \theta_0) \leq A_n$, where A_n is chosen such that $P(M_n \mid \theta_0) = \alpha$. Then $\{M_n\}$ is an asymptotically most powerful test of the hypothesis $\theta = \theta_0$, provided that the parameter θ is restricted to values $\leq \theta_0$.*

We omit the proof since it is entirely analogous to that of Theorem 1.

THEOREM 3: *Let M_n be the region consisting of all points which satisfy at least one of the inequalities*

$$\sqrt{n}\,(\hat{\theta}_n - \theta_0) \leq -A_n, \qquad \sqrt{n}\,(\hat{\theta}_n - \theta_0) \geq A_n.$$

The constant $A_n > 0$ is chosen such that $P(M_n \mid \theta_0) = \alpha$. Then $\{M_n\}$ is an asymptotically most powerful unbiased test of the hypothesis $\theta = \theta_0$.

PROOF: Assume that there exists a sequence $\{W_n\}$ ($n = 1, 2, \cdots$, ad inf.) of regions such that

$$P(W_n \mid \theta_0) = \alpha \tag{81}$$

$$\lim_{n=\infty} g(W_n) = \alpha \tag{82}$$

and

$$\limsup_{n\to\infty} L(W_n, M_n) = \delta > 0. \tag{83}$$

We shall deduce a contradiction from this assumption. On account of (83) there exists a subsequence $\{n'\}$ of $\{n\}$ such that

$$\lim_{n=\infty} \{P(W_{n'} \mid \theta_{n'}) - P(M_{n'} \mid \theta_{n'})\} = \delta. \tag{84}$$

The expression

$$(85) \qquad (\theta_{n'} - \theta_0)\sqrt{n'} = \mu_{n'}$$

must be bounded. The proof of this statement is omitted, since it is analogous to the proof of the similar statement about (63). Hence there exists a subsequence $\{n''\}$ of $\{n'\}$ such that

$$(86) \qquad \lim_{n=\infty} \mu_{n''} = \mu.$$

Denote by $F_n(z)$ the probability of the intersection of W_n with the region $(\hat{\theta}_n - \theta_0)\sqrt{n} < z$ under the hypothesis $\theta = \theta_0$. Consider a subsequence $\{n'''\}$ of $\{n''\}$ such that $F_{n'''}(z)$ converges with $n \to \infty$ towards a function $F(z)$. The existence of such a sequence $\{n'''\}$ can be proved in the same way as the similar statement in the proof of Theorem 1. Hence on account of Proposition 2 and (86) we have

$$(87) \qquad \lim_{n=\infty} P(W_{n'''} \mid \theta_{n'''}) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu z)/c} \, dF(z)$$

and

$$(88) \qquad \lim_{n=\infty} P(M_{n'''} \mid \theta_{n'''}) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu z)/c} \, d\Phi(z)$$

where

$$(89) \qquad \Phi(z) = \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2/c} \, dt \quad \text{for} \quad z \leq -z_0,$$

$$(90) \qquad \Phi(z) = \Phi(-z_0) \quad \text{for} \quad -z_0 \leq z \leq z_0$$

$$(91) \qquad \Phi(z) = \Phi(-z_0) + \frac{1}{\sqrt{2\pi c}} \int_{z_0}^{z} e^{-\frac{1}{2}t^2/c} \, dt \quad \text{for} \quad z > z_0,$$

and

$$(92) \qquad \Phi(-z_0) = \tfrac{1}{2}\alpha.$$

From (84), (87) and (88) it follows that

$$(93) \qquad \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu z)/c} \, d[F(z) - \Phi(z)] = \delta.$$

Consider a normally distributed variate y with means ν and variance c. Let B an unbiased critical region of size α for testing the hypothesis $\nu = 0$ by a single observation on y, i.e. B is a subset of the real axis $[-\infty, +\infty]$. Denote by $D(v)$ the intersection of B with the region $C(v)$ defined by the inequality $y < v$. Denote by $H(v)$ the probability of $D(v)$ under the hypothesis $\nu = 0$. Then the power of the test B with respect to the alternative $\nu = \mu$ is given by

$$(94) \qquad \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2 - 2\mu v)/c} \, dH(v).$$

If the region B consists of all points which satisfy at least one of the inequalities $y \leq -v_0$, $y \geq v_0$, and if $v_0 > 0$ is chosen such that the size of B is equal to α, then $H(v) = \Phi(v)$, where $\Phi(v)$ is defined by the equations (89)–(92). Since the latter test is a uniformly most powerful unbiased test,[5] for any μ the inequality

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2-2\mu v)/c}\, d[H(v) - \Phi(v)] \leq 0 \tag{95}$$

holds. Let

$$\psi(v) = \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{v} e^{-\frac{1}{2}t^2/c}\, dt.$$

It is obvious that

$$H(v_2) - H(v_1) \leq \psi(v_2) - \psi(v_1) \quad \text{for} \quad v_2 > v_1, \tag{96}$$

$$\int_{-\infty}^{\infty} dH(v) = \alpha \tag{97}$$

and

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2-2\mu v)/c}\, dH(v) \text{ has a minimum for } \mu = 0, \tag{98}$$

On the other hand, if $K(v)$ is a monotonically non-decreasing non-negative function of v such that

$$K(v_2) - K(v_1) \leq \psi(v_2) - \psi(v_1) \text{ for } v_2 > v_1, \tag{96'}$$

$$\int_{-\infty}^{\infty} dK(v) = \alpha, \tag{97'}$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\mu^2-2\mu v)/c}\, dK(v) \text{ has a minimum for } \mu = 0, \tag{98'}$$

then there exists a sequence $\{B^{(i)}\}$ $(i = 1, 2, \cdots$, ad inf.) of unbiased regions of size α such that

$$\lim_{i=\infty} H^{(i)}(v) = K(v)$$

uniformly in v. Since (95) holds for $H(v) = H^{(i)}(v)$ $(i = 1, 2, \cdots$, ad inf.), and since

$$H^{(i)}(v_2) - H^{(i)}(v_1) \leq \psi(v_2) - \psi(v_1) \text{ for } v_2 > v_1,$$

it is easy to see that (95) holds also for $H(v) = K(v)$. Hence for any monotonically non-decreasing non-negative function $K(v)$ for which (96′), (97′), and (98′) are fulfilled, also (95) must be fulfilled if we substitute $K(v)$ for $H(v)$.

[5] J. Neyman and E. S. Pearson, l. c., p. 29.

Since $F(v)$ is a distribution function which satisfies (96'), (97') and (98'), we have a contradiction to (93). This proves Theorem 3.

5. Appendix. *Proof of the uniform consistency of* $\hat{\theta}_n$. It will be shown here that under certain conditions on the density function $f(x, \theta)$, Assumption 1, i.e. uniform consistency of $\hat{\theta}_n$, can be proved.

For any open subset ω of the θ-axis we denote by $\varphi(x, \omega)$ the least upper bound, and by $\psi(x, \omega)$ the greatest lower bound of $\frac{\partial^2 \log f(x, \theta)}{\partial \theta^2}$ with respect to θ in the set ω. For any function $\lambda(x)$ we denote by $E_\theta \lambda(x)$ the expected value of $\lambda(x)$ under the assumption that θ is the true value of the parameter, i.e.

$$E_\theta \lambda(x) = \int_{-\infty}^{\infty} \lambda(x) f(x, \theta)\, dx.$$

Denote furthermore by $P(\hat{\theta}_n \epsilon\, \omega \mid \theta)$ the probability that $\hat{\theta}_n$ will fall in ω under the assumption that θ is the true value of the parameter. Finally denote by Ω the parameter space and assume that Ω is either the whole real axis or a subset of it.

PROPOSITION 3. *$\hat{\theta}_n$ is a uniformly consistent estimate of θ, i.e. for any positive k*

$$\lim_{n=\infty} P(-k < \hat{\theta}_n - \theta < k \mid \theta) = 1$$

uniformly for all θ in Ω, if the following two conditions are fulfilled:

Condition I. For all values θ in Ω

$$\int_{-\infty}^{\infty} \frac{\partial f(x, \theta)}{\partial \theta}\, dx = \int_{-\infty}^{\infty} \frac{\partial^2 f(x, \theta)}{\partial \theta^2}\, dx = 0.$$

Condition II. For any value θ in Ω there exists an open interval $\omega(\theta)$ containing θ and having the following three properties:

II_a. $$\lim_{n=\infty} P(\hat{\theta}_n \epsilon\, \omega(\theta) \mid \theta] = 1$$

uniformly for all θ in Ω.

II_b. *$E_\theta \varphi^2[x, \omega(\theta)]$ is a bounded function of θ in Ω, and the least upper bound A of $E_\theta \varphi[x, \omega(\theta)]$ with respect to θ in Ω is negative.*

II_c. *$E_\theta \psi[x, \omega(\theta)]$ is a bounded function of θ in the set Ω.*

Condition I means simply that we may differentiate under the integral sign. In fact

$$\int_{-\infty}^{\infty} f(x, \theta) = 1$$

identically in θ. Hence

$$\frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} f(x, \theta)\, dx = \frac{\partial^2}{\partial \theta^2} \int_{-\infty}^{\infty} f(x, \theta)\, dx = 0.$$

Differentiating under the integral sign, we obtain Condition I.

In case that $\omega(\theta)$ is the whole axis Condition II_a reduces to the condition that $\hat{\theta}_n$ exists.

In order to prove Proposition 3, we show first that for any positive η

$$\lim_{n=\infty} P\left[\left(-\eta < \frac{1}{n}\sum_{\alpha=1}^{n} \frac{\partial \log f(x_\alpha, \theta)}{\partial\theta} < \eta\right) \middle| \theta\right] = 1 \tag{99}$$

uniformly for all θ in Ω. We have on account of Condition I

$$E_\theta \frac{\partial \log f(x, \theta)}{\partial\theta} = E_\theta \frac{\partial f(x, \theta)}{\partial\theta}\Big/ f(x, \theta) = \int_{-\infty}^{\infty} \frac{\partial f(x, \theta)}{\partial\theta}\, dx = 0. \tag{100}$$

Since

$$\frac{\partial^2 \log f(x, \theta)}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left[\frac{\partial f(x, \theta)}{\partial\theta}\Big/ f(x, \theta)\right] = \frac{\partial^2 f(x, \theta)}{\partial\theta^2}\Big/ f(x, \theta) - \left\{\frac{\partial f(x, \theta)}{\partial\theta}\Big/ [f(x, \theta]^2\right\}^2$$

we have on account of Condition I

$$E_\theta \left(\frac{\partial \log f(x, \theta)}{\partial\theta}\right)^2 = - E_\theta \frac{\partial^2 \log f(x, \theta)}{\partial\theta^2}. \tag{101}$$

According to Condition II $E_\theta\psi[x, \omega(\theta)] < 0$ and is a bounded function of θ. Since $E_\theta \frac{\partial^2 \log f(x, \theta)}{\partial\theta^2} < 0$ and $> E_\theta\psi[x, \omega(\theta)]$, the left hand side of (101), i.e. the variance of $\frac{\partial \log f(x, \theta)}{\partial\theta}$, is a bounded function of θ. From this and the equation (100) we obtain easily (99). Consider the Taylor expansion

$$\frac{1}{n}\sum_\alpha \frac{\partial \log f(x_\alpha, \theta)}{\partial\theta} = (\theta - \hat{\theta}_n)\frac{1}{n}\sum_\alpha \frac{\partial^2 \log f(x_\alpha, \theta'_n)}{\partial\theta^2}, \tag{102}$$

where θ'_n lies in the interval $[\theta, \hat{\theta}_n]$. Let ϵ be an arbitrary positive number and denote by $Q_n(\theta)$ the region defined by the inequality

$$\left|\frac{1}{n}\sum_\alpha \frac{\partial \log f(x_\alpha, \theta)}{\partial\theta}\right| \leq \epsilon. \tag{103}$$

On account of (99) we have

$$\lim_{n=\infty} P[Q_n(\theta) \mid \theta] = 1 \tag{104}$$

uniformly for all θ in Ω.

Denote by $R_n(\theta)$ the region defined by the inequality

$$\frac{1}{n}\sum_\alpha \varphi[x_\alpha, \omega(\theta)] < \tfrac{1}{2}A < 0. \tag{105}$$

On account of Condition II_b

$$\lim_{n=\infty} P[R_n(\theta) \mid \theta] = 1 \tag{106}$$

uniformly for all θ in Ω. Denote by $B_n(\theta)$ the region in which $\hat{\theta}_n \epsilon \omega(\theta)$. Since in $B_n(\theta)$

$$\frac{1}{n} \sum \frac{\partial^2 \log f(x_\alpha, \theta_n')}{\partial \theta^2} \leq \frac{1}{n} \sum \varphi[x_\alpha, \omega(\theta)]$$

we have in the intersection $R_n'(\theta)$ of $R_n(\theta)$ and $B_n(\theta)$

$$\left| \frac{1}{n} \sum \frac{\partial^2 \log f(x_\alpha, \theta_n')}{\partial \theta^2} \right| > \left| \frac{A}{2} \right|. \tag{107}$$

Denote by $U_n(\theta)$ the intersection of $Q_n(\theta)$ and $R_n'(\theta)$. It is obvious that

$$\lim_{n=\infty} P[U_n(\theta) \mid \theta] = 1 \tag{108}$$

uniformly for all θ in Ω. From (102), (103) and (107) we get that in $U_n(\theta)$

$$| \theta - \hat{\theta}_n | \leq \frac{\epsilon}{|\frac{1}{2}A|} = \frac{2\epsilon}{|A|}. \tag{109}$$

Hence on account of (108)

$$\lim_{n=\infty} P\left(| \theta - \hat{\theta}_n | < \frac{2\epsilon}{|A|} \middle| \theta \right) = 1$$

uniformly for all θ in Ω. Since ϵ can be chosen arbitrarily, Proposition 3 is proved.

Conditions I and II are sufficient but not necessary for the uniform consistency of $\hat{\theta}_n$. For sufficiently small $\omega(\theta)$ the conditions II_b and II_c are rather weak. In fact, on account of (101) we have

$$E_\theta \frac{\partial^2 \log f(x, \theta)}{\partial \theta^2} < 0.$$

Hence for sufficiently small intervals $\omega(\theta)$, under certain continuity conditions, also $E_\theta \varphi[x, \omega(\theta)]$ will be negative. However, in some cases may be difficult to verify II_a for small $\omega(\theta)$. On the other hand, for sufficiently large $\omega(\theta)$ (certainly for $\omega(\theta) = [-\infty, +\infty]$) II_a can easily be verified, but the conditions II_b and II_c might be unnecessarily strong. In cases where II_b or II_c does not hold for $\omega(\theta) = [-\infty, +\infty]$ and the validity of II is not apparent, the following Lemma may be useful:

LEMMA: *Proposition* 3 *remains valid if we substitute for Condition II the conditions*

II′. *Denote by* T_n *the set of all points at which* $\hat{\theta}_n$ *exists and*

$$\sum_\alpha \frac{\partial}{\partial \theta} \log f(x_\alpha, \theta^*) = 0 \tag{110}$$

has at most one solution in θ^*. *Then* $\lim_{n=\infty} P[T_n \mid \theta] = 1$ *uniformly for all* θ *in* Ω, *and*

II″. *There exists a positive* k *such that for* $\omega(\theta) = I(\theta) = (\theta - k, \theta + k)$ *the following two conditions hold*:

II_b''. *$E_\theta \varphi^2[x, I(\theta)]$ is a bounded function of θ in Ω and the least upper bound A of $E_\theta\varphi[x, I(\theta)]$ with respect to θ in Ω is negative.*

II_c''. *$E_\theta\psi[x, I(\theta)]$ is a bounded function of θ in the set Ω.* In cases where II_b or II_c is not fulfilled for $\omega(\theta) = [-\infty, +\infty]$ the verification of II and II″ may be easier than that of II.

Our Lemma can be proved as follows: Consider the Taylor expansion

$$(111) \qquad \frac{1}{n}\Sigma\frac{\partial}{\partial\theta}\log f(x_\alpha, \theta^*) = \frac{1}{n}\Sigma\frac{\partial}{\partial\theta}\log f(x_\alpha, \theta) + (\theta^* - \theta)\frac{1}{n}\Sigma\frac{\partial^2}{\partial\theta^2}\log f(x_\alpha, \theta')$$

where θ' lies in $[\theta, \theta^*]$. Denote by $V_n(\theta)$ the region defined by

$$(112) \qquad \frac{1}{n}\Sigma\varphi[x_\alpha, I(\theta)] < \tfrac{1}{2}A < 0.$$

On account of II_b'' we have

$$(113) \qquad \lim_{n=\infty} P[V_n(\theta)\,|\,\theta] = 1$$

uniformly for all θ in Ω. Let $W_n(\theta)$ be the region defined by

$$(114) \qquad \left|\frac{1}{n}\Sigma\frac{\partial}{\partial\theta}\log f(x_\alpha, \theta)\right| < \epsilon.$$

From Condition I and Condition II_c'' it follows easily that

$$(115) \qquad \lim_{n=\infty} P[W_n(\theta)\,|\,\theta] = 1$$

uniformly for all θ in Ω. For all values θ^* in the interval $I(\theta)$ we have

$$(116) \qquad \frac{1}{n}\Sigma\varphi[x_\alpha, I(\theta)] \geq \frac{1}{n}\Sigma\frac{\partial^2}{\partial\theta^2}\log f(x_\alpha, \theta').$$

Because of (112) and (116) we have in $V_n(\theta)$

$$(117) \qquad \frac{1}{n}\Sigma\frac{\partial^2}{\partial\theta^2}\log f(x_\alpha, \theta') < \tfrac{1}{2}A < 0$$

for all values θ^* in the interval $I(\theta)$. Let ϵ be less than $|\tfrac{1}{2}kA|$. Then in the intersection $W_n'(\theta)$ of the regions $V_n(\theta)$ and $W_n(\theta)$ we obviously have on account of (114) that the values of the left hand side of (111) for $\theta^* = \theta + k$ and $\theta^* = \theta - k$ will be of opposite sign. Hence at any point of $W_n'(\theta)$ the equation (110) has at least one root which lies in the interval $I(\theta)$. Since (110) has at most one root in T_n and since $\hat\theta_n$ is a root of (110), we get that at any point of the intersection $W_n''(\theta)$ of $W_n'(\theta)$ and T_n, $\hat\theta_n$ lies in $I(\theta)$. Since

$$(118) \qquad \lim_{n=\infty} P[W''(\theta)\,|\,\theta] = 1 \quad \text{uniformly for all } \theta \text{ in } \Omega,$$

also

$$(119) \qquad \lim_{n=\infty} P[\hat\theta_n \,\epsilon\, I(\theta)\,|\,\theta] = 1$$

uniformly for all θ in Ω. The relation (119) combined with the conditions II_b'' and II_c'' is equivalent to Condition II. Hence our Lemma is proved.