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Asymptotics and sharp bounds in the Poisson approximation to the Poisson-binomial distribution

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The Poisson-binomial distribution is approximated by a Poisson law with respect to a new multi-metric (difference metric) unifying a broad class of probability metrics between discrete distributions. The accompanying non-metric situation is also considered leading to moderate- and large-deviation results. Using the Charlier B expansion and Fourier arguments, sharp bounds and asymptotic relations are given.

Keywords: Charlier B expansion; large deviations; moderate deviations; Poisson approximation; Poisson-binomial distribution; probability metrics

1. Introduction

Let S_n be the sum of independent Bernoulli random variables $X_1, \ldots, X_n$ with success probabilities $P(X_j = 1) = 1 - P(X_j = 0) = p_j \in [0, 1]$ for $1 \leqslant j \leqslant n$. We investigate the approximation of the distribution P^{S_n} of S_n by a Poisson law $\mathscr{P}(t)$ with mean $t \in (0, \infty)$ and also by finite signed measures derived from an expansion due to Charlier (1905). As a measure of accuracy, a new multi-metric (difference metric) is introduced (see formula (1)) unifying a broad class of probability metrics between discrete distributions. Further, the accompanying non-metric situation is investigated, leading to moderate- and large-deviation results. The task is to give sharp bounds and asymptotic relations. The method used is based on work by Shorgin (1977), Deheuvels and Pfeifer (1986a; 1986b; 1988), Deheuvels *et al*. (1989) and Roos (1995). For other publications on Poisson approximations, see, for example, Barbour (1987), Barbour *et al*. (1992; 1995), Chen and Choi (1992), Deheuvels (1992) and Prohorov (1953).

We proceed with the definition of the difference metric. Some notation is needed. Let $\mathbb{Z}_+ = \{0, 1, \ldots\}$ and $\mathbb{R}^{\mathbb{Z}_+} = \{f | f : \mathbb{Z}_+ \to \mathbb{R}\}$. For $f \in \mathbb{R}^{\mathbb{Z}_+}$, let $\|f\|_p$ $(p \in [1, \infty])$ be the p-norm of f and set $f(m) = 0$ for $m < 0$. In this paper, we define the difference operator $\Delta : \mathbb{R}^{\mathbb{Z}_+} \to \mathbb{R}^{\mathbb{Z}_+}$ by $\Delta f(m) = f(m-1) - f(m)$ for $f \in \mathbb{R}^{\mathbb{Z}_+}$ and $m \in \mathbb{Z}_+$; for the inverse Δ^{-1}, we have $\Delta^{-1} f(m) = -\sum_{j=0}^{m} f(j)$. The difference metric between two finite signed measures Q_1 and Q_2 concentrated on $\mathbb{Z}_+$ is defined by

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$$d_p^{(i,j)}(Q_1, Q_2) = \left\| \left(\sum_{u=m-j+1}^{m} \Delta^i (f_{Q_1} - f_{Q_2})(u) \right)_{m \in \mathbb{Z}_+} \right\|_p, \tag{1}$$

where $i \in \mathbb{Z}$, $j \in \mathbb{N}$, $p \in [1, \infty]$, and $f_{Q_1}, f_{Q_2} \in \mathbb{R}^{\mathbb{Z}_+}$ are the counting densities of Q_1, Q_2. If we consider the restriction of $d_p^{(i,j)}$ to the set of all probability measures concentrated on $\mathbb{Z}_+$, we get the total variation distance $\frac{1}{2}d_1^{(0,1)}$, the p-metric between distribution functions $d_p^{(-1,1)}$, the Kolmogorov metric $d_\infty^{(-1,1)}$, the Fortet–Mourier metric $d_1^{(-1,1)}$ and the j-point metric $d_\infty^{(0,j)}$. Here, the j-point metric indicates the largest difference between the probabilities of half-open intervals of length j.

2. The bounds for the difference metric

For $k \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, and $t \in [0, \infty)$, let $\pi(\cdot, t) \in \mathbb{R}^{\mathbb{Z}_+}$ with $\pi(m, t) = \mathscr{P}(t)(\{m\}) = \mathrm{e}^{-t} t^m / m!$ and write $\Delta^k \pi(m, t) = (\Delta^k \pi(\cdot, t))(m)$. Here and throughout the rest of the paper, we let $0^0 = 1$. Let

$$p_k(x, t) = \sum_{j=0}^{k} \binom{k}{j} \binom{x}{j} j! \, (-t)^{k-j}, \qquad t, x \in \mathbb{R}, \; k \in \mathbb{Z}_+ \tag{2}$$

be the Charlier polynomial of degree k. The following theorem is the principal tool in the argument of this paper.

Theorem 1. *For $m \in \mathbb{Z}_+$ and $t \in (0, \infty)$,*

$$P(S_n = m) = \sum_{k=0}^{\infty} a_k(t) \Delta^k \pi(m, t), \tag{3}$$

where $a_k(t) = (1/k!) \sum_{m=0}^{\infty} P(S_n = m) p_k(m, t)$, $k \in \mathbb{Z}_+$.

For the proof of a more general theorem, see Schmidt (1933). The series (3) is called the Charlier (B) expansion of P^{S_n}. The coefficients $a_k(t)$ are called Charlier coefficients of P^{S_n}. For further papers on the Charlier expansion, see Boas (1949) and the references therein. We now give a review of some well-known relations for $p_k(x, t)$ and $\Delta^k \pi(m, t)$.

Lemma 1. (a) *For $x, t \in \mathbb{R}$ and $k \in \mathbb{N}$,*

$$k p_{k-1}(x-1, t) = p_k(x, t) - p_k(x-1, t), \tag{4}$$

$$p_{k+1}(x, t) = (x - k - t) p_k(x, t) - t k p_{k-1}(x, t). \tag{5}$$

(b) *For $t \in (0, \infty)$, $k, m \in \mathbb{Z}_+$, and $z \in \mathbb{C}$,*

$$t^k \Delta^k \pi(m, t) = \pi(m, t) p_k(m, t), \tag{6}$$

$$\sum_{j=0}^{\infty} \Delta^k \pi(j, t) z^j = \exp(t(z-1))(z-1)^k, \tag{7}$$

$$\Delta^k \pi(m,\, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-\mathrm{i}xm + t(\mathrm{e}^{\mathrm{i}x} - 1))(\mathrm{e}^{\mathrm{i}x} - 1)^k \,\mathrm{d}x. \tag{8}$$

In what follows, let $\lambda_k = \sum_{j=1}^n p_j^k$ for $k \in \mathbb{N}$, $\lambda = \lambda_1 > 0$, and $\eta(t) = 2\lambda_2 + (\lambda - t)^2$ for $t \in (0,\, \infty)$. The following lemma is devoted to the Charlier coefficients of P^{S_n}.

Lemma 2. *Let* $k \in \mathbb{N}$ *and* $t \in (0,\, \infty)$. *Further, let* $I_0(x) = \sum_{m=0}^{\infty}(x/2)^{2m}/(m!)^2$ *be the* modified Bessel function of the first kind and order 0, $\beta(x) = \mathrm{e}^{-x^2/4}I_0(x)$, $x \in \mathbb{R}$. *Then*

$$a_k(t) = \frac{1}{k}\left(a_{k-1}(t)(\lambda - t) + \sum_{j=0}^{k-2}(-1)^{k-j-1}a_j(t)\lambda_{k-j}\right), \tag{9}$$

$$|a_k(t)| \leqslant \left(\frac{\eta(t)\,\mathrm{e}}{2k}\right)^{k/2}\beta\left(\frac{|\lambda - t|\sqrt{2k}}{\sqrt{\eta(t)}}\right). \tag{10}$$

Proof. Let $H(z) = \prod_{j=1}^n (1 + p_j(z-1))$, $z \in \mathbb{C}$, be the probability generating function of S_n. As in Schmidt (1933, p. 141), $\sum_{k=0}^{\infty} a_k(t)z^k = \mathrm{e}^{-tz}H(z+1)$ for $z \in \mathbb{C}$. Let $h(z) = \ln(H(z+1)) - tz$ and $g(z) = \exp(h(z))$ for $|z| < 1$. Then (9) follows from

$$\begin{aligned} a_k(t) &= \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z^k}g(z)\bigg|_{z=0} = \frac{1}{k!}\sum_{j=0}^{k-1}\left(\binom{k-1}{j}\frac{\mathrm{d}^j}{\mathrm{d}z^j}g(z)\frac{\mathrm{d}^{k-j}}{\mathrm{d}z^{k-j}}h(z)\right)\bigg|_{z=0} \\ &= \frac{1}{k}\sum_{j=0}^{k-1}\frac{a_j(t)}{(k-1-j)!}\frac{\mathrm{d}^{k-j}}{\mathrm{d}z^{k-j}}h(z)\bigg|_{z=0}. \end{aligned}$$

Let $\alpha \in (0,\, \infty)$. By Cauchy's theorem,

$$a_k(t) = \frac{1}{2\pi\alpha^k}\int_0^{2\pi}\exp(-\mathrm{i}kx - \alpha t\mathrm{e}^{\mathrm{i}x})\prod_{j=1}^n(1 + p_j\alpha\mathrm{e}^{\mathrm{i}x})\,\mathrm{d}x.$$

By $1 + x \leqslant \mathrm{e}^x$ and $I_0(x) = (1/[2\pi])\int_0^{2\pi}\exp(x\cos y)\,\mathrm{d}y$ for $x \in \mathbb{R}$, this leads to

$$\begin{aligned} |a_k(t)| &\leqslant \frac{1}{2\pi\alpha^k}\int_0^{2\pi}\exp(-\alpha t\cos x)\prod_{j=1}^n(1 + 2\alpha p_j\cos x + R^2p_j^2)^{1/2}\,\mathrm{d}x \\ &\leqslant Y(\alpha)\beta(\alpha(\lambda - t)), \end{aligned}$$

where $Y(\alpha) := \alpha^{-k}\exp(\alpha^2\eta(t)/4)$ attains its minimum for $\alpha = \alpha_0 := \sqrt{2k/\eta(t)}$. Relation (10) is proved by substituting α_0 for α. □

Note that Shorgin (1977) showed (9) and (10) in the case $\lambda = t$. Using (9), we derive

$$a_0(t) = 1, \qquad a_1(t) = \lambda - t, \qquad a_2(t) = \frac{(\lambda - t)^2 - \lambda_2}{2}. \tag{11}$$

For the rest of this paper, let $\beta(x)$ as in Lemma 2. It is clear that $\beta(x_1) \leqslant \beta(x_2)$ for $0 \leqslant x_2 \leqslant x_1$, and that $0 < \beta(|x|) = \beta(x) \leqslant 1$ for $x \in \mathbb{R}$. For the bounds for the difference metric, we need the following lemma.

Lemma 3. *For* $k \in \mathbb{N}$, $t \in (0, \infty)$, *and* $p \in [1, \infty]$,

$$\|\Delta^k \pi(\cdot, t)\|_p \leqslant \frac{\sqrt{\mathrm{e}}}{2}\left(1 + \sqrt{\frac{\pi}{2}}\right)(2k)^{1/p}\,\mathrm{e}^{-1/(2p)}\left(\frac{k}{t\mathrm{e}}\right)^{(k+1)/2-1/(2p)}. \tag{12}$$

Proof. Shorgin (1977; see the proof of his Lemma 6) proved (12) for $p = \infty$. For $p = 1$, (12) can be shown by using (23) (which can be proved independently) and the inequality $1 + x \leqslant \mathrm{e}^x$, $x \in \mathbb{R}$. The assertion is shown by using the convexity theorem: $\|f\|_p \leqslant \|f\|_{q'}^s \|f\|_q^{1-s}$ if $1 \leqslant q < p < q' \leqslant \infty$, $f \in \mathbb{R}^{\mathbb{Z}_+}$, $0 < s < 1$, and $1/p = s/q' + (1-s)/q$. $\square$

For $x \in \mathbb{R}$, let $\lfloor x \rfloor, \lceil x \rceil \in \mathbb{Z}$ be defined by $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$. For $t \in (0, \infty)$, let $\tilde{\theta}(t) = \eta(t)/(2t)$. We now give the main result of this section.

Theorem 2. *Let* $t \in (0, \infty)$, $k \in \mathbb{Z}_+$, $s \in \{k, k+1, k+2, \ldots\}$, $i \in \{-s, -s+1, -s+2, \ldots\}$, $j \in \mathbb{N}$, $p \in [1, \infty]$, $r = (i+1)/2 - 1/(2p)$,

$$C_1 = \frac{j(1 + \sqrt{\pi/2})}{2^{1-1/p}(s+i+1)^{\lceil r+1/p \rceil - r - 1/p}}, \qquad C_2 = \frac{j(2\pi)^{1/4}\exp(1/(24(s+1)))2^{(s+1)/2+i}}{(s+1)^{1/4}\sqrt{s!}},$$

and

$$\nu(t) = \beta\left(\frac{|\lambda - t|\sqrt{2(s+1)}}{\sqrt{\eta(t)}}\right).$$

Let $Q(k, t)$ *denote the finite signed measure concentrated on* $\mathbb{Z}_+$ *with the counting density* $f_{Q(k,t)}(m) = \sum_{v=0}^{k} a_v(t)\Delta^v \pi(m, t)$, $m \in \mathbb{Z}_+$. *Then* $d_p^{(i,j)}(P^{S_n}, Q(k, t)) = H + R$, *where*

$$H = \left\| \sum_{u=k+1}^{s+j-1} \left[\sum_{v=k+1}^{\min\{u,s\}} a_v(t) \binom{j}{u - v + 1} \right] \Delta^{u+i} \pi(\cdot, t) \right\|_p \tag{13}$$

and

$$|R| \leqslant \frac{C_1 \nu(t)}{\tilde{\theta}(t)^{i/2} t^r} \sum_{v=s+i+1}^{\infty} \tilde{\theta}(t)^{v/2} v^{\lceil r+1/p \rceil}, \tag{14}$$

$$|R| \leqslant C_2 \nu(t) \eta(t)^{(s+1)/2}\left(1 + \sqrt{2\eta(t)}\right)\exp(2\eta(t)). \tag{15}$$

Proof. By

$$\sum_{u=m-j+1}^{m} f(u) = \sum_{w=0}^{j-1} \binom{j}{w+1} \Delta^w f(m), \qquad f \in \mathbb{R}^{\mathbb{Z}_+},\ m \in \mathbb{Z}_+,$$

it is easy to see that, for $m \in \mathbb{Z}_+$,

$$\sum_{u=m-j+1}^{m} \Delta^i(f_{P^{S_n}} - f_{Q(k,t)})(u) = \sum_{v=k+1}^{\infty} a_v(t) \sum_{w=0}^{j-1} \binom{j}{w+1} \Delta^{w+v+i}\pi(m, t) = H_m + R_m,$$

where

$$\begin{aligned} H_m &= \sum_{v=k+1}^{s} a_v(t) \sum_{w=0}^{j-1} \binom{j}{w+1} \Delta^{w+v+i}\pi(m, t) \\ &= \sum_{u=k+1}^{s+j-1} \left[\sum_{v=k+1}^{\min\{u,\, s\}} a_v(t) \binom{j}{u-v+1} \right] \Delta^{u+i}\pi(m, t), \end{aligned}$$

$$R_m = \sum_{v=s+1}^{\infty} a_v(t) \sum_{w=0}^{j-1} \binom{j}{w+1} \Delta^{w+v+i}\pi(m, t) = \sum_{u=m-j+1}^{m} \Delta^i(f_{P^{S_n}} - f_{Q(s,t)})(u).$$

Hence $d_p^{(i,j)}(P^{S_n}, Q(k, t)) = \|(H_m + R_m)_{m \in \mathbb{Z}_+}\|_p$ and $H = \|(H_m)_{m \in \mathbb{Z}_+}\|_p$. Relation (14) can be shown by using

$$|R| \leqslant \|(R_m)_{m\in\mathbb{Z}_+}\|_p \leqslant j \left\| \sum_{v=s+1}^{\infty} a_v(t) \Delta^{v+i}\pi(\cdot, t) \right\|_p \leqslant j \sum_{v=s+1}^{\infty} |a_v(t)| \, \|\Delta^{v+i}\pi(\cdot, t)\|_p, \tag{16}$$

and the inequalities (10), (12), and $1 + x \leqslant e^x$, $x \in \mathbb{R}$. Relation (15) can be proved by using (16), (10), the inequality $\|\Delta^v \pi(\cdot, t)\|_p \leqslant 2^v$ ($v \in \mathbb{Z}_+$), Stirling's formula (see Feller 1968)

$$v! = \sqrt{2\pi}\, v^{v+1/2} \exp(\vartheta_v - v), \qquad \vartheta_v \in \left[\frac{1}{12v+1}, \frac{1}{12v} \right], \, v \in \mathbb{N}, \tag{17}$$

and

$$\sum_{m=v}^{\infty} \frac{x^m}{\sqrt{m!}} \leqslant \frac{x^v}{\sqrt{v!}} \sum_{m=0}^{\infty} \frac{x^m}{\sqrt{m!}} \binom{m}{\lfloor m/2 \rfloor}^{1/2} \leqslant \frac{x^v}{\sqrt{v!}} (x+1)\exp(x^2) \tag{18}$$

for $v \in \mathbb{Z}_+$, $x \in [0, \infty)$. □

Note that $Q(0, t) = \mathscr{P}(t)$, $Q(1, \lambda) = \mathscr{P}(\lambda)$, and that Barbour (1987) used other signed measures for the total variation distance. Only the first two of his signed measures coincide with $Q(0, \lambda)$ and $Q(2, \lambda)$. Observe that always $\nu(t) \leqslant 1$.

3. Evaluation of the norm term

In what follows, the norm term H in Theorem 2 is evaluated in the cases $p = 1$ and $p = \infty$. Using these formulae, upper and lower estimates of the corresponding distances can be derived. The following two propositions are generalizations of results by Deheuvels and Pfeifer (1986a; 1986b) and Roos (1995). The proofs are easy and therefore omitted.

Proposition 1. *Let* $q(x) = \sum_{m=0}^{k} c_m t^{k-m} p_{m+1}(x, t)$, *where* $t \in (0, \infty)$, $k \in \mathbb{Z}_+$, *and* $c_0, \ldots, c_k \in \mathbb{R}$, $c_k \neq 0$. *Then* $q(x)$ *has at least one zero in* $(0, \infty)$. *If* $q(x)$ *has exactly* $u \in \{1, \ldots, k+1\}$ *different zeros in* $[0, \infty)$, *denoted by* $x_1(t) < \ldots < x_u(t)$, *then*

$$\left\| \sum_{m=0}^{k} c_m \Delta^m \pi(\cdot, t) \right\|_\infty = \max_{1 \leqslant v \leqslant u} \left| \sum_{m=0}^{k} c_m \Delta^m \pi(\lfloor x_v(t) \rfloor, t) \right|. \tag{19}$$

Proposition 2. *Let* $q(x) = \sum_{m=0}^{k} c_m t^{k-m} p_m(x, t)$, *where* $t \in (0, \infty)$, $k \in \mathbb{Z}_+$, *and* $c_0, \ldots, c_k \in \mathbb{R}$, $c_k \neq 0$. *If, under considerations of multiplicity,* $q(x)$ *has exactly* $u \in \{0, \ldots, k\}$ *zeros in* $[0, \infty)$, *denoted by* $x_1(t) \leqslant \ldots \leqslant x_u(t)$ (*if* $u \geqslant 1$), *then*

$$\left\| \sum_{m=0}^{k} c_m \Delta^m \pi(\cdot, t) \right\|_1 = \left| (-1)^u c_0 + 2 \sum_{m=0}^{k} c_m \sum_{v=1}^{u} (-1)^v \Delta^{m-1} \pi(\lfloor x_v(t) \rfloor, t) \right|. \tag{20}$$

From the theory of orthogonal polynomials it is known that the zeros of the Charlier polynomials $p_k(x, t)$, $k \in \mathbb{N}$, $t \in (0, \infty)$, are real, simple and located in the interval $(0, \infty)$. The preceeding propositions lead to the following corollaries.

Corollary 1. *Let* $k \in \mathbb{Z}_+$, $t \in (0, \infty)$, *and* $0 < x_1(t) < \ldots < x_{k+1}(t)$ *be the zeros of* $p_{k+1}(x, t)$. *Then*

$$\|\Delta^k \pi(\cdot, t)\|_\infty = \max_{1 \leqslant v \leqslant k+1} |\Delta^k \pi(\lfloor x_v(t) \rfloor, \mathrm{t})|. \tag{21}$$

Corollary 2. *Let* $k \in \mathbb{N}$, $t \in (0, \infty)$, *and* $0 < x_1(t) < \ldots < x_k(t)$ *be the zeros of* $p_k(x, t)$. *Then*

$$\|\Delta^k \pi(\cdot, t)\|_1 = 2 \left| \sum_{v=1}^{k} (-1)^v \Delta^{k-1} \pi(\lfloor x_v(t) \rfloor, t) \right| = 2 \sum_{v=1}^{k} |\Delta^{k-1} \pi(\lfloor x_v(t) \rfloor, t)|. \tag{22}$$

Proof. The first equality follows from Proposition 2. The second equality is proved if it is shown that $p_{k-1}(\lfloor x_v(t) \rfloor, t)$ alternates in sign as v varies from 1 through k. But this is a consequence of the following lemma. □

Note that the inequalities

$$\|\Delta^k \pi(\cdot, t)\|_1 \leqslant 2k \|\Delta^{k-1} \pi(\cdot, t)\|_\infty \leqslant k \|\Delta^k \pi(\cdot, t)\|_1, \qquad k \in \mathbb{N}, t \in (0, \infty), \tag{23}$$

follow from (21) and (22), where equalities hold for $k = 1$. The first inequality of (23) is used in the proof of (12). The following lemma is needed to complete the proof of Corollary 2.

Lemma 4. *Let* $t \in (0, \infty)$ *and* $0 < x_{k,1}(t) < \ldots < x_{k,k}(t)$ *be the zeros of* $p_k(x, t)$ for $k \in \mathbb{N}$. *Then* $x_{k+1,v}(t) < x_{k,v}(t) < x_{k+1,v+1}(t) - 1$ *for* $k \in \mathbb{N}$ *and* $v \in \{1, \ldots, k\}$.

Note that the relation above without the '-1' is the well-known separation theorem for the zeros of the Charlier polynomials.

Proof. By induction over k, it can be shown that $x_{k,\upsilon+1}(t) - x_{k,\upsilon}(t) > 1$ for $k \in \{2, 3, \ldots\}$ and $\upsilon \in \{1, \ldots, k-1\}$. Here the separation theorem and (4) are used. Now the assertion can easily be proved. □

In the following lemma, some additional properties of the norm term are given.

Lemma 5. *Let* $k \in \mathbb{Z}_+$, $c_0, \ldots, c_k \in \mathbb{R}$, $c_k \neq 0$, *and* $p \in [1, \infty]$. *Then the norm* $\|\sum_{m=0}^{k} c_m \Delta^m \pi(\cdot, t)\|_p$ *is a* $(0, \infty)$*-valued, continuous function of* $t \in [0, \infty)$.

Proof. The assertion can easily be shown by using

$$|p_k(m, t)| \leqslant \sum_{w=0}^{k} \binom{k}{w} m^w t^{k-w} = (m+t)^k, \qquad k, m \in \mathbb{Z}_+, \ t \in (0, \infty), \tag{24}$$

Minkowski's inequality, (6), (7) and (19). □

4. Asymptotic relations for the norm term

In what follows, let

$$H_k(x) = k! \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(-1)^m (2x)^{k-2m}}{(k-2m)!\, m!} \tag{25}$$

be the Hermite polynomial of degree $k \in \mathbb{Z}_+$. We need the well-known relations

$$H_{k+1}(x) = 2x\, H_k(x) - 2k\, H_{k-1}(x), \qquad k \in \mathbb{N}, \ x \in \mathbb{R}, \tag{26}$$

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2} = \frac{(-1)^k}{\sqrt{2\pi} 2^{k/2}} \mathrm{e}^{-x^2/2} H_k(x/\sqrt{2}), \qquad k \in \mathbb{Z}_+, \ x \in \mathbb{R}. \tag{27}$$

For $p \in [1, \infty]$, let $\|\varphi_k\|_p$ be the p-norm of φ_k.

Proposition 3. *Let* $k \in \mathbb{Z}_+$ *and* $b: (0, \infty) \times \mathbb{R} \to \mathbb{R}$ *be bounded. For* $t \to \infty$,

$$\sup_{x \in \mathbb{R}} |t^{(k+1)/2} \Delta^k \pi(\lfloor t + x\sqrt{t} + b(t, x) \rfloor, t) - (-1)^k \varphi_k(x)| = \mathcal{O}(t^{-1/2}), \tag{28}$$

$$\sup_{x \in \mathbb{R}} (1 + x^2) |t^{(k+1)/2} \Delta^k \pi(\lfloor t + x\sqrt{t} + b(t, x) \rfloor, t) - (-1)^k \varphi_k(x)| = \mathcal{O}(t^{-1/2}). \tag{29}$$

Proof. We use Fourier techniques as in Petrov (1975). Let $k \in \mathbb{Z}_+$, $x \in \mathbb{R}$, $t \in (0, \infty)$, *and* $m = \lfloor t + x\sqrt{t} + b(t, x) \rfloor \geqslant 0$. Then $m = t + x\sqrt{t} + \tilde{b}(t, x)$, where $\tilde{b}: (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is bounded. We write $\tilde{b}$ for $\tilde{b}(t, x)$. Using (8) and

$$\varphi_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\mathrm{i}xy - y^2/2)(-\mathrm{i}y)^k \, \mathrm{d}y,$$

it is easy to see that

$$
\begin{aligned}
&2\pi|t^{(k+1)/2}\Delta^k\pi(m,\ t)-(-1)^k\varphi_k(x)| \\
&\quad\leqslant t^{(k+1)/2}\int_{-\pi}^{\pi}\mathrm{e}^{-ty^2/2}|\exp(t[\mathrm{e}^{\mathrm{i}y}-1-\mathrm{i}y+y^2/2]-\mathrm{i}y\tilde{b})(\mathrm{e}^{\mathrm{i}y}-1)^k-(\mathrm{i}y)^k|\,\mathrm{d}y+I_1 \\
&\quad\leqslant t^{(k+1)/2}(I_2+I_3+I_4)+I_1,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= 2\int_{\pi\sqrt{t}}^{\infty}\mathrm{e}^{-y^2/2}y^k\,\mathrm{d}y, \\
I_2 &= \int_{-\pi}^{\pi}\mathrm{e}^{-ty^2/2}|\exp(t[\cos y-1+y^2/2])-1|\,|\mathrm{e}^{\mathrm{i}y}-1|^k\,\mathrm{d}y, \\
I_3 &= \int_{-\pi}^{\pi}\mathrm{e}^{-ty^2/2}|(\mathrm{e}^{\mathrm{i}y}-1)^k-(\mathrm{i}y)^k|\,\mathrm{d}y, \\
I_4 &= \int_{-\pi}^{\pi}\mathrm{e}^{-ty^2/2}|\exp(\mathrm{i}[t(\sin y-y)-y\tilde{b}])-1|\,|\mathrm{y}|^k\,\mathrm{d}y.
\end{aligned}
$$

Using calculus, it is possible to show that, for $t\to\infty$, $I_1=\mathscr{O}(\exp(-\pi^2t/2)t^{(k-1)/2})$, $I_2=\mathscr{O}(t^{-(k+3)/2})$, $I_3=\mathscr{O}(t^{-(k+2)/2})$, and $I_4=\mathscr{O}(t^{-(k+2)/2})$. For I_2, the inequality

$$
I_2\leqslant 2^{k+3}\left(\frac{\pi^2}{4}-1\right)t\int_0^{\pi/2}\exp(-2t\sin^2 y)\sin^{k+4}y\,\mathrm{d}y
$$

and Shorgin's (1977; see the proof of his Lemma 6) estimate of the integral are used. For I_3, we use $|(\mathrm{e}^{\mathrm{i}y}-1)^k-(\mathrm{i}y)^k|\leqslant k|y|^{k+1}\mathrm{e}^{k|y|}$ and similar estimates for I_4. Hence

$$
\sup_{x\in A(t)}|t^{(k+1)/2}\Delta^k\pi(\lfloor t+x\sqrt{t}+b(t,\ x)\rfloor,\ t)-(-1)^k\varphi_k(x)|=\mathscr{O}(t^{-1/2}),\qquad t\to\infty,
$$

where $A(t)=\{x\in\mathbb{R}|\lfloor t+x\sqrt{t}+b(t,\ x)\rfloor\geqslant 0\}$. The proof of (28) is easily completed. To prove (29), it suffices to estimate $T=x^2|t^{(k+1)/2}\Delta^k\pi(m,\ t)-(-1)^k\varphi_k(x)|$ uniformly in $x\in A(t)$. Using (5), (6), (26), (27), and $x=(m-t-\tilde{b})/\sqrt{t}$, we obtain

$$
\begin{aligned}
x^2\Delta^k\pi(m,\ t) = {}& t\Delta^{k+2}\pi(m,\ t)+[2k-2\tilde{b}+1]\Delta^{k+1}\pi(m,\ t)+[2k+1+t^{-1}(k-\tilde{b})^2]\Delta^k\pi(m,\ t) \\
&+t^{-1}k[2k-2\tilde{b}-1]\Delta^{k-1}\pi(m,\ t)+k(k-1)t^{-1}\Delta^{k-2}\pi(m,\ t),
\end{aligned}
$$

and $x^2\varphi_k(x)=\varphi_{k+2}(x)+(2k+1)\varphi_k(x)+k(k-1)\varphi_{k-2}(x)$, where $\varphi_{-2}(x)=\varphi_{-1}(x)=0$. Hence

$$T \leqslant |t^{(k+3)/2}\Delta^{k+2}\pi(m,\, t) - (-1)^{k+2}\varphi_{k+2}(x)| + [2k+2|\tilde{b}|+1]t^{(k+1)/2}\|\Delta^{k+1}\pi(\cdot,\, t)\|_\infty$$
$$+ (k-\tilde{b})^2 t^{(k-1)/2}\|\Delta^k\pi(\cdot,\, t)\|_\infty + [2k+1]|t^{(k+1)/2}\Delta^k\pi(m,\, t) - (-1)^k\varphi_k(x)|$$
$$+ k[2k+2|\tilde{b}|+1]t^{(k-1)/2}\|\Delta^{k-1}\pi(\cdot,\, t)\|_\infty$$
$$+ k(k-1)|t^{(k-1)/2}\Delta^{k-2}\pi(m,\, t) - (-1)^{k-2}\varphi_{k-2}(x)|.$$

Using the estimates (12), (28), and $\|\pi(\cdot,\, t)\|_\infty \leqslant (2te)^{-1/2}$ for $t \in (0, \infty)$ (see Deheuvels and Pfeifer 1988), the proof is easily completed. □

Proposition 4. *Let $p \in [1, \infty]$ and $k \in \mathbb{Z}_+$. Then*

$$\exists M \in (0, \infty)\ \forall t \in (0, \infty) :\ |t^{(k+1)/2-1/(2p)}\|\Delta^k\pi(\cdot,\, t)\|_p - \|\varphi_k\|_p| \leqslant \frac{M}{\sqrt{t}}. \qquad (30)$$

Proof. For sufficiently large t the assertion is shown by using (28), (29) and

$$t^{-1/(2p)}\|\Delta^k\pi(\cdot,\, t)\|_p = \left(\int_{-\infty}^{\infty} |\Delta^k\pi(\lfloor t + x\sqrt{t}\rfloor,\, t)|^p\,\mathrm{d}x\right)^{1/p}, \qquad p \in [1, \infty).$$

For small t, Lemma 5 is used. □

5. Asymptotic results for the difference metric

For the following theorem, we consider a triangular scheme: we let n and $X_1, \ldots, X_n$ depend on an additional parameter $l \in \mathbb{N}$ and assume that $l \to \infty$. Then the following quantities also depend on l: S_n, $p_1, \ldots, p_n$, λ_k for $k \in \mathbb{N}$, $\eta(t)$, and $\tilde{\theta}(t)$ for $t \in (0, \infty)$. Let $\theta = \tilde{\theta}(\lambda) = \lambda_2/\lambda$. Note that $\theta \leqslant 1$. Sometimes we write $\theta^{(l)}$ for θ. We now present the main result of this section.

Theorem 3. *Let $i \in \{-2, -1, \ldots\}$, $j \in \mathbb{N}$, $p \in [1, \infty]$ be independent of l. Further, let*

$$H_p^{(i,j)}(t) = \left\| \sum_{u=2}^{j+1} \binom{j}{u-1} \Delta^{u+i}\pi(\cdot,\, t) \right\|_p, \qquad t \in [0, \infty).$$

If $\limsup_{l\to\infty}\theta^{(l)} < 1$, *then*

$$d_p^{(i,j)}(P^{S_n}, \mathscr{P}(\lambda)) = \frac{\lambda_2}{2} H_p^{(i,j)}(\lambda)\left[1 + \mathscr{O}\left(\min\left\{\frac{\lambda_3}{\lambda_2\sqrt{\lambda}} + \theta,\, \frac{\lambda_3}{\lambda_2} + \lambda_2\right\}\right)\right], \qquad (31)$$

and

$$d_p^{(i,j)}(P^{S_n}, \mathscr{P}(\lambda)) = \frac{j\|\varphi_{i+2}\|_p\theta}{2\lambda^{(i+1)/2-1/(2p)}}\left[1 + \mathscr{O}\left(\min\left\{1,\, \frac{1}{\sqrt{\lambda}} + \theta\right\}\right)\right]. \qquad (32)$$

If $a \in [0, \infty)$ is independent of l and $\lambda_2 = \mathscr{O}(1)$, then

$$d_p^{(i,j)}(P^{S_n}, \mathscr{P}(\lambda)) = \frac{\lambda_2}{2} H_p^{(i,j)}(a)\left[1 + \mathscr{O}\left(\min\left\{1, \frac{\lambda_3}{\lambda_2} + \lambda_2 + |\lambda - a|\right\}\right)\right]. \tag{33}$$

Proof. Letting $k = 0$, $s = 3$, and $t = \lambda$ in Theorem 2, we get $d_p^{(i,j)}(P^{S_n}, \mathscr{P}(\lambda)) = (\lambda_2/2)H_p^{(i,j)}(\lambda) + R$, where

$$|R| \leqslant \frac{\lambda_3}{3}\left\| \sum_{u=3}^{j+2} \binom{j}{u-2} \Delta^{u+i}\pi(\cdot, \lambda)\right\|_p + |R_1|$$

and the following two estimates hold:

$$|R_1| = \mathscr{O}\left(\frac{\lambda_2^2}{\lambda^{(i+5)/2-1/(2p)}}\right) \qquad \text{if } \limsup_{l\to\infty} \theta^{(l)} < 1,$$

$$|R_1| = \mathscr{O}(\lambda_2^2) \qquad \text{if } \lambda_2 = \mathscr{O}(1).$$

Using the triangular inequality, (12) and Lemma 5, we obtain

$$|R| = \mathscr{O}\left(\frac{\lambda_3}{\lambda^{(i+4)/2-1/(2p)}} + \frac{\lambda_2^2}{\lambda^{(i+5)/2-1/(2p)}}\right) \qquad \text{if } \limsup_{l\to\infty} \theta^{(l)} < 1, \tag{34}$$

$$|R| = \mathscr{O}(\lambda_3 + \lambda_2^2) \qquad \text{if } \lambda_2 = \mathscr{O}(1). \tag{35}$$

Because of (30), three constants M_1, M_2, $M_3 \in (0, \infty)$ exist such that $M_1 > 1$ and $M_2 \leqslant t^{(i+3)/2-1/(2p)} H_p^{(i,j)}(t) \leqslant M_3$ for $t \in (M_1, \infty)$. By Lemma 5, $0 < \inf_{t\in[0,M_1]} H_p^{(i,j)}(t) =: M_4$. Let $A_1 = \{l \in \mathbb{N} | \lambda > M_1\}$, $A_2 = \{l \in \mathbb{N} | M_1 \geqslant \lambda \geqslant 1\}$, and $A_3 = \mathbb{N}\backslash(A_1 \cup A_2)$. For (31), we may assume $\sup_{l\in\mathbb{N}} \theta^{(l)} < 1$. By (34) and (35), we obtain, for $l \in A_k$ $(k \in \{1, 2, 3\})$,

$$\frac{|R|}{\lambda_2 H_p^{(i,j)}(\lambda)} = \mathscr{O}\left(\min\left\{\frac{\lambda_3}{\lambda_2\sqrt{\lambda}} + \theta, \frac{\lambda_3}{\lambda_2} + \lambda_2\right\}\right).$$

Relation (31) is proved. Relations (32) and (33) are easily shown by similar arguments. □

It is easy to show that (31) and (32) remain valid if $i + 1 = 1/p$ and the condition $\limsup_{l\to\infty} \theta^{(l)} < 1$ is dropped. Hence (32) is a generalization of results of Prohorov (1953, Theorem 2), Deheuvels and Pfeifer (1986a; 1986b; 1988), and Roos (1995) concerning the Poisson approximation of the binomial and Poisson-binomial distributions with respect to the total variation distance, the Kolmogorov metric, the Fortet–Mourier metric, and the one-point metric. It should be mentioned that, as has been observed by Barbour *et al.* (1992, p. 2), the statement of Prohorov's Theorem 2 is inaccurate. A correct version, in our notation, is: $d_1^{(0,1)}(\mathscr{B}(n, p), \mathscr{P}(np)) = \sqrt{2/(\pi e)}\,p[1 + \mathscr{O}(\min\{1, [np]^{-1/2} + p\})]$, where $\mathscr{B}(n, p)$ denotes the binomial distribution with parameter n and success probability p. In Prohorov's version, the '$+p$' is missing, which invalidates his result, for example, for $p = 1$, $n \to \infty$.

For easier $\mathscr{O}$-terms in (31) and (33), observe the following relations:

$$\min\left\{\frac{\lambda_3}{\lambda_2\sqrt{\lambda}}+\theta, \frac{\lambda_3}{\lambda_2}+\lambda_2\right\} \leqslant 2\min\{\sqrt{\theta}, \sqrt{\lambda_2}\},$$

$$\frac{\lambda_3}{\lambda_2}+\lambda_2+|\lambda-a| \leqslant (1+\sqrt{M})(\sqrt{\lambda_2}+|\lambda-a|) \qquad \text{if } \lambda_2 \leqslant M.$$

6. Non-metric considerations

In this section, we are interested in relations for $\Delta^i(f_{P^{S_n}}-f_{\mathscr{P}(t)})(m)$, where $i \in \{-1, 0, \ldots\}$, $t \in (0, \infty)$, and $m \in \mathbb{Z}_+$. The first result is a consequence of Theorem 2. Here, we consider the triangular scheme as introduced before Theorem 3. Further, let $m \in \mathbb{Z}_+$ and $t \in (0, \infty)$ also depend on l.

Theorem 4. *Let $i \in \{-1, 0, \ldots\}$, $a \in \mathbb{R}$, $m_0 \in \mathbb{Z}_+$ be independent of l.*

(a) *If $\lambda \to \infty$, $m = \lambda + a\sqrt{\lambda} + \mathscr{O}(1)$, $\theta \to 0$, $(\lambda - t)\sqrt{\lambda}/\lambda_2 \to 0$, and $\lambda = t + \mathscr{O}(1)$, then*

$$\frac{\lambda^{(i+3)/2}}{\lambda_2}\Delta^i(f_{P^{S_n}}-f_{\mathscr{P}(t)})(m) \to \frac{(-1)^{i+1}}{2}\varphi_{i+2}(a).$$

(b) *If $m = m_0$, $\lambda \to a \in [0, \infty)$, $t \to a$, $\lambda_2/t^{i+4} \to 0$, and $(\lambda - t)/\lambda_2 \to 0$, then*

$$\frac{1}{\lambda_2}\Delta^i(f_{P^{S_n}}-f_{\mathscr{P}(t)})(m) \to -\frac{1}{2}\Delta^{i+2}\pi(m_0, a).$$

Proof. First note that both in (a) and (b), $\tilde{\theta}(t) \to 0$. Hence Theorem 2 yields $\Delta^i(f_{P^{S_n}} - f_{\mathscr{P}(t)})(m) = \tilde{H} + \tilde{R}$, where

$$\tilde{H} = (\lambda - t)\Delta^{i+1}\pi(m, t) + \frac{(\lambda-t)^2-\lambda_2}{2}\Delta^{i+2}\pi(m, t),$$

$$|\tilde{R}| = |\Delta^i(f_{P^{S_n}} - f_{Q(2,t)})(m)| \leqslant d_\infty^{(i,1)}(P^{S_n}, Q(2, t)) = \mathscr{O}\left(\frac{\tilde{\theta}(t)^{3/2}}{t^{(i+1)/2}}\right).$$

In case (a), we have $m = t + a\sqrt{t} + \mathscr{O}(1)$ and $t \to \infty$. Using (28), we obtain $t^{(k+1)/2}\Delta^k\pi(m, t) \to (-1)^k\varphi_k(a)$ for $k \in \mathbb{Z}_+$. Now it is easy to prove that $\tilde{H}\lambda^{(i+3)/2}/\lambda_2 \to 2^{-1}(-1)^{i+1}\varphi_{i+2}(a)$ and $\tilde{R}\lambda^{(i+3)/2}/\lambda_2 \to 0$, as required.

In case (b), the relations $\tilde{H}/\lambda_2 \to -2^{-1}\Delta^{i+2}\pi(m_0, a)$ and $\tilde{R}/\lambda_2 \to 0$ are easily shown, completing the proof. □

Theorem 5. *Let $t \in (0, \infty)$, $k \in \mathbb{Z}_+$, $s \in \{k, k+1, \ldots\}$, $i \in \{-s-1, -s, -s+1, \ldots\}$, $m \in \mathbb{Z}_+$, $V(m, t) = (m/t + 1)\sqrt{\eta(t)/2}$,*

$$C_3 = \frac{\exp(1/(24(s+1)))(2\pi)^{1/4}}{(s+1)^{1/4}\sqrt{s!}}, \qquad \nu(t) = \beta\left(\frac{|\lambda - t|\sqrt{2(s+1)}}{\sqrt{\eta(t)}}\right).$$

Then $\Delta^i(f_{P^{S_n}} - f_{Q(k,t)})(m) = H' + R'$, where $H' = \sum_{u=k+1}^{s} a_u(t)\Delta^{u+i}\pi(m, t)$ and

$$|R'| \leqslant C_3 \nu(t)\pi(m,\,t)(m/t+1)^i V(m,\,t)^{s+1}(1+V(m,\,t))\exp(V(m,\,t)^2).$$

Proof. First note that, by (6) and (24),

$$|\Delta^k \pi(m,\,t)| \leqslant \pi(m,\,t)\left(\frac{m}{t}+1\right)^k, \qquad k,\, m \in \mathbb{Z}_+,\ t \in (0,\,\infty). \tag{36}$$

The assertion is easily shown by using (10), (17), (18), and (36). □

For the following result, we use the triangular scheme as considered for Theorem 4.

Theorem 6. *Let $i \in \{-1, 0, \ldots\}$ be independent of l.*

(a) *If $\lambda/m \to 0$, $m \to \infty$, $m\sqrt{\lambda_2}/\lambda \to 0$ and $(\lambda - t)/\lambda_2 \to 0$, then*

$$\Delta^i(f_{P^{S_n}} - f_{\mathscr{P}(t)})(m) \sim -\frac{\lambda_2}{2}\pi(m,\,t)\left(\frac{m}{t}\right)^{2+i}.$$

(b) If $m/\lambda \to a \in [0,\,\infty)$, $\lambda \to \infty$, $\lambda_2 \to 0$ *and* $(\lambda - t)/\lambda_2 \to 0$, *then*

$$\frac{\Delta^i(f_{P^{S_n}} - f_{\mathscr{P}(t)})(m)}{\lambda_2 \pi(m,\,t)} \to -\frac{1}{2}(a-1)^{i+2}.$$

For the proof, the following lemma is needed.

Lemma 6. (a) *Under the assumptions in Theorem 6(a), we have $t/m \to 0$, $(\lambda - t)t/(\lambda_2 m) \to 0$, $p_k(m,\,t)/m^k \to 1$ for all $k \in \mathbb{Z}_+$, $\lambda_2 \to 0$, $\lambda - t \to 0$, $(\lambda - t)^2/\lambda_2 \to 0$, $m\sqrt{\lambda_2}/t \to 0$ and $(\lambda - t)m/t \to 0$.*

(b) *Under the assumptions in Theorem 6(b), we have $m/t \to a$, $\lambda - t \to 0$, $t \to \infty$, $p_k(m,\,t)t^{-k} \to (a-1)^k$ for all $k \in \mathbb{Z}_+$, $(\lambda - t)^2/\lambda_2 \to 0$, $m\sqrt{\lambda_2}/t \to 0$ and $(\lambda - t)m/t \to 0$.*

Proof of Theorem 6. By Theorem 5, $\Delta^i(f_{P^{S_n}} - f_{\mathscr{P}(t)})(m) = H' + R'$, where

$$H' = \pi(m,\,t)\left[\frac{\lambda - t}{t^{i+1}} p_{i+1}(m,\,t) + \frac{1}{2t^{i+2}}((\lambda - t)^2 - \lambda_2)p_{i+2}(m,\,t)\right],$$

$$R' = \mathscr{O}(\pi(m,\,t)y^{i+3}x^{3/2}(1 + y\sqrt{x})\exp(xy^2)),$$

$x = \eta(t)/2$ and $y = m/t + 1$. Using Lemma 6, the assertions are easily proved. □

In what follows, let F (or G) denote the distribution function of P^{S_n} (or $\mathscr{P}(t)$). To obtain the following results on large and moderate deviations, set $i = -1$ in Theorems 4 and 6.

Corollary 3. (a) *Under the assumptions in Theorem 4(a),*

$$\frac{1 - G(m)}{\theta}\left(\frac{1 - F(m)}{1 - G(m)} - 1\right) \to \frac{-a\mathrm{e}^{-a^2/2}}{2\sqrt{2\pi}}. \tag{37}$$

(b) *Under the assumptions in Theorem 4(b),*

$$\frac{1 - G(m_0)}{\lambda_2}\left(\frac{1 - F(m)}{1 - G(m)} - 1\right) \to \frac{-\Delta^1 \pi(m_0, a)}{2}. \tag{38}$$

(c) *Under the assumptions in Theorem 6(a),*

$$\frac{1 - F(m)}{1 - G(m)} - 1 \sim -\frac{\lambda_2}{2}\left(\frac{m}{t}\right)^2. \tag{39}$$

(d) *Under the assumptions in Theorem 6(b),*

$$\frac{1 - G(m)}{\lambda_2 \pi(m, t)}\left(\frac{1 - F(m)}{1 - G(m)} - 1\right) \to \frac{1 - a}{2}. \tag{40}$$

Note that Chen and Choi (1992, Corollary 2.4) obtained (39) under more restrictive assumptions in the case $t = \lambda$. (They overlooked the required assumption $m \to \infty$ in their corollary. Their assertion does not hold without this condition). For $t = \lambda$, Barbour *et al.* (1995, Corollary 4.3) proved (39) under more general assumptions as in Corollary 3(c).

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