## Asymptotics in Poisson order statistics

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ORDER STATISTICS
by
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## ASYMPTOTICS IN POISSON ORDER STATISTICS

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#### Abstract

In order statistics sums involving incomplete gamma functions are met. The asymptotic behaviour of such sums is studied, going beyond the results obtainable by the central limit theorem.


## 1. Introduction

Some colleagues*) of the author have posed the following problem:
Determine the asymptotic behaviour for $\mu \rightarrow \infty$ of the sums

$$
\begin{align*}
& S(\mu, m, n):=\sum_{k=0}^{\infty} I^{m}(\mu, k)(1-I(\mu, k))^{n},  \tag{1}\\
& T(\mu, m, n):=\sum_{k=0}^{\infty}(\mu-k) I^{m}(\mu, k)(1-I(\mu, k))^{n},
\end{align*}
$$

where $m$ and $n$ are positive integers and

$$
\begin{equation*}
I(\mu, k):=(k!)^{-1} \int_{0}^{\mu} e^{-t} t^{k} d t \quad\left(\mu>0, k \in \mathbb{N}_{0}\right) . \tag{3}
\end{equation*}
$$

This problem arose in the study of the expectation and variance of the order statistics in a random sample from the Poisson distribution with large mean $\mu$. In section 2 we present the results. A brief description of the derivation is given in section 3. The details of the derivation are given in sections 4 to 8 .

[^0]
## 2. Results

The sums $S$ and $T$ have the following asymptotic behaviour:

$$
\begin{align*}
& S(\mu, m, n)=A(m, n) \mu^{1 / 2}+B(m, n)+O\left(\mu^{-1 / 2}\right) \quad(\mu \rightarrow \infty)  \tag{4}\\
& T(\mu, m, n)=C(m, n) \mu+D(m, n) \mu^{1 / 2}+O(1) \quad(\mu \rightarrow \infty)
\end{align*}
$$

where

$$
\begin{equation*}
A(m, n)=\sqrt{2} \int_{-\infty}^{\infty} f^{m}(x) f^{n}(-x) d x \tag{6}
\end{equation*}
$$

$$
\begin{equation*}
B(m, n)=-2 / 3 \int_{-\infty}^{\infty} x f^{m}(x) f^{n}(-x) d x \tag{7}
\end{equation*}
$$

$$
\begin{equation*}
C(m, n)=-3 B(m, n) \tag{8}
\end{equation*}
$$

$$
\begin{equation*}
D(m, n)=\sqrt{2} \int_{-\infty}^{\infty} f^{m}(x) f^{4}(-x)\left(2 / 3-x^{2}\right) d x \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
f(x):=\pi^{-1 / 2} \int_{-\infty}^{x} e^{-s^{2}} d s \quad(x \in \mathbb{R}) \tag{10}
\end{equation*}
$$

Clearly f can be expressed in the errorfunction but formulas (6) to (10) do not become simpler in doing so. Some coefficients are

$$
\begin{align*}
& A(1,1)=2 A(1,2)=2 A(2,1)=\pi^{-1 / 2}  \tag{11}\\
& B(n, n)=0(n \in \mathbb{N}) \\
& B(1,2)=-B(2,1)=B(1,3)=-B(3,1)=\frac{\sqrt{3}}{12} \pi^{-1} \\
& D(1,1)=2 D(1,2)=2 D(2,1)=\frac{1}{4} \pi^{-1 / 2} .
\end{align*}
$$

## 3. Sketch of the derivation

The results are obtained by taking the following steps.
(i) The sums are approximated by sums over $|u-k| \leq \mu^{2 / 3}$ with an error of $O\left(e^{-c \mu^{2 / 3}}\right) \quad(\mu \rightarrow \infty)$ where c is some positive constant.
(ii) For $k \in\left[\mu-c \mu^{2 / 3}, \mu+c \mu^{2 / 3}\right]$ the asymptotic behaviour of $I(\mu, k)$ for $\mu \rightarrow \infty$ is determined. Let $x \in \mathbb{R}$ be defined by

$$
x=k^{1 / 2}\left|h\left(\mu k^{-1}-1\right)\right|^{1 / 2} \operatorname{sgn}(\mu-k)
$$

where $h(s):=-s+\log (1+s)(s>-1)$.
Roughly $x=(\mu-k)(2 \mu)^{-1 / 2}$. Then $I(\mu, k)$ has a complete asymptotic expansion in powers of $\mu^{-1 / 2}$ wich is uniform with respect to $x \in \mathbb{R}, x=O\left(\mu^{1 / 6}\right)(\mu \rightarrow \infty)$.

$$
I(\mu, k) \approx f(x)-2^{1 / 2} 3^{-1} \pi^{-1 / 2} e^{-x^{2}} \mu^{-1 / 2}+\ldots(\mu \rightarrow \infty)
$$

where $f$ is defined by (10), i.e.

$$
\begin{aligned}
& \forall_{N \in \mathbb{N}} \forall_{c>0} \exists_{A>0} \exists_{B>0} \forall_{\mu>A} \forall_{k \in \mathbb{N}} \\
& |k-\mu| \leq c \mu^{2 / 3} \Rightarrow\left|I(\mu, k)-\left\{f(x)+e^{-x^{2}} \sum_{l=1}^{N} q_{l}(x) \mu^{-l / 2}\right\}\right| \leq B \mu^{-(N+1) / 2} .
\end{aligned}
$$

(iii) The sums $\sum_{|\mu-k| \leq \mu^{23}}$ are approximated by integrals $\int_{|\mu-k| \leq \mu^{23}} \cdots d k$ with an error which, for every positive number $r$, is $O\left(\mu^{-r}\right)(\mu \rightarrow \infty)$. Then these integrals are transformed into integrals over $x$ and then approximated by integrals $\int_{-\infty}^{\infty} d x$ with errors of the kind $O\left(e^{-c \mu^{13}}\right)(\mu \rightarrow \infty)$.

## 4. The truncation of the sum

The function $I(\mu, k)$ interpreted as a function of the real variable $k \in[0, \infty)$ is decreasing on $[0, \infty)$. This statement follows from

$$
\frac{d}{d k} I(\mu, k)=(\Gamma(k+1))^{-2} \int_{0}^{\mu} d t \int_{\mu}^{\infty} e^{-t-\tau} t^{k} \tau^{k} \log \left(t \tau^{-1}\right) d \tau<0
$$

Let $k \leq \mu-\mu^{2 / 3}:=a$. Then substituting $t=k(1+s)$ we have
$1-I(\mu, k) \leq 1-I(\mu, a)=(\Gamma(a+1))^{-1} e^{-a} a^{a+1} \int_{a^{-1}{ }_{\mu-1}}^{\infty} e^{a h(s)} d s \quad$,
where $h(s):=-s+\log (1+s)$. The function $h$ is concave and negative on $(-1, \infty)$, whence $h(s) \leq h\left(\mu a^{-1}-1\right)+\left(s-\mu a^{-1}+1\right) h^{\prime}\left(\mu a^{-1}-1\right)\left(s \geq \mu a^{-1}-1\right)$.
It follows that

$$
\begin{gathered}
0<e^{-a} a^{a+1}(\Gamma(a+1))^{-1} \int_{a^{-1} \mu-1}^{\infty} e^{a h(s)} d s \leq e^{-a} a^{a+1}(\Gamma(a+1))^{-1} e^{a h\left(a^{-1} \mu-1\right)}\left(a h^{\prime}\left(\mu a^{-1}-1\right)\right)^{-1} \\
\leq \mu^{1 / 2} e^{-1 / \mu^{\prime \prime}} \quad(\mu \geq 8) .
\end{gathered}
$$

Hence, for both of the sums in (1) and (2)

$$
0<\sum_{0 \leq k \leq a}<\mu \cdot \mu^{3 / 2} e^{-\frac{3 n}{2} \mu^{1 / 3}}=O\left(e^{-\mu^{1 / 3}}\right)(\mu \rightarrow \infty)
$$

Let $k \geq \mu+\mu^{2 / 3}$. Then $I(\mu, k)=e^{-k} k^{k+1}(\Gamma(k+1))^{-1} \int_{-1}^{k^{-1} \mu-1} e^{k h(s)} d s$. Since $h(s) \leq-1 / 2 s^{2}$ on $(-1,0]$ we have

$$
\begin{aligned}
I(\mu, k) & \leq e^{-k} k^{k+1}(\Gamma(k+1))^{-1} \int_{-1}^{k^{-1} \mu-1} e^{-1 / 2 k s^{2}} d s \\
& \leq e^{-k} k^{k+1}(\Gamma(k+1))^{-1} \int_{1-k^{-1} \mu}^{\infty} e^{-1 / 2 k s^{2}} d s \\
& \leq e^{k} k^{k+1}(\Gamma(k+1))^{-1} \int_{1-k^{-1} \mu}^{\infty} e^{-1 / 2 k\left(1-k^{-1} \mu\right) s} d s \\
& \leq \mu^{-1 / 6} e^{-1 / 2 k\left(\mu^{1 / 3}+1\right)^{-2}} \quad(\mu \geq 8)
\end{aligned}
$$

Hence, for both of the sums

$$
0<\sum_{k \geq \mu+\mu^{23}} \leq \mu^{-1 / 6} \sum_{k \geq \mu+\mu^{23}} k e^{-1 / 2 k m\left(\mu^{1 / 3}+1\right)^{-2}} \leq 2 \mu e^{-1 / 3 m \mu^{1 / 3}}(\mu \geq 8)
$$

Hence, in both cases

$$
\sum_{k=0}^{\infty}=\sum_{|\mu-k| \leq \mu^{23}}+O\left(e^{-c \mu^{13}}\right)(\mu \rightarrow \infty)
$$

where $c$ is a positive number.

## 5. The incomplete gamma function

As we have seen already the substitution

$$
\begin{equation*}
t=k(1+s) \tag{12}
\end{equation*}
$$

in the integral representation of $I(\mu, k)$ gives

$$
\begin{equation*}
I(\mu, k)=(k!)^{-1} e^{-k} k^{k+1} \int_{-1}^{k^{-1} \mu-1} e^{k h(s)} d s \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
h(s):=-s+\log (1+s) \quad(s>-1) . \tag{14}
\end{equation*}
$$

We introduce a new integration variable $y$ by

$$
\begin{equation*}
y:=k^{1 / 2}|h(s)|^{1 / 2} \operatorname{sgn}(s) \quad(s>-1) . \tag{15}
\end{equation*}
$$

Then we get

$$
\begin{equation*}
I(\mu, k)=(k!)^{-1} e^{-k} k^{k+1 / 2} \int_{-\infty}^{x} e^{-y^{2}} \frac{d s}{d y} d y \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
x:=k^{1 / 2}\left|h\left(k^{-1} \mu-1\right)\right|^{1 / 2} \operatorname{sgn}(\mu-k) . \tag{17}
\end{equation*}
$$

To study the transformation (15) we introduce first in (13)

$$
\begin{equation*}
\tau=|h(s)|^{1 / 2} \operatorname{sgn}(s) \tag{18}
\end{equation*}
$$

Then

$$
\begin{align*}
\sqrt{2} & \tau \tag{19}
\end{align*}=s\left(1-2 / 3 s+2 / 4 s^{2}-2 / 5 s^{3}+\ldots\right)^{1 / 2} \quad(|s|<1), ~, ~(|s|<1) \quad, ~ \$ s-1 / 3 s^{2}+7 / 36 s^{3}+\cdots \quad, \quad .
$$

the radius of convergence being one since $h(s) / s^{2}$ has no zeros inside the unit circle. By the Bürmann-Lagrange theorem we can expand $s$ as a powerseries in $\tau$ with a positive radius of convergence, say $\rho$.
We calculate

$$
\begin{equation*}
s=\sqrt{2} \tau+2 / 3 \tau^{2}+\frac{\sqrt{2}}{18} \tau^{3}-2 / 135 \tau^{4}+\cdots \quad(|\tau|<\rho) . \tag{20}
\end{equation*}
$$

$$
\begin{equation*}
\frac{d s}{d \tau}=\sum_{l=0}^{\infty} c_{l} \tau^{l}=\sqrt{2}+4 / 3 \tau+\frac{\sqrt{2}}{6} \tau^{2}-8 / 135 \tau^{3}+\cdots \quad(|\tau|<\rho) \tag{21}
\end{equation*}
$$

The transformation (18) changes the integral (13) into

$$
\begin{equation*}
I(\mu, k)=(k!)^{-1} e^{-k} k^{k+1} \int_{-\infty}^{k^{-12} x} e^{-k \tau^{2}} \frac{d s}{d \tau} d \tau . \tag{22}
\end{equation*}
$$

We shall study the asymptotic behaviour of this integral for $k \rightarrow \infty$ and fixed $x \in \mathbb{R}$. Therefore we shall denote the right hand of (22) by $I(x, k)$.
In order to use (21) we must truncate the interval of integration. Now it is easily shown that

$$
\left|\int_{|\tau| \geq 1 / 2 \rho} e^{-k \tau^{2}} \frac{d s}{d \tau} d \tau\right|=O\left(e^{-c k^{1 / 3}}\right) \quad(k \rightarrow \infty),
$$

where $c$ is a positive number.
Inside the circle $|\tau| \leq 1 / 2 \rho$ the powerseries in (21) is also an asymptotic series, i.e. for every $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\frac{d s}{d \tau}=\sqrt{2}+4 / 3 \tau+\frac{\sqrt{2}}{6} \tau^{2}-8 / 135 \tau^{3}+\cdots+c_{N-1} \tau^{N-1}+O\left(\tau^{N}\right) \quad(|\tau| \leq 1 / 2 \rho) \tag{23}
\end{equation*}
$$

Hence

$$
\begin{align*}
& \tilde{I}(x, k)=(k!)^{-1} e^{-k} k^{k+1} \int_{|\tau| \leq 1 / 2 \rho, \tau<k^{-12} x} e^{-k \tau^{2}}\left(\sqrt{2}+\frac{4}{3} \tau+\ldots+c_{N-1} \tau^{N-1}+O\left(\tau^{N}\right)\right) d \tau  \tag{24}\\
&+O\left[e^{-c k^{1 / 2}}\right](k \rightarrow \infty)
\end{align*}
$$

Now we change the lower bound into $-\infty$ and, evenually, the upper bound into $k^{-1 / 2} x$, thereby making an error of the kind $O\left(e^{-c k^{k 3}}\right)(k \rightarrow \infty)$ uniformly in $x \in \mathbb{R}$.
Then we substitute $\tau=k^{-1 / 2} y$ and we get

$$
\begin{align*}
\tilde{I}(x, k) & =(k!)^{-1} e^{-k} k^{k+1 / 2} \int_{-\infty}^{x} e^{-y^{2}}\left(\sqrt{2}+4 / 3 k^{-1 / 2} y+\frac{\sqrt{2}}{6} k^{-1} y^{2}+\cdots\right.  \tag{25}\\
& \left.+c_{N-1} k^{-\frac{N-1}{2}} y^{N-1}+O\left(k^{-N / 2} y^{N}\right)\right) d y \\
& +O\left(e^{-c k^{1 / 3}}\right) \quad(k \rightarrow \infty) \quad \text { uniformly in } x \in \mathbb{R}
\end{align*}
$$

Since $\int_{-\infty}^{x} e^{-y^{2}} O\left(k^{-N / 2} y^{N}\right) d y=O\left(k^{-N / 2}\right) \quad(k \rightarrow \infty)$ uniformly in $x \in \mathbb{R}$, we have

$$
\begin{align*}
\tilde{I}(x, k) & =\frac{g(k)}{\sqrt{2 \pi}} \int_{-\infty}^{x} e^{-y^{2}}\left(\sqrt{2}+4 / 3 k^{-1 / 2} y+\cdots+c_{N-1} k^{-(N-1) / 2} y^{N-1}\right) d y  \tag{26}\\
& +O\left(k^{-N / 2}\right) \quad(k \rightarrow \infty)
\end{align*}
$$

uniformly in $x \in \mathbb{R}$, where

$$
\begin{equation*}
g(k):=(2 \pi)^{1 / 2}(k!)^{1} e^{-k} k^{k+1 / 2} . \tag{27}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\tilde{I}(x, k)=g(k) \sum_{l=0}^{N-1} c_{l} f_{l}(x) k^{-l / 2}+O\left(k^{-N / 2}\right)(k \rightarrow \infty) \text { uniformly in } x \in \mathbb{R}, \tag{28}
\end{equation*}
$$

where the $c_{l}^{\prime} s$ are given by (21) and

$$
\begin{equation*}
f_{l}(x):=(2 \pi)^{-1 / 2} \int_{-\infty}^{x} y^{l} e^{-y^{2}} d y \quad\left(l \in \mathbb{N}_{0}\right) . \tag{29}
\end{equation*}
$$

Integration by parts gives

$$
\begin{equation*}
f_{l}(x)=e^{-x^{2}} p_{l}(x)+\frac{1+(-1)^{I}}{2} \Gamma\left(\frac{l+1}{2}\right)(2 \pi)^{-1 / 2} f(x) \quad(l \in I N) \tag{30}
\end{equation*}
$$

where $f$ is defined by (10) and

$$
\begin{equation*}
p_{l}(x):=-2^{-3 / 2} \pi^{-1 / 2} \Gamma\left(\frac{l+1}{2}\right) \sum_{s=0}^{|(l-1) / 2|}\left(\Gamma\left(\frac{l+1}{2}-s\right)\right)^{-1} x^{l-1-2 s} \quad(l \in \mathbb{N}) \tag{31}
\end{equation*}
$$

Letting $x \rightarrow \infty$ we find that $f_{l}(\infty)=0$ if $l$ is odd and $f_{l}(\infty)=\Gamma\left(\frac{l+1}{2}\right)(2 \pi)^{-1 / 2}$ if $l$ is even.
Since $\lim _{\mu \rightarrow \infty} I(\mu, k)=1$ we get the asymptotic series for $(g(k))^{-1}$ by letting $x \rightarrow \infty$ in (28). We have

$$
\begin{equation*}
1 \approx g(k) \sum_{l=0}^{\infty} c_{l} f_{l}(\infty) k^{-l / 2} \quad(k \rightarrow \infty) \tag{32}
\end{equation*}
$$

Comparing (28) (30) and (32) we see that the factor with which $f(x)$ occurs in (28) has the same asymptotic expansion as the factor $g(k)$ in (32). Hence

$$
\begin{equation*}
\bar{l}(x, k)) \approx f(x)+g(k) \sum_{l=1}^{\infty} c_{l} e^{-x^{2}} p_{l}(x) k^{-1 / 2}(k \rightarrow \infty) \text { uniformly in } x \in \mathbb{R} \tag{33}
\end{equation*}
$$

(i.e. after truncation the (absolute) error is smaller than $c k^{-(N+1) / 2}$ with $c$ independent of $x$ ).

According to the results of section 4 we restrict ourselves to values of $k \in\left[\mu-\mu^{2 / 3}, \mu+\mu^{2 / 3}\right]$. Then $x=O\left(\mu^{1 / 6)} \quad(\mu \rightarrow \infty)\right.$.
In order to get an asymptotic scries for $I(\mu, k)$ for $\mu \rightarrow \infty$ we have to express $k$ as a function of $\mu$ and $x$. From (17) we have $k h\left(\mu k^{-1}-1\right)=-x^{2}, \operatorname{sgn} x=\operatorname{sgn}(\mu-k)$. Putting $k=\mu(1-\psi)$ and $\mu^{-1 / 2} x=z$ we get

$$
\begin{equation*}
\psi+(1-\psi) \log (1-\psi)=z^{2} \tag{34}
\end{equation*}
$$

whence

$$
\begin{equation*}
\sum_{i=2}^{\infty} \frac{\psi^{i}}{(i-1) i}=z^{2} \quad(|\psi|<1), \operatorname{sgn} z=\operatorname{sgn} \psi \tag{35}
\end{equation*}
$$

Then

$$
\begin{equation*}
\psi\left(1+2 \sum_{i=3}^{\infty} \frac{\psi^{i}}{(i-1) i}\right)^{1 / 2}=\sqrt{2} z \tag{36}
\end{equation*}
$$

By the Bürmann-Lagrance inversion theorem there is a postive number $r$ such that

$$
\begin{equation*}
\psi=\sum_{i=1}^{\infty} d_{i} z^{i} \quad(|z|<r) \tag{37}
\end{equation*}
$$

Hence

$$
\begin{equation*}
k=\mu-\sum_{i=1}^{\infty} d_{i} \mu^{1-i / 2} x^{i} \quad(|x|<r \sqrt{\mu}) \tag{38}
\end{equation*}
$$

Clearly, for $\mu$ sufficiently large, $x$ is within the range of convergence since $x=O\left(\mu^{1 / 6}\right) \quad(\mu \rightarrow \infty)$.
A few coefficients $d_{i}$ are calculated.

$$
\begin{equation*}
d_{1}=\sqrt{2}, d_{2}=-1 / 3, d_{3}=\frac{-\sqrt{2}}{36}, d_{4}=-\frac{23}{90} \tag{39}
\end{equation*}
$$

We need also expansions for $k^{-1 / 2}, k^{-1}$ and $k^{-3 / 2}$ :
Clearly, there is a positive number $r_{0}$ such that for all $\alpha \in \mathbb{R}$ the function $(1-\psi)^{\alpha}$ has a powerseries expansion in $z$ which converges for $|z|<r_{0}$; if

$$
\begin{equation*}
(1-\psi)^{\alpha}=\sum_{j=0}^{\infty} a_{j}(\alpha) z^{j} \quad\left(|z|<r_{0}\right) \tag{40}
\end{equation*}
$$

then

$$
\begin{equation*}
k^{\alpha}=\mu^{\alpha}(1-\psi)^{\alpha}=\sum_{j=0}^{\infty} a_{j}(\alpha) \mu^{\alpha-i / 2} x^{j} \quad\left(|x|<r_{0} \mu^{1 / 2}\right) \tag{41}
\end{equation*}
$$

We calculate

$$
\begin{align*}
& k^{-1 / 2}=\mu^{-1 / 2}+2^{-1 / 2} \mu^{-1} x+\frac{7}{12} \mu^{-3 / 2} x^{2}+13 \frac{\sqrt{2}}{36} \mu^{-2} x^{3}+\cdots  \tag{42}\\
& k^{-1}=\mu^{-1}+2^{1 / 2} \mu-3 / 2 x+5 / 3 \mu^{-2} x^{2}+\cdots  \tag{43}\\
& k^{-3 / 2}=\mu^{-3 / 2}+3.2^{-1 / 2} \mu^{-2} x+\cdots \tag{44}
\end{align*}
$$

An asymptotic expansion for $g(k)$ can be determined from (32).

$$
\begin{equation*}
g(k)=\sum_{l=0}^{\infty} g_{l} k^{-l / 2} \quad(k \rightarrow \infty) \tag{45}
\end{equation*}
$$

Some coefficients $g_{l}$ are

$$
\begin{equation*}
g_{0}=1, g_{2}=-1 / 12 \text { and } g_{l}=0 \text { if } l=\text { odd. } \tag{46}
\end{equation*}
$$

Substituting (46) into (33) and then using (41) we get a complete asymptotic series for $I(\mu, k)$.

$$
\begin{align*}
& I(\mu, k) \approx f(x)+e^{-x^{2}} \sum_{l=1}^{\infty} q_{l}(x) \mu^{-l / 2} \quad(\mu \rightarrow \infty)  \tag{47}\\
& \text { uniformly in } x=O\left(\mu^{1 / 6}\right) \quad(\mu \rightarrow \infty) .
\end{align*}
$$

The $q_{l}$ 's are polynominals. We calculate

$$
\begin{align*}
& q_{1}(x)=-2^{1 / 2} 3^{-1} \pi^{-1 / 2}  \tag{48}\\
& q_{2}(x)=-\frac{5}{12} \pi^{-1 / 2} x . \tag{49}
\end{align*}
$$

## 6. The replacement of the sum by an integral

We have already

$$
\begin{equation*}
\frac{d}{d k} I(\mu, k)=(\Gamma(k+1))^{-2} \int_{0}^{\mu} d t \int_{0}^{\infty} e^{-t-\tau}(t \tau)^{k} \log \left(t \tau^{-1)} d \tau .\right. \tag{50}
\end{equation*}
$$

By induction one can prove easily that

$$
\begin{align*}
& \frac{d^{l}}{d k^{l}} I(\mu, k)=(\Gamma(k+1))^{-l-1} \int_{0}^{\mu} d t \int_{0}^{\infty} \cdots  \tag{51}\\
& \int_{0}^{\infty} e^{-t-\tau_{1}-\cdots-\tau_{l}}\left(t \tau_{1} \cdots \tau_{l}\right)^{k} L\left(t, \tau_{1}, \cdots, \tau_{l}\right) d \tau_{1} \cdots d \tau_{l}
\end{align*}
$$

where the functions $L\left(\tau_{0}, \tau_{1}, \cdots, \tau_{l}\right)$ are defined by

$$
\begin{equation*}
L\left(\tau_{0}\right)=1 \tag{52}
\end{equation*}
$$

$$
\begin{equation*}
L\left(\tau_{0}, \tau_{1}, \cdots, \tau_{l+1}\right)=L\left(\tau_{0}, \tau_{1}, \cdots, \tau_{l}\right) \log \left(\tau_{0} \tau_{1} \cdots \tau_{l} \tau_{l} \tau_{l+1}^{l-1}\right) \quad(l=0,1, \ldots) . \tag{53}
\end{equation*}
$$

With methods simular to those used in the treatment of $I(\mu, k)$ in section 5 we can prove easily that

$$
\begin{equation*}
\left|\frac{d^{l}}{d k^{l}} I(\mu, k)\right|=O\left(e^{-c \mu^{13}}\right) \quad\left(\mu \rightarrow \infty,|\mu-k| \geq \mu^{2 / 3}\right) . \tag{54}
\end{equation*}
$$

Furthermore, if we restrict ourselves to values of $k$ such that $|\mu-k| \leq \mu^{2 / 3}$, then the integrals in (51) can be replaced by $\int_{\mu-2 \mu^{2 s}}^{\mu+2 \int^{\mu}+2 \mu^{2 s}} \cdots \int_{\mu-2 \mu^{23}}^{\mu+2 \mu^{23}}$ with an error of the kind $O\left(e^{-c \mu^{2 / 3}}\right)(\mu \rightarrow \infty)$. Then it follows from (51) that for $|\mu-k| \leq \mu^{2 / 3}$

$$
\begin{equation*}
\left|\frac{d^{l}}{d k^{l}} I(\mu, k)\right|<M^{l}:=\max \left\{\left|L\left(t, \tau_{1}, \cdots, \tau_{l}\right)\right| \quad \mid \mu-2 \mu^{2 / 3} \leq t, \cdots, \tau_{l} \leq \mu+2 \mu^{2 / 3}\right\} \tag{55}
\end{equation*}
$$

$$
+O\left(e^{c \mu^{13}}\right)(\mu \rightarrow \infty)
$$

From (52) and (53) it follows that

$$
M_{l+1} \leq M_{l} \log \left[\frac{\mu+2 \mu^{2 / 3}}{\mu-2 \mu^{2 / 3}}\right)^{l+1} \quad\left(l \in \mathbb{N}_{0}\right),
$$

whence, by (54) and (55), we have for all $k \in \mathbb{N}$

$$
\begin{equation*}
\left|\frac{d^{l}}{d k^{l}} I(\mu, k)\right|=O\left(\mu^{-l / 3}\right)(\mu \rightarrow \infty)\left(l \in \mathbb{N}_{0}\right) \tag{56}
\end{equation*}
$$

Now we apply the Euler-Maclaurin sumformula: For every fixed $r \in \mathbb{N}$ wc have

$$
\begin{align*}
& \sum_{k=p}^{q} f(k)=\int_{p}^{q} f(x) d x+1 / 2 f(q)+1 / 2 f(p)  \tag{57}\\
& +\sum_{l=1}^{r} \frac{B_{2 l}}{(2 l)!}\left[f^{(2 l-1)}(q)-f^{(2 l-1)}(p)\right) \\
& +O\left[\int_{p}^{q}\left|f^{(2 r)}(x)\right| d x\right](q \geq p) .
\end{align*}
$$

Taking $f(k)=I^{m}(\mu, k)(1-I(\mu, k))^{n}, p=\left\lceil\mu-\mu^{2 / 3}\right\rceil, q=\left\lfloor\mu+\mu^{2 / 3}\right\rfloor$,
we find, for every $r \in N$, using (54), (56) and (57)

$$
\begin{align*}
& \sum_{|k-\mu| \leq \mu^{23}} I^{m}(\mu, k)(1-I(\mu, k))^{n}=\int_{\mu-\mu^{23}}^{\mu+\mu^{23}} I^{m}(\mu, k)(1-I(\mu, k))^{n} d k  \tag{58}\\
& +O\left(\mu^{-2 / 3 r+2 / 3}\right)(\mu \rightarrow \infty)
\end{align*}
$$

Simularly

$$
\begin{align*}
& \sum_{|k-\mu| \leq \mu^{23}}(u-k) I^{m}(\mu, k)(1-I(\mu, k))^{n}=\int_{\mu-\mu^{23}}^{\mu+\mu^{23}}(\mu-k) I^{m}(\mu, k)(1-I(\mu, k))^{n} d k  \tag{59}\\
& +O\left(\mu^{-2 / 3 r+4 / 3}\right)(\mu \rightarrow \infty)
\end{align*}
$$

## 7. The asymptotic behaviour of $S$ and $T$

According to section 6 the sums $\sum_{|\mu-k| \leq \mu^{23}}$ can be replaced by integrals with small errors which are, for every $r>0, O\left(\mu^{-r}\right)(\mu \rightarrow \infty)$. Then these integrals are transformed into integrals
$\int_{-A(\mu)}^{B(\mu)} \cdots \frac{d k}{d x} d x$ where $A(\mu)$ and $B(\mu)$ are asymptotically equivalent with $1 / 2 \sqrt{2} \mu^{1 / 6} \quad(\mu \rightarrow \infty)$

By means of (38) and (47) we get complete asymptotic expansions for the two integrands.

$$
\begin{align*}
& I^{m}(\mu, k)(1-I(\mu, k))^{n} \frac{d k}{d x} \approx \sum_{l=-1}^{\infty} s_{l}(x) \mu^{-1 / 2}(\mu \rightarrow \infty)  \tag{60}\\
& (\mu-k) I^{m}(\mu, k)(1-I(\mu, k))^{n} \frac{d k}{d x} \approx \sum_{l=-2}^{\infty} t_{l}(x) \mu^{-1 / 2} \quad(\mu \rightarrow \infty) \tag{61}
\end{align*}
$$

both uniformly in $x \in \mathbb{R}, x=O\left(\mu^{1 / 6}\right)(\mu \rightarrow \infty)$. The functions $s_{l}(x)$ and $t_{l}(x)$ are absolutely integrable over $(-\infty, \infty)$. Now we proceed as follows. Let $N \in \mathbb{N}$. Then

$$
\begin{aligned}
& \int_{\mu-\mu^{23}}^{\mu+\mu^{23}} I^{m}(\mu, k)(1-I(\mu, k))^{n} \frac{d k}{d x} d x \\
& =\sum_{l=-1}^{N} \mu^{-l / 2} \int_{-A(\mu)}^{B(\mu)} s_{l}(x) d x+O\left[\mu^{-\frac{N}{2} \frac{1}{2}+\frac{1}{6}}\right](\mu \rightarrow \infty) .
\end{aligned}
$$

It is easily seen that the functions $s_{l}(x)$ are of the form $\bar{p}\left(x, e^{-x^{2}}, f(x)\right) e^{-x^{2}}+\bar{q}\left(x, e^{-x^{2}}, f(x)\right) f(x)(1-f(x))$, where $\bar{p}$ and $\bar{q}$ are polynomials. Hence $\int_{x<-A(\mu)}\left|s_{l}(x)\right| d x$ and $\int_{x>B(\mu)}\left|s_{l}(x)\right| d x$ are $O\left(e^{-c \mu^{1 / 3}}\right)(\mu \rightarrow \infty)$.
It follows that $S(\mu, m, n)$ has a complete asymptotic expansion

$$
\begin{equation*}
S(\mu, m, n) \approx \sum_{l=-1}^{\infty} \mu^{-l / 2} \int_{-\infty}^{\infty} s_{l}(x) d x \quad(\mu \rightarrow \infty) \tag{62}
\end{equation*}
$$

A simular argument holds for $T(\mu, m, n)$.

$$
\begin{equation*}
T(\mu, m, n)=\sum_{l=-2}^{\infty} \mu^{-1 / 2} \int_{-\infty}^{\infty} t_{l}(x) d x \quad(\mu \rightarrow \infty) . \tag{63}
\end{equation*}
$$

We calculate

$$
\begin{equation*}
s_{0}(x)=2 / 3 \pi^{-1 / 2}\left[n f^{m}(x) f^{n-1}(-x)-m f^{m-1}(x) f^{n}(-x)\right] e^{-x^{2}}-2 / 3 x f^{m}(x) f^{n}(-x) \tag{65}
\end{equation*}
$$

$$
\begin{equation*}
s_{-1}(x)=\sqrt{2} f^{m}(x) f^{n}(-x) \tag{64}
\end{equation*}
$$

$$
\begin{equation*}
t_{-2}(x)=2 x f^{m}(x) f^{n}(-x) \tag{66}
\end{equation*}
$$

$$
\begin{equation*}
t_{-1}(x)= \tag{67}
\end{equation*}
$$

$$
2^{3 / 2} 3^{-1} \pi^{-1 / 2}\left[n f^{m}(x) f^{n-1}(-x)-m f^{m-1}(x) f^{n}(-x)\right] x e^{x^{2}}-2^{1 / 2} x^{2} f^{m}(x) f^{n}(-x)
$$

The term with factor $e^{-x^{2}}$ in the right hand of (65) gives 0 upon integration.
Integration by parts of the term with factor $x e^{-x^{2}}$ in the right hand of (67) gives $2 / 3 \sqrt{2} \int_{-\infty}^{\infty} f^{m}(x) f^{n}(-x) d x$.

## 8. Computation of some coefficients

$$
\begin{aligned}
A(1,1) & =\sqrt{2} \int_{-\infty}^{\infty} f(x)(1-f(x)) d x=-\sqrt{2} \pi^{-1 / 2} \int_{-\infty}^{\infty} x(1-2 f(x)) e^{-x^{2}} d x \\
& =2 \sqrt{2} \pi^{-1 / 2} \int_{-\infty}^{\infty} f(x) x e^{-x^{2}} d x=\sqrt{2} \pi^{-1} \int_{-\infty}^{\infty} e^{-2 x^{2}} d x=\pi^{-1 / 2} . \\
A(1,2) & =A(2,1)=1 / 2 \sqrt{2} \int_{-\infty}^{\infty}\left(f(x) f^{2}(-x)+f(-x) f^{2}(x)\right) d x \\
& =1 / 2 \sqrt{2} \int_{-\infty}^{\infty} f(x) f(-x) d x=1 / 2 A(1,1) \\
B(1,1) & =-2 / 3 \int_{-\infty}^{\infty} x f(x) f(-x) d x=0 . \\
B(1,2) & =-B(2,1)=-2 / 3 \int_{-\infty}^{\infty} x f(x) f^{2}(-x)(f(x)+f(-x)) d x \\
& =-2 / 3 \int_{-\infty}^{\infty} x f(x) f^{3}(-x) d x=B(1,3)=-B(3,1) \\
B(1,2) & =-2 / 3 \int_{-\infty}^{\infty} x\left(f(x)-2 f^{2}(x)+f^{3}(x)\right) d x \\
& =1 / 3 \pi^{-1 / 2} \int_{-\infty}^{\infty} x^{2} e^{-x^{2}}\left(1-4 f(x)+3 f^{2}(x)\right) d x \\
& =1 / 6 \pi^{-1 / 2} \int_{-\infty}^{\infty} e^{-x^{2}}\left(1-4 f(x)+3 f^{2}(x)+\pi^{-1 / 2} x e^{-x^{2}}(-4+6 f(x))\right) d x \\
& =1 / 6 \int_{-\infty}^{\infty}\left(1-4 f(x)+3 f^{2}(x)\right) f^{\prime}(x) d x+\pi^{-1} \int_{-\infty}^{\infty} x e^{-2 x^{2}} f(x) d x \\
& =1 / 4 \pi^{-3 / 2} \int_{-\infty}^{\infty} e^{-3 x^{2}} d x=\frac{\sqrt{3}}{12} \pi^{-1} .
\end{aligned}
$$

$$
D(1,1)=2 / 3 A(1,1)-\sqrt{2} \int_{-\infty}^{\infty} x^{2} f(x) f(-x) d x
$$

$$
=2 / 3 A(1,1)-2 \frac{\sqrt{2}}{3} \pi^{-1 / 2} \int_{-\infty}^{\infty} x^{3} e^{-x^{2}} f(x) d x
$$

$$
\begin{aligned}
& =2 / 3 A(1,1)-2 \frac{\sqrt{2}}{3} \pi^{-1 / 2} \int_{-\infty}^{\infty} x e^{-2 x^{2}} f(x) d x-\frac{\sqrt{2}}{3} \pi^{-1} \int_{-\infty}^{\infty} x^{2} e^{-2 x^{2}} d x \\
& =2 / 3 A(1,1)-\frac{5}{12} \pi^{-1 / 2}=1 / 4 \pi^{-1 / 2} . \\
D(1,2) & =D(2,1)=2 / 3 A(1,2)-1 / 2 \sqrt{2} \int_{-\infty}^{\infty} x^{2}\left(f(x) f^{2}(-x)+f^{2}(x) f(-x)\right) d x= \\
& =2 / 3 A(1,2)-1 / 2 \sqrt{2} \int_{-\infty}^{\infty} x^{2} f(x) f(-x) d x \\
& =2 / 3 A(1,2)+1 / 2 D(1,1)-1 / 3 A(1,1)=1 / 8 \pi^{-1 / 2} .
\end{aligned}
$$


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