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RANA 88-12
July 1988

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Reports on Applied and Numerical Analysis
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ABSTRACT

In order statistics sums involving incomplete gamma functions are met. The asymptotic behaviour of such sums is studied, going beyond the results obtainable by the central limit theorem.

1. Introduction

Some colleagues*) of the author have posed the following problem:
Determine the asymptotic behaviour for $\mu \to \infty$ of the sums

$$(1) \qquad S(\mu, m, n) := \sum_{k=0}^{\infty} I^m(\mu, k)(1 - I(\mu, k))^n \quad ,$$

$$(2) \qquad T(\mu, m, n) := \sum_{k=0}^{\infty} (\mu - k) I^m(\mu, k)(1 - I(\mu, k))^n \quad ,$$

where m and n are positive integers and

$$(3) \qquad I(\mu, k) := (k!)^{-1} \int_0^{\mu} e^{-t} t^k dt \quad (\mu > 0, k \in \mathbb{N}_0) \quad .$$

This problem arose in the study of the expectation and variance of the order statistics in a random sample from the Poisson distribution with large mean μ. In section 2 we present the results. A brief description of the derivation is given in section 3. The details of the derivation are given in sections 4 to 8.

*) F.W. Steutel and D.A. Overdijk, Department of Mathematics, Eindhoven University of Technology.

2. Results

The sums S and T have the following asymptotic behaviour:

(4) $$S(\mu, m, n) = A(m,n)\mu^{1/2} + B(m,n) + O(\mu^{-1/2}) \quad (\mu \to \infty) \quad ,$$

(5) $$T(\mu,m,n) = C(m,n)\mu + D(m,n)\mu^{1/2} + O(1) \quad (\mu \to \infty) \quad ,$$

where

(6) $$A(m,n) = \sqrt{2} \int_{-\infty}^{\infty} f^m(x) f^n(-x)\,dx \quad ,$$

(7) $$B(m,n) = -2/3 \int_{-\infty}^{\infty} x f^m(x) f^n(-x)\,dx \quad ,$$

(8) $$C(m,n) = -3B(m,n) \quad ,$$

(9) $$D(m,n) = \sqrt{2} \int_{-\infty}^{\infty} f^m(x) f^n(-x)(2/3 - x^2)\,dx \quad ,$$

and

(10) $$f(x) := \pi^{-1/2} \int_{-\infty}^{x} e^{-s^2}\,ds \quad (x \in \mathbb{R}) \quad .$$

Clearly f can be expressed in the errorfunction but formulas (6) to (10) do not become simpler in doing so. Some coefficients are

(11) $$A(1,1) = 2A(1,2) = 2A(2,1) = \pi^{-1/2}$$

$$B(n,n) = 0 \quad (n \in \mathbb{N})$$

$$B(1,2) = -B(2,1) = B(1,3) = -B(3,1) = \frac{\sqrt{3}}{12}\pi^{-1}$$

$$D(1,1) = 2D(1,2) = 2D(2,1) = \tfrac{1}{4}\pi^{-1/2} \quad .$$

3. Sketch of the derivation

The results are obtained by taking the following steps.

(i) The sums are approximated by sums over $|u-k| \le \mu^{2/3}$ with an error of $O(e^{-c\mu^{1/3}})$ $(\mu \to \infty)$ where c is some positive constant.

(ii) For $k \in [\mu - c\,\mu^{2/3}, \mu + c\,\mu^{2/3}]$ the asymptotic behaviour of $I(\mu, k)$ for $\mu \to \infty$ is determined. Let $x \in \mathbb{R}$ be defined by

$$x = k^{1/2} \left| h(\mu k^{-1} - 1) \right|^{1/2} \operatorname{sgn}(\mu - k)$$

where $h(s) := -s + \log(1+s)$ $(s > -1)$.
Roughly $x = (\mu - k)\ (2\mu)^{-1/2}$. Then $I(\mu,k)$ has a complete asymptotic expansion in powers of $\mu^{-1/2}$ wich is uniform with respect to $x \in \mathbb{R}$, $x = O(\mu^{1/6})$ $(\mu \to \infty)$.

$$I(\mu, k) \approx f(x) - 2^{1/2}\, 3^{-1} \pi^{-1/2}\, e^{-x^2} \mu^{-1/2} + \ldots (\mu \to \infty),$$

where f is defined by (10), i.e.

$$\forall_{N \in \mathbb{N}}\ \forall_{c>0}\ \exists_{A>0}\ \exists_{B>0}\ \forall_{\mu>A}\ \forall_{k \in \mathbb{N}}$$

$$|k - \mu| \le c\, \mu^{2/3} \Rightarrow \left| I(\mu, k) - \{f(x) + e^{-x^2} \sum_{l=1}^{N} q_l(x)\, \mu^{-l/2}\} \right| \le B\, \mu^{-(N+1)/2}.$$

(iii) The sums $\sum_{|\mu-k| \le \mu^{2/3}}$ are approximated by integrals $\int_{|\mu-k| \le \mu^{2/3}} \cdots dk$ with an error which, for every positive number r, is $O(\mu^{-r})$ $(\mu \to \infty)$. Then these integrals are transformed into integrals over x and then approximated by integrals $\int_{-\infty}^{\infty} dx$ with errors of the kind $O(e^{-c\mu^{1/3}})$ $(\mu \to \infty)$.

4. The truncation of the sum

The function $I(\mu, k)$ interpreted as a function of the real variable $k \in [0,\infty)$ is decreasing on $[0,\infty)$. This statement follows from

$$\frac{d}{dk} I(\mu, k) = (\Gamma(k+1))^{-2} \int_0^{\mu} dt \int_{\mu}^{\infty} e^{-t-\tau} t^k \tau^k \log(t\tau^{-1})\, d\tau < 0 \quad .$$

Let $k \le \mu - \mu^{2/3} := a$. Then substituting $t = k(1+s)$ we have

$$1 - I(\mu, k) \le 1 - I(\mu, a) = (\Gamma(a+1))^{-1} e^{-a} a^{a+1} \int_{a^{-1}\mu - 1}^{\infty} e^{ah(s)} ds \quad ,$$

where $h(s) := -s + \log(1+s)$. The function h is concave and negative on $(-1,\infty)$, whence $h(s) \le h(\mu a^{-1} - 1) + (s - \mu a^{-1} + 1)\, h'(\mu a^{-1} - 1)$ $(s \ge \mu a^{-1} - 1)$.

It follows that

$$0 < e^{-a} a^{a+1} (\Gamma(a+1))^{-1} \int_{a^{-1}\mu - 1}^{\infty} e^{ah(s)} ds \le e^{-a} a^{a+1} (\Gamma(a+1))^{-1} e^{ah(a^{-1}\mu - 1)}\ (a\, h'(\mu a^{-1} - 1))^{-1}$$

$$\le \mu^{1/2} e^{-\frac{1}{2}\mu^{1/3}} \quad (\mu \ge 8) \quad .$$

Hence, for both of the sums in (1) and (2)

$$0 < \sum_{0 \le k \le a} < \mu \, . \, \mu^{3/2} \, e^{-\frac{3n}{2}\mu^{1/3}} = O(e^{-\mu^{1/3}}) \;\; (\mu \to \infty) \quad .$$

Let $k \ge \mu + \mu^{2/3}$. Then $I(\mu,k) = e^{-k} k^{k+1} (\Gamma(k+1))^{-1} \int_{-1}^{k^{-1}\mu - 1} e^{kh(s)} ds$. Since $h(s) \le -1/2s^2$ on (-1, 0] we have

$$I(\mu,k) \le e^{-k} k^{k+1} (\Gamma(k+1))^{-1} \int_{-1}^{k^{-1}\mu-1} e^{-1/2ks^2} ds$$

$$\le e^{-k} k^{k+1} (\Gamma(k+1))^{-1} \int_{1-k^{-1}\mu}^{\infty} e^{-1/2ks^2} ds$$

$$\le e^{k} k^{k+1} (\Gamma(k+1))^{-1} \int_{1-k^{-1}\mu}^{\infty} e^{-1/2k(1-k^{-1}\mu)s} ds$$

$$\le \mu^{-1/6} \, e^{-1/2k(\mu^{1/3}+1)^{-2}} \quad (\mu \ge 8) \quad .$$

Hence, for both of the sums

$$0 < \sum_{k \ge \mu + \mu^{2/3}} \le \mu^{-1/6} \sum_{k \ge \mu + \mu^{2/3}} k \, e^{-\frac{1}{2}km(\mu^{1/3}+1)^{-2}} \le 2\mu e^{-1/3m\mu^{1/3}} \quad (\mu \ge 8) \quad .$$

Hence, in both cases

$$\sum_{k=0}^{\infty} = \sum_{|\mu - k| \le \mu^{2/3}} + O(e^{-c\mu^{1/3}}) \;\; (\mu \to \infty)$$

where c is a positive number.

5. The incomplete gamma function

As we have seen already the substitution

(12) $\qquad t = k(1+s)$

in the integral representation of $I(\mu,k)$ gives

(13) $\qquad I(\mu,k) = (k!)^{-1} e^{-k} k^{k+1} \int_{-1}^{k^{-1}\mu-1} e^{kh(s)} ds \quad ,$

where

(14) $\qquad h(s) := -s + \log(1+s) \quad (s > -1).$

We introduce a new integration variable y by

(15) $\qquad y := k^{1/2} \, | h(s) |^{1/2} \, \mathrm{sgn}(s) \quad (s > -1).$

Then we get

(16) $$I(\mu,k) = (k!)^{-1} e^{-k} k^{k+1/2} \int_{-\infty}^{x} e^{-y^2} \frac{ds}{dy} dy \quad ,$$

where

(17) $\qquad x := k^{1/2} \, | h(k^{-1}\mu - 1) |^{1/2} \mathrm{sgn}(\mu - k) \quad .$

To study the transformation (15) we introduce first in (13)

(18) $\qquad \tau = | h(s) |^{1/2} \mathrm{sgn}(s) \quad .$

Then

(19) $$\begin{aligned} \sqrt{2}\,\tau &= s\,(1 - 2/3s + 2/4s^2 - 2/5s^3 + \ldots)^{1/2} \quad (|s|<1) \quad , \\ &= s - 1/3s^2 + 7/36s^3 + \cdots \quad (|s|<1) \quad , \end{aligned}$$

the radius of convergence being one since $h(s)/s^2$ has no zeros inside the unit circle. By the Bürmann-Lagrange theorem we can expand s as a powerseries in τ with a positive radius of convergence, say ρ.

We calculate

(20) $$s = \sqrt{2}\,\tau + 2/3\,\tau^2 + \frac{\sqrt{2}}{18}\tau^3 - 2/135\,\tau^4 + \cdots \quad (|\tau| < \rho) \quad .$$

(21) $$\frac{ds}{d\tau} = \sum_{l=0}^{\infty} c_l \tau^l = \sqrt{2} + 4/3\tau + \frac{\sqrt{2}}{6}\tau^2 - 8/135\tau^3 + \cdots \quad (|\tau| < \rho) \quad .$$

The transformation (18) changes the integral (13) into

(22) $$I(\mu,k) = (k!)^{-1} \, e^{-k} k^{k+1} \int_{-\infty}^{k^{-1/2}x} e^{-k\tau^2} \, \frac{ds}{d\tau} d\tau \quad .$$

We shall study the asymptotic behaviour of this integral for $k \to \infty$ and fixed $x \in \mathbb{R}$. Therefore we shall denote the right hand of (22) by $\tilde{I}(x,k)$.

In order to use (21) we must truncate the interval of integration. Now it is easily shown that

$$\left| \int_{|\tau| \geq 1/2\rho} e^{-k\tau^2} \frac{ds}{d\tau} \, d\tau \right| = O(e^{-ck^{1/3}}) \quad (k \to \infty) \quad ,$$

where c is a positive number.

Inside the circle $|\tau| \leq 1/2\rho$ the powerseries in (21) is also an asymptotic series, i.e. for every $N \in \mathbb{N}$ we have

(23) $$\frac{ds}{d\tau} = \sqrt{2} + 4/3\tau + \frac{\sqrt{2}}{6}\tau^2 - 8/135\tau^3 + \cdots + c_{N-1}\,\tau^{N-1} + O(\tau^N) \quad (|\tau| \leq 1/2\rho)$$

Hence

(24) $$\tilde{I}(x,k) = (k!)^{-1} e^{-k} k^{k+1} \int\limits_{|\tau|\leq 1/2\rho,\ \tau<k^{-1/2}x} e^{-k\tau^2}(\sqrt{2} + \frac{4}{3}\tau + \ldots + c_{N-1}\tau^{N-1} + O(\tau^N))d\tau$$

$$+O\left[e^{-ck^{1/3}}\right] \quad (k\to\infty) \quad .$$

Now we change the lower bound into $-\infty$ and, evenually, the upper bound into $k^{-1/2}x$, thereby making an error of the kind $O(e^{-ck^{1/3}})$ $(k\to\infty)$ uniformly in $x \in \mathbb{R}$.
Then we substitute $\tau = k^{-1/2}y$ and we get

(25) $$\tilde{I}(x,k) = (k!)^{-1} e^{-k} k^{k+1/2} \int\limits_{-\infty}^{x} e^{-y^2}(\sqrt{2} + 4/3k^{-1/2}y + \frac{\sqrt{2}}{6}k^{-1}y^2 + \cdots$$

$$+c_{N-1}k^{-\frac{N-1}{2}}\,y^{N-1} + O(k^{-N/2}y^N))\,dy$$

$$+O(e^{-ck^{1/3}}) \quad (k\to\infty) \quad \text{uniformly in } x\in\mathbb{R}.$$

Since $\int\limits_{-\infty}^{x} e^{-y^2} O(k^{-N/2}y^N)\,dy = O(k^{-N/2})$ $(k\to\infty)$ uniformly in $x\in\mathbb{R}$, we have

(26) $$\tilde{I}(x,k) = \frac{g(k)}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-y^2}(\sqrt{2} + 4/3k^{-1/2}y + \cdots + c_{N-1}k^{-(N-1)/2}y^{N-1})\,dy$$

$$+O(k^{-N/2}) \quad (k\to\infty)$$

uniformly in $x\in\mathbb{R}$, where

(27) $$g(k) := (2\pi)^{1/2}(k!)^{1} e^{-k} k^{k+1/2} \quad .$$

Hence

(28) $$\tilde{I}(x,k) = g(k) \sum_{l=0}^{N-1} c_l f_l(x) k^{-l/2} + O(k^{-N/2}) \ (k\to\infty) \text{ uniformly in } x\in\mathbb{R},$$

where the c_l's are given by (21) and

(29) $$f_l(x) := (2\pi)^{-1/2} \int\limits_{-\infty}^{x} y^l e^{-y^2} dy \quad (l \in \mathbb{N}_0) \quad .$$

Integration by parts gives

(30) $$f_l(x)=e^{-x^2}p_l(x)+\frac{1+(-1)^l}{2}\Gamma(\frac{l+1}{2})(2\pi)^{-1/2}f(x) \quad (l\in \mathbb{N}) \quad .$$

where f is defined by (10) and

(31) $$p_l(x):=-2^{-3/2}\pi^{-1/2}\Gamma(\frac{l+1}{2})\sum_{s=0}^{\lfloor (l-1)/2\rfloor}(\Gamma(\frac{l+1}{2}-s))^{-1}x^{l-1-2s} \quad (l\in \mathbb{N}) \quad .$$

Letting $x\to\infty$ we find that $f_l(\infty)=0$ if l is odd and $f_l(\infty)=\Gamma(\frac{l+1}{2})(2\pi)^{-1/2}$ if l is even.

Since $\lim_{\mu\to\infty} I(\mu,k)=1$ we get the asymptotic series for $(g(k))^{-1}$ by letting $x\to\infty$ in (28). We have

(32) $$1\approx g(k)\sum_{l=0}^{\infty}c_l f_l(\infty)k^{-l/2} \quad (k\to\infty) \quad .$$

Comparing (28) (30) and (32) we see that the factor with which $f(x)$ occurs in (28) has the same asymptotic expansion as the factor $g(k)$ in (32). Hence

(33) $$\bar{I}(x,k))\approx f(x)+g(k)\sum_{l=1}^{\infty}c_l e^{-x^2}p_l(x)k^{-l/2} \quad (k\to\infty) \text{ uniformly in } x\in \mathbb{R}.$$

(i.e. after truncation the (absolute) error is smaller than $c\ k^{-(N+1)/2}$ with c independent of x). According to the results of section 4 we restrict ourselves to values of $k\in[\mu-\mu^{2/3},\mu+\mu^{2/3}]$. Then $x=O(\mu^{1/6})$ $(\mu\to\infty)$.

In order to get an asymptotic series for $I(\mu,k)$ for $\mu\to\infty$ we have to express k as a function of μ and x. From (17) we have $kh(\mu k^{-1}-1)=-x^2$, $\operatorname{sgn}x=\operatorname{sgn}(\mu-k)$. Putting $k=\mu(1-\psi)$ and $\mu^{-1/2}x=z$ we get

(34) $$\psi+(1-\psi)\log(1-\psi)=z^2$$

whence

(35) $$\sum_{i=2}^{\infty}\frac{\psi^i}{(i-1)i}=z^2 \quad (|\psi|<1),\ \operatorname{sgn}z=\operatorname{sgn}\psi.$$

Then

(36) $$\psi(1+2\sum_{i=3}^{\infty}\frac{\psi^i}{(i-1)i})^{1/2}=\sqrt{2}\ z \quad .$$

By the Bürmann-Lagrance inversion theorem there is a postive number r such that

$$\psi = \sum_{i=1}^{\infty} d_i z^i \quad (|z|<r) \quad . \tag{37}$$

Hence

$$k = \mu - \sum_{i=1}^{\infty} d_i \mu^{1-i/2} x^i \quad (|x|<r\sqrt{\mu}). \tag{38}$$

Clearly, for μ sufficiently large, x is within the range of convergence since $x=O(\mu^{1/6})$ $(\mu\to\infty)$.
A few coefficients d_i are calculated.

$$d_1=\sqrt{2}\ , d_2=-1/3\ , d_3=\frac{-\sqrt{2}}{36}\ ,\ d_4=-\frac{23}{90} \quad . \tag{39}$$

We need also expansions for $k^{-1/2}$, k^{-1} and $k^{-3/2}$:

Clearly, there is a positive number r_0 such that for all $\alpha\in\mathbb{R}$ the function $(1-\psi)^\alpha$ has a power-series expansion in z which converges for $|z|<r_0$; if

$$(1-\psi)^\alpha = \sum_{j=0}^{\infty} a_j(\alpha) z^j \quad (|z|<r_0) \tag{40}$$

then

$$k^\alpha = \mu^\alpha (1-\psi)^\alpha = \sum_{j=0}^{\infty} a_j(\alpha) \mu^{\alpha-i/2} x^j \quad (|x|<r_0\, \mu^{1/2}) \quad . \tag{41}$$

We calculate

$$k^{-1/2} = \mu^{-1/2} + 2^{-1/2}\mu^{-1}x + \frac{7}{12}\mu^{-3/2}x^2 + 13\frac{\sqrt{2}}{36}\mu^{-2}x^3 + \cdots \tag{42}$$

$$k^{-1} = \mu^{-1} + 2^{1/2}\mu{-3/2}x + 5/3\mu^{-2}x^2 + \cdots \tag{43}$$

$$k^{-3/2} = \mu^{-3/2} + 3.2^{-1/2}\mu^{-2}x + \cdots \tag{44}$$

An asymptotic expansion for $g(k)$ can be determined from (32).

$$g(k) \approx \sum_{l=o}^{\infty} g_l k^{-l/2} \quad (k\to\infty) \tag{45}$$

Some coefficients g_l are

$$g_0=1,\ g_2=-1/12 \text{ and } g_l=0 \text{ if } l=\text{odd}. \tag{46}$$

Substituting (46) into (33) and then using (41) we get a complete asymptotic series for $I(\mu,k)$.

(47) $$I(\mu,k) \simeq f(x) + e^{-x^2} \sum_{l=1}^{\infty} q_l(x)\mu^{-l/2} \qquad (\mu\to\infty)$$

uniformly in $x = O(\mu^{1/6}) \quad (\mu\to\infty)$.

The q_l's are polynominals. We calculate

(48) $$q_1(x) = -2^{1/2}3^{-1}\pi^{-1/2} \quad .$$

(49) $$q_2(x) = -\frac{5}{12}\pi^{-1/2}x \quad .$$

6. The replacement of the sum by an integral

We have already

(50) $$\frac{d}{dk}I(\mu,k) = (\Gamma(k+1))^{-2} \int_0^{\mu} dt \int_0^{\infty} e^{-t-\tau}(t\tau)^k \log(t\tau^{-1})d\tau \quad .$$

By induction one can prove easily that

(51) $$\frac{d^l}{dk^l}I(\mu,k) = (\Gamma(k+1))^{-l-1} \int_0^{\mu} dt \int_0^{\infty} \cdots$$

$$\int_0^{\infty} e^{-t-\tau_1-\cdots-\tau_l}(t\tau_1\cdots\tau_l)^k L(t,\tau_1,\cdots,\tau_l)d\tau_1\cdots d\tau_l \quad ,$$

where the functions $L(\tau_0,\tau_1,\cdots,\tau_l)$ are defined by

(52) $$L(\tau_0) = 1$$

(53) $$L(\tau_0,\tau_1,\cdots,\tau_{l+1}) = L(\tau_0,\tau_1,\cdots,\tau_l)\log(\tau_0\tau_1\cdots\tau_l\tau_l\tau_{l+1}^{-l-1}) \quad (l=0,1,...) \quad .$$

With methods simular to those used in the treatment of $I(\mu,k)$ in section 5 we can prove easily that

(54) $$\left|\frac{d^l}{dk^l}I(\mu,k)\right| = O(e^{-c\mu^{1/3}}) \quad (\mu\to\infty\,,\ |\mu-k|\ge\mu^{2/3}) \quad .$$

Furthermore, if we restrict ourselves to values of k such that $|\mu-k|\le\mu^{2/3}$, then the integrals in (51) can be replaced by $\int_{\mu-2\mu^{2/3}}^{\mu}\int_{\mu+2\mu^{2/3}}^{\mu+2\mu^{2/3}}\cdots\int_{\mu-2\mu^{2/3}}^{\mu+2\mu^{2/3}}$ with an error of the kind $O(e^{-c\mu^{1/3}})$ $(\mu\to\infty)$. Then it follows from (51) that for $|\mu-k|\le\mu^{2/3}$

(55) $$\left|\frac{d^l}{dk^l}I(\mu,k)\right| < M^l := \max\{|L(t,\tau_1,\cdots,\tau_l)| \ \ |\mu-2\mu^{2/3}\le t,\cdots,\tau_l\le\mu+2\mu^{2/3}\}$$

$$+O(e^{c\mu^{1/3}}) \ (\mu\to\infty)$$

From (52) and (53) it follows that

$$M_{l+1} \le M_l \log \left[\frac{\mu+2\mu^{2/3}}{\mu-2\mu^{2/3}} \right]^{l+1} \ (l\in \mathbb{N}_0),$$

whence, by (54) and (55), we have for all $k\in \mathbb{N}$

$$(56) \qquad \left| \frac{d^l}{dk^l} I(\mu,k) \right| = O(\mu^{-l/3}) \ (\mu\to\infty) \ (l\in \mathbb{N}_0).$$

Now we apply the Euler-Maclaurin sumformula: For every fixed $r\in \mathbb{N}$ we have

$$(57) \qquad \sum_{k=p}^{q} f(k) = \int_p^q f(x)dx + 1/2 f(q) + 1/2 f(p)$$

$$+ \sum_{l=1}^{r} \frac{B_{2l}}{(2l)!} \left[f^{(2l-1)}(q) - f^{(2l-1)}(p) \right]$$

$$+ O\left[\int_p^q |f^{(2r)}(x)|\,dx \right] \quad (q\ge p) \quad .$$

Taking $f(k) = I^m(\mu,k)(1-I(\mu,k))^n$, $p = \left\lceil \mu-\mu^{2/3} \right\rceil$, $q = \left\lfloor \mu+\mu^{2/3} \right\rfloor$,

we find, for every $r\in \mathbb{N}$, using (54), (56) and (57)

$$(58) \qquad \sum_{|k-\mu|\le\mu^{2/3}} I^m(\mu,k)\ (1-I(\mu,k))^n = \int_{\mu-\mu^{2/3}}^{\mu+\mu^{2/3}} I^m(\mu,k)(1-I(\mu,k))^n\,dk$$

$$+O(\mu^{-2/3r+2/3}) \quad (\mu\to\infty)$$

Simularly

$$(59) \qquad \sum_{|k-\mu|\le\mu^{2/3}} (u-k) I^m(\mu,k)(1-I(\mu,k))^n = \int_{\mu-\mu^{2/3}}^{\mu+\mu^{2/3}} (\mu-k) I^m(\mu,k)\ (1-I(\mu,k))^n\,dk$$

$$+O(\mu^{-2/3r+4/3})(\mu\to\infty) \quad .$$

7. The asymptotic behaviour of S and T

According to section 6 the sums $\sum_{|\mu-k|\le\mu^{2/3}}$ can be replaced by integrals with small errors which are, for every $r>0$, $O(\mu^{-r})(\mu\to\infty)$. Then these integrals are transformed into integrals $\int_{-A(\mu)}^{B(\mu)} \cdots \frac{dk}{dx}dx$ where $A(\mu)$ and $B(\mu)$ are asymptotically equivalent with $1/2\sqrt{2}\ \mu^{1/6} \ (\mu\to\infty)$

By means of (38) and (47) we get complete asymptotic expansions for the two integrands.

(60) $$I^m(\mu,k)(1-I(\mu,k))^n \frac{dk}{dx} \approx \sum_{l=-1}^{\infty} s_l(x)\mu^{-l/2} \quad (\mu\to\infty) \quad ,$$

(61) $$(\mu-k)I^m(\mu,k)(1-I(\mu,k))^n \frac{dk}{dx} \approx \sum_{l=-2}^{\infty} t_l(x)\mu^{-l/2} \quad (\mu\to\infty) \quad ,$$

both uniformly in $x\in I\!R$, $x = O(\mu^{1/6})$ $(\mu \to \infty)$. The functions $s_l(x)$ and $t_l(x)$ are absolutely integrable over $(-\infty,\infty)$. Now we proceed as follows. Let $N\in I\!N$. Then

$$\int_{\mu-\mu^{2/3}}^{\mu+\mu^{2/3}} I^m(\mu,k)(1-I(\mu,k))^n \frac{dk}{dx} dx$$

$$= \sum_{l=-1}^{N} \mu^{-l/2} \int_{-A(\mu)}^{B(\mu)} s_l(x)dx + O\left(\mu^{-\frac{N}{2}-\frac{1}{2}+\frac{1}{6}}\right) \quad (\mu\to\infty).$$

It is easily seen that the functions $s_l(x)$ are of the form $\bar{p}(x,e^{-x^2},f(x))\ e^{-x^2} + \tilde{q}(x,e^{-x^2},f(x))\ f(x)(1-f(x))$, where $\bar{p}$ and $\tilde{q}$ are polynomials. Hence $\int_{x<-A(\mu)} |s_l(x)|\,dx$ and $\int_{x>B(\mu)} |s_l(x)|\,dx$ are $O(e^{-c\mu^{1/3}})$ $(\mu\to\infty)$.

It follows that $S(\mu,m,n)$ has a complete asymptotic expansion

(62) $$S(\mu,m,n) \approx \sum_{l=-1}^{\infty} \mu^{-l/2} \int_{-\infty}^{\infty} s_l(x)dx \quad (\mu\to\infty)$$

A simular argument holds for $T(\mu,m,n)$.

(63) $$T(\mu,m,n) \approx \sum_{l=-2}^{\infty} \mu^{-l/2} \int_{-\infty}^{\infty} t_l(x)dx \quad (\mu\to\infty).$$

We calculate

(64) $$s_{-1}(x) = \sqrt{2}f^m(x)f^n(-x)$$

(65) $$s_0(x) = 2/3\pi^{-1/2}\left[n\,f^m(x)f^{n-1}(-x) - m\,f^{m-1}(x)f^n(-x)\right] e^{-x^2} - 2/3xf^m(x)f^n(-x)$$

(66) $$t_{-2}(x) = 2\,x\,f^m(x)f^n(-x)$$

(67) $$t_{-1}(x) =$$

$$2^{3/2}3^{-1}\pi^{-1/2}\left[nf^m(x)f^{n-1}(-x) - m\,f^{m-1}(x)f^n(-x)\right] xe^{x^2} - 2^{1/2}x^2f^m(x)f^n(-x)$$

The term with factor e^{-x^2} in the right hand of (65) gives 0 upon integration.

Integration by parts of the term with factor xe^{-x^2} in the right hand of (67) gives $2/3\sqrt{2} \int_{-\infty}^{\infty} f^m(x)f^n(-x)dx.$

8. Computation of some coefficients

$$A(1,1)=\sqrt{2}\int_{-\infty}^{\infty} f(x)(1-f(x))dx=-\sqrt{2}\pi^{-1/2}\int_{-\infty}^{\infty} x(1-2f(x))e^{-x^2}dx$$

$$=2\sqrt{2}\pi^{-1/2}\int_{-\infty}^{\infty} f(x)xe^{-x^2}dx=\sqrt{2}\pi^{-1}\int_{-\infty}^{\infty} e^{-2x^2}dx=\pi^{-1/2}\quad .$$

$$A(1,2)=A(2,1)=1/2\sqrt{2}\int_{-\infty}^{\infty}(f(x)f^2(-x)+f(-x)f^2(x))dx$$

$$=1/2\sqrt{2}\int_{-\infty}^{\infty} f(x)f(-x)dx=1/2A(1,1)\quad .$$

$$B(1,1)=-2/3\int_{-\infty}^{\infty} xf(x)f(-x)dx=0\quad .$$

$$B(1,2)=-B(2,1)=-2/3\int_{-\infty}^{\infty} xf(x)f^2(-x)(f(x)+f(-x))dx$$

$$=-2/3\int_{-\infty}^{\infty} xf(x)f^3(-x)dx=B(1,3)=-B(3,1)\quad .$$

$$B(1,2)=-2/3\int_{-\infty}^{\infty} x(f(x)-2f^2(x)+f^3(x))dx$$

$$=1/3\pi^{-1/2}\int_{-\infty}^{\infty} x^2e^{-x^2}(1-4f(x)+3f^2(x))dx$$

$$=1/6\pi^{-1/2}\int_{-\infty}^{\infty} e^{-x^2}(1-4f(x)+3f^2(x)+\pi^{-1/2}xe^{-x^2}(-4+6f(x)))dx$$

$$=1/6\int_{-\infty}^{\infty}(1-4f(x)+3f^2(x))f'(x)dx+\pi^{-1}\int_{-\infty}^{\infty} xe^{-2x^2}f(x)dx$$

$$=1/4\pi^{-3/2}\int_{-\infty}^{\infty} e^{-3x^2}dx=\frac{\sqrt{3}}{12}\pi^{-1}\quad .$$

$$D(1,1)=2/3A(1,1)-\sqrt{2}\int_{-\infty}^{\infty} x^2f(x)f(-x)dx$$

$$=2/3A(1,1)-2\frac{\sqrt{2}}{3}\pi^{-1/2}\int_{-\infty}^{\infty} x^3e^{-x^2}f(x)dx$$

$$=2/3A(1,1)-2\frac{\sqrt{2}}{3}\pi^{-1/2}\int\limits_{-\infty}^{\infty} x\,e^{-2x^2} f(x)dx - \frac{\sqrt{2}}{3}\pi^{-1}\int\limits_{-\infty}^{\infty} x^2 e^{-2x^2} dx$$

$$=2/3A(1,1)-\frac{5}{12}\pi^{-1/2} = 1/4\,\pi^{-1/2} \quad .$$

$$D(1,2)=D(2,1)=2/3A(1,2)-1/2\sqrt{2}\int\limits_{-\infty}^{\infty} x^2(f(x)f^2(-x)+f^2(x)f(-x))dx =$$

$$=2/3A(1,2)-1/2\sqrt{2}\int\limits_{-\infty}^{\infty} x^2 f(x)f(-x)dx$$

$$=2/3A(1,2)+1/2D(1,1)-1/3A(1,1)=1/8\pi^{-1/2}.$$