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Asymptotics of a τ-Function Arising in the Two-Dimensional Ising Model*

Craig A. Tracy

Department of Mathematics and Institute of Theoretical Dynamics, University of California, Davis, CA 95616, USA

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Abstract. The short-distance assymptotics of the τ-function associated to the 2-point function of the two-dimensional Ising model is computed as a function of the integration constant defined from the long-distance behavior of the τ-function. The result is expressible in terms of the Barnes double gamma function (equivalently, the Barnes G-function).

1. Introduction and Summary of Results

If $\xi = \xi(T)$ is the correlation length, T is temperature, and $\langle \sigma_{00}\sigma_{MN}\rangle$ is the spin–spin correlation function for the two-dimensional Ising model,[1] then in the scaling limit, defined by

$$\xi \to \infty, \quad R = (M^2 + N^2)^{1/2} \to \infty,$$

such that

$$t = \frac{R}{\xi} \quad \text{is fixed,}$$

it is known [3, 6] (see also [4]) that

$$\lim R^{1/4} \langle \sigma_{00}\sigma_{MN}\rangle = F_{\pm}(t), \tag{1.1}$$

where $+(-)$ denotes the limit is taken above (below) the critical temperature T_c. Furthermore, the scaling functions $F_{\pm}(t)$ are given by [3, 6]

$$F_{\pm}(t) = 2^{3/8} t^{1/4} \tau_{\pm}\left(t, \frac{1}{\pi}\right), \tag{1.2}$$

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[1] For simplicity of presentation we assume that the horizontal and vertical interactions are equal

where

$$\tau_\pm(t,\lambda)=\begin{Bmatrix}\sinh(\frac{1}{2}\psi(t,\lambda))\\ \cosh(\frac{1}{2}\psi(t,\lambda))\end{Bmatrix}\exp\left(\frac{1}{4}\int_t^\infty s\left((\sinh\psi(s,\lambda))^2-\left(\frac{d\psi}{ds}\right)^2\right)ds\right), \tag{1.3}$$

and $\psi(t,\lambda)$ satisfies the differential equation

$$\frac{d^2\psi}{dt^2}+\frac{1}{t}\frac{d\psi}{dt}=\frac{1}{2}\sinh(2\psi), \tag{1.4}$$

subject to the boundary condition

$$\psi(t,\lambda)\sim 2\lambda K_0(t) \quad \text{as} \quad t\to+\infty. \tag{1.5}$$

Here $K_0(t)$ is the Bessel function and we assume $0<\pi\lambda\leqq 1$. It is known [3] that for $0<\pi\lambda<1$,

$$\psi(t,\lambda)=-\sigma\log t-\log B+o(1) \quad \text{as} \quad t\to 0^+ \tag{1.6}$$

with

$$\sigma=\sigma(\lambda)=\frac{2}{\pi}\arcsin(\pi\lambda), \tag{1.7}$$

$$B=2^{-3\sigma}\frac{\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(\frac{1+\sigma}{2}\right)}, \tag{1.8}$$

and $\Gamma(z)$ the gamma function.

Using the short distance behavior (1.6) of $\psi(t,\lambda)$ in (1.3), it is easy to show that

$$\tau_\pm(t,\lambda)\sim\tau_0 t^{-(1-\sigma/2)\sigma/2} \quad \text{as} \quad t\to 0^+, \tag{1.9}$$

where τ_0 is independent of t but will depend upon the parameter λ. However, the "constant" τ_0 will not be determined by this elementary analysis. It is shown in [3, 6] that (1.9) is also valid (with $\sigma=1$) for $\pi\lambda=1$. It is the purpose of this paper to give τ_0 as a function of λ, or equivalently in view of (1.7), τ_0 as a function of σ.

To state our result for τ_0, we first recall both the definition and some properties of the Barnes double gamma function [1], $\Gamma_2(s)$:

$$\frac{1}{\Gamma_2(s+1)}:=(2\pi)^{s/2}e^{-s/2-(1/2)(\gamma+1)s^2}\prod_{k=1}^{\infty}\left(1+\frac{s}{k}\right)^k e^{-s+s^2/2k},$$

where γ is Euler's constant and $\Gamma_2(s)$ satisfies [1]

$$\begin{aligned}\Gamma_2(s+1)&=\Gamma_2(s)/\Gamma(s),\\ \Gamma_2(1)&=1,\\ \Gamma_2(\tfrac{1}{2})&=\exp(-\tfrac{3}{2}\zeta'(-1)+\tfrac{1}{4}\log\pi-\tfrac{1}{24}\log 2),\end{aligned} \tag{1.10}$$

where $\zeta'(-1)$ is the derivative of the Riemann zeta function evaluated at minus one.[2] Another common notation is $G(s)=1/\Gamma_2(s)$ which is called the Barnes G-function. In this paper we prove

[2] $\zeta'(-1)=-0.16542\,11437\,00450\,92925\ldots$. Some authors use the Glaisher, or Kinkelin–Glaisher, constant A instead of $\zeta'(-1)$ – the two are related by $\log A=\frac{1}{12}-\zeta'(-1)$. Note the 1/12 is missing in (6.109) of [6] which came about from a sign error in (6.107) of [6]. Thus (6.110) must be corrected but (6.104) of [6] is correct – which is the equation checked numerically in [6]

Theorem 1. *Let* $s=(1-\sigma)/2$, *then*

$$\tau_0 = e^{3\zeta'(-1)-(3s^2+1/6)\log 2}\Gamma_2(1+s)\Gamma_2(1-s).$$

The proof of Theorem 1 uses techniques very similar to the proof of the connection formulae in [3].

We now apply Theorem 1 to the Ising model. If one first sets $T=T_c$, then Wu [5] has shown that

$$\langle\sigma_{00}\sigma_{NN}\rangle|_{T=T_c} = \frac{e^{3\zeta'(-1)+(1/12)\log 2}}{N^{1/4}}\left(1+O\left(\frac{1}{N^2}\right)\right) \quad \text{as} \quad N\to\infty. \tag{1.11}$$

Now part of the scaling hypothesis is that the constant calculated at $T=T_c$ in the leading asymptotic behavior of the spin–spin correlation function, i.e. (1.11), should equal $F_\pm(0)$ – in the language of asymptotic expansions, the two asymptotic expansions should match. Setting $s=0$ ($\pi\lambda=1$) in Theorem 1 and using (1.9) for $\pi\lambda=1$ in (1.2) we see that the two constants are indeed equal (note $R=\sqrt{2}N$). This closes a gap in the proof of the scaling hypothesis of the spin–spin correlation function in the analysis given in [3, 4, 6].

2. Reduction to Integral Equations

In their generalization of the Ising field theory to holonomic quantum fields, Sato, Miwa, and Jimbo [4] also discussed a neutral bosonic theory (the Ising model, of course, is a fermionic theory) and associated to this bosonic theory a τ-function which in the case of the 2-point function and in the notation of Sect. 1 is given by (we set their parameter $l=l_1-l_2=0$)

$$\tau_B(t,\lambda)=\exp\left(-\frac{1}{2}\int_t^\infty s\left((\sinh\psi(s,\lambda))^2-\left(\frac{d\psi}{ds}\right)^2\right)ds\right). \tag{2.1}$$

Just as in the Ising case (see (1.14b) in [3]), τ_B has a representation as an infinite series of integrals (see SMJ (4.5.31)):

$$\tau_B(t,\lambda)=\exp(E(t,\lambda))=\exp\left(\sum_{k=1}^\infty \frac{\lambda^{2k}}{k}e_{2k}(t)\right), \tag{2.2}$$

where

$$e_k(t)=\int_0^\infty dx_1\cdots\int_0^\infty dx_k\prod_{j=1}^k \frac{\exp(-\frac{1}{2}t(x_j+x_j^{-1}))}{x_j+x_{j+1}}, \tag{2.3}$$

with $x_{k+1}=x_1$ in the above integral. As in the Ising case, the integrals are of the form of an iterated integral operator; and hence, both $\tau_\pm$ and τ_B can be expressed as infinite product formulae in terms of the unknown eigenvalues of the corresponding integral operator. In the Appendix we show that the series (2.2) converges for all $0\leqq\pi\lambda<1$ and for all $t>0$.

We will show that

$$e_k(t)=\alpha_k\log\left(\frac{1}{t}\right)+\beta_k+o(1) \quad \text{as} \quad t\to 0^+, \tag{2.4}$$

$$E(t,\lambda)=\alpha(\lambda)\log\left(\frac{1}{t}\right)+\beta(\lambda)+o(1) \quad \text{as} \quad t\to 0^+, \tag{2.5}$$

where

$$\alpha(\lambda)=\sum_{k=1}^{\infty}\frac{\lambda^{2k}}{k}\alpha_{2k},\quad \beta(\lambda)=\sum_{k=1}^{\infty}\frac{\lambda^{2k}}{k}\beta_{2k}. \tag{2.6}$$

To begin, we take the Mellin transform of $e_k(t)$:

$$\begin{aligned}\hat{e}_k(z)&=\int_0^\infty t^{z-1}e_k(t)dt,\quad \Re(z)>1,\\ &=2^{z-1}\Gamma(z)\int_0^\infty dx_1\cdots\int_0^\infty dx_k\prod_{j=1}^{k}(x_j+x_{j+1})^{-1}\left[x_1+\cdots+x_k+\frac{1}{x_1}+\cdots+\frac{1}{x_k}\right]^{-z}\\ &=2^{z-1}\Gamma(z)\int_0^1 d\delta_1\int_0^{1-\delta_1}d\delta_2\cdots\int_0^{1-\delta_1-\cdots-\delta_{k-2}}d\delta_{k-1}\prod_{j=1}^{k}(\delta_j+\delta_{j+1})^{-1}\\ &\quad\cdot\int_0^\infty d\rho\,\rho^{z-1}[\rho^2+\Delta_k(\delta)]^{-z},\end{aligned} \tag{2.7}$$

where

$$\Delta_k(\delta)=\frac{1}{\delta_1}+\cdots+\frac{1}{\delta_k},$$

and we made the change of variables

$$\begin{aligned}\rho&=x_1+\cdots+x_k,\\ x_j&=\rho\delta_j,\end{aligned} \tag{2.8}$$

with $\delta_k=1-\delta_1-\cdots-\delta_{k-1}$, in going from the first multiple integral to the second (note the Jacobian is ρ^{k-1}). Using the integral

$$\int_0^\infty\frac{\rho^{z-1}}{(A+B\rho^2)^z}d\rho=\frac{1}{2}A^{-z/2}B^{-z/2}\frac{\Gamma^2(z/2)}{\Gamma(z)},$$

$\hat{e}_k(z)$ becomes

$$\begin{aligned}\hat{e}_k(z)&=2^{z-1}\Gamma^2(z/2)\int_0^1 d\delta_1\int_0^{1-\delta_1}d\delta_2\cdots\int_0^{1-\delta_1-\cdots-\delta_{k-2}}d\delta_{k-1}\prod_{j=1}^{k}(\delta_j+\delta_{j+1})^{-1}(\Delta_k(\delta))^{-z/2}\\ &=2^{z-1}\Gamma^2(z/2)\mathscr{I}_k(z),\end{aligned} \tag{2.9}$$

where $\mathscr{I}_k(z)$ is defined by the last equation. This last expression for $\hat{e}_k(z)$ provides an analytic continuation to $\Re(z)>0$ and gives the boundary values on the imaginary axis except at $z=0$ where we see that $\hat{e}_k(z)$ has a pole of order 2. This expression might give the analytic continuation for $\Re(z)>-2$, but we have not proved this. For $k=2$ (2.9) can be evaluated to give

$$\hat{e}_2(z)=2^{z-1}\Gamma^2(z/2)\Gamma^2(z/2+1)/\Gamma(z+2).$$

We now proceed to calculate the principal part of $\hat{e}_k(z)$ at $z=0$.

Proposition 2.1. *For* $k=1,2,\ldots$ *we have*

$$\mathscr{I}_k(0)=\frac{\pi^{k-2}}{2}B\left(\frac{1}{2},\frac{k}{2}\right)=\frac{k}{4}\sigma_k,$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function, and σ_k is defined by the last equality.

Putting $z = 0$ into the definition of $\mathscr{I}_k(z)$ and comparing with (4.30) of [3], the result follows immediately from Lemma 4.2 in [3].

To find $\mathscr{I}'_k(0)$ we consider the integral for $k = 2, 3, \ldots,$

$$J_k = \int_0^\infty dx_1 \cdots \int_0^\infty dx_k e^{-x_1 - x_k} \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} \log\left(\frac{(x_1 + \cdots + x_k)\Delta_k(x)}{x_1 + x_k}\right) \tag{2.10}$$

and make the change of variables (2.8) to find

$$J_k = \gamma \mathscr{I}_k(0) - 2\mathscr{I}'_k(0). \tag{2.11}$$

Thus it suffices to look at J_k. We break J_k into the two pieces

$$J_k = J_k^{(1)} + J_k^{(2)} \tag{2.12}$$

with

$$J_k^{(1)} = \int_0^\infty dx_1 \cdots \int_0^\infty dx_k e^{-x_1 - x_k} \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} \log \Delta_k(x) \tag{2.13}$$

and

$$J_k^{(2)} = \int_0^\infty dx_1 \cdots \int_0^\infty dx_k e^{-x_1 - x_k} \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} \log\left(\frac{x_1 + \cdots + x_k}{x_1 + x_k}\right). \tag{2.14}$$

Using the identity

$$\log\left(\frac{x}{y}\right) = \int_0^\infty \frac{d\xi}{\xi}(e^{-\xi y} - e^{-\xi x}),$$

we break each $J_k^{(i)}$ into two pieces

$$J_k^{(i)} = \lim_{\varepsilon \to 0} (J_k'^{(i)}(\varepsilon) - J_k''^{(i)}(\varepsilon)), \quad i = 1, 2, \tag{2.15}$$

where for $i = 1$,

$$J_k'^{(1)}(\varepsilon) = \int_\varepsilon^\infty \frac{d\xi}{\xi} e^{-\xi} \int_0^\infty dx_1 \cdots \int_0^\infty dx_k e^{-x_1} \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} e^{-x_k}, \tag{2.16}$$

$$\begin{aligned} J_k''^{(1)}(\varepsilon) &= \int_\varepsilon^\infty \frac{d\xi}{\xi} \int_0^\infty dx_1 \cdots \int_0^\infty dx_k e^{-x_1} \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} e^{-x_k} \exp(-\xi \Delta_k(x)) \\ &= \int_\varepsilon^\infty d\xi \int_0^\infty d\mu_1 \cdots \int_0^\infty d\mu_k \frac{\exp(-\xi/x_1)}{x_1} \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} \frac{\exp(-\xi/x_k)}{x_k}, \end{aligned} \tag{2.17}$$

and we made the change of variables $x_j \to \xi/x_j$ in obtaining that last integral and have written $d\mu_j = e^{-x_j} dx_j$. Similarly for $i = 2$ we have

$$J_k'^{(2)}(\varepsilon) = \int_\varepsilon^\infty \frac{d\xi}{\xi} \int_0^\infty dx_1 \cdots \int_0^\infty dx_k e^{-(1+\xi)x_1} \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} e^{-(1+\xi)x_k}, \tag{2.18}$$

and

$$J_k''^{(2)}(\varepsilon) = \int_\varepsilon^\infty \frac{d\xi}{\xi} \int_0^\infty dx_1 \cdots \int_0^\infty dx_k \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} e^{-x_1 - x_k} \exp(-\xi(x_1 + \cdots + x_k))$$
$$= \int_0^{1/\varepsilon} d\zeta \int_0^\infty d\mu_1 \cdots \int_0^\infty d\mu_k e^{-\zeta x_1} \prod_{j=1}^{k-1} (x_j + x_{j+1})^{-1} e^{-\zeta x_k}, \tag{2.19}$$

where we made the change of variables $\xi x_j \to x_j$ and then $\zeta = 1/\xi$ in obtaining the last integral.

Again referring to the proof of Lemma 4.2 in [3] we obtain

$$J_k'^{(1)}(\varepsilon) = \frac{k}{4}\sigma_k\left(\log\frac{1}{\varepsilon} - \gamma\right) + o(1), \quad \varepsilon \to 0^+,$$

$$J_k'^{(2)}(\varepsilon) = \frac{k}{4}\sigma_k \log\frac{1}{\varepsilon} + o(1), \quad \varepsilon \to 0^+. \tag{2.20}$$

We now define the integral operator K on the Hilbert space $L^2(0, \infty; e^{-x}dx) = L^2(0, \infty; d\mu)$,

$$(Kf)(x) := \int_0^\infty \frac{1}{x+y} f(y) d\mu(y). \tag{2.21}$$

The (generalized) eigenvalues and eigenfunctions of the operator K can be determined from the Mehler–Fock transform (see, e.g., Lemmas 4.3 and 4.4 in [3]):

$$K\chi_p = \lambda_p \chi_p, \quad p \geqq 0, \tag{2.22}$$

where

$$\lambda_p = \pi \operatorname{sech} \pi p, \quad 0 \leqq p < \infty, \tag{2.23}$$

and

$$\chi_p(x) = \left(\frac{p \sinh \pi p}{2\pi}\right)^{1/2} \int_1^\infty e^{-(\xi - 1)x/2} P_{-1/2+ip}(\xi) d\xi$$
$$= \frac{1}{\pi}\left(\frac{2p \sinh \pi p}{x}\right)^{1/2} e^{x/2} K_{ip}\left(\frac{x}{2}\right), \tag{2.24}$$

where $P_{-1/2+ip}(\xi)$ is the Legendre function, $K_{ip}(x)$ is the K-Bessel function of imaginary order, and we used GR 7.1415 [2] to evaluate the integral. The normalization is chosen so that if

$$g(x) = \int_0^\infty \chi_p(x) \hat{g}(p) dp,$$

then

$$\hat{g}(p) = \int_0^\infty \chi_p(x) g(x) d\mu(x).$$

Thus we have for any $f, g \in L^2(0, \infty; d\mu)$,

$$(f, K^j g) = \int_0^\infty \overline{\hat{f}(p)} \lambda_p^j \hat{g}(p) dp, \quad j = 1, 2, \ldots . \tag{2.25}$$

Defining

$$\hat{f}_1(p,\xi)=\int_0^\infty \chi_p(x)\frac{e^{-\xi/x}}{x}\,d\mu(x),$$

$$\hat{f}_2(p,\xi)=\int_0^\infty \chi_p(x)e^{-\xi x}\,d\mu(x), \tag{2.26}$$

(2.17) and (2.19) become

$$J_k''^{(1)}(\varepsilon)=\int_\varepsilon^\infty d\xi\int_0^\infty dp\,|\hat{f}_1(p,\xi)|^2\lambda_p^{k-1},$$

$$J_k''^{(2)}(\varepsilon)=\int_0^{1/\varepsilon} d\zeta\int_0^\infty dp\,|\hat{f}_2(p,\zeta)|^2\lambda_p^{k-1}. \tag{2.27}$$

The integrals (2.26) can be evaluated:

$$\hat{f}_1(p,\xi)=2\left(\frac{2p\sinh\pi p}{\pi\xi}\right)^{1/2}K_{2ip}(2\sqrt{\xi}),$$

$$\hat{f}_2(p,\xi)=\lambda_p\left(\frac{2p\sinh\pi p}{\pi}\right)^{1/2}P_{-1/2+ip}(2\xi+1). \tag{2.28}$$

We record here the integral

$$\frac{1}{\pi}\int_0^\infty \lambda_p^k\,dp=\frac{k}{4}\sigma_k. \tag{2.29}$$

3. Functions $\alpha(\lambda)$ and $\beta(\lambda)$ in Terms of a Single Integral

Comparing (2.26) with (4.67) of [3], we see that

$$J_k''^{(2)}(\varepsilon)=\frac{k}{4}B_k^{(2)}(\varepsilon),$$

where the right-hand side is in the notation of [3]. Thus using (4.84) of [3] we obtain

$$J_k''^{(2)}(\varepsilon)=\frac{k}{4}\sigma_k\log\left(\frac{1}{\varepsilon}\right)+\frac{1}{\pi}\int_0^\infty \lambda_p^k\left[2\log 2+\Re\psi(ip)-\Re\psi\left(\frac{1}{2}+ip\right)\right]dp$$
$$-\frac{\pi^k}{4}+o(1)\quad\text{as}\quad \varepsilon\to 0^+, \tag{3.1}$$

where $\psi(z)=d\log\Gamma(z)/dz$.

We will now show that

$$J_k''^{(1)}(\varepsilon)=\frac{k}{4}\sigma_k\log\left(\frac{1}{\varepsilon}\right)-\frac{\pi^k}{4}+\frac{2}{\pi}\int_0^\infty \lambda_p^k\Re\psi(2ip)\,dp+o(1)\quad\text{as}\quad\varepsilon\to 0^+, \tag{3.2}$$

and hence combining the four J's (and using $2\psi(2x)=\psi(x)+\psi(1/2+x)+2\log 2$)

we will get

$$J_k = -\frac{1}{4}k\gamma\sigma_k - k\log 2\,\sigma_k + \frac{1}{2}\pi^k - \frac{2}{\pi}\int_0^\infty \lambda_p^k \Re\psi(ip)dp. \tag{3.3}$$

To begin, we recall (2.27, 2.28) and interchange the order of integration, to see that we must examine

$$F(\varepsilon) := \int_\varepsilon^\infty \xi^{-1}|K_{2ip}(2\sqrt{\xi})|^2 d\xi \tag{3.4}$$

for small ε. Hence we introduce the Mellin transform

$$\begin{aligned}\hat{F}(z) &= \int_0^\infty \varepsilon^{z-1}F(\varepsilon)d\varepsilon \\ &= \frac{1}{z}\int_0^\infty \xi^{z-1}|K_{2ip}(2\sqrt{\xi})|^2 d\xi \\ &= \frac{\Gamma^2(z)\Gamma(z-2ip)\Gamma(z+2ip)}{4z\Gamma(2z)} \qquad (3.5)\\ &= \frac{\sqrt{\pi}}{2^{2z+1}z}\frac{\Gamma(z)\Gamma(z+2ip)\Gamma(z-2ip)}{\Gamma(z+\frac{1}{2})}, \qquad (3.6)\end{aligned}$$

where to obtain (3.5) we used GR 6.5764 [2] and (3.6) follows from applying the gamma function duplication formula. Equation (3.6) provides the analytic continuation of $\hat{F}(z)$ into the left-half plane where we see it has on the imaginary axis a double pole at $z=0$ and simple poles at $z=\pm 2ip$. In the inverse Mellin transform, we now deform the contour into the left-half plane picking up the residues at the poles on the imaginary axis. The integral along the vertical direction in the left-half plane is $o(1)$ as $\varepsilon \to 0^+$. This result when used in (2.27) gives an asymptotic expansion for $J_k''^{(1)}(\varepsilon)$. The integrals which result from the residues of the poles at $z=\pm 2ip$ can be evaluated in the limit $\varepsilon\to 0^+$ by appealing to the Riemann–Lebesgue lemma to see that the nonzero contribution to the integrals is in the neighborhood of $p \sim 0$. The double pole requires expanding the Γ-functions about $z=0$, hence the appearance of the ψ function. The result of this calculation is (3.2).

We can now calculate the principal part of $\hat{e}_k(z)$ at $z=0$. Proposition 2.1, (2.11), and (3.3) shows that (2.9) has the Laurent expansion

$$\hat{e}_k(z) = e_{-2,k}\frac{1}{z^2} + e_{-1,k}\frac{1}{z} + \cdots, \tag{3.7}$$

where

$$\begin{aligned} e_{-2,k} &= \frac{k}{2}\sigma_k, \\ e_{-1,k} &= \frac{k}{4}(-\gamma + 2\log 2)\sigma_k - J_k. \qquad (3.8)\end{aligned}$$

Using the inverse Mellin transform and deforming the contour to the imaginary

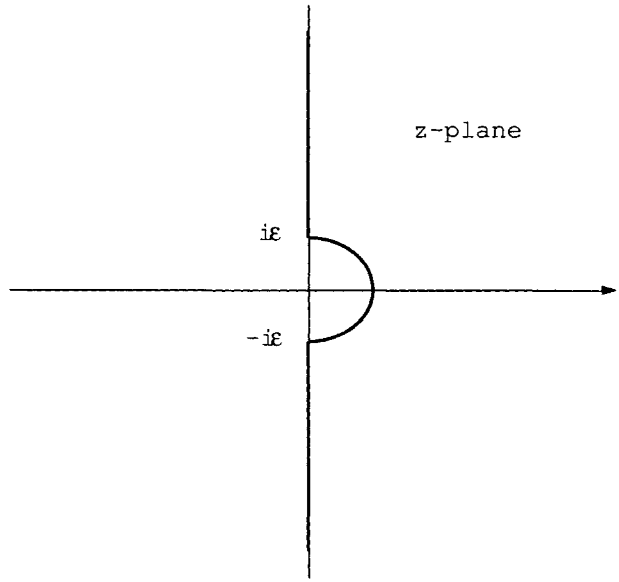

Fig. 1. Contour used in inverse Mellin transform

axis as shown in Fig. 1, we see that in the limit $\varepsilon \to 0^+$ (note the $O(1/\varepsilon)$ term from the small semi-circle is cancelled by the $O(1/\varepsilon)$ term from the integrals on the imaginary axis):

$$e_k(t) = -e_{-2,k}\log t + e_{-1,k} + \frac{1}{\pi}\int_0^\infty \cos(y\log t)\left[\Re(\hat{e}_k(iy)) + \frac{e_{-2,k}}{y^2}\right]dy$$
$$+\frac{1}{\pi}\int_0^\infty \sin(y\log t)\left[\Im(\hat{e}_k(iy)) + \frac{e_{-1,k}}{y}\right]dy. \tag{3.9}$$

We now let $t \to 0^+$ in (3.9), use the Riemann–Lebesgue lemma to conclude that the integrals are $o(1)$, and hence (2.4) holds with (using (3.3) and (3.8))

$$\alpha_k = \frac{k}{2}\sigma_k,$$
$$\beta_k = \frac{3}{2}k\log 2\,\sigma_k - \frac{1}{2}\pi^k + \frac{2}{\pi}\int_0^\infty \lambda_p^k \Re\psi(ip)dp. \tag{3.10}$$

We now *define* the functions $\alpha(\lambda)$ and $\beta(\lambda)$ by (2.6). Then an elementary calculation shows

$$\alpha(\lambda) = \sum_{k=1}^\infty \sigma_{2k}\lambda^{2k}, \tag{3.11}$$

$$\beta(\lambda) = 3\log 2\,\alpha(\lambda) + \tfrac{1}{2}\log(1-(\pi\lambda)^2) - \frac{2}{\pi}\int_0^\infty \log\left(1 - \frac{\pi^2\lambda^2}{\cosh^2 \pi p}\right)\Re\psi(ip)dp. \tag{3.12}$$

Result (2.5) will now follow once we show the error estimate $o(1)$ in (2.4) remains $o(1)$ when summed over k. We postpone the error estimate analysis to Sect. 6 where we show that indeed (2.5) is true. It can be shown that

$$\alpha(\lambda) = \tfrac{1}{2}\sigma^2(\lambda), \tag{3.13}$$

where

$$\sigma(\lambda) = \sum_{k=0}^{\infty} \sigma_{2k+1}\lambda^{2k+1} = \frac{2}{\pi}\arcsin \pi\lambda, \tag{3.14}$$

and σ_k are defined in Proposition 2.1.

4. Evaluation of an Integral

In this section we evaluate

$$I(\sigma) = \frac{1}{\pi}\int_0^{\infty} \log\left(1 - \frac{\sin^2\frac{\pi}{2}\sigma}{\cosh^2 \pi p}\right)\Re\psi(ip)dp \tag{4.1}$$

which appears in the expression for $\beta(\lambda)$. Differentiating $I(\sigma)$ with respect to σ results in

$$\frac{dI}{d\sigma} = -\sin\frac{\pi}{2}\sigma\cos\frac{\pi}{2}\sigma\int_0^{\infty}\left(\cosh^2\pi p - \sin^2\frac{\pi}{2}\sigma\right)^{-1}\Re\psi(ip)dp. \tag{4.2}$$

To evaluate $dI/d\sigma$ we define

$$f(z) = \frac{z}{\cosh^2 \pi z - \sin^2\frac{\pi}{2}\sigma}\psi(iz),$$

and evaluate $\int_{\mathscr{C}} f(z)dz$ where the contour $\mathscr{C}$ is shown in Fig. 2 (the ε is chosen small enough so that the two poles of f lying on the imaginary axis between 0

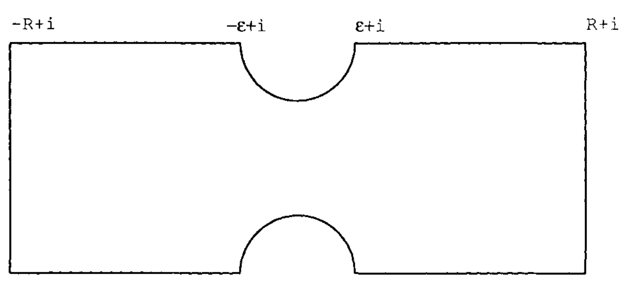

Fig. 2. Contour $\mathscr{C}$

and i are inside $\mathscr{C}$). An elementary calculation shows that

$$\sum \text{Residues}(f) = \left(2\pi\cos\frac{\pi}{2}\sigma\sin\frac{\pi}{2}\sigma\right)^{-1}\left[\frac{\pi}{2}\tan\frac{\pi}{2}\sigma - \frac{\sigma}{2}\left(\psi\left(\frac{1+\sigma}{2}\right)+\psi\left(\frac{1-\sigma}{2}\right)\right)\right].$$

In evaluating the integral over various portions of the contour $\mathscr{C}$, one makes use of the fact that for $p\in\mathbf{R}$, $\Re\psi(ip)$ ($\Im\psi(ip)$) is an even (odd) function of p, and one relates the values of ψ on the upper horizontal contour to those on the lower horizontal contour by the functional equation for ψ. Doing this results in (after letting $R\to\infty$ and $\varepsilon\to 0$)

$$\frac{dI}{d\sigma} = \sin\frac{\pi}{2}\sigma\cos\frac{\pi}{2}\sigma\int_0^\infty\left(\cosh^2\pi p - \sin^2\frac{\pi}{2}\sigma\right)^{-1}dp - \frac{\pi}{4}\tan\frac{\pi}{2}\sigma$$
$$-\frac{\sigma}{4}\left[\psi\left(\frac{1+\sigma}{2}\right)+\psi\left(\frac{1-\sigma}{2}\right)\right]. \tag{4.3}$$

The integral appearing in (4.3) has the value $\left(2\sin\frac{\pi}{2}\sigma\cos\frac{\pi}{2}\sigma\right)^{-1}\sigma$ so that (4.3) becomes

$$\frac{dI}{d\sigma} = \frac{\sigma}{2} - \frac{\pi}{4}\tan\frac{\pi}{2}\sigma - \frac{\sigma}{4}\left[\psi\left(\frac{1+\sigma}{2}\right)+\psi\left(\frac{1-\sigma}{2}\right)\right]. \tag{4.4}$$

Since $I(0)=0$, we obtain upon integrating (4.4),

$$I(\sigma) = \frac{\sigma^2}{4} + \frac{1}{2}\log\cos\frac{\pi}{2}\sigma - \frac{1}{4}\int_0^\sigma\left(x\psi\left(\frac{1+x}{2}\right)+x\psi\left(\frac{1-x}{2}\right)\right)dx$$
$$= \frac{\sigma^2}{4} + \frac{1}{2}\log\cos\frac{\pi}{2}\sigma - \frac{\sigma}{2}\log\frac{\Gamma\left(\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{1-\sigma}{2}\right)}$$
$$+\int_0^{\sigma/2}\left(\log\Gamma\left(\frac{1}{2}+x\right) - \log\Gamma\left(\frac{1}{2}-x\right)\right)dx, \tag{4.5}$$

where we integrated by parts and made an obvious change of variables to obtain the last equality. Now Alexeiewsky's integral [1] is

$$\int_0^z \log\Gamma(x+a)dx = \frac{z}{2}\log 2\pi - \frac{z}{2}(z+2a-1) + (z+a-1)\log\Gamma(z+a)$$
$$+\log\frac{\Gamma_2(z+a)}{\Gamma_2(a)} - (a-1)\log\Gamma(a), \tag{4.6}$$

where $\Gamma_2(z)$ is the Barnes double gamma function. Using this we obtain our final expression

$$I(\sigma) = \log\cos\frac{\pi}{2}\sigma + \log\left(\Gamma_2\left(\frac{1+\sigma}{2}\right)\Gamma_2\left(\frac{1-\sigma}{2}\right)\right) - 2\log\Gamma_2\left(\frac{1}{2}\right). \tag{4.7}$$

5. Final Result

Referring to (1.3), (1.6), (2.1), (2.5), and (3.13), we obtain for $0<\sigma<1$,

$$\log\tau_-(t,\lambda) = -\frac{1}{2}\log\left(\cosh\frac{1}{2}\psi(t,\lambda)\right)-\frac{1}{2}\log\tau_B(t,\lambda)$$

$$= -\frac{\sigma}{2}\left(1-\frac{\sigma}{2}\right)\log t-\log 2-\frac{1}{2}\log B-\frac{1}{2}\beta(\lambda)+o(1), \quad \text{as} \quad t\to 0^+. \quad (5.1)$$

Calling the constant term in the above expansion $\log\tau_0$, using (1.8), (3.12), (4.1) and (4.7), $\log\tau_0$ becomes

$$\log\tau_0 = -\log 2-\frac{1}{2}\log\pi+3\log 2\ \frac{\sigma}{2}\left(1-\frac{\sigma}{2}\right)+\log\Gamma\left(\frac{1+\sigma}{2}\right)$$

$$+\log\cos\frac{\pi}{2}\sigma+\log\left(\Gamma_2\left(\frac{1+\sigma}{2}\right)\Gamma_2\left(\frac{1-\sigma}{2}\right)\right)-2\log\Gamma_2\left(\frac{1}{2}\right). \quad (5.2)$$

Using (1.10) and standard identities for the Γ-function, Theorem 1 now follows.

There is one subtle point in applying Theorem 1 to the Ising case $\sigma=1$; namely, that in the expansion (1.6) of $\psi(t,\lambda)$ there is a term of order $t^{2-2\sigma}$ (there are no terms of order $t^{n-n\sigma}, n\geqq 3$ in $\eta(t/2,\lambda)=e^{-\psi(t,\lambda)}$ see [3] for discussion of this point). Now both $\log B$ and β are divergent as $\sigma\to 1$, but the sum (which is what appears in (5.2)) is not. However it is possible that the $t^{2-2\sigma}$ term in the expansion (5.1) could contribute to the constant as $\sigma\to 1$. This can be settled by a local analysis by computing, for $0<\sigma<1$, $d\log\tau_\pm(t,\lambda)/dt$ as $t\to 0^+$ using expression (1.3) and the asymptotic expansion of $\psi(t,\lambda)$ to higher order (see (1.10) in [3] with $v=0$). A computation shows the term of order $t^{1-2\sigma}$ in $d\log\tau_\pm(t,\lambda)/dt$ makes no contribution as $\sigma\to 1$. Thus we are allowed to simply set $\sigma=1$ ($s=0$) in Theorem 1.

6. Error Estimates

As discussed in Sect. 3, we must show that the integrals appearing in (3.9) remain $o(1)$ as $t\to 0^+$ when summed over k in (2.2). Referring to (2.9) for $z=iy, y\in\mathbf{R}$, a straightforward estimate that eliminates the term $[\Delta_k(\delta)]^{-iy/2}$ shows that the resulting k-dimensional integral is bounded by

$$\frac{k}{8}\left|\Gamma\left(\frac{iy}{2}\right)\right|^2\sigma_k.$$

This shows that the series

$$\sum_{k=1}^{\infty}\frac{1}{k}\hat{e}_{2k}(iy)\lambda^{2k}$$

converges uniformly in λ in compact subsets of $(0,1/\pi)$ and uniformly in y in compact subsets of $(0,\infty)$. Thus in (3.9) the summation over k may be brought inside the integral. Standard estimates of $\Gamma(iy)$ for large y shows that the integral is uniformly convergent near the upper endpoint. For small y the integral is term-by-term improper so we put in a small ε at the lower endpoint. Again since the convergence of the series is uniform in y and λ, we may examine the small y behavior of the summed integral by taking the limit of small y inside the integral and inside the series. But these terms are constructed termwise in k to cancel any singularity at $y=0$. Thus we may remove the small ε cutoff at the lower endpoint to obtain a uniformly convergent integral at the lower endpoint. Thus we may apply the Riemann–Lebesgue lemma to the summed integrals to conclude that they are $o(1)$ as $t \to 0^+$.

Appendix

We define the integral operator K on the Hilbert space $L^2(0, \infty; de_t)$ by

$$(Kf)(x) = \int_0^\infty \frac{1}{x+y} f(y) de_t(y), \quad t > 0,$$

where

$$de_t(x) = \exp(-\tfrac{1}{2}t(x+x^{-1}))dx.$$

We denote by ϕ_j and λ_j the normalized eigenvectors and eigenvalues, respectively. We take $\lambda_0 \geqq \lambda_1 \geqq \cdots$. Hilbert's inequality states that if $f \in L^2(0, \infty)$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y} dx\, dy < \pi \int_0^\infty f^2(x) dx.$$

Now

$$\begin{aligned}\lambda_0(t) &= (\phi_0, K\phi_0)\\ &= \int_0^\infty dx e^{-(1/2)t(x+1/x)} \int_0^\infty dy e^{-(1/2)t(y+1/y)} \frac{\phi_0(x)\phi_0(y)}{x+y}.\end{aligned}$$

Letting $f(x) = \exp(-\frac{1}{2}t(x+1/x))\phi_0(x)$, we have $f \in L^2(0, \infty)$ since $\phi_0 \in L^2(0, \infty; de_t)$. Thus applying Hilbert's inequality we have

$$\begin{aligned}\lambda_0(t) &< \pi \int_0^\infty e^{-t(x+1/x)} \phi_0^2(x) dx\\ &< \pi \int_0^\infty e^{-(1/2)t(x+1/x)} \phi_0^2(x) dx = \pi,\end{aligned}$$

since $\phi_0(x)$ is normalized in $L^2(0, \infty; de_t)$. Since this holds for all $t > 0$ we have

$$\sup_{t>0} \lambda_0(t) \leqq \pi.$$

The largest eigenvalue $\lambda_0(t)$ satisfies for all $\phi \in L^2(0, \infty, de_t)$,

$$B_\phi(t) := \frac{(\phi, K\phi)}{(\phi, \phi)} \leqq \lambda_0(t).$$

We now choose

$$\phi(x) = \frac{1}{\sqrt{x}}.$$

Using

$$(x+y)^{-1}=\int_0^\infty e^{-(x+y)\xi}d\xi,$$

we find

$$(\phi,K\phi)=\int_0^\infty\left[\int_0^\infty \frac{1}{x^{1/2}}e^{-(t/2+\xi)x-(1/2)t/x}dx\right]^2 d\xi.$$

Using

$$K_p(z)=\frac{1}{2}\left(\frac{z}{2}\right)^p\int_0^\infty \frac{1}{\vartheta^{p+1}}e^{-\vartheta-z^2/4\vartheta}d\vartheta,$$

we find

$$(\phi,K\phi)=4\int_t^\infty (K_{1/2}(u))^2du,$$

where we made the change of variables $u^2=t(t+2\xi)$. Using the fact that the Bessel function

$$K_{1/2}(x)=\sqrt{\frac{\pi}{2x}}e^{-x},$$

we can conclude upon integration by parts

$$(\phi,K\phi)=2\pi e^{-2t}\log(1/t)+\pi\int_t^\infty \log x e^{-2x}dx.$$

A similar calculation gives

$$(\phi,\phi)=2K_0(t).$$

Putting these two expressions together gives the final result for B_ϕ. Elementary arguements show that

$$\sup_{t>0} B_\phi(t)=\pi,$$

so we have

$$\pi\leqq\sup_{t>0}\lambda_0(t),$$

and hence

$$\sup_{t>0}\lambda_0(t)=\pi.$$

Referring to (2.2) and (2.3) we see that if the series converges (which it clearly does for large enough t)

$$E(t,\lambda)=-\sum_{j\geqq 0}\log(1-\lambda^2\lambda_j^2).$$

The above series can be expanded in a power series in λ provided that $|\lambda_0\lambda|<1$. But for $0<\pi\lambda<1$ we have

$$|\lambda_0(t)\lambda|\leqq\sup_{t>0}\lambda_0(t)\lambda=\pi\lambda<1.$$

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