Asymptotics of Arbitrary Order for a Thin Elastic Clamped Plate, II. Analysis of the Boundary Layer Terms

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Abstract. *This paper is the last of a series of two, where we study the asymptotics of the displacement in a thin clamped plate as its thickness tends to* 0 *. In Part I, relying on the structure at infinity of the solutions of certain model problems posed on unbounded domains, we proved that the combination of a polynomial Ansatz (outer expansion) and of a boundary layer Ansatz (inner expansion) yields a complete multi-scale asymptotics of the displacement and optimal estimates in energy norm. The "profiles" for the boundary layer terms are solutions of such model problems. In this paper, adapting Saint-Venant's principle to our framework, we prove the results which we used in Part I.*

Investigating more precisely the structure of the boundary layer terms, we go further in the analysis performed in Part I: the introduction of edge layer terms along the intersections of the clamped face with the top and the bottom of the plate respectively, allows estimates in higher order norms. These edge layer terms are constructed with the help of stable asymptotics, and are the singular parts of the boundary layer terms. As a by-product of all these investigations, we obtain expansions and estimates for the stress tensor in various anisotropic norms, and also estimates in L^∞ *- norm form the displacement field.*

To appear in *Asymptotic Analysis*

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INTRODUCTION

Boundary value problems in thin plates appear as singularly perturbed problems with respect to the thickness parameter ε, and the boundary conditions on the part of the boundary transverse to the mean surface give rise in general to boundary layer terms.

More precisely, let us introduce thin plates as a family of three dimensional domains of the form $\Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon)$ where the two-dimensional domain ω is the mean surface of the plates; an interesting question is the investigation of the asymptotic behavior as $\varepsilon \to 0$ of the solution u^ε of the linear elasticity system on Ω^ε. In general the boundary conditions on the lateral side $\Gamma_0^\varepsilon = \partial\omega \times (-\varepsilon, +\varepsilon)$ cannot be satisfied by any polynomial Ansatz for u^ε and as usual in singularly perturbed problems, see Il'in [14], there appears a boundary layer in the neighborhood of Γ_0^ε. To our knowledge, the only situation where there is no boundary layer, is the case of periodic boundary conditions on the lateral side Γ_0^ε (when ω is a rectangle, see J. C. Paumier [23]): indeed the periodicity conditions imposed on the right hand sides cancel any boundary layer terms.

Boundary layer terms can be roughly described by saying that they behave like $e^{-cr/\varepsilon}$, where r is the distance to Γ_0^ε. For isotropic materials, it is known that the possible values of the constant c are determined by the Papkovich-Fadle eigenfunctions, see R. D. Gregory & F. Y. Wan [13]. Even two-dimensional models for thin plates, as the Reissner-Mindlin model, are singularly perturbed problems: see the papers by D. N. Arnold & R. S. Falk [2, 3] where the boundary layer terms of the Reissner-Mindlin model have been exhibited.

In the part I of this paper [11] we constructed an expansion of the solution u^ε as the sum of an outer (polynomial) part and inner (boundary layer) part, with estimates in H^1 and L^2 of the error between truncated expansions and the solution itself. The remaining part of the proof is that relating to the behavior at infinity of the solutions of some model problems on a half-strip, which may be referred as Saint-Venant problems. In this part II, we prove that such solutions can be split into the sum of a rigid displacement and an exponentially decreasing term (this result was stated in [11] and referred as Theorem 3.2 and Corollary 3.4).

We also extend the results of [11] by the investigation of estimates in other norms: error on the strain and the stress tensors, or estimates in L^∞ or H^2 norms for instance, which requires the splitting of the solutions of different problems in regular and singular parts in a *stable* way, *cf* [9], with respect to the curvilinear abscissa s along the boundary $\partial\omega$, which can be considered as a parameter. The outcome is that the singular part of the expansion of the three-dimensional solution is concentrated in the boundary layer terms.

ACKNOWLEDGEMENTS

The authors are glad to thank P. G. Ciarlet and M. Costabel for valuable discussions and remarks.

This work is part of the Human Capital and Mobility Program *"Shells: Mathematical Modeling and Analysis, Scientific Computing"* of the Commission of the European Communities (Contract n° ERBCHRXCT940536).

1 THE ORIGIN OF BOUNDARY LAYER TERMS. OUTLINE OF THE PAPER

1.1 The scaled plate problem

In the plate $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$, where $\omega \subset \mathbb{R}^2$ is a bounded plane domain with a smooth boundary, we investigate the behavior as $\varepsilon \to 0$ of the displacement u^ε in the case when the plate Ω^ε is clamped along its lateral face $\Gamma_0^\varepsilon = \partial\omega \times (-\varepsilon, \varepsilon)$ and when the governing equations are those of the linearized elasticity. Denoting by A the rigidity matrix and by $e(u^\varepsilon) = (e_{ij}(u^\varepsilon))_{ij}$ the linearized strain tensor defined by $e_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$, the displacement u^ε of the clamped plate corresponding to volume forces f^ε is the unique solution of the variational elasticity problem:

$$u^\varepsilon \in V(\Omega^\varepsilon), \tag{1.1a}$$

$$\forall v^\varepsilon \in V(\Omega^\varepsilon), \qquad \int_{\Omega^\varepsilon} A\, e(u^\varepsilon) : e(v^\varepsilon) = \int_{\Omega^\varepsilon} f^\varepsilon \cdot v^\varepsilon, \tag{1.1b}$$

where the variational space $V(\Omega^\varepsilon)$ is defined as

$$V(\Omega^\varepsilon) = \{v = (v_1, v_2, v_3) \in H^1(\Omega^\varepsilon)^3 \mid \quad v = 0 \quad \text{on} \quad \Gamma_0^\varepsilon\}.$$

The 9×9 rigidity tensor A, with coefficients A_{ijkl}, satisfies the usual symmetry relations $A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}$ and is supposed to be uniformly positive definite, namely:

$$\forall\, x^\varepsilon \in \overline{\Omega}^\varepsilon,\ \forall (t_{ij}) \in \mathbb{R}^9 \text{ s.t. } t_{ij} = t_{ji}, \quad A_{ijkl}(x^\varepsilon)\, t_{kl} t_{ij} \geq c\, t_{ij}^2 \tag{1.2}$$

where $c > 0$ is a positive constant. Moreover, we assume that the coefficients of A do not depend on the "vertical" coordinate x_3 and smoothly depend on the in-plane variables (x_1, x_2) and, as in [12], [24] that

$$\left\{ \begin{array}{lcll} A_{\alpha\beta\gamma 3} & = & 0 & \forall \alpha, \beta, \gamma \in \{1, 2\} \\ A_{\alpha 333} & = & 0 & \forall \alpha \in \{1, 2\}. \end{array} \right. \tag{1.3}$$

The relations (1.3) are satisfied for any isotropic material, and allow the uncoupling of the system of elasticity into bending and membrane problems.

Problem (1.1) is studied with the help of a scaling in order to set the problem on the reference set $\Omega = \omega \times (-1, 1)$, which is the image of Ω^ε through a dilatation along the normal direction to the plane containing ω:

$$x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) \in \Omega^\varepsilon \quad \longmapsto \quad x = (x_1, x_2, x_3) = (x_1^\varepsilon, x_2^\varepsilon, \varepsilon^{-1} x_3^\varepsilon) \in \Omega.$$

The corresponding "scaled" linearized strain tensor denoted by $\kappa(\varepsilon)(v)$ is defined for any function $v \in H^1(\Omega)^3$ by:

$$\kappa_{\alpha\beta}(\varepsilon)(v) = e_{\alpha\beta}(v), \quad \kappa_{\alpha 3}(\varepsilon)(v) = \varepsilon^{-1}\, e_{\alpha 3}(v), \quad \kappa_{33}(\varepsilon)(v) = \varepsilon^{-2}\, e_{33}(v), \tag{1.4a}$$

where we use the convention that the Greek indices α and β span the set $\{1,2\}$. Correspondingly, the convenient scaling of the components of u^ε writes

$$u_\alpha(\varepsilon)(x) = u^\varepsilon_\alpha(x^\varepsilon) \quad \text{and} \quad u_3(\varepsilon)(x) = \varepsilon\, u^\varepsilon_3(x^\varepsilon), \tag{1.4b}$$

for which the resulting scaled displacement $u(\varepsilon)$ satisfies $e(u^\varepsilon)(x^\varepsilon) = \kappa(\varepsilon)(u(\varepsilon))(x)$. The corresponding canonical scaling for the body forces writes

$$f_\alpha(\varepsilon)(x) := f^\varepsilon_\alpha(x^\varepsilon) \quad \text{and} \quad f_3(\varepsilon)(x) := \varepsilon^{-1}\, f^\varepsilon_3(x^\varepsilon). \tag{1.4c}$$

Our analysis holds if the scaled body force $f(\varepsilon)$ satisfies an asymptotic property like

$$f(\varepsilon)(x) \simeq f^0(x) + \varepsilon\, f^1(x) + \varepsilon^2\, f^2(x) + \cdots + \varepsilon^k\, f^k(x) + \cdots \tag{1.5}$$

Then the scaled displacement $u(\varepsilon)$ solves the new problem:

$$u(\varepsilon) \in V(\Omega), \tag{1.6a}$$

$$\forall v \in V(\Omega), \qquad \int_\Omega A\, \kappa(\varepsilon)(u(\varepsilon)) : \kappa(\varepsilon)(v) = \int_\Omega f(\varepsilon)\cdot v, \tag{1.6b}$$

where

$$V(\Omega) = \{ v \in H^1(\Omega)^3 \mid \quad v = 0 \quad \text{on} \quad \Gamma_0 = \partial\omega \times (-1,1)\}. \tag{1.7}$$

1.2 Multi-scale asymptotics

In part I of this work, we exhibited an algorithm of construction of a multi-scale asymptotics for $u(\varepsilon)$. Our result can be compared with many others, relating to various problems, e.g. the homogeneization of periodic elastic structures [22], related problems in thin plates [2, 3], [25], [21]. The outcome of this algorithm is an asymptotics for $u(\varepsilon)$ of the form

$$u(\varepsilon)(x) \simeq \Psi^0(x, \frac{r}{\varepsilon}) + \varepsilon\, \Psi^1(x, \frac{r}{\varepsilon}) + \varepsilon^2\, \Psi^2(x, \frac{r}{\varepsilon}) + \cdots + \varepsilon^k\, \Psi^k(x, \frac{r}{\varepsilon}) + \cdots$$

with r the distance to the clamped part of the boundary (r is also the distance to $\partial\omega$ in the in-plane variables (x_1, x_2)) and

$$\Psi^0(x,t) \;=\; u^0(x), \tag{1.8a}$$

$$\Psi^k(x,t) \;=\; u^k(x) - \chi(r)\, w^k(t,s,x_3) \quad \text{for} \quad k \geq 1 \tag{1.8b}$$

$$\text{with} \quad \begin{cases} t = r\,\varepsilon^{-1}, \quad s \text{ is a curvilinear abscissa along } \partial\omega, \\ w^k(t,s,x_3) \text{ is uniformly exponentially decreasing as } t \to +\infty, \\ \chi \text{ is a cut-off function equal to 1 in a neighborhood of } r = 0. \end{cases} \tag{1.8c}$$

In this expansion, the *polynomial* or *outer* part is $\sum_{k\geq 0}\varepsilon^k u^k$, whereas the remaining sum $\sum_{k\geq 1}\varepsilon^k w^k(\frac{r}{\varepsilon}, s, x_3)$ is the *boundary layer* or *inner* part.

The necessity of such a multi-scale Ansatz was proved in Part I [11], where was explained how the Dirichlet traces of the polynomial part of the Ansatz could be compensated by those of the boundary layer part. Roughly speaking, the equations inside Ω and the Neumann boundary conditions involved in (1.6) can be solved by a polynomial Ansatz. Thus the remaining part has to solve a non homogeneous Dirichlet problem, with homogeneous Neumann and interior right hand sides.

1.3 Reduced-normal problems

As already hinted, we introduce smooth local coordinates (r, s) in a tubular neighborhood $\mathcal{V}$ of $\partial\omega$: r is the distance to $\partial\omega$ and the curvilinear abscissa s belongs to S the disjoint union $L_1\mathbb{S}^1 \dot\cup \cdots \dot\cup L_I\mathbb{S}^1$, where $L_1, \ldots, L_I$ are the lengths of the connected components of $\partial\omega$. The change of variables $(x_1, x_2, x_3) \mapsto (r, s, x_3)$ maps $\mathcal{V}$ onto a domain $\widetilde{\Omega} = (0, \rho) \times S \times (-1, 1)$ with lateral face $\widetilde{\Gamma}_0 = \{0\} \times S \times (-1, 1)$ and the problem solved by the terms w_k has the form

$$w(\varepsilon) \in H^1(\widetilde{\Omega})^3 \quad \text{and} \quad w(\varepsilon) = h(\varepsilon) \quad \text{on} \quad \widetilde{\Gamma}_0, \tag{1.9a}$$

$$\forall v \in V(\widetilde{\Omega}), \qquad \int_{\widetilde{\Omega}} \widetilde{A}\,\tilde{\kappa}(\varepsilon)(w(\varepsilon)) : \tilde{\kappa}(\varepsilon)(v) + \widetilde{\mathcal{A}}'(w(\varepsilon), v) = 0, \tag{1.9b}$$

where the 9×9 matrix $\widetilde{A} = \widetilde{A}(r, s)$ has the same features as A (positivity, smoothness and property (1.3) on the coefficients), $\tilde{\kappa}(\varepsilon)$ is the scaled strain tensor in the variables (r, s, x_3) and $\widetilde{\mathcal{A}}'$ is a first order integro-differential form: if $\boldsymbol{i}$ and $\boldsymbol{j}$ denote multi-indices in $\mathbb{N}^3$:

$$\widetilde{\mathcal{A}}'(w, v) = \sum_{|\boldsymbol{i}|+|\boldsymbol{j}|\leq 1} \int_{\widetilde{\Omega}} c_{\boldsymbol{ij}}(r, s)\,\partial^{\boldsymbol{i}} w\,\partial^{\boldsymbol{j}} v.$$

Keeping in mind that r is now a fast variable as x_3^ε, we set $t = r\,\varepsilon^{-1}$ — like $x_3 = x_3^\varepsilon\,\varepsilon^{-1}$. Thus $\partial_t = \varepsilon\,\partial_r$ and for purposes of homogeneity we introduce the following change in the displacements

$$(w_r, w_s, w_3) \longmapsto (\varphi_t, \varphi_s, \varphi_3) = (\varepsilon\, w_r, \varepsilon\, w_s, w_3). \tag{1.10}$$

Now, problem (1.9) is transformed into the problem on $\widetilde{\Sigma} := \mathbb{R}^+ \times S \times (-1, 1)$:

$$\varphi \in H^1(\widetilde{\Sigma})^3 \quad \text{and} \quad \varphi = g \quad \text{on} \quad \widetilde{\Gamma}_0, \tag{1.11a}$$

$$\forall v \in V(\widetilde{\Sigma}), \quad \int_{\widetilde{\Sigma}} \widetilde{A}(\varepsilon t, s)\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) + \sum_{k=1}^{4} \varepsilon^k\, \widetilde{\mathcal{A}}^k(\varphi, v) = 0, \tag{1.11b}$$

where $\widetilde{\mathcal{A}}^k$ is a second order integro-differential form:

$$\widetilde{\mathcal{A}}^k(\varphi, v) = \sum_{|\boldsymbol{i}|,\,|\boldsymbol{j}|\leq 1} \int_{\widetilde{\Sigma}} c^k_{\boldsymbol{ij}}(\varepsilon t, s)\,\partial^{\boldsymbol{i}}\varphi\,\partial^{\boldsymbol{j}} v.$$

With the help of the Taylor expansion of the coefficients $c^k_{\boldsymbol{ij}}$ in $t=0$, we obtain for problem (1.11) the formal expansion

$$\varphi \in H^1(\widetilde{\Sigma})^3 \quad \text{and} \quad \varphi = g \quad \text{on} \quad \widetilde{\Gamma}_0, \tag{1.12a}$$

$$\forall v \in V(\widetilde{\Sigma}), \quad \int_{\widetilde{\Sigma}} \widetilde{A}(0,s)\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) + \sum_{k\geq 1} \varepsilon^k \, \mathcal{B}^k(\varphi,\, v) = 0, \tag{1.12b}$$

where for any $k \geq 1$, $\mathcal{B}^k$ is a second order integro-differential form:

$$\mathcal{B}^k(\varphi,\, v) = \sum_{|\boldsymbol{i}|,\, |\boldsymbol{j}| \leq 1} \int_{\widetilde{\Sigma}} b^k_{\boldsymbol{ij}}(t,s)\, \partial^{\boldsymbol{i}} \varphi \, \partial^{\boldsymbol{j}} v$$

whose coefficients $b^k_{\boldsymbol{ij}}$ are polynomials of t with smooth coefficients in s.

Thus, the only slow variable is now s and we see that the "principal part" (in the sense of the powers of ε) of this problem does not involve any more the tangential derivation ∂_s: thus the variable s is a mere parameter. The following definition is now natural (it was referred as Definition 3.1 in [11]).

Definition 1.1 The matrix $\widetilde{A}(r,s)$ being the matrix transformed from A by the change of variables $(x_1, x_2) \mapsto (r,s)$, we denote by $B(s)$ the matrix

$$B(s) = \widetilde{A}(0,s).$$

The matrix $B(s)$ is positive definite, depends smoothly on $s \in S$ and its coefficients satisfy the same symmetry properties as A. Let Σ^+ and γ_0 denote the semi-infinite strip and its lateral face:

$$\Sigma^+ = \{(t, x_3) \mid t > 0,\ -1 < x_3 < 1\} \quad \text{and} \quad \gamma_0 = \{(t, x_3) \mid t = 0,\ -1 < x_3 < 1\}.$$

The space of admissible displacements on Σ^+ is

$$V(\Sigma^+) = \{v \in H^1(\Sigma^+)^3 \mid \quad v = 0 \ \text{ on } \ \gamma_0\}. \tag{1.13}$$

For each s fixed in S, the *reduced-normal problem* is the mixed Dirichlet-Neumann problem in Σ^+:

$$\varphi \in H^1(\Sigma^+)^3 \qquad \text{and} \quad \varphi\Big|_{\gamma_0} = g, \tag{1.14a}$$

$$\forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = 0, \tag{1.14b}$$

with unknown $\varphi = (\varphi_t, \varphi_s, \varphi_3)$ and data $g = (g_t, g_s, g_3)$. ■

Thus the canonical reduced-normal problem has zero interior (in Σ^+) and Neumann (on $\mathbb{R}^+ \times \{-1, 1\}$) data. But, due to the presence of the operators $\mathcal{B}^k$ in formula (1.12b), the profiles w^k appearing in (1.8b) are associated to functions φ^k according to (1.10) which have to solve a recursive system of the form:

$$\begin{cases} \varphi^k(s)\Big|_{\gamma_0} = g^k(s) \\ \forall v \in V(\Sigma^+),\ \displaystyle\int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi^k) : e(\partial_t, 0, \partial_3)(v) = -\int_{\Sigma^+} \sum_{j=1}^{k-1} \mathcal{B}^j(\varphi^{k-j},\, v) \end{cases} \tag{1.15}$$

for $k \geq 0$, with some (smooth) trace functions $g^k(s)$. Thus we are lead to consider non zero interior right-hand sides and we introduce Sobolev spaces with an exponential weight:

Definition 1.2 Let $\eta \in \mathbb{R}$. For $m \geq 0$ let $H^m_\eta(\Sigma^+)$ be the space of functions v such that $e^{\eta t} v$ belongs to $H^m(\Sigma^+)$. Let $V'_\eta(\Sigma^+)$ be the space of distributions $w = (w_t, w_s, w_3)$ such that $e^{\eta t} w$ belongs to the dual space of $V(\Sigma^+)$. ■

Let B satisfying

$$\begin{cases} B \text{ is a } 9 \times 9 \text{ positive definite matrix with constant coefficients,} \\ B \text{ has the symmetry properties } B_{ijkl} = B_{jikl} = B_{ijlk} = B_{klij}. \end{cases} \tag{1.16}$$

The general form of problem (1.14) is

$$\varphi \in H^1(\Sigma^+)^3 \quad \text{and} \quad \varphi\Big|_{\gamma_0} = g\Big|_{\gamma_0}, \tag{1.17a}$$

$$\forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = \int_{\Sigma^+} f\, v, \tag{1.17b}$$

for $g \in H^{1/2}(\gamma_0)^3$ and $f \in V'_\eta(\Sigma^+)$ for an $\eta > 0$.

We are going to prove that the solutions of these Dirichlet-Neumann problems are the sum of a term exponentially decreasing at infinity and of a rigid displacement associated to the strain tensor $e(\partial_t, 0, \partial_3)$. It is straightforward that this space $\mathcal{R}$ of rigid displacements is the space of dimension 4:

$$\mathcal{R} = \{R = (R_t, R_s, R_3) = (c_t, c_s, c_3) + c_n\,(-x_3, 0, t) \mid c_t, c_s, c_3, c_n \in \mathbb{R}\}. \tag{1.18}$$

Theorem 1.3 *Let $\eta > 0$. For all $g \in H^{1/2}(\gamma_0)^3$ and for all $f \in V'_\eta(\Sigma^+)$ there exists a unique rigid displacement $R = R(g, f) \in \mathcal{R}$ so that the problem*

$$\varphi \in H^1(\Sigma^+)^3 \quad \textit{and} \quad \varphi\Big|_{\gamma_0} = g + R\Big|_{\gamma_0}, \tag{1.19a}$$

$$\forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = \int_{\Sigma^+} f\, v, \tag{1.19b}$$

has a (unique) solution. Moreover, there exists $\eta_0 > 0$, which only depends on the coefficients of B, such that if $\eta < \eta_0$, this solution φ belongs to $H^1_\eta(\Sigma^+)^3$.

Remark 1.4 • The optimal value of η_0 is given in (4.13) below. It is the real part of the "first" singularities at $t = -\infty$ of the Neumann problem on the whole strip $\mathbb{R} \times (-1, +1)$.

• If we define

$$\boldsymbol{K}^1_\eta(\Sigma^+) = \bigcap_{\tilde\eta < \eta} H^1_{\tilde\eta}(\Sigma^+)^3 \tag{1.20}$$

and

$$\boldsymbol{K}^{-1}_\eta(\Sigma^+) = \bigcap_{\tilde\eta < \eta} V'_{\tilde\eta}(\Sigma^+) \tag{1.21}$$

Theorem 1.3 gives that if f belongs to $\boldsymbol{K}^{-1}_{\eta_0}(\Sigma^+)$ then the solution φ of problem (1.19) belongs to $\boldsymbol{K}^1_{\eta_0}(\Sigma^+)$.

- As a consequence of Theorem 1.3, we will prove that the profiles φ^k solution of (1.15) belong to $\mathscr{C}^\infty(S, \boldsymbol{K}^1_{\eta_0}(\Sigma^+))$, see Corollary 4.13. In fact, they even belong to a smaller space: for each $s \in S$, they behave like the product of $e^{-\eta_0 t}$ by a polynomial as $t \to +\infty$, see (4.27).

$\blacksquare$

This result can be interpreted as the solution of a *Saint-Venant problem*, where the solution is exponentially decreasing far from the support of the data (which is here the Dirichlet data). Among an abundant literature, we quote the book [22] by O.A. Oleinik, A.S. Shamaev & G.A. Yosifian for periodic structures, the paper [19] by A. Mielke for unbounded thin plates and the paper [25] by C. Schwab for the simpler situation of scalar second-order operators.

The key-argument is a "Saint-Venant principle", adapted from [22], proved in §3, combined with arguments about operator-valued pseudo-differential operators (§4), which are classical in the theory of corner problem: *cf* the reference paper by V.A. Kondrat'ev [15]. Before doing that, we prove in §2 the corresponding results for the Laplace operator Δ by very simple arguments: the comparison between the Laplacian and the elasticity system is quite interesting.

We end this work by estimates in higher order norms relying on an edge decomposition of the boundary layer terms (there appear now edge layer terms) in §5 and we conclude in §6 by results for the strain and stress tensors.

2 A SIMPLE EXAMPLE: THE LAPLACIAN

The study of the asymptotics of solutions of the heat equilibrium problem in the family of thin plates Ω^ε (Laplace equation inside Ω^ε, Dirichlet condition on Γ_0^ε and Neumann conditions on $\partial\Omega^\varepsilon \setminus \Gamma_0^\varepsilon$) would lead to the canonical reduced-normal problem on Σ^+ corresponding to (1.14)

$$\begin{cases} (\partial_{tt} + \partial_{33})w = 0, & \text{on } \Sigma^+ \\ \partial_3 w = 0, & \text{on } x_3 = \pm 1 \\ w = g, & \text{on } t = 0. \end{cases} \tag{2.1}$$

Taking advantage of the possibility of separation of variables for Δ, we expand the Dirichlet data g in a basis of eigenvectors of the Neumann problem on the interval $(-1, 1)$:

$$g \;=\; \sum_{\ell \geq 0} g_\ell^+ \cos \ell\pi x_3 \;+\; \sum_{\ell \geq 1} g_\ell^- \sin(\ell - \tfrac{1}{2})\pi x_3$$

It is easy to see that the unique temperate solution of the Dirichlet-Neumann problem (2.1) is given by

$$w \;=\; \sum_{\ell \geq 0} g_\ell^+ \cos \ell\pi x_3 \, e^{-\ell\pi t} \;+\; \sum_{\ell \geq 1} g_\ell^- \sin(\ell - \tfrac{1}{2})\pi x_3 \, e^{-(\ell - \frac{1}{2})\pi t}.$$

Therefore, w is exponentially decaying when $t \to +\infty$ if and only if $g_0^+ = 0$, i.e.

$$\int_{-1}^{+1} g(x_3)\, dx_3 = 0.$$

Relying on this result, we can solve the general problem corresponding to (1.17)

$$\begin{cases} (\partial_{tt} + \partial_{33})w = f, & \text{on } \Sigma^+ \\ \partial_3 w = 0, & \text{on } x_3 = \pm 1 \\ w = g, & \text{on } t = 0, \end{cases} \tag{2.2}$$

with an exponentially decreasing right hand side f in $H_\eta^{-1}(\Sigma_+)$, by

- the solution of the problem with $f = 0$,
- the odd extension $\tilde{f}$ of f to the full strip $\Sigma = \mathbb{R} \times (-1, 1)$, in order to solve (2.2) with $g = 0$,
- the expansion of $\tilde{f}$ in the above basis of eigenvectors of the Neumann problem on the interval $(-1, 1)$ which yields coefficients $f_\ell^+(t)$ and $f_\ell^-(t)$,
- a Fourier-Laplace transform on these coefficients

$$\hat{f}_\ell^\pm(\tau, x_3) = \int_{-\infty}^{+\infty} e^{-t\tau}\, f_\ell^\pm(t, x_3)\, dt, \qquad \text{for } \operatorname{Re}\tau = -\eta$$

  which yields the elementary equations

$$\begin{aligned} &\Big(\tau^2 - \ell^2\pi^2\Big)\hat{w}_\ell^+(\tau) = \hat{f}_\ell^+(\tau), && \operatorname{Re}\tau = -\eta \\ &\Big(\tau^2 - (\ell - \tfrac{1}{2})^2\pi^2\Big)\hat{w}_\ell^-(\tau) = \hat{f}_\ell^-(\tau), && \operatorname{Re}\tau = -\eta, \end{aligned}$$

- the inverse Fourier-Laplace transform.

The outcome is the following statement corresponding to Theorem 1.3

Theorem 2.1 *Let $\eta > 0$. For all $g \in H^{1/2}(\gamma_0)$ and for all $f \in H_\eta^{-1}(\Sigma^+)$ there exists a unique constant c_0 so that the problem*

$$\begin{cases} (\partial_{tt} + \partial_{33})w = f, & \textit{on } \Sigma^+ \\ \partial_3 w = 0, & \textit{on } x_3 = \pm 1 \\ w = g + c_0, & \textit{on } t = 0, \end{cases} \tag{2.3}$$

has a solution in $H^1(\Sigma^+)$. Moreover, if $\eta < \frac{\pi}{2}$, this solution w belongs to $H_\eta^1(\Sigma^+)$.

Of course $c_0 = -\frac{1}{2}\int_{-1}^{+1} g(x_3)\, dx_3$.

3 SAINT-VENANT PRINCIPLE

3.1 The main result

This section is devoted to the proof of the Saint-Venant Principle, about the behavior of solutions of the homogeneous elasticity system on any strip $\Sigma(0,L) = (-1,1) \times (0,L)$:

$$\varphi \in H^1(\Sigma(0,L))^3 \quad \text{and} \quad \varphi\Big|_{t=0} = 0, \tag{3.1a}$$

$$\forall v \in V(\Sigma(0,L)), \quad \int_{\Sigma(0,L)} B\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = 0, \tag{3.1b}$$

where $V(\Sigma(0,L))$ is the subspace of $H^1(\Sigma(0,L))^3$ of triples v with null traces on $t=0$ and $t=L$ and B is a 9×9 positive definite rigidity matrix with constant coefficients.

Moreover, we introduce the condition of zero flux against rigid displacements through γ_0 , namely:

$$\forall R \in \mathcal{R}, \qquad \Phi(\varphi \,|\, R) := \int_{-1}^{+1} \sigma_{it}(\varphi)(0, x_3)\, R_i(0, x_3)\, dx_3 = 0, \tag{3.2}$$

where $\sigma_{it}(\varphi)_{i=1,2,3}$ are the components of the normal stress $Be(\partial_t, 0, \partial_3)(\varphi)\boldsymbol{n}$ at $t=0$, and $\mathcal{R}$ is the set (1.18) of rigid displacements related to the strain tensor $e(\partial_t, 0, \partial_3)$.

Lemma 3.1 *If the displacement* φ *is solution of* (3.1)*, then for any rigid displacement* R *we have the conservation of flux:*

$$\forall t \in [0, L], \quad \Phi(\varphi \,|\, R) = \int_{-1}^{+1} \sigma_{it}(\varphi)(t, x_3)\, R_i(t, x_3)\, dx_3.$$

Proof. As φ satisfies (3.1b), Green's formula yields for all $t \in (0, L]$

$$\int_{\Sigma(0,t)} B\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(R) = \int_{-1}^{+1} \sigma_{it}(\varphi)(t, \cdot)\, R_i(t, \cdot) - \sigma_{it}(\varphi)(0, \cdot)\, R_i(0, \cdot).$$

Since $e(\partial_t, 0, \partial_3)(R) = 0$, the conclusion follows immediately. ■

The Saint-Venant Principle expresses that a solution of the homogeneous elasticity system with zero flux on the semi-infinite strip is exponentially increasing. The following statement and proof are inspired by the corresponding theorem in [22].

Theorem 3.2 *(Saint-Venant Principle). Any solution* φ *of the problem* (3.1) *with condition* (3.2) *satisfies the estimate:*

$$\int_{\Sigma(0,r)} |e(\partial_t, 0, \partial_3)(\varphi)|^2 \, dt\, dx_3 \quad \leq \quad e^{-A(L-r)} \int_{\Sigma(0,L)} |e(\partial_t, 0, \partial_3)(\varphi)|^2 \, dt\, dx_3 \tag{3.3}$$

where $A > 0$ *is a positive constant independent of the parameters* r *and* L *.*

Proof. In equation (3.1b) we use a convenient test-function. To this end, we introduce $\Psi = \Psi(t)$ defined by :

$$\Psi(t) = \begin{cases} e^{A(L-r)} & \text{when} \quad 0 \le t \le r \\ e^{A(L-t)} & \text{when} \quad r \le t \le L \\ 1 & \text{when} \quad L \le t \end{cases} \tag{3.4}$$

where A is a positive constant which will be determined later on. The definition of Ψ and the boundary condition (3.1a) yield that $v = (\Psi - 1)\varphi$ belongs to the test space $V(\Sigma(0,L))$. Using this v in (3.1b), we get:

$$\int_{\Sigma(0,L)} B\, e(\partial_t, 0, \partial_3)(\varphi) : (\Psi - 1)\, e(\partial_t, 0, \partial_3)(\varphi) = -\int_{\Sigma(r,L)} \sigma_{it}(\varphi)\, \partial_t \Psi\, \varphi_i \tag{3.5}$$

as a consequence of the identity $\partial_t \Psi = 0$ when $0 \le t \le r$ by construction of Ψ. We divide the interval $\Sigma(r,L)$ into the truncated segments $\Sigma^1_{r+s} = \Sigma(r+s, r+s+1)$ of length 1. Then, it may be noticed that $\partial_t \Psi = -Ae^{A(L-t)} = -A\Psi$ in Σ^1_{r+s}, which yields the estimate:

$$\Big| \int_{\Sigma^1_{r+s}} \sigma_{it}(\varphi)\, \partial_t \Psi\, \varphi_i \Big| \le A \Big| \int_{r+s}^{r+s+1} \Psi \int_{-1}^{+1} \sigma_{it}(\varphi)\, (\varphi_i - R_i)\, dx_3\, dt \Big| \tag{3.6}$$

for any fixed element $R \in \mathcal{R}$ because of the assumption that the flux is zero against the rigid displacements and the conservation of flux (Lemma 3.1). It follows:

$$\Big| \int_{\Sigma^1_{r+s}} \sigma_{it}(\varphi)\, \partial_t \Psi\, \varphi_i \Big| \le A\, e^{A(L-r-s)} \Big| \int_{\Sigma^1_{r+s}} \sigma_{it}(\varphi)\, (\varphi_i - R_i) \Big| \tag{3.7}$$

since we have the inequality

$$e^{A(L-r-s-1)} \le \Psi \le e^{A(L-r-s)} \quad \text{in} \quad \Sigma^1_{r+s}.$$

Thus:

$$\Big| \int_{\Sigma^1_{r+s}} \sigma_{it}(\varphi)\, \partial_t \Psi\, \varphi_i \Big| \le A\, e^{A(L-r-s)}\, \|e(\partial_t, 0, \partial_3)(\varphi)\|_{L^2(\Sigma^1_{r+s})}\, \|\varphi - R\|_{L^2(\Sigma^1_{r+s})}. \tag{3.8}$$

But as a consequence of Korn's and Poincaré's inequalities applied in the rectangles Σ^1_{r+s} whose length and width are independent of the parameters r and s, the rigid displacement R may be chosen so as to satisfy the estimate:

$$\|\varphi - R\|_{L^2(\Sigma^1_{r+s})} \le C\, \|e(\partial_t, 0, \partial_3)(\varphi)\|_{L^2(\Sigma^1_{r+s})} \tag{3.9}$$

which yields, with (3.8):

$$\begin{aligned} \Big| \int_{\Sigma^1_{r+s}} \sigma_{it}(\varphi)\, \partial_t \Psi\, \varphi_i \Big| &\le C\, A\, e^{A(L-r-s)}\, \|e(\partial_t, 0, \partial_3)(\varphi)\|^2_{L^2(\Sigma^1_{r+s})} \\ &= C\, A \int_{\Sigma^1_{r+s}} e^A\, e^{A(L-r-s-1)}\, |e(\partial_t, 0, \partial_3)(\varphi)|^2 \\ &\le C\, A\, e^A \int_{\Sigma^1_{r+s}} \Psi\, |e(\partial_t, 0, \partial_3)(\varphi)|^2. \end{aligned} \tag{3.10}$$

Summing over the Σ^1_{r+s}, $(s=0,\dots,L-r-1)$ we find :

$$\Big|\int_{\Sigma(r,L)} \sigma_{it}(\varphi)\,\partial_t\Psi\,\varphi_i\Big| \le C\,A\,e^A \int_{\Sigma(r,L)} \Psi\,|e(\partial_t,0,\partial_3)(\varphi)|^2 \tag{3.11}$$

and whence, substituting this inequality into (3.5) :

$$\int_{\Sigma(0,L)} (\Psi-1)\;B\,e(\partial_t,0,\partial_3)(\varphi) : e(\partial_t,0,\partial_3)(\varphi) \le C\,A\,e^A \int_{\Sigma(r,L)} \Psi\,|e(\partial_t,0,\partial_3)(\varphi)|^2.$$

But the coercivity of the operator B yields :

$$\int_{\Sigma(0,L)} (\Psi-1)\;|e(\partial_t,0,\partial_3)(\varphi)|^2 \le C\,A\,e^A \int_{\Sigma(r,L)} \Psi\,|e(\partial_t,0,\partial_3)(\varphi)|^2, \tag{3.12}$$

which gives:

$$\int_{\Sigma(0,L)} (\Psi-1)\;|e(\partial_t,0,\partial_3)(\varphi)|^2 - C\,A\,e^A \int_{\Sigma(r,L)} \Psi\,|e(\partial_t,0,\partial_3)(\varphi)|^2 \le 0.$$

Keeping in mind the expression of Ψ in the set $\Sigma(0,r)$ — *cf* (3.4), and choosing A such that $C\,A\,e^A \le 1$, we immediately deduce (3.3). ■

3.2 The exponential growth

Now, we may reformulate the Saint-Venant principle in a way more convenient for our purposes.

Corollary 3.3 *Let φ be a solution of problem* (3.1) *on $\Sigma(0,L)$ for any L, i.e., the displacement φ is solution of the homogeneous Dirichlet-Neumann problem on the semi-infinite strip $\Sigma^+ = \Sigma(0,\infty)$. Moreover, we assume that φ satisfies the flux condition* (3.2). *Then, the following alternative holds for all $0<\eta<A/2$:*

$$\forall \eta \in (0, A/2), \qquad \begin{cases} \textit{either} \quad e^{-\eta t}\, e(\partial_t,0,\partial_3)(\varphi) \notin L^2(\Sigma^+), \\ \textit{or} \qquad e(\partial_t,0,\partial_3)(\varphi) \equiv 0. \end{cases}$$

where $A>0$ is the constant appearing in Theorem 3.2.

Proof. If $e(\partial_t,0,\partial_3)(\varphi)$ is not identically 0, there exists $r>0$ such that

$$\int_{\Sigma(0,r)} |e(\partial_t,0,\partial_3)(\varphi)|^2 > 0.$$

Then, inequality (3.3) shows that there exists a constant $c>0$ such that

$$\forall L > r, \qquad c\,e^{AL} \le \int_{\Sigma(0,L)} |e(\partial_t,0,\partial_3)(\varphi)|^2.$$

Let $0<\eta<A/2$. Since we have $e^{-2\eta L} \le e^{-2\eta t}$ on $\Sigma(0,L)$, we deduce that

$$\forall L > r, \qquad c\,e^{(A-2\eta)L} \le \int_{\Sigma(0,L)} e^{-2\eta t}\,|e(\partial_t,0,\partial_3)(\varphi)|^2.$$

Since $c\,e^{(A-2\eta)L}$ is unbounded when $L\to+\infty$, $e^{-\eta t}\,e(\partial_t,0,\partial_3)(\varphi)$ does not belong to $L^2(\Sigma^+)$. ■

4 BEHAVIOR AT INFINITY OF SOLUTIONS IN THE HALF-STRIP

Let B be a matrix satisfying (1.16). The aim of this section is to determine the behavior at infinity of the solutions of the two-dimensional problem (1.17) in the half-strip Σ^+. We are going to prove Theorem 1.3, i.e. that such solutions are exponentially decreasing towards rigid displacements as the distance to the clamped face becomes "large". The main idea is to perform a splitting of φ similar to the one that occurs when one looks for the singularities of a boundary problem around a corner, *cf* [15].

4.1 Fredholm operators in weighted spaces

For any $\eta \in \mathbb{R}$ and $m = \pm 1$, we are going to use the following weighted Sobolev space on $\Sigma = \mathbb{R} \times (-1,1)$:

$$H^m_\eta(\Sigma) = \{v \in \mathscr{D}'(\Sigma)^3 \mid e^{\eta t} v \in H^m(\Sigma)\}, \tag{4.1a}$$

and on $\Sigma^+ = \mathbb{R}^+ \times (-1,1)$:

$$H^1_\eta(\Sigma^+) = \{v \in \mathscr{D}'(\Sigma^+)^3 \mid e^{\eta t} v \in H^1(\Sigma^+)\}, \tag{4.1b}$$

$$V_\eta(\Sigma^+) = \{v \in \mathscr{D}'(\Sigma^+)^3 \mid e^{\eta t} v \in V(\Sigma^+)\}, \tag{4.1c}$$

$$V'_\eta(\Sigma^+) = \{v \in \mathscr{D}'(\Sigma^+)^3 \mid e^{\eta t} v \in V'(\Sigma^+)\}, \tag{4.1d}$$

with $V'(\Sigma^+)$ the dual space of $V(\Sigma^+)$. Then $V'_\eta(\Sigma^+)$ is the dual space of $V_{-\eta}(\Sigma^+)$. The corresponding elasticity operators are:

Definition 4.1 *Let $\mathcal{B}_\eta$ be the operator defined by*

$$\mathcal{B}_\eta : V_\eta(\Sigma^+) \ni \varphi \longmapsto \mathcal{B}_\eta \varphi \in V'_\eta(\Sigma^+) \tag{4.2}$$

where $\mathcal{B}_\eta \varphi$ stands for the element of the dual space $V'_\eta(\Sigma^+)$ defined as:

$$\forall v \in V_{-\eta}(\Sigma^+), \qquad \langle \mathcal{B}_\eta \varphi, v \rangle_{V'_\eta \times V_{-\eta}} = \int_{\Sigma^+} B\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v), \tag{4.3}$$

and similarly we define $\widetilde{\mathcal{B}}_\eta :\ H^1_\eta(\Sigma)^3 \to H^{-1}_\eta(\Sigma)^3$.

Remark 4.2 It is easily seen that the adjoint of $\mathcal{B}_\eta$ is $\mathcal{B}_{-\eta}$. ■

The Fredholm properties of the operator $\mathcal{B}_\eta$ are closely linked to those of $\widetilde{\mathcal{B}}_\eta$: we can prove that

Lemma 4.3 *For any $\eta \in \mathbb{R}$, if $\widetilde{\mathcal{B}}_\eta$ is a Fredholm operator, then $\mathcal{B}_\eta$ is a Fredholm operator, i.e. the dimension of* $\operatorname{Ker} \mathcal{B}_\eta$ *and the codimension of* $\operatorname{Im} \mathcal{B}_\eta$ *are finite.*

The Fredholm properties of $\widetilde{\mathcal{B}}_\eta$ depend on the invertibility of its operator-valued symbol $\hat{\mathcal{B}}$ in the complex plane through the partial Fourier–Laplace transform with respect to the variable t (which corresponds to the Mellin transform for corner problems):

$$\tau \longmapsto \hat{\varphi}(\tau, x_3) = \int_{-\infty}^{+\infty} e^{-t\tau}\, \varphi(t, x_3)\, dt. \tag{4.4}$$

Any $\varphi \in H^1_\eta(\Sigma)^3$ has a Fourier–Laplace transform well defined in $H^1(-1,1)^3$ for $\operatorname{Re}\tau = -\eta$.

Definition 4.4 *For every complex number τ, we define:*

$$\hat{\mathcal{B}}(\tau) : H^1(-1,1)^3 \longrightarrow [H^1(-1,1)^3]', \qquad \hat{\varphi} \longmapsto \hat{\mathcal{B}}(\tau)\hat{\varphi} \tag{4.5}$$

as follows:

$$\langle \hat{\mathcal{B}}(\tau)\hat{\varphi}, \hat{v}\rangle_{[H^1]'\times H^1} = \int_{-1}^{+1} B\, e(\tau, 0, \partial_3)\hat{\varphi} : e(-\tau, 0, \partial_3)\hat{v}\, dx_3. \tag{4.6}$$

We know from the general theory, *cf* M.S. Agranovich & M.I. Vishik [1] that there exists a discrete subset $\operatorname{Sp}(\mathcal{B})$ of $\mathbb{C}$, such that:

$$\forall \tau \notin \operatorname{Sp}(\mathcal{B}), \quad \hat{\mathcal{B}}(\tau) \text{ is an isomorphism.} \tag{4.7}$$

Moreover, $\tau \to \hat{\mathcal{B}}(\tau)^{-1}$ is meromorphic with its poles in $\operatorname{Sp}(\mathcal{B})$ and in every strip of the form $\eta_1 < \operatorname{Re}\tau < \eta_2$:

$$\{\tau \in \mathbb{C} \mid \quad \eta_1 < \operatorname{Re}\tau < \eta_2\} \,\cap\, \operatorname{Sp}(\mathcal{B}) \quad \text{is finite.} \tag{4.8}$$

Similarly to [15] we can show that:

Lemma 4.5 *For any $\eta \in \mathbb{R}$ such that $\operatorname{Re}\tau = -\eta$ does not intersect $\operatorname{Sp}(\mathcal{B})$, the operator $\widetilde{\mathcal{B}}_\eta$ is an isomorphism.*

As a consequence of Lemmas 4.3 and 4.5

Proposition 4.6 *For any $\eta \in \mathbb{R}$ such that $\operatorname{Re}\tau = -\eta$ does not intersect $\operatorname{Sp}(\mathcal{B})$, $\mathcal{B}_\eta$ is a Fredholm operator.*

Here are some additional properties of the above operators.

Proposition 4.7

$$\forall \eta \geq 0, \qquad \operatorname{Ker}\mathcal{B}_\eta = \{0\}. \tag{4.9}$$

Proof. It may be noticed that for all $\eta \geq 0$:

$$V_{-\eta}(\Sigma^+) \supset V_\eta(\Sigma^+). \tag{4.10}$$

Therefore, if $w \in \operatorname{Ker}\mathcal{B}_\eta$ for some $\eta \geq 0$, then we may take as test function $v = w$ in (4.3). From the positivity of the matrix B, it follows that:

$$e(\partial_t, 0, \partial_3)w = 0.$$

Thus w belongs to the space $\mathcal{R}$ of rigid displacements. Since moreover w satisfies zero Dirichlet boundary condition for $t=0$, it coincides with the null displacement. This yields the result. ■

We also need extra information about the symbol $\hat{\mathcal{B}}$.

Proposition 4.8 *For all* $\tau \neq 0$ *such that* $\operatorname{Re}\tau = 0$, $\hat{\mathcal{B}}(\tau)$ *is an isomorphism.*

Proof. We have, for every $\hat{w} \in \operatorname{Ker}\hat{\mathcal{B}}(\tau)$:

$$\int_{-1}^{+1} B\, e(\tau, 0, \partial_3)\hat{w} : e(-\tau, 0, \partial_3)\overline{\hat{w}} = 0,$$

If $\operatorname{Re}\tau = 0$ we have $\overline{\tau} = -\tau$, whence:

$$\int_{-1}^{+1} B\, e(\tau, 0, \partial_3)\hat{w} : \overline{e(\tau, 0, \partial_3)\hat{w}} = 0.$$

As B is positive definite with all its coefficients real, it follows that $e(\tau, 0, \partial_3)\hat{w} = 0$. Therefore, $\hat{w} = 0$ as soon as $\tau \neq 0$ and we get the result. ■

If $\tau = 0$, the kernel $\operatorname{Ker}\hat{\mathcal{B}}(\tau)$ is generated by the constants $\hat{w} = (a_t, a_s, a_3)$. Thus $\hat{\mathcal{B}}(\tau)^{-1}$ has a pole in $\tau = 0$ and the following space $\mathcal{P}$ is not reduced to 0:

Definition 4.9 The space $\mathcal{P}$ is the space of functions defined on the whole strip $\Sigma = \mathbb{R} \times (-1, 1)$ by:

$$\mathcal{P} = \Big\{ W \mid \quad W(t, x_3) = \operatorname*{Res}_{\tau=0} e^{t\tau}\, \hat{\mathcal{B}}(\tau)^{-1}\, \hat{f}(\tau, x_3), \quad \hat{f} \text{ holomorphic} \Big\}. \tag{4.11}$$

■

From the general theory [1], we know that $\mathcal{P}$ *is finite dimensional.* Its elements have clearly the form $\sum_{q=0}^{Q} t^q\, w_q(x_3)$. Conversely, we also classically have

Lemma 4.10

$$\begin{aligned}\mathcal{P} = \Big\{ W(t, x_3) = \sum_{q=0}^{Q} t^q\, w_q(x_3) \mid \\ \forall v \in \mathscr{D}(\Sigma), \quad \int_\Sigma B\, e(\partial_t, 0, \partial_3)(W) : e(\partial_t, 0, \partial_3)(v) = 0 \Big\}\end{aligned} \tag{4.12}$$

and the elements of $\mathcal{P}$ *are polynomial.*

This last property comes from the fact that the coefficients of B are constant. One should calculate "Jordan chains" for $\tau = 0$ to determine $\mathcal{P}$, see [19] for the computation of such a space on $\mathbb{R}^2 \times (-1, 1)$.

4.2 Proof of Theorem 1.3

(i) First step: Solution up to polynomials. Let η_0 be the largest positive real number such that

$$\forall \eta \in (-\eta_0, 0) \cup (0, \eta_0), \qquad \operatorname{Sp}\mathcal{B} \cap \{\operatorname{Re}\tau = \eta\} = \emptyset. \tag{4.13}$$

As a consequence of property (4.8) of $\operatorname{Sp}\mathcal{B}$:

$$\eta_0 > 0. \tag{4.14}$$

Lemma 4.11 *Let $\eta \in (0, \eta_0)$ and a right hand side $f \in V'_\eta(\Sigma^+)$ be given. Then (i) there exists a displacement φ belonging to $\cap_{\bar\eta<0} V_{\bar\eta}(\Sigma^+)$ such that:*

$$\mathcal{B}_{-\eta}\varphi = f, \tag{4.15}$$

(ii) there exists W in the finite dimensional space $\mathcal{P}$ of polynomials (cf Definition 4.9 *and Lemma* 4.10*) such that*

$$\varphi + W \in H^1_\eta(\Sigma^+)^3. \tag{4.16}$$

Proof. As the right hand sides belongs to $V'_\eta(\Sigma^+)$, it also belongs to any space $V'_{\bar\eta}(\Sigma^+)$, for $\bar\eta < \eta$. Let us fix $\bar\eta = -\eta$. By Proposition 4.6 and the definition (4.13) of η_0, $\mathcal{B}_{-\eta}$ is Fredholm, by Proposition 4.7 $\operatorname{Ker}\mathcal{B}_\eta$ is reduced to $\{0\}$ and moreover (cf Remark 4.2) $\mathcal{B}^\star_{-\eta} = \mathcal{B}_\eta$. Therefore, by the Fredholm alternative, $\mathcal{B}_{-\eta}$ is onto and we have the existence of φ.

From the general theory of corner problems, *cf* [15], we may use the Cauchy formula to obtain:

$$\varphi - \frac{1}{2i\pi}\int_\gamma e^{t\tau}\,\hat{\mathcal{B}}(\tau)^{-1}\,\hat f(\tau, x_3)\,d\tau \in H^1_\eta(\Sigma^+)^3,$$

where

- γ is a simple contour in the complex plane surrounding all the poles of $\hat{\mathcal{B}}(\tau)^{-1}$ contained in the strip $-\eta < \operatorname{Re}\tau < \eta$,
- $\hat f$ the Fourier-Laplace transform of $\mathcal{B}_{-\eta}(\xi\varphi)$,
- ξ is a smooth cut-off such that $\xi(0) = 0$ and $\xi(t) = 1$ for $t \geq 1$.

By (4.13) and Proposition 4.8, this set of poles surrounded by γ is reduced to $\{0\}$ and by Remark 4.10, this residue is an element W of $\mathcal{P}$. The result then follows. ■

(*ii*) Second step: the relation between the space $\mathcal{P}$ and the subspace $\mathcal{R}$ of rigid displacements. If $\eta \in (0, \eta_0)$, the operator $\mathcal{B}_{-\eta}$ is onto — *cf* (4.15), but we can expect that its kernel is not reduced to $\{0\}$. By the previous lemma applied with the right hand side $f = 0$, we deduce that there exists a subspace $\mathcal{T} \subset \mathcal{P}$ of polynomials such that:

$$\forall\eta \in (0, \eta_0), \qquad \operatorname{Ker}\mathcal{B}_{-\eta} = \{T + X(T) \mid \quad T \in \mathcal{T}, \quad X(T) \in H^1_\eta(\Sigma^+)^3\} \tag{4.17}$$

where X is a linear map defined on $\mathcal{T}$. Thus the elements of $\operatorname{Ker}\mathcal{B}_{-\eta}$ are the sum of a polynomial and an exponentially decreasing function. The key-argument for the proof of Theorem 1.3 is:

Proposition 4.12 *We have*

$$\dim\mathcal{T} = \dim\mathcal{R} = 4 \quad \textit{and} \quad \mathcal{P} = \mathcal{T} \oplus \mathcal{R}. \tag{4.18}$$

Proof. Now we denote the flux through the surface $t = t_0$ by

$$\Phi_{t=t_0}(\varphi \,|\, R) = \int_{-1}^{+1} \sigma_{it}(\varphi)(t_0, x_3)\, R_i(t_0, x_3)\, dx_3.$$

Using the Saint-Venant Principle under the form of Corollary 3.3, we deduce

$$\forall K \in \operatorname{Ker} \mathcal{B}_{-\eta}, \qquad \Big(\forall R \in \mathcal{R},\ \Phi_{t=0}(K \,|\, R) = 0\Big) \implies e(\partial_t, 0, \partial_3)(K) \equiv 0, \quad (4.19)$$

thus, if $\forall R \in \mathcal{R},\ \Phi_{t=0}(K \,|\, R) = 0$, then K is a rigid displacement. If, in addition, K satisfies an homogeneous condition of Dirichlet type on $\{t = 0\}$, then we necessarily have $K \equiv 0$. Therefore:

$$\forall K \in \operatorname{Ker} \mathcal{B}_{-\eta} \backslash \{0\}, \qquad \exists R \in \mathcal{R} \ \text{ s.t. } \ \Phi_{t=0}(K \,|\, R) \neq 0. \quad (4.20)$$

The flux conservation implies, with the splitting (4.17):

$$\forall t_0 > 0, \qquad \Phi_{t=0}(K \,|\, R) = \Phi_{t=t_0}(K \,|\, R) = \Phi_{t=t_0}(T \,|\, R) + \Phi_{t=t_0}(X(T) \,|\, R). \quad (4.21)$$

We may expect that $\Phi_{t=t_0}(T \,|\, R)$ behaves as a polynomial whereas $\Phi_{t=t_0}(X(T) \,|\, R)$ is exponentially decreasing as $t_0 \to +\infty$. So, the applications $t_0 \mapsto \Phi_{t=t_0}(T \,|\, R)$ and $t_0 \mapsto \Phi_{t=t_0}(X(T) \,|\, R)$ are respectively constant and reduced to zero. Thus

$$\forall K = T + X(T) \in \operatorname{Ker} \mathcal{B}_{-\eta}, \qquad \Phi(K \,|\, R) = \Phi(T \,|\, R). \quad (4.22)$$

Let $d = \dim \operatorname{Ker} \mathcal{B}_{-\eta}$, and $T_1, \ldots, T_d$ denote a basis of $\mathcal{T}$. There exists a basis $R_1, \ldots, R_4$ of $\mathcal{R}$ such that:

$$\Phi(T_j \,|\, R_k) = \delta_{jk}, \quad j = 1, \ldots, d \quad k = 1, \ldots, 4,$$

which means that $R_1, \cdots, R_4$ yield a sort of dual basis of $T_1, \ldots, T_d$. Whence

$$d \leq 4. \quad (4.23)$$

Moreover, if T is a rigid displacement, for all $R \in \mathcal{R},\ \Phi(T \,|\, R) = 0$; thus we deduce from (4.20) and (4.22) that $\mathcal{T} \cap \mathcal{R} = \{0\}$ and consequently, as $\mathcal{R} \subset \mathcal{P}$ by (4.12):

$$\mathcal{T} + \mathcal{R} = \mathcal{T} \oplus \mathcal{R} \subset \mathcal{P}. \quad (4.24)$$

It remains to show that both spaces $\mathcal{T} \oplus \mathcal{R}$ and $\mathcal{P}$ have the same dimension. Indeed, the splitting (4.16) allows to prove $\dim \mathcal{P} = \operatorname{Ind} \mathcal{B}_{-\eta} - \operatorname{Ind} \mathcal{B}_{\eta}$ (*cf* [10]), and since $\mathcal{B}_{\eta}^{\star} = \mathcal{B}_{-\eta}$:

$$\dim \mathcal{P} = 2 \operatorname{Ind} \mathcal{B}_{-\eta} = 2 \dim \operatorname{Ker} \mathcal{B}_{-\eta} = 2d. \quad (4.25)$$

Taking into account (4.24), we deduce that $d + 4 \leq 2d$, i.e. $4 \leq d$. The conclusion follows by using inequality (4.23), namely $d = 4$, which ends the proof. ■

(*iii*) Third step: Solution up to rigid displacements. Let $\eta \in (0, \eta_0)$ and a right hand side $(f, g) \in V'_{\eta}(\Sigma^+) \times H^{1/2}(\gamma_0)^3$ for problem (1.17). Let G be a lifting of g in $H^1_{\eta}(\Sigma^+)^3$. The change of unknown $\varphi \mapsto \varphi - G$ transforms the data (f, g) into $(f - \mathcal{B}_{\eta} G, 0)$.

As a mere consequence of Lemma 4.11, we have the existence of $\varphi_1 \in V_{-\eta}(\Sigma^+)$ such that $\mathcal{B}_{-\eta}\varphi_1 = f - \mathcal{B}_{\eta} G$, and the splitting of φ_1 into $\varphi_0 - W$, with $\varphi_0 \in H^1_{\eta}(\Sigma^+)^3$ and $W \in \mathcal{P}$. Proposition 4.12 yields that $W = T + R$, with $T \in \mathcal{T}$ and $R \in \mathcal{R}$. Using (4.17), we obtain that $\varphi_1 + T + X(T) = \varphi_0 + X(T) - R$ is a solution of problem

(1.17) with data $(f - \mathcal{B}_\eta G, 0)$. Setting

$$\varphi = \varphi_0 + X(T) + G,$$

we obtain a solution in $H^1_\eta(\Sigma^+)^3$ of problem (1.17) for $(f, g + R)$, i.e. a solution of problem (1.19). Theorem 1.3 is now proved.

4.3 Existence of exponentially decreasing profiles

We return to the recursive system (1.15) which must be solved by the profiles φ^k.

Corollary 4.13 *For any $k \in \mathbb{N}$, let $h^k = h^k(s)$ belong to $\mathscr{C}^\infty(S, H^{1/2}(\gamma_0)^3)$. Then there exists for any $k \in \mathbb{N}$, $\varphi^k \in \mathscr{C}^\infty(S, H^1(\Sigma^+)^3)$ and $R^k \in \mathscr{C}^\infty(S, \mathcal{R})$ solving for any $\ell \in \mathbb{N}$ the system*

$$\begin{cases} \varphi^\ell(s)\Big|_{\gamma_0} = h^\ell(s) + R^\ell(s)\Big|_{\gamma_0} \\ \forall v \in V(\Sigma^+), \ \displaystyle\int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi^\ell) : e(\partial_t, 0, \partial_3)(v) = -\int_{\Sigma^+} \sum_{j=1}^{\ell-1} \mathcal{B}^j(\varphi^{\ell-j},\, v). \end{cases} \tag{4.26}$$

Moreover, φ^k belongs to $\mathscr{C}^\infty(S, H^1_\eta(\Sigma^+)^3)$ for all $\eta < \eta_0$.

Proof. Let $\eta \in (0, \eta_0)$ be fixed. Let $\eta_0 > \eta_1 > \cdots > \eta_k > \cdots > \eta$ be a decreasing sequence. As a consequence of Theorem 1.3 for $f = 0$ and $g = h^0$, we obtain φ^0 in $\mathscr{C}^\infty(S, H^1_{\eta_1}(\Sigma^+)^3)$ solving (4.26) for $\ell = 0$.

Let us assume that φ^j are constructed in $\mathscr{C}^\infty(S, H^1_{\eta_{j+1}}(\Sigma^+)^3)$ for $j = 0, \ldots, \ell - 1$ such that (4.26) holds for $j = 0, \ldots, \ell-1$. Then, since the operators $\mathcal{B}^j$ are of order 2 with polynomial coefficients in t, $\sum_{j=1}^{\ell-1} \mathcal{B}^j(\varphi^{\ell-j}, \cdot)$ belongs to $\mathscr{C}^\infty(S, V'_{\eta_{\ell+1}}(\Sigma^+))$, and a new application of Theorem 1.3 yields the existence of φ^ℓ in $\mathscr{C}^\infty(S, H^1_{\eta_{\ell+1}}(\Sigma^+)^3)$ such that (4.26) holds for ℓ. ■

Thus the profiles φ^k belong to $\mathscr{C}^\infty(S, \cap_{\eta<\eta_0} H^1_\eta(\Sigma^+)^3)$. A closer look at the structure of $\mathcal{B}_{\eta_0}$ would allow to prove that the φ^k belong to the space $\mathscr{C}^\infty(S, \mathbb{P}_{\eta_0}(\Sigma^+)^3)$ where $\mathbb{P}_{\eta_0}(\Sigma^+)$ is the space of the functions φ admitting a splitting as

$$\varphi - \sum_{q=0}^{Q} a_q(x_3)\, t^q\, e^{-\eta_0 t} \in H^1_{\eta_0}(\Sigma^+), \tag{4.27}$$

where the functions a_q are smooth functions which depend on φ and Q is an integer which also depends on φ.

5 BOUNDARY LAYER TERMS ALONG THE EDGES OF THE PLATE

As a consequence of the previous Corollary 4.13 and of the construction algorithms exhibited in Part I [11], we obtain that the solution $u(\varepsilon)$ of problem (1.6) with a smooth right hand side f has the following asymptotics as $\varepsilon \to 0$

$$u(\varepsilon)(x) \sim u^0(x) + \varepsilon\Big(u^1(x) - \chi(r)\, w^1(\frac{r}{\varepsilon}, s, x_3)\Big) + \cdots \\ + \varepsilon^k\Big(u^k(x) - \chi(r)\, w^k(\frac{r}{\varepsilon}, s, x_3)\Big) + \cdots \tag{5.1}$$

where the displacements u^k are smooth up to the boundary of the scaled plate Ω, *see* §4.3 in [11], and the profiles w^k belong to the weighted spaces $\mathscr{C}^\infty(S, H^1_\eta(\Sigma^+)^3)$ for all $\eta < \eta_0$. The asymptotics (5.1) satisfies optimal error estimates in $H^1(\Omega)$ and $L^2(\Omega)$, in the sense that we have optimal estimates for the remainders $\bar{U}^N(\varepsilon)$ defined as

$$\bar{U}^N(\varepsilon) = u(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k\Big(u^k(x) - \chi(r)\, w^k(\frac{r}{\varepsilon}, s, x_3)\Big). \tag{5.2}$$

For all $N \in \mathbb{N}$, we have

$$\|\bar{U}^N(\varepsilon)\|_{H^1(\Omega)^3} \le C\,\varepsilon^{N+1}\Big(\|u^{N+1}\|_{H^1(\Omega)^3} + \|w^{N+1}(\frac{r}{\varepsilon})\|_{H^1(\Omega)^3} + \mathcal{O}(\varepsilon)\Big). \tag{5.3a}$$

$$\|\bar{U}^N(\varepsilon)\|_{L^2(\Omega)^3} \le C\,\varepsilon^{N+1}\Big(\|u^{N+1}\|_{L^2(\Omega)^3} + \|w^{N+1}(\frac{r}{\varepsilon})\|_{L^2(\Omega)^3} + \mathcal{O}(\varepsilon)\Big). \tag{5.3b}$$

We will see in this section that, in general, $u(\varepsilon)$ does not belong to $H^2(\Omega)$ and that the singular part of $u(\varepsilon)$ can be expanded in powers of ε as $\varepsilon \to 0$ in a way compatible with (5.1).

5.1 Singular exponents

The plates Ω^ε have two *edges* $\gamma^\varepsilon_\pm$, which are the intersections between the lateral face Γ^ε_0 and the upper and lower faces of the plate:

$$\gamma^\varepsilon_\pm = \partial\omega \times \{\pm\varepsilon\}.$$

The corresponding edges of the reference set Ω are

$$\gamma_\pm = \partial\omega \times \{\pm 1\}.$$

These edges form the junction between two different boundary conditions: Dirichlet on the lateral boundary and Neumann on the two horizontal surfaces. The opening angle of the domain Ω all along the edges is constant and equal to $\pi/2$.

Because of this change in boundary conditions, we cannot expect that the solution $u(\varepsilon)$ is regular, even if the right hand side is smooth up to the boundary. The regularity and the asymptotics of $u(\varepsilon)$ in the neighborhood of the edges is governed by the *singular exponents* of the reduced-normal problems defined for each value of the arc length $s \in S$ (*cf* Definition 1.1):

$$\varphi \in H^1(\Sigma^+)^3 \quad \text{and} \quad \varphi\Big|_{\gamma_0} = g, \tag{5.4a}$$

$$\forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = 0. \tag{5.4b}$$

We refer to [16] and [10] for the general theory of the regularity along edges, and to [17, 18] and [6, 7, 5, 9] for asymptotics along a curved edge.

To each value of the arc length $s \in S$, there correspond two sequences of complex numbers $\nu_1^{\pm}(s), \ldots, \nu_\ell^{\pm}(s)$ with

$$0 < \operatorname{Re} \nu_1^{\pm}(s) \leq \ldots \leq \operatorname{Re} \nu_\ell^{\pm}(s) \leq \ldots$$

that govern the regular and singular behavior of the solution of the mixed Dirichlet Neumann problem (5.4) near the corner $(t, x_3) = (0, \pm 1)$: let $m \in \mathbb{R}$, $m > 0$ and let φ be the solution of problem (5.4) with $g \in H^{m+1/2}(\gamma_0)^3$;
if

$$\operatorname{Re} \nu_1^{\pm}(s) > m, \tag{5.5a}$$

then

$$\varphi \in H^{m+1}(\Sigma^+)^3; \tag{5.5b}$$

if

$$\forall \ell \geq 1, \quad \operatorname{Re} \nu_\ell^{\pm}(s) \neq m, \tag{5.6a}$$

then

$$\varphi = \varphi_{\text{reg}} + \varphi_{\text{sing}}, \quad \text{with} \quad \begin{cases} \varphi_{\text{sing}} = \displaystyle\sum_{+,\,-} \chi^{\pm} \sum_{\ell,\, \operatorname{Re} \nu_\ell^{\pm}(s) < m} c_\ell^{\pm}\, \mathcal{S}_\ell^{\pm}(s), \\ \varphi_{\text{reg}} \in H^{m+1}(\Sigma^+)^3, \end{cases} \tag{5.6b}$$

where $\chi^{\pm}$ is a smooth cut-off function equal to 1 in the neighborhood of the corner $(t, x_3) = (0, \pm 1)$ of Σ^+ and the functions $\mathcal{S}_\ell^{\pm}(s)$ are singular solutions of (5.4) whose behavior in terms of

$$\rho_{\pm}(t, x_3) := \operatorname{dist}\Big((t, x_3), (0, \pm 1)\Big)$$

is

$$(\rho_{\pm})^{\nu_\ell^{\pm}(s)} \quad \text{or} \quad (\rho_{\pm})^{\nu_\ell^{\pm}(s)} \log \rho_{\pm},$$

according to the multiplicity of $\nu_\ell^{\pm}(s)$.

The complex numbers $\nu_\ell^{\pm}(s)$ are the eigenvalues of an analytic family of elasticity operators $\nu \mapsto \mathscr{B}^{\pm}(s)(\nu)$, i.e. the values of ν for which $\mathscr{B}^{\pm}(s)(\nu)$ is not invertible. For each $\nu \in \mathbb{C}$, $\mathscr{B}^{\pm}(s)(\nu)$ operates in *one* variable and is defined as follows. Let $\mathcal{B}(s)$ be the operator governing problem (5.4) and $(\rho_{\pm}, \theta_{\pm})$ be the polar coordinates centered at $(0, \pm 1)$. The changes of coordinates $(t, x_3) \mapsto (\rho_{\pm}, \theta_{\pm})$ defines for each $s \in S$ an operator $\mathscr{B}^{\pm}(s)$ such that

$$\mathscr{B}^{\pm}(s)\Big(\theta_{\pm};\ \rho_{\pm}\,\partial_{\rho_{\pm}},\ \partial_{\theta_{\pm}}\Big) = \rho_{\pm}^2\, \mathcal{B}(s)(\partial_t,\ \partial_3).$$

Then

$$\mathscr{B}^{\pm}(s)(\nu) = \mathscr{B}^{\pm}(s)\Big(\theta_{\pm};\ \nu,\ \partial_{\theta_{\pm}}\Big).$$

For an isotropic material with Lamé constants λ and μ, $\nu_\ell^+(s) = \nu_\ell^-(s) = \nu_\ell$ for all $s \in S$, where ν_ℓ are the singular exponents of the two-dimensional Lamé operator with mixed Dirichlet-Neumann conditions on an angle with opening $\pi/2$. In the following two tables, we give the values of the first three singular exponents computed by the method of [8] for a few pairs (λ, μ); note that the singular exponents depend only on the ratio λ/μ.

In Table 1, we consider the cases where $\mu > 0$ and $\lambda > 0$. We note that ν_1 is always real and less than 1. Thus, for an isotropic material, solutions $u(\varepsilon)$ do not belong to $H^2(\Omega)$ in general.

λ	μ	ν_1	ν_2	ν_3
1	0.0001	0.594637	1.671905	$\pm$ 0.808683 i
1	0.001	0.594874	1.672056	$\pm$ 0.807980 i
1	0.01	0.597220	1.673714	$\pm$ 0.801032 i
1	0.1	0.619071	1.686965	$\pm$ 0.741342 i
1	1	0.744750	1.718359	$\pm$ 0.474498 i
1	10	0.928406	1.510215	1.885198
1	100	0.990438	1.368903	1.989911
1	1000	0.999005	1.354005	1.998999

Table 1

In Table 2, we consider the situation where $\lambda < 0$ with $\lambda > -\mu$, which still ensures the ellipticity in two dimensions. We repeat the last row of Table 1 for the sake of comparison — note that the choice $(\lambda, \mu) = (1, 1000)$ is equivalent to the choice $(\lambda, \mu) = (0.001, 1)$. Now, $\operatorname{Re} \nu_1 > 1$, and we have the regularity H^2.

λ	μ	ν_1	ν_2	ν_3
0.001	1	0.999005	1.354005	1.998999
0.	1	1.000000	1.352317	2.000000
- 0.25	1	1.109950	$\pm$ 0.284274 i	2.231620
- 0.50	1	1.061964	$\pm$ 0.466259 i	2.519427
- 0.75	1	1.023139	$\pm$ 0.691068 i	3.094857 $+$ 0.512300 i
- 0.95	1	1.001794	$\pm$ 1.182371 i	3.005457 $+$ 1.176263 i
- 0.99	1	1.000106	$\pm$ 1.688153 i	3.000319 $+$ 1.687901 i
- 0.999	1	1.000001	$\pm$ 2.419603 i	3.000005 $+$ 2.419601 i

Table 2

5.2 Stable singular functions

As can be seen from these tables, branchings appear in the exponents when the parameters of the rigidity matrix change, for instance if the Lamé coefficients depend on (x_1, x_2). So, in general, the singular functions $\mathcal{S}^{\pm}_{\ell}(s)$ do not depend smoothly on the arc length s. But, relying on [5, 9], we can construct special linear combinations of the $\mathcal{S}^{\pm}_{\ell}(s)$ in order to obtain *stable singular functions* $\mathcal{S}^{\pm}_{\mathrm{stab},\,\ell}(s)$

$$s \longmapsto \mathcal{S}^{\pm}_{\mathrm{stab},\,\ell}(s) \quad \mathscr{C}^\infty \text{ with respect to } s \text{ and taking values in } \mathscr{C}^\infty(\Sigma^+).$$

Simple eigenvalues $\nu^{\pm}_{\ell}(s)$ yield directly stable singular functions.

If $\nu^{\pm}_{\ell}(s_0) = \nu^{\pm}_{\ell+1}(s_0)$ is a double eigenvalue at s_0 and if $\nu^{\pm}_{\ell}(s)$ and $\nu^{\pm}_{\ell+1}(s)$ are simple for $s \neq s_0$ in a neighborhood of s_0, then stable singular functions are given in this neighborhood by

$$\mathcal{S}^{\pm}_{\mathrm{stab},\,\ell}(s) = \mathcal{S}^{\pm}_{\ell}(s) + \mathcal{S}^{\pm}_{\ell+1}(s) \quad \text{and} \quad \mathcal{S}^{\pm}_{\mathrm{stab},\,\ell+1}(s) = \frac{\mathcal{S}^{\pm}_{\ell}(s) - \mathcal{S}^{\pm}_{\ell+1}(s)}{s - s_0}.$$

Using these stable singular functions, we have (the spaces $H^m_\eta(\Sigma^+)$ are those introduced in Definition 1.2):

Proposition 5.1 *Let* $g = g(s)$ *and* $f = f(s)$ *be functions in* $\mathscr{C}^\infty(S)$ *with values in* $H^{m+1/2}(\gamma_0)^3$ *and* $H^{m-1}_\eta(\Sigma^+)^3$ *respectively, where* $0 < \eta < \eta_0$. *If*

$$\forall s \in S, \quad \forall \ell \geq 1, \qquad \mathrm{Re}\, \nu^{\pm}_{\ell}(s) \neq m, \tag{5.7}$$

and if for all s, $\varphi(s)$ *is solution of*

$$\varphi \in H^1(\Sigma^+)^3 \qquad \text{and} \quad \varphi\Big|_{\gamma_0} = g(s), \tag{5.8a}$$

$$\forall v \in V(\Sigma^+), \qquad \int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) = \int_{\Sigma^+} f(s)\, v, \tag{5.8b}$$

then

$$\varphi = \varphi_{\mathrm{reg}} + \varphi_{\mathrm{sing}}, \quad \text{with} \quad \begin{cases} \varphi_{\mathrm{sing}}(s) = \displaystyle\sum_{+,-} \chi^{\pm} \sum_{\ell,\, \mathrm{Re}\,\nu^{\pm}_{\ell}(s) < m} c^{\pm}_{\ell}(s)\, \mathcal{S}^{\pm}_{\mathrm{stab},\,\ell}(s), \\ \varphi_{\mathrm{reg}} \in \mathscr{C}^\infty(S,\, H^{m+1}_\eta(\Sigma^+)^3), \end{cases} \tag{5.9}$$

with smooth coefficients $c^{\pm}_{\ell} \in \mathscr{C}^\infty(S)$.

Now we consider problem (1.11) on $\widetilde{\Sigma} := \mathbb{R}^+ \times S \times (-1, +1)$

$$\varphi \in H^1(\widetilde{\Sigma})^3 \quad \text{and} \quad \varphi = g \quad \text{on} \quad \widetilde{\Gamma}_0, \tag{5.10a}$$

$$\forall v \in V(\widetilde{\Sigma}), \quad \begin{cases} \displaystyle\int_{\widetilde{\Sigma}} \widetilde{A}(\varepsilon t, s)\, e(\partial_t, 0, \partial_3)(\varphi) : e(\partial_t, 0, \partial_3)(v) \\ \displaystyle\qquad\qquad + \sum_{k=1}^{4} \varepsilon^k\, \widetilde{\mathcal{A}}^k(\varphi,\, v) = \int_{\widetilde{\Sigma}} f\, v. \end{cases} \tag{5.10b}$$

We note that this problem *depends on* ε through its coefficients, but that this dependence is now *regular*.

As is well-known in the edge analysis, the asymptotics near the edges is governed by the reduced-normal part of the operator "frozen" at the edges. Since we start with smooth data, all along the construction of the profiles w^k we remain in spaces of functions which are smooth with respect to the tangential variable (the arc length s) and the following statement is convenient for our purposes:

Proposition 5.2 *Let* $g \in \mathscr{C}^\infty(S,\, H^{m+1/2}(\gamma_0)^3)$ *and* $f \in \mathscr{C}^\infty(S,\, H^{m-1}_\eta(\Sigma^+)^3)$*, for* $0 < m \le 1$ *and* $0 < \eta < \eta_0$*. If hypothesis* (5.7) *holds and if* $\varphi(\varepsilon)$ *is solution of problem* (5.10)*, then expansion* (5.9) *still holds with smooth coefficients* $c^\pm_\ell(\varepsilon) \in \mathscr{C}^\infty(S)$*. Moreover, the dependence with respect to the small parameter* ε *is uniform: for all* $n \in \mathbb{N}$

$$
\begin{aligned}
&\exists\, C_n(g,f), C'_n > 0, \quad \forall \varepsilon \in (0,1),\\
&\qquad \|c^\pm_\ell(\varepsilon)\|_{H^n(S)} + \|\chi(\varepsilon t)\, \varphi_{\mathrm{reg}}(\varepsilon)\|_{H^n(S, H^{m+1}(\Sigma^+))}\\
&\qquad\qquad \le C_n(g,f) + C'_n \|\varphi(\varepsilon)\|_{L^2(S, H^1(\Sigma^+))},
\end{aligned}
$$

where χ *is a cut-off function equal to* 1 *in a neighborhood of* $t = 0$*.*

The proof combines a classical analysis in the neighborhood of the edges, in the region where $t \in (0,1)$ for instance, and uniform a priori estimates in the region where $1 < t < \rho/\varepsilon$.

Remark 5.3 The assumption $m \le 1$ serves only to avoid the dependence on ε of the stable singular functions. To handle larger values of m, we have to introduce the "shadows" $\mathcal{U}^{\pm,k}_{\mathrm{stab},\ell}(s)$ of the stable singular parts

$$
\mathcal{U}^{\pm,0}_{\mathrm{stab},\ell}(s) := c^\pm_\ell(s)\, \mathcal{S}^\pm_{\mathrm{stab},\ell}(s)
$$

which are defined by induction as solutions of (compare with (1.15))

$$
\left\{
\begin{aligned}
&\mathcal{U}^{\pm,k}_{\mathrm{stab},\ell}(s)\Big|_{\gamma_0} = 0 \quad \text{and} \quad \forall v \in V(\Sigma^+),\\
&\int_{\Sigma^+} B(s)\, e(\partial_t, 0, \partial_3)(\mathcal{U}^{\pm,k}_{\mathrm{stab},\ell}) : e(\partial_t, 0, \partial_3)(v) = -\int_{\Sigma^+} \sum_{j=1}^{k-1} \mathcal{B}^j(\mathcal{U}^{\pm,k-j}_{\mathrm{stab},\ell},\, v).
\end{aligned}
\right.
\tag{5.11}
$$

With these new functions, the expansion of the solution φ of problem (5.10) can be written as

$$
\varphi_{\mathrm{sing}}(s) = \sum_{+,-} \chi^\pm \sum_{\ell,\, \mathrm{Re}\,\nu^\pm_\ell(s) < m} \sum_{k=0}^{[m-1]} \varepsilon^k\, \mathcal{U}^{\pm,k}_{\mathrm{stab},\ell}(s).
$$

More precisely, there are functions $\mathcal{S}^{\pm,k,d}_{\mathrm{stab},\ell}(s)$ which do not depend on φ, such that

$$
\varphi_{\mathrm{sing}} = \sum_{+,-} \chi^\pm \sum_{\ell,\, \mathrm{Re}\,\nu^\pm_\ell(s) < m} \left(c^\pm_\ell\, \mathcal{S}^\pm_{\mathrm{stab},\ell} + \sum_{k=1}^{[m-1]} \varepsilon^k \sum_{d=0}^{[m-1]} \partial^d_s c^\pm_\ell\, \mathcal{S}^{\pm,k,d}_{\mathrm{stab},\ell} \right).
\tag{5.12}
$$

■

5.3 Expansion of $u(\varepsilon)$ near the edges

Let us recall that problem (1.6) is equivalent to problem (5.10) with $g = 0$, as far as the behavior in an annulus neighborhood $r < \rho$ of the clamped part Γ_0 of the boundary is concerned, *cf* §1.3. Thus we can deduce from Proposition 5.2:

Theorem 5.4 *Let $u(\varepsilon)$ be the solution of problem* (1.6) *and let $0 < m \leq 1$. Then $u(\varepsilon)$ admits the splitting*

$$u(\varepsilon) = u_{\mathrm{reg}}(\varepsilon) + u_{\mathrm{sing}}(\varepsilon), \tag{5.13a}$$

with

$$\begin{cases} u_{\mathrm{sing}}(\varepsilon)(r, s, x_3) = \sum\limits_{+,\,-} \sum\limits_{\ell,\, \mathrm{Re}\,\nu_\ell^\pm(s) \leq m} c_\ell^\pm(\varepsilon)(s)\, \chi^\pm(\frac{r}{\varepsilon}, x_3)\, \mathcal{S}^\pm_{\mathrm{stab},\,\ell}(\frac{r}{\varepsilon}, s, x_3), \\ u_{\mathrm{reg}}(\varepsilon) \in \mathscr{C}^\infty(S,\, H^{m+1}(\Omega)^3). \end{cases} \tag{5.13b}$$

Remark 5.5 We can drop the assumption (5.7) because of the smoothness of the right hand side $f(\varepsilon)$. ■

As already hinted, in the expansion (5.1) of $u(\varepsilon)$, the only singular terms are the boundary layer terms w^k. Proposition 5.1 applied to w^k yield (by induction over k) that

Lemma 5.6 *The boundary layer terms w^k admit the splitting*

$$w^k = w^k_{\mathrm{reg}} + w^k_{\mathrm{sing}}, \quad \text{with} \quad \begin{cases} w^k_{\mathrm{sing}}(s) = \sum\limits_{+,\,-} \sum\limits_{\ell,\, \mathrm{Re}\,\nu_\ell^\pm(s) \leq m} c_\ell^{\pm,\,k}(s)\, \chi^\pm\, \mathcal{S}^\pm_{\mathrm{stab},\,\ell}(s), \\ w^k_{\mathrm{reg}} \in \mathscr{C}^\infty(S,\, H^{m+1}_\eta(\Sigma^+)^3), \end{cases} \tag{5.14}$$

for all $0 < m \leq 1$ and for all $\eta < \eta_0$.

Applying once more the result of Proposition 5.2 to the remainder $\bar{U}^N(\varepsilon)$ defined in (5.2) and using the technique of pushing forward the asymptotics in order to obtain sharp estimates of the error, we can prove our final result:

Theorem 5.7 *The splitting* (5.13) *of $u(\varepsilon)$ admits the following asymptotic expansion in powers of ε: For all $N \in \mathbb{N}$*

$$c_\ell^\pm(\varepsilon) = \sum_{k=1}^{N} \varepsilon^k\, c_\ell^{\pm,\,k} + \mathcal{O}(\varepsilon^{N+1}) \tag{5.15}$$

where $\mathcal{O}(\varepsilon^{N+1})$ is in the sense of $\mathscr{C}^\infty(S)$ and the coefficients $c_\ell^{\pm,\,k}$ are those of (5.14)*, and*

$$\|u_{\mathrm{reg}}(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k\, u^k + \chi(r) \sum_{k=1}^{N} \varepsilon^k\, w^k_{\mathrm{reg}}(\frac{r}{\varepsilon}, s, x_3)\|_{H^{m+1}(\Omega)^3} \leq C\, \varepsilon^{N-m+1/2} \tag{5.16}$$

where w^k_{reg} is defined in (5.14)*.*

Of course we could extend this result to larger values of m by making use of the shadow singular functions $\mathcal{S}^{\pm,\,k,\,d}_{\mathrm{stab},\,\ell}$.

6 ESTIMATES FOR THE DISPLACEMENT AND THE STRAIN AND STRESS TENSORS

Without further development, we can deduce from all our results reliable estimates for the displacement field $u(\varepsilon)$ itself in various Sobolev spaces, but also for objects depending linearly from $u(\varepsilon)$, namely the scaled strain tensor $\kappa(\varepsilon)(u(\varepsilon))$ and the scaled stress tensor $\sigma(\varepsilon)$.

More interesting than estimates relative to the remainder $\bar{U}^N$ defined in (5.2) are estimates relative to the possibility of approximation of $u(\varepsilon)$ by polynomials of ε with coefficients depending on x only. So we introduce

$$\check{U}^N(\varepsilon) = u(\varepsilon) - \sum_{k=0}^{N} \varepsilon^k \, u^k(x). \tag{6.1}$$

Another question is the approximation of $u(\varepsilon)$ by hierarchical models, i.e. where the "discrete spaces" are displacements with a polynomial behavior in x_3. Since the boundary layer terms are not polynomial in the variable x_3 in general, this question is closely linked to the estimates of $\check{U}^N$.

6.1 Error estimates for the displacement field

We deduce from the results of the previous section that for any Sobolev norm $\|\cdot\|_{\mathcal{N}(\Omega)}$ such that $\|u(\varepsilon)\|_{\mathcal{N}(\Omega)}$ is finite for all smooth data f, we have — compare with (5.3a) and (5.3b):

$$\|\check{U}^N(\varepsilon)\|_{\mathcal{N}(\Omega)} \leq C\Big(\varepsilon^{N+1} + \varepsilon^K \, \|w^K(\frac{r}{\varepsilon})\|_{\mathcal{N}(\Omega)}\Big), \tag{6.2}$$

where w^K is the first non-zero boundary layer term (in general, $K = 1$ for the in-plane components and $K = 2$ for the third component). As $\|w(\frac{r}{\varepsilon})\|_{H^\mu(\Omega)}$ is $\mathcal{O}(\varepsilon^{-\mu+1/2})$ for any $\mu < \mu_0$, with — *cf* (5.5a)-(5.5b):

$$\mu_0 = \sup_{s\in S} \sup_{+,-} \operatorname{Re} \nu_1^{\pm}(s) + 1, \tag{6.3}$$

we obtain that, for the in-plane components

$$\|\check{U}_*^N(\varepsilon)\|_{H^\mu(\Omega)^2} \leq C(\varepsilon^{N+1} + \varepsilon^{\frac{3}{2}-\mu}), \tag{6.4a}$$

and for the transverse component

$$\|\check{U}_3^N(\varepsilon)\|_{H^\mu(\Omega)} \leq C(\varepsilon^{N+1} + \varepsilon^{\frac{5}{2}-\mu}). \tag{6.4b}$$

The smaller is μ, the better is the estimate.

Similarly, for the L^∞ norm, we obtain

$$\|\check{U}_*^N(\varepsilon)\|_{L^\infty(\Omega)^2} \leq C(\varepsilon^{N+1} + \varepsilon) = \mathcal{O}(\varepsilon), \tag{6.5a}$$

and

$$\|\check{U}_3^N(\varepsilon)\|_{H^\mu(\Omega)} \leq C(\varepsilon^{N+1} + \varepsilon^2). \tag{6.5b}$$

Thus the estimate of $\check{U}_*^N(\varepsilon)$, resp. $\check{U}_3^N(\varepsilon)$, does not improve if $N > 0$, resp. $N > 1$.

6.2 Error estimates for the scaled strain tensor

The scaled strain tensor is defined in (1.4a). We are interested by its L^2 norm. We have to consider the behavior of the first terms in the asymptotics. As w_3^1 is zero, we obtain that

$$\|\kappa(\varepsilon)(\varepsilon\, w^1)\|_{L^2(\Omega)^6} = \mathcal{O}(\varepsilon^{1/2}) \tag{6.6}$$

and

$$\|\kappa(\varepsilon)(\varepsilon^2\, w^2)\|_{L^2(\Omega)^6} = \mathcal{O}(\varepsilon^{1/2}). \tag{6.7}$$

We recall from [11] that u^0 and u^1 are Kirchhoff-Love displacements, i.e. cancel e_{33} and $e_{\alpha 3}$. Thus

$$\|\kappa(\varepsilon)(\varepsilon\, u^1)\|_{L^2(\Omega)^6} = \mathcal{O}(\varepsilon). \tag{6.8}$$

Finally for $k \geq 2$, u^k is the sum of a Kirchhoff-Love displacement and of a term v^k which does not cancel e_{33} in general, thus

$$\|\kappa(\varepsilon)(\varepsilon^k\, v^k)\|_{L^2(\Omega)^6} = \mathcal{O}(\varepsilon^{k-2}). \tag{6.9}$$

Whence, $\kappa(\varepsilon)(\check{U}^0)$ and $\kappa(\varepsilon)(\check{U}^1)$ do not tend to 0 in general as $\varepsilon \to 0$, but

$$\|\kappa(\varepsilon)(\check{U}^2)\|_{L^2(\Omega)^6} = \mathcal{O}(\varepsilon^{1/2}). \tag{6.10}$$

6.3 Error estimates for the scaled stress tensor

Following [4], we define the scaled stress tensor $\sigma(\varepsilon) = \big(\sigma_{kl}(\varepsilon)\big)_{1\leq k,l\leq 3}$ by

$$\sigma_{\alpha\beta} = A_{\alpha\beta ij}\kappa_{ij}(\varepsilon)(u(\varepsilon)), \quad \sigma_{\alpha 3} = \varepsilon^{-1}\, A_{\alpha 3ij}\kappa_{ij}(\varepsilon)(u(\varepsilon)), \quad \sigma_{33} = \varepsilon^{-2}\, A_{33ij}\kappa_{ij}(\varepsilon)(u(\varepsilon)).$$

Thus, the equation (1.6b) is equivalent to $\partial_k\sigma_{kl}(\varepsilon) = f_l$ in Ω and $n_k\sigma_{kl}(\varepsilon) = 0$ on $\omega \times \{\pm 1\}$. We have

$$\begin{aligned}
\sigma_{\alpha\beta}(\varepsilon) &= \varepsilon^{-2}\, A_{\alpha\beta 33}\, e_{33}(u(\varepsilon)) + A_{\alpha\beta\gamma\delta}\, e_{\gamma\delta}(u(\varepsilon)), && \text{(6.11a)}\\
\sigma_{\alpha 3}(\varepsilon) &= 2\varepsilon^{-2}\, A_{\alpha 3\gamma 3}\, e_{\gamma 3}(u(\varepsilon)), && \text{(6.11b)}\\
\sigma_{33}(\varepsilon) &= \varepsilon^{-4}\, A_{3333}\, e_{33}(u(\varepsilon)) + \varepsilon^{-2}\, A_{\gamma\delta 33}\, e_{\gamma\delta}(u(\varepsilon)). && \text{(6.11c)}
\end{aligned}$$

Inserting the asymptotic expansion (1.8) into each term of (6.11), we obtain

$$\begin{aligned}
\sigma_{\alpha\beta}(\varepsilon) &= \sum_{k\geq 0} \varepsilon^k\, \sigma^k_{\alpha\beta} + \chi(r) \sum_{k\geq 0} \varepsilon^k\, \Xi^k_{\alpha\beta}(\frac{r}{\varepsilon}, s, x_3), && \text{(6.12a)}\\
\sigma_{\alpha 3}(\varepsilon) &= \sum_{k\geq 0} \varepsilon^k\, \sigma^k_{\alpha 3} + \chi(r) \sum_{k\geq -1} \varepsilon^k\, \Xi^k_{\alpha 3}(\frac{r}{\varepsilon}, s, x_3), && \text{(6.12b)}\\
\sigma_{33}(\varepsilon) &= \sum_{k\geq 0} \varepsilon^k\, \sigma^k_{33} + \chi(r) \sum_{k\geq -2} \varepsilon^k\, \Xi^k_{33}(\frac{r}{\varepsilon}, s, x_3), && \text{(6.12c)}
\end{aligned}$$

where the σ^k_{ij} are smooth tensors and the Ξ^k_{ij} are exponentially decreasing tensors. The expression of the first terms σ^0_{ij} is similar to that given in [4, Ch.3]. We note the strong influence of boundary layer terms which arise at the degree 0 for $\sigma_{\alpha\beta}$, the degree -1 for $\sigma_{\alpha 3}$, and the degree -2 for σ_{33}.

Combining (6.12) with the structure of the boundary layer terms in the neighborhood of the edges of the plate, we arrive to the (generically optimal) estimates

$$\|\sigma_{\alpha\beta}(\varepsilon) - \sigma^0_{\alpha\beta}\|_{L^2(\Omega)} \leq C\,\varepsilon^{1/2} \tag{6.13a}$$

$$\|\sigma_{\alpha 3}(\varepsilon) - \sigma^0_{\alpha 3}\|_{H^1\left((-1,+1),\,H^{-1}(\omega)\right)} \leq C\,\varepsilon^{1/2} \tag{6.13b}$$

$$\|\sigma_{33}(\varepsilon) - \sigma^0_{33}\|_{H^2\left((-1,+1),\,H^{-2}(\omega)\right)} \leq C\,\varepsilon^{1/2}. \tag{6.13c}$$

We can also see that, in a generic way:

$$\|\sigma_{\alpha 3}(\varepsilon)\|_{L^2(\Omega)} \geq C\varepsilon^{-1/2} \quad \text{and} \quad \|\sigma_{33}(\varepsilon)\|_{L^2(\Omega)} \geq C\varepsilon^{-3/2}. \tag{6.14}$$

All these estimates are in accordance with the results quoted in [4, Ch.3].

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