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ASYMPTOTICS OF ORTHONORMAL POLYNOMIALS IN THE PRESENCE OF A DENUMERABLE SET OF MASS POINTS

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ABSTRACT. Let σ be a positive measure whose support is an interval E plus a denumerable set of mass points which accumulate at the boundary points of E only. Under the assumptions that the mass points satisfy Blaschke's condition and that the absolutely continuous part of σ satisfies Szegö's condition, asymptotics for the orthonormal polynomials on and off the support are given. So far asymptotics were only available if the set of mass points is finite.

1. INTRODUCTION

Henceforth in this paper let $E = [-2, 2]$ and let σ be a measure which has a decomposition of the form

$$\sigma = \mu + \nu = \mu_{a.c.} + \mu_{s.} + \nu, \tag{1.1}$$

where μ is a measure with $\operatorname{supp}(\mu_{a.c.}) = [-2, 2]$ and $\operatorname{supp}(\mu_{s.}) \subset [-2, 2]$ and ν is a point measure supported on $X = \{x_k\} \subset \mathbb{R} \setminus [-2, 2]$, where the accumulation points of X are boundary points of E. As usual, $\mu_{a.c.}$ denotes the absolutely continuous part of μ and $\mu_{s.}$ the singular part. By $P_n(x) = P_n(x, \sigma)$ we denote the polynomial of degree n orthonormal with respect to σ, i.e.:

$$\int P_n(x)P_m(x)\, d\sigma(x) = \delta_{n,m}. \tag{1.2}$$

It is well known that $\{P_n\}$ satisfies a three–term recurrence relation

$$zP_n(z) = p_nP_{n-1}(z) + q_nP_n(z) + p_{n+1}P_{n+1}(z), \quad n = 1, 2, \ldots, \tag{1.3}$$

with initial data

$$p_0P_0(z) = 1, \quad zP_0(z) = q_0P_0(z) + p_1P_1(z).$$

One of the main problems is to find an explicit or at least an asymptotic representation of the orthonormal polynomials, of the minimum deviation $\prod_{j=0}^{n} p_j =: 1/r_n$ and the recurrence coefficients. For the special case that $\sigma = \mu_{a.c.}$ and that

$$\int_{-2}^{2} |\log \mu'_{a.c.}(x)| \frac{dx}{\sqrt{4 - x^2}} < \infty, \tag{1.4}$$

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condition (1.4) is nowadays called Szegö's condition; Szegö has given such asymptotic representations. In the forties Kolmogorov and Krein have shown that Szegö's asymptotic formulas hold true if an arbitrary singular measure μ_s with supp $(\mu_{s.}) \subset E$ is added (see e.g. [7], [2]). Another extension of Szegö's results has been given by Gonchar [3], Nevai [4] and Nikishin [6], who found asymptotics for the case that supp $(\nu) \subset \mathbb{R} \setminus E$ is a finite set and $\mu_{a.c.}$ satisfies Szegö's condition. In this paper we derive asymptotics for measures σ of the form (1.1) under the assumptions that $\mu_{a.c.}$ satisfies Szegö's condition and that the mass points $\{x_k\}$ satisfy Blaschke's condition, i.e.,

$$\sum_{x_k \in X} \sqrt{x_k^2 - 4} < \infty. \tag{1.5}$$

As an easy consequence of our results we obtain that $\lim p_n = 1$ and $\lim q_n = 0$. In this connection it might be worth mentioning that $\lim p_n = 1$ and $\lim q_n = 0$ imply, by Weyl's Theorem on compact perturbations, that $\text{supp}(\sigma) = E \cup X$, where X is a finite or countable set of points outside E which can accumulate at the boundary points of this interval only; σ denotes the orthogonality measure associated with the recurrence coefficients $\{p_n\}$ and $\{q_n\}$. If

$$\sum_{n=0}^{\infty} n(|p_n - 1| + |q_n|) < \infty,$$

then X is finite (see [1]).

As usual it is convenient to transform the problem to the unit circle $\mathbb{T}$. Therefore let us put $z = \zeta + 1/\zeta$, $\zeta \in \mathbb{D}$

$$B(\zeta) = \prod_{\zeta_k \in Z} \frac{\zeta_k - \zeta}{1 - \zeta_k \zeta} \frac{|\zeta_k|}{\zeta_k}, \quad Z = \left\{ \zeta_k \in \mathbb{D}, \zeta_k = \frac{x_k - \sqrt{x_k^2 - 4}}{2} \right\}. \tag{1.6}$$

Then condition (1.5) becomes the standard Blaschke condition

$$\sum_{\zeta_k \in Z} (1 - |\zeta_k|^2) < \infty.$$

Further let the transformed measure $\tilde{\mu}$ be given by

$$\int_{-2}^{2} F(x)\, d\mu(x) = \int_{\mathbb{T}} F(z(t))\, d\tilde{\mu}(t). \tag{1.7}$$

Since $\mu_{a.c.}$ satisfies Szegö's condition it follows that $\tilde{\mu}'_{a.c.}$ has a representation of the form

$$\tilde{\mu}'_{a.c.} = \mu'_{a.c.}(2\cos\phi)\pi|2\sin\phi| = |D(t)|^2 \quad \text{for a.e. } t = e^{i\phi} \in \mathbb{T},$$

where

$$\log D(\zeta) = \frac{1}{2} \int_{\mathbb{T}} \frac{t + \zeta}{t - \zeta} \log\{\tilde{\mu}'_{a.c.}(t)\}\, dm(t) \tag{1.8}$$

and dm denotes the Lebesgue measure on $\mathbb{T}$. Note that (1.7) becomes

$$\int_{-2}^{2} F(x)\, d\mu(x) = \int_{\mathbb{T}} F(z(t))|D(t)|^2\, dm(t) + \int_{\mathbb{T}} F(z(t))\, d\tilde{\mu}_{s.}(t).$$

2. The result

Theorem. *Let a measure $\sigma = \mu + \nu$ satisfy* (1.4) *and* (1.5)*. Associate with the measure σ the functions $B(\zeta)$ and $D(\zeta)$ by* (1.6), (1.8)*. Then the minimum deviation and the orthonormal polynomials $P_n(z,\sigma) = r_n(\sigma)z^n + \dots$ have the following asymptotic behavior ($n \to \infty$):*

$$\frac{r_n D(0)}{B(0)} \to \frac{1}{\sqrt{2}}, \tag{2.1}$$

$$D(t)P_n(z(t)) = \frac{t^{-n}B(t) + t^n B(\bar{t})D(t)/D(\bar{t})}{\sqrt{2}} + o(1) \text{ in } L^2, \tag{2.2}$$

$$\|P_n\|_{L^2_{d\sigma_{s.}}} = \left\{ \sum_{x_k \in X} |P_n(x_k)|^2 \nu_k + \int_{-2}^{2} |P_n(x)|^2 \, d\mu_{s.}(x) \right\}^{1/2} \to 0, \tag{2.3}$$

and

$$P_n(z(\zeta))\zeta^n = \frac{B(\zeta)}{\sqrt{2}D(\zeta)} + o(1) \tag{2.4}$$

uniformly on compact subsets of $\mathbb{D}$.

The proof of the theorem will be divided into several steps. The main part deals with the statement (2.1). First we show an upper estimate.

Lemma. *Under the assumptions of the previous Theorem, we have*

$$\varlimsup_{n\to\infty} \frac{r_n D(0)}{B(0)} \le \frac{1}{\sqrt{2}}. \tag{2.5}$$

Furthermore, (2.1) *implies* (2.2) *and* (2.3)*.*

Proof. Put $s(t) = D(t)/D(\bar{t})$ and consider the norm of the following function:

$$\begin{aligned} &\left\| D(t)P_n(z(t)) - \frac{t^{-n}B(t) + t^n s(t)B(\bar{t})}{\sqrt{2}} \right\|_{L^2}^2 + \int |P_n(x)|^2 \, d\sigma_{s.}(x) \\ &= \int_{\mathbb{T}} |P_n(z(t))|^2 |D(t)|^2 \, dm(t) + \sum_{x_k \in X} |P_n(x_k)|^2 \nu_k + \int_{-2}^{2} |P_n(x)|^2 \, d\mu_{s.}(x) \\ &+ \frac{1}{2} \left\| t^{-n}B(t) + t^n s(t)B(\bar{t}) \right\|_{L^2}^2 - 2\mathrm{Re} \left\langle D(t)P_n(z(t)), \frac{t^{-n}B(t) + t^n s(t)B(\bar{t})}{\sqrt{2}} \right\rangle. \end{aligned} \tag{2.6}$$

To prove (2.5) we only use the fact that this norm is non–negative. From the estimate we get it follows immediately that (2.1) implies (2.2) and (2.3).

First of all

$$\begin{aligned} &\int_{\mathbb{T}} |P_n(z(t))|^2 |D(t)|^2 \, dm(t) + \int_{-2}^{2} |P_n(x)|^2 \, d\mu_{s.}(x) \\ &\qquad + \sum_{x_k \in X} |P_n(x_k)|^2 \nu_k = \|P_n\|^2_{L^2_{d\sigma}} = 1. \end{aligned} \tag{2.7}$$

Since

$$\left\langle t^{-n}B(t), t^n s(t)B(\bar{t}) \right\rangle \to 0 \quad n \to \infty,$$

we have

$$\frac{1}{2}\left\|t^{-n}B(t)+t^{n}s(t)B(\bar t)\right\|^2_{L^2}=1+o(1),\quad n\to\infty. \tag{2.8}$$

Now we note that $D(t)P_n(z(t))$ possesses the following symmetry property:

$$s(t)D(\bar t)P_n(z(\bar t))=D(t)P_n(z(t)).$$

Therefore, it is orthogonal to any function of the form $g(t)-s(t)g(\bar t)$. Thus,

$$\left\langle D(t)P_n(z(t)),\frac{t^{-n}B(t)+t^{n}s(t)B(\bar t)}{\sqrt{2}}\right\rangle=\sqrt{2}\left\langle D(t)P_n(z(t)),t^{-n}B(t)\right\rangle.$$

Let $\{Z_N\}_N$ be an exhaustion of Z by finite subsets and let B_N be the finite Blaschke product with zeros Z_N. Then

$$\begin{aligned}\left\langle D(t)P_n(z(t)),t^{-n}B(t)\right\rangle\\ =\left\langle D(t)P_n(z(t)),t^{-n}(B(t)-B_N(t))\right\rangle+\left\langle D(t)P_n(z(t)),t^{-n}B_N(t)\right\rangle\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&\left|\left\langle D(t)P_n(z(t)),t^{-n}(B(t)-B_N(t))\right\rangle\right|\\ &\qquad\le\|D(t)P_n(z(t))\|\|B(t)-B_N(t)\|\le\|B(t)-B_N(t)\|.\end{aligned}$$

To evaluate the second term in (2.9) we apply the Cauchy Theorem:

$$\begin{aligned}\langle D(t)P_n(z(t)),t^{-n}B_N(t)\rangle&=\int_{\mathbb{T}}\frac{D(t)P_n(z(t))}{B_N(t)}t^n\frac{dt}{2\pi it}\\ &=\frac{D(0)}{B_N(0)}r_n+\sum\frac{D(\zeta_k)}{B_N'(\zeta_k)}P_n(x_k)\zeta_k^{n-1}.\end{aligned}$$

For the last term we have an estimate

$$\left|\sum\frac{D(\zeta_k)}{B_N'(\zeta_k)}P_n(x_k)\zeta_k^{n-1}\right|^2\le\left\{\sum_{\zeta_k\in Z_N}\left|\frac{D(\zeta_k)}{B_N'(\zeta_k)}\right|^2\frac{|\zeta_k^{n-1}|^2}{\nu_k}\right\}\left\{\sum_{x_k\in X}|P_n(x_k)|^2\nu_k\right\}.$$

So, first choosing N big enough and then n we conclude that

$$\left\langle D(t)P_n(z(t)),\frac{t^{-n}B(t)+t^{n}s(t)B(\bar t)}{\sqrt{2}}\right\rangle=\sqrt{2}\frac{D(0)}{B(0)}r_n+o(1). \tag{2.10}$$

Substituting (2.7), (2.8), (2.10) in (2.6) we obtain the final result

$$\begin{aligned}1+1-2\sqrt{2}\frac{D(0)}{B(0)}r_n+o(1)&=\left\|D(t)P_n(z(t))-\frac{t^{-n}B(t)+t^{n}s(t)B(\bar t)}{\sqrt{2}}\right\|^2_{L^2}\\ &\quad+\int_{-2}^{2}|P_n(x)|^2\,d\mu_{s.}(x)+\sum_{x_k\in X}|P_n(x_k)|^2\nu_k\ge 0.\end{aligned}$$

□

Remark. The proof of the Lemma shows that $r_n\to 0$ as $n\to\infty$ if

$$\sum_{x_k\in X}\sqrt{x_k^2-4}=\infty$$

and if (1.4) is satisfied.

Proof of (2.1) *and* (2.4). As usual, first we assume that the weight function $|D|$ is bounded from below. To simplify notation, assume $|D| \geq 2$. Let $|D_\epsilon|$ be a smooth function such that $|D_\epsilon| \geq 1$ and

$$\int_{\mathbb{T}} ||D|^2 - |D_\epsilon|^2|\, dm < \epsilon \quad (\epsilon > 0). \tag{2.11}$$

Now, for $\eta > 0$ with $\max |D_\epsilon(t)| \leq 1/\eta$, we choose a finite system of intervals $E_{s.} \subset \mathbb{T}$ such that

$$\int_{\mathbb{T}\setminus E_{s.}} d\tilde{\mu}_{s.} \leq \eta, \quad |E_{s.}| = \int_{E_{s.}} dm \leq \eta. \tag{2.12}$$

Let $\tilde{E}_{s.}$ with $|\tilde{E}_{s.}| \leq 2\eta$ be another system of intervals which arises by proper extension of each interval of $E_{s.}$. We also fix vicinities of ± 1 of the form

$$E_\pm = \{t \in \mathbb{T}, |t \pm 1|^2 \leq \eta/2\}, \quad \tilde{E}_\pm = \{t \in \mathbb{T}, |t \pm 1|^2 \leq \eta\}.$$

Let us define a smooth function $|F_{\epsilon,\eta}(t)|$ in the following way. It coincides with $1/|D_\epsilon(t)|$ on $\mathbb{T} \setminus (\tilde{E}_{s.} \cup \tilde{E}_+ \cup \tilde{E}_-)$ and equals to η on $E_{s.} \setminus (\tilde{E}_+ \cup \tilde{E}_-)$. Further, it coincides with $|t \pm 1|^2$, when $t \in E_\pm$, and is such that

$$|t \pm 1|^2 \leq |F_{\epsilon,\eta}(t)| \leq \frac{1}{|D_\epsilon(t)|} \quad \text{for} \quad t \in \tilde{E}_\pm \setminus E_\pm,$$

$$\eta \leq |F_{\epsilon,\eta}(t)| \leq \frac{1}{|D_\epsilon(t)|} \quad \text{for} \quad t \in \tilde{E}_{s.} \setminus (E_{s.} \cup \tilde{E}_+ \cup \tilde{E}_-).$$

Hence, by the above settings

$$\begin{aligned} 0 \leq \log \frac{1}{F_{\epsilon,\eta}(0)} - \log D_\epsilon(0) &\leq \int_{(\tilde{E}s. \cup \tilde{E}_+ \cup \tilde{E}_-)} \log \left| \frac{1}{F_\epsilon(t) D_\epsilon(t)} \right| dm \\ \leq \int_{\tilde{E}_+} \log \frac{1}{|t-1|^2}\, dm &+ \int_{\tilde{E}_-} \log \frac{1}{|t+1|^2}\, dm + \int_{\tilde{E}_{s.}} \log \frac{1}{\eta}\, dm \\ = o(1), \quad \eta \to 0. \end{aligned}$$

So, taking into account (2.11),

$$F_{\epsilon,\eta}(0) = 1/D(0) + o(1), \quad \eta \to 0, \epsilon \to 0. \tag{2.13}$$

Further, we note that the Blaschke product oscillates only in vicinities of the points ± 1, moreover,

$$\sup\{|B'(t)||t^2 - 1|^2, t \in \mathbb{T}\} < \infty.$$

Therefore, $(BF_{\epsilon,\eta})' = B' F_{\epsilon,\eta} + BF'_{\epsilon,\eta} \in L^\infty$, and the Fourier series of $BF_{\epsilon,\eta}$ converges to this function uniformly on $\mathbb{T}$. Let

$$(BF_{\epsilon,\eta})(t) = Q_{n,\epsilon,\eta}(t) + t^{n+1} g_{n,\epsilon,\eta}(t),\ g_{n,\epsilon,\eta}(t) \in H^\infty,$$

where

$$Q_{n,\epsilon,\eta}(t) = q_{0,\epsilon,\eta} + \cdots + q_{n,\epsilon,\eta} t^n.$$

We claim that the polynomial

$$P_{n,\epsilon,\eta}(z) = \frac{\zeta^{-n} Q_{n,\epsilon,\eta}(\zeta) + \zeta^n Q_{n,\epsilon,\eta}(1/\zeta)}{\sqrt{2}}$$

is a suitable approximation to the extremal one, when η and ϵ are small and n is big.

Let us estimate the norm of the given polynomial. For the absolutely continuous part of the measure we have

$$\|D_\epsilon(t)P_{n,\epsilon,\eta}(z(t))\|_{L^2} \leq \left\| D_\epsilon(t)\frac{F_{\epsilon,\eta}(t)B(t)t^{-n}+F_{\epsilon,\eta}(\bar t)B(\bar t)t^n}{\sqrt{2}}\right\| + \left\| D_\epsilon(t)\frac{tg_{n,\epsilon,\eta}(t)+\bar t g_{n,\epsilon,\eta}(\bar t)}{\sqrt{2}}\right\|.$$

Since

$$\|g_{n,\epsilon,\eta}\|_{L^\infty}\to 0, \quad \langle D_\epsilon(t)F_{\epsilon,\eta}(t)B(t)t^{-n}, D_\epsilon(t)F_{\epsilon,\eta}(\bar t)B(\bar t)t^n\rangle \to 0\ (n\to\infty),$$

and $|D_\epsilon(t)F_{\epsilon,\eta}(t)|\leq 1$, we have $\|D_\epsilon(t)P_{n,\epsilon,\eta}(z(t))\|_{L^2}\leq 1+o(1)$. Since $P_{n,\epsilon,\eta}$ is uniformly bounded, using (2.11), we get

$$\|D(t)P_{n,\epsilon,\eta}(z(t))\|_{L^2}\leq 1+C\epsilon+o(1). \tag{2.14}$$

For the singular measure $\mu_{s.}$ we have

$$\begin{aligned}\|P_{n,\epsilon,\eta}\|_{L^2_{\tilde\mu}} &= \|P_{n,\epsilon,\eta}(z(t))\|_{L^2_{\tilde\mu_{s.}}}\\ &\leq \left\|\frac{F_{\epsilon,\eta}(t)B(t)t^{-n}+F_{\epsilon,\eta}(\bar t)B(\bar t)t^n}{\sqrt{2}}\right\|_{L^2_{\tilde\mu_{s.}}} + \left\|\frac{tg_{n,\epsilon,\eta}(t)+\bar t g_{n,\epsilon,\eta}(\bar t)}{\sqrt{2}}\right\|_{L^2_{\tilde\mu_{s.}}}\\ &\leq \sqrt{2}\,\|F_{\epsilon,\eta}(t)\|_{L^2_{\tilde\mu_{s.}}} + \sqrt{2}\,\|g_{n,\epsilon,\eta}(t)\|_{L^2_{\tilde\mu_{s.}}}.\end{aligned}$$

Due to (2.12),

$$\begin{aligned}\int_{\mathbb{T}}|F_{\epsilon,\eta}(t)|^2\,d\mu_{s.} &\leq \int_{E_{s.}}|F_{\epsilon,\eta}(t)|^2\,d\mu_{s.} + \int_{\mathbb{T}\setminus E_{s.}}|F_{\epsilon,\eta}(t)|^2\,d\mu_{s.}\\ &\leq \eta^2\int_{\mathbb{T}}d\mu_{s.}+\eta.\end{aligned}$$

Again using $\|g_{n,\epsilon,\eta}\|_{L^\infty}\to 0$, $n\to\infty$, we get

$$\|P_{n,\epsilon,\eta}(z(t))\|_{L^2_{\tilde\mu_{s.}}} = C\sqrt{\eta}+o(1), \quad n\to\infty. \tag{2.15}$$

At last, for the discrete measure ν we have

$$\sqrt{2}\|P_{n,\epsilon,\eta}(z)\|_{L^2_{d\nu}} \leq \left\{\sum|\zeta_k^{-n}Q_{n,\epsilon,\eta}(\zeta_k)|^2\nu_k\right\}^{1/2} + \left\{\sum|\zeta_k^{n}Q_{n,\epsilon,\eta}(1/\zeta_k)|^2\nu_k\right\}^{1/2}.$$

Since $B(\zeta_k)=0$, we are able to rewrite the first term in the form

$$\left\{\sum|\zeta_k^{-n}(Q_{n,\epsilon,\eta}(\zeta_k)-B(\zeta_k)F_{\epsilon,\eta}(\zeta_k))|^2\nu_k\right\}^{1/2} \leq \left\{\sum|\zeta_k g_{n,\epsilon,\eta}(\zeta_k)|^2\nu_k\right\}^{1/2}\to 0.$$

To show that the second term also tends to 0, we note that the sequence of polynomials

$$\zeta^n Q_{n,\epsilon,\eta}(1/\zeta) = q_{0,\epsilon,\eta}\zeta^n+\cdots+q_{n,\epsilon,\eta}$$

is bounded uniformly on $\bar{\mathbb{D}}$, since $|Q_{n,\epsilon,\eta}(t)|=|t^nQ_{n,\epsilon,\eta}(1/t)|$ is bounded uniformly on $\mathbb{T}$. So it is enough to show that $\zeta_k^nQ_{n,\epsilon,\eta}(1/\zeta_k)$ tends to 0 when k is fixed. Let $\delta>0$, choose n_0 such that $|\zeta_k|^{n_0}\leq\delta$ and m_0 such that

$$\left\{\sum_{m_0+1}^{\infty}|q_{j,\epsilon,\eta}|^2\right\}^{1/2}\leq\delta.$$

Then, for $n \geq n_0 + m_0$ we have

$$|q_{0,\epsilon,\eta}\zeta_k^n + \cdots + q_{n-n_0,\epsilon,\eta}\zeta_k^{n_0}| \leq \frac{|\zeta_k^{n_0}|}{\sqrt{1-|\zeta_k|^2}} \left\{ \sum_0^\infty |q_{j,\epsilon,\eta}|^2 \right\}^{1/2} \leq C(\zeta_k)\delta$$

and

$$|q_{n-n_0+1,\epsilon,\eta}\zeta_k^{n_0-1} + \cdots + q_{n,\epsilon,\eta}| \leq \left\{ \sum_{m_0+1}^\infty |q_{j,\epsilon,\eta}|^2 \right\}^{1/2} \frac{1}{\sqrt{1-|\zeta_k|^2}} \leq C(\zeta_k)\delta.$$

Thus $\zeta_k^n Q_{n,\epsilon,\eta}(1/\zeta_k)$ indeed tends to 0 as $n \to \infty$.

Using the extremal property of the orthonormal polynomial and the fact that

$$P_{n,\epsilon,\eta}(z) = \frac{(BF_{\epsilon,\eta})(0)}{\sqrt{2}} z^n + \dots,$$

we get with the help of (2.14), (2.15)

$$r_n \geq \frac{B(0)F_{\epsilon,\eta}(0)/\sqrt{2}}{\|P_{n,\epsilon,\eta}\|_{L^2_{d\sigma}}} \geq \frac{B(0)F_{\epsilon,\eta}(0)/\sqrt{2}}{1 + C\epsilon + C\sqrt{\eta} + o(1)}, \quad n \to \infty.$$

Therefore, recall (2.13),

$$\overline{\lim_{n\to\infty}}\, r_n \geq \frac{B(0)}{\sqrt{2}D(0)}.$$

To get rid of the assumption that $|D|$ is bounded from below we use the following standard trick. For given $D(t)$ define $|D_\epsilon(t)|^2 = |D(t)|^2 + \epsilon^2,\ t \in \mathbb{T}$ ($\epsilon > 0$) and

$$\int |P(x)|^2\, d\sigma_\epsilon(x) = \int_{\mathbb{T}} |P(z(t))|^2 |D_\epsilon(t)|^2\, dm + \int |P(x)|^2\, d\sigma_{s.}(x).$$

Note that $|D_\epsilon(t)|$ is bounded from below. Since $\|P\|_{L^2_{d\sigma}} \leq \|P\|_{L^2_{d\sigma_\epsilon}}$, we have

$$r_n(\sigma) \geq \frac{r_n(\sigma_\epsilon)}{\|P_n(x,\sigma_\epsilon)\|_{L^2_{d\sigma}}} \geq \frac{r_n(\sigma_\epsilon)}{\|P_n(x,\sigma_\epsilon)\|_{L^2_{d\sigma_\epsilon}}} \geq r_n(\sigma_\epsilon).$$

Therefore,

$$\underline{\lim_{n\to\infty}}\, r_n(\sigma) \geq \frac{B(0)}{\sqrt{2}D_\epsilon(0)}.$$

Since $D_\epsilon(0) \to D(0),\ \epsilon \to 0$, using (2.5) we get

$$\frac{B(0)}{\sqrt{2}D(0)} \geq \overline{\lim_{n\to\infty}}\, r_n(\sigma) \geq \underline{\lim_{n\to\infty}}\, r_n(\sigma) \geq \frac{B(0)}{\sqrt{2}D(0)},$$

and (2.1) is proved.

We derive (2.4) from (2.2) in a standard way. First of all,

$$\left| \left\langle D(t)P_n(z(t)) - \frac{t^{-n}B(t) + t^n s(t) B(\bar{t})}{\sqrt{2}}, \frac{t^{-n}}{1 - t\bar{\zeta}} \right\rangle \right| \leq \left\| D(t)P_n(z(t)) - \frac{t^{-n}B(t) + t^n s(t) B(\bar{t})}{\sqrt{2}} \right\|_{L^2} \frac{1}{\sqrt{1-|\zeta|^2}}.$$

Since

$$\left\langle D(t)P_n(z(t)) - \frac{t^{-n}B(t)}{\sqrt{2}}, \frac{t^{-n}}{1-t\bar{\zeta}} \right\rangle = \zeta^n D(\zeta)P_n(z(\zeta)) - \frac{B(\zeta)}{\sqrt{2}}$$

and

$$\left\langle \frac{s(t)t^{2n}B(\bar{t})}{\sqrt{2}}, \frac{1}{1-t\bar{\zeta}} \right\rangle \to 0, \quad n \to \infty,$$

uniformly on compact subsets in $\mathbb{D}$, assertion (2.4) is also proved. □

Remark. In this paper we were interested in an analogue of Szegö's Theorem, that is, in asymptotics for the orthonormal polynomials on and off the interval $[-2, 2]$, when infinitely many mass points appear outside the interval. We have been informed by the referee that there is an interesting conjecture by P. Nevai [4, Conjecture 2.7] related to the problem considered in this paper. He conjectured, without any rate at which the masspoints are allowed to converge to -2 or 2, that for measures of the type (1.1) with $\mu_{a.c.} > 0$ a.e. on $[-2, 2]$ the associated Jacobi matrix is a compact perturbation of a constant Jacobi matrix. Thus a positive answer would give a generalization of Rakmanov's Theorem.

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