ASYMPTOTICS OF PLANCHEREL MEASURES FOR SYMMETRIC GROUPS

ALEXEI BORODIN, ANDREI OKOUNKOV, AND GRIGORI OLSHANSKI

1. INTRODUCTION

1.1. **Plancherel measures.** Given a finite group G, by the corresponding Plancherel measure we mean the probability measure on the set $G^\wedge$ of irreducible representations of G which assigns to a representation $\pi \in G^\wedge$ the weight $(\dim \pi)^2/|G|$. For the symmetric group $S(n)$, the set $S(n)^\wedge$ is the set of partitions λ of the number n, which we shall identify with Young diagrams with n squares throughout this paper. The Plancherel measure on partitions λ arises naturally in representation–theoretic, combinatorial, and probabilistic problems. For example, the Plancherel distribution of the first part of a partition coincides with the distribution of the longest increasing subsequence of a uniformly distributed random permutation [31].

We denote the Plancherel measure on partitions of n by M_n,

$$(1.1) \qquad M_n(\lambda) = \frac{(\dim \lambda)^2}{n!}, \quad |\lambda| = n\,,$$

where $\dim \lambda$ is the dimension of the corresponding representation of $S(n)$. The asymptotic properties of these measures as $n \to \infty$ have been studied very intensively; see the References and below.

In the seventies, Logan and Shepp [23] and, independently, Vershik and Kerov [40, 42] discovered the following measure concentration phenomenon for M_n as $n \to \infty$. Let λ be a partition of n and let i and j be the usual coordinates on the diagrams, namely, the row number and the column number. Introduce new coordinates u and v by

$$u = \frac{j-i}{\sqrt{n}}\,, \quad v = \frac{i+j}{\sqrt{n}}\,,$$

that is, we flip the diagram, rotate it $135°$ as in Figure 1, and scale it by the factor of $n^{-1/2}$ in both directions.

After this scaling, the Plancherel measures M_n converge as $n \to \infty$ (see [23, 40, 42] for precise statements) to the delta measure supported on the following shape:

$$\{|u| \le 2, |u| \le v \le \Omega(u)\}\,,$$

where the function $\Omega(u)$ is defined by

$$\Omega(u) = \begin{cases} \frac{2}{\pi}\left(u\arcsin(u/2) + \sqrt{4-u^2}\right), & |u| \le 2\,, \\ |u|\,, & |u| > 2\,. \end{cases}$$
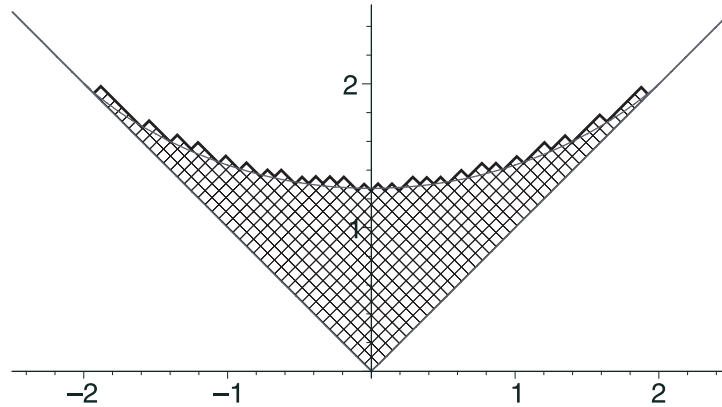
FIGURE 1. The limit shape of a typical diagram.

The function Ω is plotted in Figure 1. As explained in detail in [22], this limit shape Ω is closely connected to Wigner's semicircle law for distribution of eigenvalues of random matrices; see also [19, 20, 21].

From a different point of view, the connection with random matrices was observed in [3, 4], and also in earlier papers [16, 28, 29]. In [3], Baik, Deift, and Johansson made the following conjecture. They conjectured that in the $n \to \infty$ limit and after proper scaling the joint distribution of λ_i, $i = 1, 2, \dots$, becomes identical to the joint distribution of properly scaled largest eigenvalues of a Gaussian random Hermitian matrix (which form the so-called Airy ensemble; see Section 1.4). They proved this for the individual distribution of λ_1 and λ_2 in [3] and [4], respectively. A combinatorial proof of the full conjecture was given by one of us in [25]. It was based on an interplay between maps on surfaces and ramified coverings of the sphere.

In this paper we study the local structure of a typical Plancherel diagram both in the bulk of the limit shape Ω and on its edge, where by the study of the edge we mean the study of the behavior of λ_1, λ_2, and so on.

We employ an analytic approach based on an exact formula in terms of Bessel functions for the correlation functions of the so-called *poissonization* of the Plancherel measures M_n (see Theorem 1 in the following subsection), and the so-called *depoissonization* techniques (see Section 1.4).

The exact formula in Theorem 1 is a limit case of a formula from [8]; see also the recent papers [26, 27] for more general results. The use of poissonization and depoissonization is very much in the spirit of [3, 16, 39] and represents a well–known statistical mechanics principle of the equivalence of canonical and grand canonical ensembles.

Our main results are the following two. In the bulk of the limit shape Ω, we prove that the local structure of a Plancherel typical partition converges to a determinantal point process with the discrete sine kernel; see Theorem 3. This result is parallel to the corresponding result for random matrices. On the edge of the limit shape, we give an analytic proof of the Baik-Deift-Johansson conjecture; see Theorem 4. These results will be stated in Sections 1.3 and 1.4 of the present Introduction, respectively.

Simultaneously and independently, results equivalent to our Theorems 2 and 4 were obtained by K. Johansson [17].

1.2. **Poissonization and correlation functions.** For $\theta > 0$, consider the *poissonization* M^θ of the measures M_n:

$$M^\theta(\lambda) = e^{-\theta} \sum_n \frac{\theta^n}{n!} M_n(\lambda) = e^{-\theta} \theta^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 .$$

This is a probability measure on the set of all partitions. Our first result is the computation of the correlation functions of the measures M^θ.

By correlation functions we mean the following. By definition, set

$$\mathcal{D}(\lambda) = \{\lambda_i - i\} \subset \mathbb{Z} .$$

Also, following [41], define the *modified Frobenius coordinates* $\mathrm{Fr}(\lambda)$ of a partition λ by

$$(1.2) \qquad \begin{aligned} \mathrm{Fr}(\lambda) &= \left(\mathcal{D}(\lambda) + \tfrac{1}{2} \right) \triangle \left(\mathbb{Z}_{\leq 0} - \tfrac{1}{2} \right) \\ &= \left\{ p_1 + \tfrac{1}{2}, \ldots, p_d + \tfrac{1}{2}, -q_1 - \tfrac{1}{2}, \ldots, -q_d - \tfrac{1}{2} \right\} \subset \mathbb{Z} + \tfrac{1}{2} , \end{aligned}$$

where $\triangle$ stands for the symmetric difference of two sets, d is the number of squares on the diagonal of λ, and p_i's and q_i's are the usual Frobenius coordinates of λ. Recall that p_i is the number of squares in the ith row to the right of the diagonal, and q_i is number of squares in the ith column below the diagonal. The equality (1.2) is a well–known combinatorial fact discovered by Frobenius; see Ex. I.1.15(a) in [24]. Note that, in contrast to $\mathrm{Fr}(\lambda)$, the set $\mathcal{D}(\lambda)$ is infinite and, moreover, it contains all but finitely many negative integers.

The sets $\mathcal{D}(\lambda)$ and $\mathrm{Fr}(\lambda)$ have the following nice geometric interpretation. Let the diagram λ be flipped and rotated $135°$ as in Figure 1, but not scaled. Denote by ω_λ a piecewise linear function with $\omega'_\lambda = \pm 1$ whose graph is given by the upper boundary of λ completed by the lines

$$v = |u| , \quad u \notin [-\lambda'_1, \lambda_1] .$$

Then

$$k \in \mathcal{D}(\lambda) \Leftrightarrow \omega'_\lambda \big|_{[k,k+1]} = -1 .$$

In other words, if we consider ω_λ as a history of a walk on $\mathbb{Z}$, then $\mathcal{D}(\lambda)$ are those moments when a step is made in the negative direction. It is therefore natural to call $\mathcal{D}(\lambda)$ the *descent set* of λ. As we shall see, the correspondence $\lambda \mapsto \mathcal{D}(\lambda)$ is a very convenient way to encode the local structure of the boundary of λ.

The halves in the definition of $\mathrm{Fr}(\lambda)$ have the following interpretation: one splits the diagonal squares in half and gives half to the rows and half to the columns.

Definition 1.1. The correlation functions of M^θ are the probabilities that the sets $\mathrm{Fr}(\lambda)$ or, similarly, $\mathcal{D}(\lambda)$ contain a fixed subset X. More precisely, we set

$$(1.3) \qquad \rho^\theta(X) = M^\theta \left(\{\lambda \,|\, X \subset \mathrm{Fr}(\lambda)\} \right) , \quad X \subset \mathbb{Z} + \tfrac{1}{2} ,$$

$$(1.4) \qquad \varrho^\theta(X) = M^\theta \left(\{\lambda \,|\, X \subset \mathcal{D}(\lambda)\} \right) , \quad X \subset \mathbb{Z} .$$

Theorem 1. *For any* $X = \{x_1, \ldots, x_s\} \subset \mathbb{Z} + \tfrac{1}{2}$ *we have*

$$\rho^\theta(X) = \det \left[\mathsf{K}(x_i, x_j) \right]_{1 \leq i,j \leq s} ,$$

where the kernel K *is given by the following formula:*

$$(1.5) \qquad \mathsf{K}(x,y) = \begin{cases} \sqrt{\theta}\, \dfrac{\mathsf{k}_+(|x|,|y|)}{|x|-|y|}\,, & xy > 0\,, \\[2mm] \sqrt{\theta}\, \dfrac{\mathsf{k}_-(|x|,|y|)}{x-y}\,, & xy < 0\,. \end{cases}$$

The functions $\mathsf{k}_\pm$ *are defined by*

$$(1.6) \qquad \mathsf{k}_+(x,y) = J_{x-\frac{1}{2}}\, J_{y+\frac{1}{2}} - J_{x+\frac{1}{2}}\, J_{y-\frac{1}{2}}\,,$$

$$(1.7) \qquad \mathsf{k}_-(x,y) = J_{x-\frac{1}{2}}\, J_{y-\frac{1}{2}} + J_{x+\frac{1}{2}}\, J_{y+\frac{1}{2}}\,,$$

where $J_x = J_x(2\sqrt{\theta})$ *is the Bessel function of order* x *and argument* $2\sqrt{\theta}$. *The diagonal values* $\mathsf{K}(x,x)$ *are determined by the l'Hospital rule.*

This theorem is established in Section 2.1; see also Remark 1.2 below. By the complementation principle (see Sections A.3 and 2.2), Theorem 1 is equivalent to the following

Theorem 2. *For any* $X = \{x_1, \ldots, x_s\} \subset \mathbb{Z}$ *we have*

$$(1.8) \qquad \varrho^\theta(X) = \det\left[\mathsf{J}(x_i, x_j)\right]_{1 \le i,j \le s}.$$

Here the kernel J *is given by the following formula:*

$$(1.9) \qquad \mathsf{J}(x,y) = \mathsf{J}(x,y;\theta) = \sqrt{\theta}\, \frac{J_x\, J_{y+1} - J_{x+1}\, J_y}{x-y}\,,$$

where $J_x = J_x(2\sqrt{\theta})$. *The diagonal values* $\mathsf{J}(x,x)$ *are determined by the l'Hospital rule.*

Remark 1.2. Theorem 1 is a limit case of Theorem 3.3 of [8]. For the reader's convenience a direct proof of it is given in Section 2. Various limit cases of the results of [8] are discussed in [9]. By different methods, the formula (1.8) was obtained by K. Johansson [17].

A representation–theoretic proof of a more general formula than Theorem 3.3 of [8] has been subsequently given in [27, 26]; see also [7].

Remark 1.3. Observe that all Bessel functions involved in the above formulas are of integer order. Also note that the ratios like $\mathsf{J}(x,y)$ are entire functions of x and y because J_x is an entire function of x. In particular, the values $\mathsf{J}(x,x)$ are well defined. Various denominator–free formulas for the kernel J are given in Section 2.1.

1.3. **Asymptotics in the bulk of the spectrum.** Given a sequence of subsets

$$X(n) = \{x_1(n) < \cdots < x_s(n)\} \subset \mathbb{Z}\,,$$

where $s = |X(n)|$ is some fixed integer, we call this sequence *regular* if the limits

$$(1.10) \qquad a_i = \lim_{n\to\infty} \frac{x_i(n)}{\sqrt{n}}\,,$$

$$(1.11) \qquad d_{ij} = \lim_{n\to\infty} (x_i(n) - x_j(n))$$

exist, finite or infinite. Here $i, j = 1, \ldots, s$. Observe that if d_{ij} is finite, then $d_{ij} = x_i(n) - x_j(n)$ for $n \gg 0$.

In the case when $X(n)$ can be represented as $X(n) = X'(n) \cup X''(n)$ and the distance between $X'(n)$ and $X''(n)$ goes to ∞ as $n \to \infty$ we shall say that the sequence *splits*; otherwise, we call it *nonsplit*. Obviously, $X(n)$ is nonsplit if and only if all $x_i(n)$ stay at a finite distance from each other.

Define the correlation functions $\varrho(n, \cdot)$ of the measures M_n by the same rule as in (1.4):

$$\varrho(n, X) = M_n\left(\{\lambda \,|\, X \subset \mathcal{D}(\lambda)\}\right).$$

We are interested in the limit of $\varrho(n, X(n))$ as $n \to \infty$. This limit will be computed in Theorem 3 below. As we shall see, if $X(n)$ splits, then the limit correlations factor accordingly.

Introduce the following *discrete sine kernel* which is a translation invariant kernel on the lattice $\mathbb{Z}$,

$$\mathsf{S}(k, l; a) = \mathsf{S}(k - l, a), \quad k, l \in \mathbb{Z},$$

depending on a real parameter a:

$$\mathsf{S}(k, a) = \frac{\sin(\arccos(a/2) \cdot k)}{\pi k}, \quad k \in \mathbb{Z}.$$

Note that $\mathsf{S}(k, a) = \mathsf{S}(-k, a)$ and for $k \geq 1$ we have

$$\mathsf{S}(k, a) = \frac{\sqrt{4 - a^2}}{2\pi} \frac{U_{k-1}(a/2)}{k},$$

where U_k is the Tchebyshev polynomials of the second kind. We agree that

$$\mathsf{S}(0, a) = \frac{\arccos(a/2)}{\pi}, \quad \mathsf{S}(\infty, a) = 0$$

and also that

$$\mathsf{S}(k, a) = \begin{cases} 0, & a \geq 2 \text{ or } a \leq -2 \text{ and } k \neq 0, \\ 1, & a \leq -2 \text{ and } k = 0. \end{cases}$$

The following result describes the local structure of a Plancherel typical partition.

Theorem 3. *Let $X(n) \subset \mathbb{Z}$ be a regular sequence and let the numbers a_i, d_{ij} be defined by (1.10), (1.11). If $X(n)$ splits, that is, if $X(n) = X'(n) \cup X''(n)$ and the distance between $X'(n)$ and $X''(n)$ goes to ∞ as $n \to \infty$, then*

$$(1.12) \qquad \lim_{n \to \infty} \varrho(n, X(n)) = \lim_{n \to \infty} \varrho(n, X'(n)) \cdot \lim_{n \to \infty} \varrho(n, X''(n)).$$

If $X(n)$ is nonsplit, then

$$(1.13) \qquad \lim_{n \to \infty} \varrho(n, X(n)) = \det\left[\mathsf{S}(d_{ij}, a)\right]_{1 \leq i, j \leq s},$$

where S is the discrete sine kernel and $a = a_1 = a_2 = \ldots$.

We prove this theorem in Section 3.

Remark 1.4. Notice that, in particular, Theorem 3 implies that, as $n \to \infty$, the shape of a typical partition λ near any point of the limit curve Ω is described by a stationary random process. For distinct points on the curve Ω these random processes are independent.

Remark 1.5. By complementation (see Sections A.3 and 3.2), one obtains from Theorem 3 an equivalent statement about the asymptotics of the following correlation functions:

$$\rho(n, X) = M_n \left(\{ \lambda \,|\, X \subset \text{Fr}(\lambda) \} \right) .$$

Remark 1.6. The discrete sine kernel was studied before (see [44, 45]), mainly as a model case for the continuous sine kernel. In particular, the asymptotics of Toeplitz determinants built from the discrete sine kernel was obtained by H. Widom [45] who was answering a question of F. Dyson. We thank S. Kerov for pointing out this reference.

Remark 1.7. Note that, in particular, Theorem 3 implies that the limit density (the 1-point correlation function) is given by

(1.14) $$\boldsymbol{\varrho}(\infty, a) = \begin{cases} \frac{1}{\pi} \arccos(a/2) \,, & |a| \le 2 \,, \\ 0 \,, & a > 2 \,, \\ 1 \,, & a < -2 \,. \end{cases}$$

This is in agreement with the Logan-Shepp-Vershik-Kerov result about the limit shape Ω. More concretely, the function Ω is related to the density (1.14) by

$$\boldsymbol{\varrho}(\infty, u) = \frac{1 - \Omega'(u)}{2} \,,$$

which can be interpreted as follows. Approximately, we have

$$\# \left\{ i \;\middle|\; \frac{\lambda_i - i}{\sqrt{n}} \in [u, u + \Delta u] \right\} \approx \sqrt{n} \, \boldsymbol{\varrho}(\infty, u) \, \Delta u \,.$$

Set $w = \dfrac{i}{\sqrt{n}}$. Then the above relation reads $\Delta w \approx \boldsymbol{\varrho}(\infty, u) \, \Delta u$ and it should be satisfied on the boundary $v = \Omega(u)$ of the limit shape. Since $v = u + 2w$, we conclude that

$$\boldsymbol{\varrho}(\infty, u) \approx \frac{dw}{du} = \frac{1 - \Omega'}{2} \,,$$

as was to be shown.

Remark 1.8. The discrete sine kernel S becomes especially nice near the diagonal, that is, where $a = 0$. Indeed,

$$\mathsf{S}(x, 0) = \begin{cases} 1/2 \,, & x = 0 \,, \\ (-1)^{(x-1)/2} \big/ (\pi x) \,, & x = \pm 1, \pm 3, \dots \,, \\ 0 \,, & x = \pm 2, \pm 4, \dots \,. \end{cases}$$

1.4. **Behavior near the edge of the spectrum and the Airy ensemble.** The discrete sine kernel $\mathsf{S}(k, a)$ vanishes if $a \ge 2$. Therefore, it follows from Theorem 3 that the limit correlations $\lim \boldsymbol{\varrho}(n, X(n))$ vanish if $a_i \ge 2$ for some i. However, as will be shown below in Proposition 4.1, after a suitable scaling near the edge $a = 2$, the correlation functions $\boldsymbol{\varrho}^\theta$ converge to the correlation functions given by the Airy kernel [12, 36]

$$\mathsf{A}(x, y) = \frac{A(x)A'(y) - A'(x)A(y)}{x - y} \,.$$

Here $A(x)$ is the Airy function:

$$(1.15) \qquad A(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{u^3}{3} + xu\right) \, du.$$

In fact, the following more precise statement is true about the behavior of the Plancherel measure near the edge $a = 2$. By symmetry, everything we say about the edge $a = 2$ applies to the opposite edge $a = -2$.

Consider the random point process on $\mathbb{R}$ whose correlation functions are given by the determinants

$$\rho_k^{\mathrm{Airy}}(x_1, \ldots, x_k) = \det\left[\mathsf{A}(x_i, x_j)\right]_{1 \le i,j \le k},$$

and let

$$\zeta = (\zeta_1 > \zeta_2 > \zeta_3 > \ldots) \in \mathbb{R}^\infty$$

be its random configuration. We call the random variables ζ_i's the *Airy ensemble*. It is known [12, 36] that the Airy ensemble describes the behavior of the (properly scaled) 1st, 2nd, and so on largest eigenvalues of a Gaussian random Hermitian matrix. The distribution of individual eigenvalues was obtained by Tracy and Widom in [36] in terms of certain Painlevé transcendents.

It has been conjectured by Baik, Deift, and Johansson that the random variables

$$\widetilde{\lambda} = \left(\widetilde{\lambda}_1 \ge \widetilde{\lambda}_2 \ge \ldots\right), \quad \widetilde{\lambda}_i = n^{1/3}\left(\frac{\lambda_i}{n^{1/2}} - 2\right),$$

converge, in distribution and together with all moments, to the Airy ensemble. They verified this conjecture for individual distribution of λ_1 and λ_2 in [3] and [4], respectively. In particular, in the case of λ_1, this generalizes the result of [40, 42] that $\frac{\lambda_1}{\sqrt{n}} \to 2$ in probability as $n \to \infty$. The computation of $\lim \frac{\lambda_1}{\sqrt{n}}$ was known as the Ulam problem; different solutions to this problem were given in [1, 16, 32]; see also the survey [2].

Convergence of all expectations of the form

$$(1.16) \qquad \left\langle \prod_{k=1}^r \sum_{i=1}^\infty e^{t_k \widetilde{\lambda}_i} \right\rangle, \quad t_1, \ldots, t_r > 0, \quad r = 1, 2, \ldots,$$

to the corresponding quantities for the Airy ensemble was established in [25]. The proof in [25] was based on a combinatorial interpretation of (1.16) as the asymptotics in a certain enumeration problem for random surfaces.

In the present paper we use different ideas to prove the following

Theorem 4. *As $n \to \infty$, the random variables $\widetilde{\lambda}$ converge, in joint distribution, to the Airy ensemble.*

This is done in Section 4 using methods described in the next subsection. The result stated in Theorem 4 was independently obtained by K. Johansson in [17]. See, for example, [13] for an application of Theorem 4.

1.5. Poissonization and depoissonization.

We obtain Theorems 3 and 4 from Theorem 1 using the so-called depoissonization techniques. We recall that the fundamental idea of depoissonization is the following.

Given a sequence $b_1, b_2, b_3, \ldots$ its *poissonization* is, by definition, the function

$$(1.17) \qquad B(\theta) = e^{-\theta} \sum_{k=1}^{\infty} \frac{\theta^k}{k!} \, b_k \, .$$

Provided the b_k's grow not too rapidly this is an entire function of θ. In combinatorics, it is usually called the exponential generating function of the sequence $\{b_k\}$. Various methods of extracting asymptotics of sequences from their generating functions are classically known and widely used (see for example [39] where such methods are used to obtain the limit shape of a typical partition under various measures on the set of partitions).

A probabilistic way to look at the generating function (1.17) is the following. If $\theta \geq 0$, then $B(\theta)$ is the expectation of b_η where $\eta \in \{0, 1, 2, \ldots\}$ is a Poisson random variable with parameter θ. Because η has mean θ and standard deviation $\sqrt{\theta}$, one expects that

$$(1.18) \qquad B(n) \approx b_n \, , \quad n \to \infty \, ,$$

provided the variations of b_k for $|k - n| \leq \mathrm{const} \sqrt{n}$ are small. One possible regularity condition on b_n which implies (1.18) is monotonicity. In a very general and very convenient form, a depoissonization lemma for nonincreasing nonnegative b_n was established by K. Johansson in [16]. We use this lemma in Section 4 to prove Theorem 4.

Another approach to depoissonization is to use a contour integral

$$(1.19) \qquad b_n = \frac{n!}{2\pi i} \int_C \frac{B(z) \, e^z}{z^n} \frac{dz}{z} \, ,$$

where C is any contour around $z = 0$. Suppose, for a moment, that b_n is constant, $b = b_n = B(z)$. The function $e^z / z^n = e^{z - n \ln z}$ has a unique critical point $z = n$. If we choose $|z| = n$ as the contour C, then only neighborhoods of size $|z - n| \leq \mathrm{const} \sqrt{n}$ contribute to the asymptotics of (1.19). Therefore, for general $\{b_n\}$, we still expect that provided the overall growth of $B(z)$ is under control and the variations of $B(z)$ for $|z - n| \leq \mathrm{const} \sqrt{n}$ are small, the asymptotically significant contribution to (1.19) will come from $z = n$. That is, we still expect (1.18) to be valid. See, for example, [15] for a comprehensive discussion and survey of this approach.

We use this approach to prove Theorem 3 in Section 3. The growth conditions on $B(z)$ which are suitable in our situation are spelled out in Lemma 3.1.

In our case, the functions $B(\theta)$ are combinations of the Bessel functions. Their asymptotic behavior as $\theta \approx n \to \infty$ can be obtained directly from the classical results on asymptotics of Bessel functions which are discussed, for example, in the fundamental Watson's treatise [43]. These asymptotic formulas for Bessel functions are derived using the integral representations of Bessel functions and the steepest descent method. The different behavior of the asymptotics in the bulk $(-2, 2)$ of the spectrum, near the edges ± 2 of the spectrum, and outside of $[-2, 2]$ is produced by the different location of the saddle point in these three cases.

1.6. Organization of the paper.
Section 2 contains the proof of Theorems 1 and 2 and also various formulas for the kernels K and J. We also discuss a difference operator which commutes with J and its possible applications.

Section 3 deals with the behavior of the Plancherel measure in the bulk of the spectrum; there we prove Theorem 3. Theorem 4 and a similar result (Theorem 5) for the poissonized measure M^θ are established in Section 4.

At the end of the paper there is an Appendix, where we collected some necessary results about Fredholm determinants, point processes, and convergence of trace class operators.

2. Correlation functions of the measures M^θ

2.1. **Proof of Theorem 1.** As noted above, Theorem 1 is a limit case of Theorem 3.3 of [8]. That theorem concerns a family $\{M_{zz'}^{(n)}\}$ of probability measures on partitions of n, where z, z' are certain parameters. When the parameters go to infinity, $M_{zz'}^{(n)}$ tends to the Plancherel measure M_n. Theorem 3.3 in [8] gives a determinantal formula for the correlation functions of the measure

$$(2.1) \qquad M_{zz'}^\xi = (1 - \xi)^t \sum_{n=1}^{\infty} \frac{(t)_n}{n!} \, \xi^n \, M_{zz'}^{(n)}$$

in terms of a certain *hypergeometric kernel*. Here $t = zz' > 0$ and $\xi \in (0, 1)$ is an additional parameter. As $z, z' \to \infty$ and $\xi = \frac{\theta}{t} \to 0$, the negative binomial distribution in (2.1) tends to the Poisson distribution with parameter θ. In the same limit, the hypergeometric kernel becomes the kernel K of Theorem 1. The Bessel functions appear as a suitable degeneration of hypergeometric functions.

Recently, these results of [8] were considerably generalized in [26], where it was shown how this type of correlation functions can be computed using simple commutation relations in the infinite wedge space.

For the reader's convenience, we present here a direct and elementary proof of Theorem 1 which uses the same ideas as in [8] plus an additional technical trick, namely, differentiation with respect to θ which kills denominators. This trick yields a denominator–free integral formula for the kernel K; see Proposition 2.7. Our proof here is a verification, not a derivation. For more conceptual approaches the reader is referred to [26, 27, 7].

Let $x, y \in \mathbb{Z} + \frac{1}{2}$. Introduce the following kernel L:

$$\mathsf{L}(x, y; \theta) = \begin{cases} 0, & xy > 0, \\ \dfrac{1}{x - y} \dfrac{\theta^{(|x|+|y|)/2}}{\Gamma(|x| + \frac{1}{2}) \, \Gamma(|y| + \frac{1}{2})}, & xy < 0. \end{cases}$$

We shall consider the kernels K and L as operators in the ℓ^2 space on $\mathbb{Z} + \frac{1}{2}$.

We recall that simple multiplicative formulas (for example, the hook formula) are known for the number $\dim \lambda$ in (1.1). For our purposes, it is convenient to rewrite the hook formula in the following determinantal form. Let $\lambda = (p_1, \dots, p_d \,|\, q_1, \dots, q_d)$ be the Frobenius coordinates of λ; see Section 1.2. We have

$$(2.2) \qquad \frac{\dim \lambda}{|\lambda|!} = \det \left[\frac{1}{(p_i + q_j + 1) \, p_i! \, q_i!} \right]_{1 \le i, j \le d}.$$

The following proposition is a straightforward computation using (2.2).

Proposition 2.1. *Let λ be a partition. Then*

$$(2.3) \qquad M^\theta(\lambda) = e^{-\theta} \det \left[\mathsf{L}(x_i, x_j; \theta) \right]_{1 \le i, j \le s},$$

where $\mathrm{Fr}(\lambda) = \{x_1, \ldots, x_s\} \subset \mathbb{Z} + \frac{1}{2}$ *are the modified Frobenius coordinates of* λ
defined in (1.2).

Let $\mathrm{Fr}_* \left(M^\theta \right)$ be the push-forward of M^θ under the map Fr. Note that the
image of Fr consists of sets $X \subset \mathbb{Z} + \frac{1}{2}$ having equally many positive and negative
elements. For other $X \subset \mathbb{Z} + \frac{1}{2}$, the right-hand side of (2.3) can be easily seen to
vanish. Therefore $\mathrm{Fr}_* \left(M^\theta \right)$ is a determinantal point process (see the Appendix)
corresponding to L, that is, its configuration probabilities are determinants of the
form (2.3).

Corollary 2.2. $\det(1 + \mathsf{L}) = e^\theta$.

This follows from the fact that M^θ is a probability measure. This is explained
in Propositions A.1 and A.4 in the Appendix. Note that, in general, one needs to
check that L is a trace class operator.[1] However, because of the special form of L,
it suffices to check a weaker claim – that L is a Hilbert–Schmidt operator, which is
immediate.

Theorem 1 now follows from general properties of determinantal point processes
(see Proposition A.6 in the Appendix) and the following

Proposition 2.3. $\mathsf{K} = \mathsf{L} \left(1 + \mathsf{L} \right)^{-1}$.

We shall need three identities for Bessel functions which are degenerations of the
identities (3.13–15) in [8] for the hypergeometric function. The first identity is due
to Lommel (see [43], Section 3.2, or [14], 7.2.(60)):

$$(2.4) \qquad J_\nu(2z) \, J_{1-\nu}(2z) + J_{-\nu}(2z) \, J_{\nu-1}(2z) = \frac{\sin \pi \nu}{\pi z} \, .$$

The other two identities are the following.

Lemma 2.4. *For any* $\nu \neq 0, -1, -2, \ldots$ *and any* $z \neq 0$ *we have*

$$(2.5) \qquad \sum_{m=0}^{\infty} \frac{1}{m+\nu} \frac{z^m}{m!} \, J_m(2z) = \frac{\Gamma(\nu) \, J_\nu(2z)}{z^\nu} \, ,$$

$$(2.6) \qquad \sum_{m=0}^{\infty} \frac{1}{m+\nu} \frac{z^m}{m!} \, J_{m+1}(2z) = \frac{1}{z} - \frac{\Gamma(\nu) \, J_{\nu-1}(2z)}{z^\nu} \, .$$

Proof. Another identity due to Lommel (see [43], Section 5.23, or [14], 7.15.(10))
reads

$$\sum_{m=0}^{\infty} \frac{\Gamma(\nu - s + m)}{\Gamma(\nu + m + 1)} \frac{z^m}{m!} \, J_{m+s}(2z) = \frac{\Gamma(\nu - s)}{\Gamma(s + 1)} \, \frac{J_\nu(2z)}{z^{\nu-s}} \, .$$

Substituting $s = 0$ we get (2.5). Substituting $s = 1$ yields

$$(2.7) \qquad \sum_{m=0}^{\infty} \frac{1}{(m+\nu)(m+\nu-1)} \frac{z^m}{m!} \, J_{m+1}(2z) = \frac{\Gamma(\nu - 1) \, J_\nu(2z)}{z^{\nu-1}} \, .$$

Let $r(\nu, z)$ be the difference of the left-hand side and the right-hand side in (2.6).
Using (2.7) and the recurrence relation

$$(2.8) \qquad J_{\nu+1}(2z) - \frac{\nu}{z} J_\nu(2z) + J_{\nu-1}(2z) = 0$$

---

[1] Actually, L is of trace class because the sum of the absolute values of its matrix elements is
finite. We are grateful to P. Deift for this remark.

we find that $r(\nu + 1, z) = r(\nu, z)$. Hence for any z it is a periodic function of ν and it suffices to show that $\lim_{\nu \to \infty} r(\nu, z) = 0$. Clearly, the left-hand side in (2.6) goes to 0 as $\nu \to \infty$. From the defining series for J_ν it is clear that

$$(2.9) \qquad J_\nu(2z) \sim \frac{z^\nu}{\Gamma(\nu + 1)}, \quad \nu \to \infty,$$

which implies that the right-hand side of (2.6) also goes to 0 as $\nu \to \infty$. This concludes the proof. $\qquad \square$

Proof of Proposition 2.3. It is convenient to set $z = \sqrt{\theta}$. Since the operator $1 + \mathsf{L}$ is invertible we have to check that

$$\mathsf{K} + \mathsf{K}\,\mathsf{L} - \mathsf{L} = 0\,.$$

This is clearly true for $z = 0$; therefore, it suffices to check that

$$(2.10) \qquad \dot{\mathsf{K}} + \dot{\mathsf{K}}\,\mathsf{L} + \mathsf{K}\dot{\mathsf{L}} - \dot{\mathsf{L}} = 0\,,$$

where $\dot{\mathsf{K}} = \frac{\partial \mathsf{K}}{\partial z}$ and $\dot{\mathsf{L}} = \frac{\partial \mathsf{L}}{\partial z}$. Using the formulas

$$(2.11) \qquad \frac{d}{dz}\, J_x(2z) = -2J_{x+1}(2z) + \frac{x}{z}\, J_x(2z)$$
$$= \; 2J_{x-1}(2z) - \frac{x}{z}\, J_x(2z)$$

one computes

$$\dot{\mathsf{K}}(x, y) = \begin{cases} J_{|x|-\frac{1}{2}}\, J_{|y|+\frac{1}{2}} + J_{|x|+\frac{1}{2}}\, J_{|y|-\frac{1}{2}}\,, & xy > 0\,, \\ \mathrm{sgn}(x)\left(J_{|x|-\frac{1}{2}}\, J_{|y|-\frac{1}{2}} - J_{|x|+\frac{1}{2}}\, J_{|y|+\frac{1}{2}} \right)\,, & xy < 0\,, \end{cases}$$

where $J_x = J_x(2z)$. Similarly,

$$\dot{\mathsf{L}}(x, y) = \begin{cases} 0\,, & xy > 0\,, \\ \mathrm{sgn}(x)\, \dfrac{z^{|x|+|y|-1}}{\Gamma(|x| + \frac{1}{2})\, \Gamma(|y| + \frac{1}{2})}\,, & xy < 0\,. \end{cases}$$

Now the verification of (2.10) becomes a straightforward application of the formulas (2.5) and (2.6), except for the occurrence of the singularity $\nu \in \mathbb{Z}_{\leq 0}$ in those formulas. This singularity is resolved using (2.4). This concludes the proof of Proposition 2.3 and Theorem 1. $\qquad \square$

2.2. **Proof of Theorem 2.** Recall that by construction

$$\mathrm{Fr}(\lambda) = \left(\mathcal{D}(\lambda) + \tfrac{1}{2}\right) \triangle \left(\mathbb{Z}_{\leq 0} - \tfrac{1}{2}\right)\,.$$

Let us check that this and Proposition A.8 imply Theorem 2. In Proposition A.8 we substitute

$$\mathfrak{X} = \mathbb{Z} + \tfrac{1}{2}\,, \quad Z = \mathbb{Z}_{\leq 0} - \tfrac{1}{2}\,, \quad K = \mathsf{K}\,.$$

By definition, set

$$\varepsilon(x) = \mathrm{sgn}(x)^{x+1/2}\,, \quad x \in \mathbb{Z} + \tfrac{1}{2}\,.$$

We have the following

Lemma 2.5. $\mathsf{K}^{\triangle}(x, y) = \varepsilon(x)\, \varepsilon(y)\, \mathsf{J}(x - \tfrac{1}{2}, y - \tfrac{1}{2})$.

It is clear that since the ε-factors cancel out of all determinantal formulas, this lemma and Proposition A.8 establish the equivalence of Theorems 1 and 2.

Proof. Using the relation

$$J_{-n} = (-1)^n J_n$$

and the definition of K one computes

$$(2.12) \qquad \mathsf{K}(x,y) = \operatorname{sgn}(x)\,\varepsilon(x)\,\varepsilon(y)\,\mathsf{J}(x - \tfrac{1}{2}, y - \tfrac{1}{2})\,, \quad x \neq y\,.$$

Clearly, the relation (2.12) remains valid for $x = y > 0$. It remains to consider the case $x = y < 0$. In this case we have to show that

$$1 - \mathsf{K}(x,x) = \mathsf{J}(x - \tfrac{1}{2}, y - \tfrac{1}{2})\,, \quad x \in \mathbb{Z}_{\leq 0} - \tfrac{1}{2}\,.$$

Rewrite it as

$$(2.13) \qquad 1 - \mathsf{J}(k,k) = \mathsf{J}(-k-1, -k-1)\,, \quad k = -x - \tfrac{1}{2} \in \mathbb{Z}_{\geq 0}\,.$$

By (2.14) this is equivalent to

$$1 - \sum_{m=0}^{\infty} (-1)^m \, \frac{(2k+m+2)_m}{\Gamma(k+m+2)\Gamma(k+m+2)} \, \frac{\theta^{k+m+1}}{m!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \, \frac{(-2k+n)_n}{\Gamma(-k+n+1)\Gamma(-k+n+1)} \, \frac{\theta^{-k+n}}{n!}\,.$$

Examine the right-hand side. The terms with $n = 0, \ldots, k-1$ vanish because then $1/\Gamma(-k+n+1) = 0$. The term with $n = k$ is equal to 1, which corresponds to 1 in the left-hand side. Next, the terms with $n = k+1, \ldots, 2k$ vanish because for these values of n, the expression $(-2k+n)_n$ vanishes. Finally, for $n \geq 2k+1$, set $n = 2k+1+m$. Then the nth term in the second sum is equal to minus the mth term in the first sum. Indeed, this follows from the trivial relation

$$-(-1)^m \, \frac{(2k+m+2)_m}{m!} = (-1)^n \, \frac{(-2k+n)_n}{n!}\,, \qquad n = 2k+1+m.$$

This concludes the proof. $\qquad\qquad\square$

2.3. Various formulas for the kernel J.

Recall that since J_x is an entire function of x, the function $\mathsf{J}(x,y)$ is entire in x and y. We shall now obtain several denominator–free formulas for the kernel J.

Proposition 2.6.

$$(2.14) \qquad \mathsf{J}(x,y;\theta) = \sum_{m=0}^{\infty} (-1)^m \, \frac{(x+y+m+2)_m}{\Gamma(x+m+2)\Gamma(y+m+2)} \, \frac{\theta^{\frac{x+y}{2}+m+1}}{m!}\,.$$

Proof. Straightforward computation using a formula due to Nielsen (see Section 5.41 of [43] or [14], formula 7.2.(48)). $\qquad\qquad\square$

Proposition 2.7. *Suppose $x + y > -2$. Then*

$$\mathsf{J}(x,y;\theta) = \frac{1}{2} \int_0^{2\sqrt{\theta}} \left(J_x(z)\,J_{y+1}(z) + J_{x+1}(z)\,J_y(z) \right) dz.$$

Proof. Follows from a computation done in the proof of Proposition 2.3,

$$\frac{\partial}{\partial \theta}\,\mathsf{J}(x,y;\theta) = \frac{1}{2\sqrt{\theta}}\,(J_x\,J_{y+1} + J_{x+1}\,J_y)\,, \quad J_x = J_x(2\sqrt{\theta})\,,$$

and the following corollary of (2.14):

$$\mathsf{J}(x, y; 0) = 0, \quad x + y > -2.$$

$\square$

Remark 2.8. Observe that by Proposition 2.7 the operator $\frac{\partial \mathsf{J}}{\partial \theta}$ is a sum of two operators of rank 1.

Proposition 2.9.

$$(2.15) \qquad \mathsf{J}(x, y; \theta) = \sum_{s=1}^{\infty} J_{x+s} J_{y+s}, \qquad J_x = J_x(2\sqrt{\theta}).$$

Proof. Our argument is similar to an argument due to Tracy and Widom; see the proof of the formula (4.6) in [36]. The recurrence relation (2.8) implies that

$$(2.16) \qquad \mathsf{J}(x+1, y+1) - \mathsf{J}(x, y) = -J_{x+1} J_{y+1}.$$

Consequently, the difference between the left-hand side and the right-hand side of (2.15) is a function which depends only on $x - y$. Let x and y go to infinity in such a way that $x - y$ remains fixed. Because of the asymptotics (2.9) both sides in (2.15) tend to zero and, hence, the difference actually is 0. $\square$

In the same way as in [36] this results in the following

Corollary 2.10. *For any $a \in \mathbb{Z}$, the restriction of the kernel J to the subset $\{a, a+1, a+2, \dots\} \subset \mathbb{Z}$ defines a nonnegative trace class operator in the ℓ^2 space on that subset.*

Proof. By Proposition 2.9, the restriction of J on $\{a, a+1, a+2, \dots\}$ is the square of the kernel $(x, y) \mapsto J_{x+y+1-a}(2\sqrt{\theta})$. Since the latter kernel is real and symmetric, the kernel J is nonnegative. Hence, it remains to prove that its trace is finite. Again, by Proposition 2.9, this trace is equal to

$$\sum_{s=1}^{\infty} s \, (J_{a+s+1}(2\sqrt{\theta}))^2.$$

This sum is clearly finite by (2.9). $\square$

Remark 2.11. The kernel J resembles a Christoffel–Darboux kernel and, in fact, the operator in $\ell^2(\mathbb{Z})$ defined by the kernel J is an Hermitian projection operator. Recall that $\mathsf{K} = \mathsf{L}(1 + \mathsf{L})^{-1}$, where L is of the form

$$\mathsf{L} = \begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}.$$

One can prove that this together with Lemma 2.5 imply that J is an Hermitian projection kernel. However, in contrast to a Christoffel–Darboux kernel, it projects to an infinite–dimensional subspace.

Note that in [17] the restriction of the kernel J to $\mathbb{Z}_+$ was obtained as a limit of Christoffel–Darboux kernels for Charlier polynomials.

2.4. **Commuting difference operator.** Consider the difference operators Δ and ∇ on the lattice $\mathbb{Z}$,

$$(\Delta f)(k) = f(k+1) - f(k), \qquad (\nabla f)(k) = f(k) - f(k-1).$$

Note that $\nabla = -\Delta^*$ as operators on $\ell^2(\mathbb{Z})$. Consider the following second order difference Sturm–Liouville operator:

$$(2.17) \qquad\qquad D = \Delta \circ \alpha \circ \nabla + \beta,$$

where α and β are operators of multiplication by certain functions $\alpha(k)$, $\beta(k)$. The operator (2.17) is self–adjoint in $\ell^2(\mathbb{Z})$. A straightforward computation shows that

$$(2.18) \quad \big[Df\big](k) = (-\alpha(k+1) - \alpha(k) + \beta(k))f(k)$$
$$+ \alpha(k)f(k-1) + \alpha(k+1)f(k+1).$$

It follows that if $\alpha(s) = 0$ for a certain $s \in \mathbb{Z}$, then the space of functions $f(k)$ vanishing for $k < s$ is invariant under D.

Proposition 2.12. *Let* $[\mathsf{J}]_s$ *denote the operator in* $\ell^2(\{s, s+1, \dots\})$ *obtained by restricting the kernel* J *to* $\{s, s+1, \dots\}$. *Then the difference Sturm–Liouville operator* (2.17) *commutes with* $[\mathsf{J}]_s$ *provided*

$$\alpha(k) = k - s, \qquad \beta(k) = -\frac{k(k+1-s-2\sqrt{\theta})}{\sqrt{\theta}} + \mathrm{const}.$$

Proof. Since $[\mathsf{J}]_s$ is the square of the operator with the kernel $J_{k+l+1-s}$, it suffices to check that the latter operator commutes with D, with the above choice of α and β. But this is readily checked using (2.18). $\qquad\square$

This proposition is a counterpart of a known fact about the Airy kernel; see [36]. Moreover, in the scaling limit when $\theta \to \infty$ and

$$k = 2\sqrt{\theta} + x\,\theta^{1/6}, \qquad s = 2\sqrt{\theta} + \varsigma\,\theta^{1/6},$$

the difference operator D becomes, for a suitable choice of the constant, the differential operator

$$\frac{d}{dx} \circ (x - \varsigma) \circ \frac{d}{dx} - x(x - \varsigma),$$

which commutes with the Airy operator restricted to $(\varsigma, +\infty)$. The above differential operator is exactly that of Tracy and Widom [36].

Remark 2.13. Presumably, this commuting difference operator can be used to obtain, as was done in [36] for the Airy kernel, asymptotic formulas for the eigenvalues of $[\mathsf{J}]_s$, where $s = 2\sqrt{\theta} + \varsigma\,\theta^{1/6}$ and $\varsigma \ll 0$. Such asymptotic formulas may be very useful if one wishes to refine Theorem 4 and to establish convergence of moments in addition to convergence of distribution functions. For individual distributions of λ_1 and λ_2 the convergence of moments was obtained, by other methods, in [3, 4].

3. Correlation functions in the bulk of the spectrum

3.1. **Proof of Theorem 3.** We refer the reader to Section 1.3 of the Introduction for the definition of a regular sequence $X(n) \subset \mathbb{Z}$ and the statement of Theorem 3. Also, in this section, we shall be working in the bulk of the spectrum, that is, we shall assume that all numbers a_i defined in (1.10) lie inside $(-2, 2)$. The edges ± 2 of the spectrum and its exterior will be treated in the next section.

In our proof, we shall follow the strategy explained in Section 1.5. Namely, in order to compute the limit of $\varrho(n, X(n))$ we shall use the contour integral

$$\varrho(n, X(n)) = \frac{n!}{2\pi i} \int_{|\theta|=n} \varrho^\theta(X(n)) \frac{e^\theta}{\theta^{n+1}} \, d\theta \,,$$

compute the asymptotics of ϱ^θ for $\theta \approx n$, and estimate $|\varrho^\theta|$ away from $\theta = n$. Both tasks will be accomplished using classical results about the Bessel functions.

We start our proof with the following lemma which formalizes the above informal depoissonization argument. The hypothesis of this lemma is very far from optimal, but it is sufficient for our purposes. For the rest of this section, we fix a number $0 < \alpha < 1/4$ which shall play an auxiliary role.

Lemma 3.1. *Let $\{f_n\}$ be a sequence of entire functions*

$$f_n(z) = e^{-z} \sum_{k \geq 0} \frac{f_{nk}}{k!} z^k, \quad n = 1, 2, \dots \,,$$

and suppose that there exist constants f_∞ and γ such that

$$(3.1) \qquad \max_{|z|=n} |f_n(z)| = O\left(e^{\gamma \sqrt{n}}\right),$$

$$(3.2) \qquad \max_{|z/n-1| \leq n^{-\alpha}} |f_n(z) - f_\infty| \, e^{-\gamma|z-n|/\sqrt{n}} = o(1) \,,$$

as $n \to \infty$. Then

$$\lim_{n \to \infty} f_{nn} = f_\infty \,.$$

Proof. By replacing $f_n(z)$ by $f_n(z) - f_\infty$, we may assume that $f_\infty = 0$. By Cauchy and Stirling formulas, we have

$$f_{nn} = (1 + o(1)) \sqrt{\frac{n}{2\pi}} \int_{|\zeta|=1} \frac{f_n(n\zeta) \, e^{n(\zeta-1)}}{\zeta^n} \frac{d\zeta}{i\zeta} \,.$$

Choose some large $C > 0$ and split the circle $|\zeta| = 1$ into two parts as follows:

$$S_1 = \left\{ \frac{C}{n^{1/4}} \leq |\zeta - 1| \right\}, \quad S_2 = \left\{ \frac{C}{n^{1/4}} \geq |\zeta - 1| \right\} \,.$$

The inequality (3.1) and the equality

$$\left| e^{n(\zeta-1)} \right| = e^{-n|\zeta-1|^2/2}$$

imply that the integral $\int_{S_1}$ decays exponentially provided C is large enough. On S_2, the inequality (3.2) applies for sufficiently large n and gives

$$\max_{z \in S_2} |f_n(n\zeta)| \, e^{-\gamma\sqrt{n}|\zeta-1|} = o(1) \,.$$

Therefore, the integral $\int_{S_2}$ is $o(\ )$ of the following integral:

$$\sqrt{n} \int_{|\zeta|=1} \frac{d\zeta}{i\zeta} \exp\left(-n\frac{|\zeta - 1|^2}{2} + \gamma\sqrt{n}|\zeta - 1| \right) \sim \int_{-\infty}^{\infty} e^{-s^2/2+\gamma|s|} \, ds \,.$$

Hence, $\int_{S_2} = o(1)$ and the lemma follows. $\qquad \qquad \square$

Definition 3.2. Denote by $\mathcal{F}$ the algebra (with respect to term-wise addition and multiplication) of sequences $\{f_n(z)\}$ which satisfy the properties (3.1) and (3.2) for some, depending on the sequence, constants f_∞ and γ. Introduce the map

$$\mathcal{L}\mathrm{im} : \mathcal{F} \to \mathbb{C}, \quad \{f_n(z)\} \mapsto f_\infty,$$

which is clearly a homomorphism.

Remark 3.3. Note that we do not require $f_n(z)$ to be entire. Indeed, the kernel J may have a square root branching; see the formula (2.14).

By Theorem 2, the correlation functions ϱ^θ belong to the algebra generated by sequences of the form

$$\{f_n(z)\} = \{\mathsf{J}(x_n, y_n; z)\}\,,$$

where the sequence $X = X(n) = \{x_n, y_n\} \subset \mathbb{Z}$ is regular which, we recall, means that the limits

$$a = \lim_{n\to\infty} \frac{x_n}{\sqrt{n}}, \quad d = \lim_{n\to\infty} (x_n - y_n)$$

exist, finite or infinite. Therefore, we first consider such sequences.

Proposition 3.4. *If $X = \{x_n, y_n\} \subset \mathbb{Z}$ is regular, then*

$$\{\mathsf{J}(x_n, y_n; z)\} \in \mathcal{F}, \quad \mathcal{L}\mathrm{im}\left(\{\mathsf{J}(x_n, y_n; z)\}\right) = \mathsf{S}(d, a)\,.$$

In the proof of this proposition it will be convenient to allow $X \subset \mathbb{C}$. For complex sequences X we shall require $a \in \mathbb{R}$; the number $d \in \mathbb{C}$ may be arbitrary.

Lemma 3.5. *Suppose that a sequence $X \subset \mathbb{C}$ is as above and, additionally, suppose that $\Im x_n, \Im y_n$ are bounded and $d \neq 0$. Then the sequence $\{\mathsf{J}(x_n, y_n; z)\}$ satisfies (3.2) with $f_\infty = \mathsf{S}(d, a)$ and certain γ.*

Proof of Lemma 3.5. We shall use Debye's asymptotic formulas for Bessel functions of complex order and large complex argument; see, for example, Section 8.6 in [43]. Introduce the function

$$F(x, z) = z^{1/4} J_x(2\sqrt{z})\,.$$

The formula (1.9) can be rewritten as

$$(3.3) \qquad \mathsf{J}(x, y; z) = \frac{F(x, z)\, F(y+1, z) - F(x+1, z)\, F(y, z)}{x - y}\,.$$

The asymptotic formulas for Bessel functions imply that

$$(3.4) \qquad F(x, z) = \frac{\cos\left(\sqrt{z}\, G(u) + \frac{\pi}{4}\right)}{H(u)^{1/2}} \left(1 + O\left(z^{-1/2}\right)\right), \quad u = \frac{x}{\sqrt{z}}\,,$$

where

$$G(u) = \frac{\pi}{2}\left(u - \Omega(u)\right), \quad H(u) = \frac{\pi}{2}\sqrt{4 - u^2}\,,$$

provided that $z \to \infty$ in such a way that u stays in some neighborhood of $(-2, 2)$; the precise form of this neighborhood can be seen in Figure 22 in Section 8.61 of [43]. Because we assume that

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{n}}, \ \lim_{n\to\infty} \frac{y_n}{\sqrt{n}} \in (-2, 2)\,,$$

and because $|z/n - 1| < n^{-\alpha}$, the ratios $x_n/\sqrt{z}$, $y_n/\sqrt{z}$ stay close to $(-2, 2)$. For future reference, we also point out that the constant in $O\left(z^{-1/2}\right)$ in (3.4) is uniform in u provided u is bounded away from the endpoints ± 2.

First we estimate $\Im\left(\sqrt{z}\,G(u)\right)$. The function G clearly takes real values on the real line. From the obvious estimate

$$\left|\Im\left(\sqrt{z}\,G(u)\right)\right| \leq \left|\Im\left(\sqrt{n}\,G(x/\sqrt{n})\right)\right| + \left|\sqrt{z}\,G(x/\sqrt{z}) - \sqrt{n}\,G(x/\sqrt{n})\right|$$

and the boundedness of G, G', and $|\Im x|$ we obtain an estimate of the form

$$(3.5) \qquad \max_{|z/n-1|\leq n^{-\alpha}} |F(x; z)| e^{-\text{const } |z-n|/\sqrt{n}} = O(1)\,.$$

If $d = \infty$, then because of the denominator in (3.3) the estimate (3.5) implies that

$$\mathsf{J}(x_n, y_n; z) = o\left(e^{\text{const } |z-n|/\sqrt{n}}\right)\,.$$

Since $\mathsf{S}(\infty, a) = 0$, it follows that in this case the lemma is established.

Assume, therefore, that d is finite. Observe that for any bounded increment Δx we have

$$(3.6) \quad F(x + \Delta x, z) = \frac{\cos\left(\sqrt{z}\,G(u) + G'(u)\,\Delta x + \frac{\pi}{4}\right)}{H(u)^{1/2}}$$
$$+ O\left(\frac{(\Delta x)^2}{\sqrt{z}}\,e^{\text{const } |z-n|/\sqrt{n}}\right)\,,$$

and, in particular, the last term is $o\left(e^{\text{const } |z-n|/\sqrt{n}}\right)$. Using the trigonometric identity

$$\cos(A)\cos(B + C) - \cos(A + C)\cos(B) = \sin(C)\sin(A - B)\,,$$

and observing that

$$G'(u) = \arccos(u/2)\,, \quad \sin(G'(u)) = \frac{\sqrt{4 - u^2}}{2} = \frac{H(u)}{\pi}\,,$$

we compute

$$F(x_n; z)\,F(y_n + 1; z) - F(x_n + 1; z)\,F(y_n; z)$$
$$= \frac{1}{\pi}\sin\left(\arccos\left(\frac{x_n}{2\sqrt{z}}\right)(x_n - y_n)\right) + o\left(e^{\text{const } |z-n|/\sqrt{n}}\right)\,.$$

Since, by hypothesis,

$$\frac{x_n}{\sqrt{z}} \to a\,, \quad (x_n - y_n) \to d\,,$$

and $d \neq 0$, the lemma follows. $\qquad\square$

Remark 3.6. Below we shall need this lemma for a variable sequence $X = \{x_n, y_n\}$. Therefore, let us spell out explicitly under what conditions on X the estimates in Lemma 3.5 remain uniform. We need the sequences $\frac{x_n}{\sqrt{n}}$ and $\frac{y_n}{\sqrt{n}}$ to converge uniformly; then, in particular, the ratios $\frac{x_n}{\sqrt{n}}$ and $\frac{y_n}{\sqrt{n}}$ are uniformly bounded away from ± 2. Also, we need $\Im x_n$ and $\Im y_n$ to be uniformly bounded. Finally, we need $|d|$ to be uniformly bounded from below.

Proof of Proposition 3.4. First, we check the condition (3.2). In the case $d \neq 0$ this was done in the previous lemma. Suppose, therefore, that $\{x_n\}$ is a regular sequence in $\mathbb{Z}_{\geq 0}$ and consider the asymptotics of $\mathsf{J}(x_n, x_n; z)$.

Because the function $\mathsf{J}(x, y; z)$ is an entire function of x and y we have

$$(3.7) \qquad\qquad \mathsf{J}(x, x; z) = \frac{1}{2\pi} \int_0^{2\pi} \mathsf{J}\left(x, x + re^{it}; z\right) dt,$$

where r is arbitrary; we shall take r to be some small but fixed number. From the previous lemma we know that

$$\mathsf{J}\left(x, x + re^{it}; z\right) = \frac{1}{\pi re^{it}} \sin\left(\omega\left(\frac{x}{\sqrt{z}}\right) re^{it}\right) + o\left(e^{\mathrm{const}\,|z-n|/\sqrt{n}}\right).$$

From the above remark it follows that this estimate is uniform in t. This implies the property (3.2) for $\mathsf{J}(x_n, x_n; z)$.

To prove the estimate (3.1) we use Schläfli's integral representation (see Section 6.21 in [43])

$$(3.8) \quad J_x(2\sqrt{z}) = \frac{1}{\pi} \int_0^\pi \cos\left(xt - 2\sqrt{z}\,\sin t\right) dt - \frac{\sin \pi x}{\pi} \int_0^\infty e^{-xt - 2\sqrt{z}\,\sinh t}\, dt,$$

which is valid for $|\arg z| < \pi$ and even for $\arg z = \pm\pi$ provided $\Re x > 0$ or $x \in \mathbb{Z}$.

If $x \in \mathbb{Z}$, then the second summand in (3.8) vanishes and the first summand is $O\left(e^{\mathrm{const}\,|z|^{1/2}}\right)$ uniformly in $x \in \mathbb{Z}$. This implies the estimate (3.1) provided $d \neq 0$.

It remains, therefore, to check (3.1) for $\mathsf{J}(x_n, x_n; z)$ where $\{x_n\} \in \mathbb{Z}$ is a regular sequence. Again, we use (3.7). Observe that since $\Re\sqrt{z} \geq 0$, the second summand in (3.8) is uniformly small provided $\Im x$ is bounded from above and $\Re x$ is bounded from below. Therefore, (3.7) produces the (3.1) estimate for $x_n \geq 1$. For $x_n \leq 0$ we use the relation (2.13) and the reccurence (2.16) to obtain the estimate. □

Proof of Theorem 3. Let $X(n)$ be a regular sequence and let the numbers a_i and d_{ij} be defined by (1.10), (1.11). We shall assume that $|a_i| < 2$ for all i. The validity of the theorem in the case when $|a_i| \geq 2$ for some i will be obvious from the results of the next section.

We have

$$(3.9) \qquad\qquad \varrho^\theta(X(n)) = e^{-\theta} \sum_{k=0}^\infty \varrho(k, X(n)) \frac{\theta^k}{k!}$$

$$(3.10) \qquad\qquad = \det\left[\mathsf{J}(x_i(n), x_j(n))\right]_{1 \leq i, j \leq s},$$

where the first line is the definition of ϱ^θ and the second is Theorem 2. From (3.9) it is obvious that ϱ^θ is entire. Therefore, we can apply Lemma 3.1 to it. It is clear that Lemma 3.1, together with Proposition 3.4, implies Theorem 3. The factorization (1.12) follows from the vanishing $\mathsf{S}(\infty, a) = 0$. □

3.2. **Asymptotics of $\rho(n, X)$.** Recall that the correlation functions $\rho(n, X)$ were defined by

$$\rho(n, X) = M_n\left(\{\lambda \,|\, X \subset \mathrm{Fr}(\lambda)\}\right), \quad X \subset \mathbb{Z} + \tfrac{1}{2}.$$

The asymptotics of these correlation functions can be easily obtained from Theorem 3 by complementation (see Sections A.3 and 2.2), and the result is the following.

Let $X(n) \subset \mathbb{Z} + \frac{1}{2}$ be a regular sequence. If it splits, then $\lim_{n\to\infty} \rho(n, X(n))$ factors as in (1.12). Suppose therefore, that $X(n)$ is nonsplit. Here one has to distinguish two cases. If $X(n) \subset \mathbb{Z}_{\geq 0} + \frac{1}{2}$ or $X(n) \subset \mathbb{Z}_{\leq 0} - \frac{1}{2}$, then we shall say that this sequence is *off-diagonal*. Geometrically, it means that $X(n)$ corresponds to modified Frobenius coordinates of only one kind: either the row ones or the column ones. For off-diagonal sequences we obtain from Theorem 3 by complementation that

$$\lim_{n\to\infty} \rho\left(n, X(n)\right) = \det\left[\mathsf{S}(d_{ij}, |a|) \right]_{1 \leq i,j \leq s},$$

where S is the discrete sine kernel and $a = a_1 = a_2 = \dots$.

If $X(n)$ is nonsplit and *diagonal*, that is, if it is nonsplit and includes both positive and negative numbers, then one has to assume additionally that the number of positive and negative elements of $X(n)$ stabilizes for sufficiently large n. In this case the limit correlations are given by the kernel

$$(3.11) \qquad \mathsf{D}(x,y) = \begin{cases} \mathsf{S}\left(x - y, 0\right), & xy > 0, \\ \dfrac{\cos\left(\frac{\pi}{2}(x+y)\right)}{\pi(x-y)}, & xy < 0. \end{cases}$$

Remark that this kernel *is not* translation invariant. Note, however, that

$$\mathsf{D}(x+1, y+1) = \operatorname{sgn}(xy)\, \mathsf{D}(x,y),$$

provided x and $x+1$ have the same sign and similarly for y. Therefore, if the subsets $X \subset \mathbb{Z} + \frac{1}{2}$ and $X + m$, $m \in \mathbb{Z}$, have the same number of positive and negative elements, then

$$\det\left[\mathsf{D}(x_i, x_j) \right]_{x_i \in X} = \det\left[\mathsf{D}(x_i + m, x_j + m) \right]_{x_i \in X}.$$

4. Edge of the spectrum: Convergence to the Airy ensemble

4.1. **Results and strategy of proof.** In this section we prove Theorem 4 which was stated in Section 1.4 of the Introduction. We refer the reader to Section 1.4 for a discussion of the relation between Theorem 4 and the results obtained in [3, 4, 25].

Recall that the Airy kernel was defined as

$$\mathsf{A}(x,y) = \frac{A(x)A'(y) - A'(x)A(y)}{x - y},$$

where $A(x)$ is the Airy function (1.15). The *Airy ensemble* is, by definition, a random point process on $\mathbb{R}$, whose correlation functions are given by

$$\rho_k^{\mathrm{Airy}}(x_1, \dots, x_k) = \det\left[\mathsf{A}(x_i, x_j) \right]_{1 \leq i,j \leq k}.$$

This ensemble was studied in [36]. We denote by $\zeta_1 > \zeta_2 > \dots$ a random configuration of the Airy ensemble. Theorem 4 says that after a proper scaling and normalization, the rows $\lambda_1, \lambda_2, \dots$ of a Plancherel random partition λ converge in joint distribution to the Airy ensemble. Namely, the random variables $\widetilde{\lambda}$,

$$\widetilde{\lambda} = \left(\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \dots \right), \quad \widetilde{\lambda}_i = n^{1/3}\left(\frac{\lambda_i}{n^{1/2}} - 2 \right),$$

converge, in joint distribution, to the Airy ensemble as $n \to \infty$.

In the proof of Theorem 4, we shall follow the strategy explained in Section 1.5 of the Introduction. First, we shall prove that under the poissonized measure M^θ

on the set of partitions λ, the random variables $\widetilde{\lambda}$ converge, in joint distribution, to the Airy ensemble as $\theta \approx n \to \infty$. This result is stated below as Theorem 5. From this, using certain monotonicity and Lemma 4.7 which is due to K. Johansson, we shall conclude that the same is true for the measures M_n as $n \to \infty$.

The proof of Theorem 5 will be based on the analysis of the behavior of the correlation functions of M^θ, $\theta \approx n \to \infty$, near the point $2\sqrt{n}$. From the expression for correlation functions of M^θ given in Theorem 1 it is clear that this amounts to the study of the asymptotics of $J_{2\sqrt{n}}(2\sqrt{\theta})$ when $\theta \approx n \to \infty$. This asymptotics is classically known and from it we shall derive the following

Proposition 4.1. *Set $r = \sqrt{\theta}$. We have*

$$r^{\frac{1}{3}} \mathsf{J}\left(2r + xr^{\frac{1}{3}}, 2r + yr^{\frac{1}{3}}, r^2\right) \to \mathsf{A}(x, y), \quad r \to +\infty\,,$$

uniformly in x and y on compact sets of $\mathbb{R}$.

The prefactor $r^{\frac{1}{3}}$ corresponds to the fact that we change the local scale near $2r$ to get nonvanishing limit correlations.

Using this and verifying certain tail estimates we obtain the following

Theorem 5. *For any fixed $m = 1, 2, \ldots$ and any $a_1, \ldots, a_m \in \mathbb{R}$ we have*

$$(4.1) \quad \lim_{\theta \to +\infty} M^\theta\left(\left\{\lambda \;\middle|\; \frac{\lambda_i - 2\sqrt{\theta}}{\theta^{\frac{1}{6}}} < a_i\,, \; 1 \le i \le m\right\}\right)$$
$$= \mathrm{Prob}\{\zeta_i < a_i\,, \; 1 \le i \le m\}\,,$$

where $\zeta_1 > \zeta_2 > \ldots$ is the Airy ensemble.

Observe that the limit behavior of $\widetilde{\lambda}$ is, obviously, identical with the limit behavior of similarly scaled 1st, 2nd, and so on maximal Frobenius coordinates.

Proofs of Proposition 4.1 and Theorem 5 are given Section 4.2. In Section 4.3, using a depoissonization argument based on Lemma 4.7 we deduce Theorem 4.

Remark 4.2. We consider the behavior of any number of first rows of λ, where λ is a Plancherel random partition. By symmetry, same results describe the behavior of any number of first columns of λ.

4.2. **Proof of Theorem 5.** Suppose we have a point process on $\mathbb{R}$ with determinantal correlation functions

$$\rho_k(x_1, \ldots, x_k) = \det[K(x_i, x_j)]_{1 \le i, j \le k}\,,$$

for some kernel $K(x, y)$. Let I be a possibly infinite interval $I \subset \mathbb{R}$. By $[K]_I$ we denote the operator in $L^2(I, dx)$ obtained by restricting the kernel on $I \times I$. Assume $[K]_I$ is a trace class operator. Then the intersection of the random configuration X with I is finite almost surely and

$$\mathrm{Prob}\{|X \cap I| = N\} = \frac{(-1)^N}{N!} \frac{d^N}{dz^N} \det\left(1 - z[K]_I\right)\bigg|_{z=1}\,.$$

In particular, the probability that $X \cap I$ is empty is equal to

$$\mathrm{Prob}\{X \cap I = \emptyset\} = \det\left(1 - [K]_I\right).$$

More generally, if $I_1, \ldots, I_m$ is a finite family of pairwise nonintersecting intervals such that the operators $[K]_{I_1}, \ldots, [K]_{I_m}$ are trace class, then

$$(4.2) \quad \text{Prob}\{|X \cap I_1| = N_1, \ldots, |X \cap I_m| = N_m\}$$

$$= \frac{(-1)^{\sum N_i}}{\prod N_i!} \frac{\partial^{N_1 + \cdots + N_m}}{\partial z_1^{N_1} \ldots \partial z_m^{N_m}} \det \left(1 - z_1[K]_{I_1} - \cdots - z_m[K]_{I_m}\right)\Big|_{z_1 = \cdots = z_m = 1}.$$

Here operators $\{[K]_{I_i}\}$ are considered to be acting in the same Hilbert space, for example, in $L^2(I_1 \sqcup I_2 \sqcup \cdots \sqcup I_m, dx)$.

In the case of intersecting intervals $I_1, \ldots, I_m$, the probabilities

$$\text{Prob}\{|X \cap I_1| = N_1, \ldots, |X \cap I_m| = N_m\}$$

are finite linear combinations of expressions of the form (4.2). Therefore, in order to show the convergence in distribution of point processes with determinantal correlation functions, it suffices to show the convergence of expressions of the form (4.2).

The formula (4.2) is discussed, for example, in [37]. See also Theorem 2 in [35]. It remains valid for processes on a lattice such as $\mathbb{Z}$ in which case the kernel K should be an operator in $\ell^2(\mathbb{Z})$.

As verified, for example, in Proposition A.11 in the Appendix, the right-hand side of (4.2) is continuous in each $[K]_{I_i}$ with respect to the trace norm. We shall show that after a suitable embedding of $\ell^2(\mathbb{Z})$ into $L^2(\mathbb{R})$ the kernel $\mathsf{J}(x, y; \theta)$ converges to the Airy kernel $\mathsf{A}(x, y)$ as $\theta \to \infty$.

Namely, we shall consider a family of embeddings $\ell^2(\mathbb{Z}) \to L^2(\mathbb{R})$, indexed by a positive number $r > 0$, which are defined by

$$(4.3) \qquad \ell^2(\mathbb{Z}) \ni \chi_k \mapsto r^{1/6} \chi_{\left[\frac{k-2r}{r^{1/3}}, \frac{k+1-2r}{r^{1/3}}\right]} \in L^2(\mathbb{R}), \quad k \in \mathbb{Z},$$

where $\chi_k \in \ell^2(\mathbb{Z})$ is the characteristic function of the point $k \in \mathbb{Z}$ and, similarly, the function on the right is the characteristic function of a segment of length $r^{-1/3}$. Observe that this embedding is isometric. Let J_r denote the kernel on $\mathbb{R} \times \mathbb{R}$ that is obtained from the kernel $\mathsf{J}(\cdot, \cdot, r^2)$ on $\mathbb{Z} \times \mathbb{Z}$ using the embedding (4.3). We shall establish the following

Proposition 4.3. *We have*

$$[\mathsf{J}_r]_{[a,\infty)} \to [\mathsf{A}]_{[a,\infty)}, \quad r \to \infty,$$

in the trace norm for all $a \in \mathbb{R}$ uniformly on compact sets in a.

This proposition immediately implies Theorem 5 as follows.

Proof of Theorem 5. Consider the left-hand side of (4.1) and choose for each a_i a pair of functions $k_i^-(r), k_i^+(r) \in \mathbb{Z}$ such that

$$\frac{k_i^-(r) - 2r}{r^{1/3}} = a_i^-(r) \le a_i \le a_i^+(r) = \frac{k_i^+(r) - 2r}{r^{1/3}}$$

and $a_i^-(r), a_i^+(r) \to a_i$ as $r \to \infty$. Then, on the one hand, the probability in the left-hand side of (4.1) lies between the corresponding probabilities for $a_i^-(r)$ and $a_i^+(r)$. On the other hand, the probabilities for $a_i^-(r)$ and $a_i^+(r)$ can be expressed in the form (4.2) for the kernel J_r and by Proposition 4.3 and continuity of the Airy kernel they converge to the corresponding probabilities given by the Airy kernel as $r \to \infty$. $\square$

Now we get to the proofs of Propositions 4.1 and 4.3 which will require some computations. Recall that the Airy function can be expressed in terms of Bessel functions as follows:

(4.4)
$$A(x) = \begin{cases} \frac{1}{\pi}\sqrt{\frac{x}{3}} K_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right), & x \geq 0, \\ \frac{\sqrt{|x|}}{3}\left[J_{\frac{1}{3}}\left(\frac{2}{3}|x|^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}|x|^{\frac{3}{2}}\right)\right], & x \leq 0 \end{cases}$$

(see Section 6.4 in [43]). Also recall that

(4.5)
$$A(x) \sim \frac{1}{2x^{1/4}\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}}, \quad x \to +\infty$$

(see, for example, the formula 7.23 (1) in [43]).

Lemma 4.4. *For any $x \in \mathbb{R}$ we have*

(4.6)
$$\left| r^{\frac{1}{3}} J_{2r+xr^{\frac{1}{3}}}(2r) - A(x) \right| = O(r^{-\frac{1}{3}}), \quad r \to \infty,$$

moreover, the constant in $O(r^{-\frac{1}{3}})$ is uniform in x on compact subsets of $\mathbb{R}$.

Proof. Assume first that $x \geq 0$. We denote

$$\nu = 2r + xr^{\frac{1}{3}}, \quad \alpha = \operatorname{arccosh}\left(1 + xr^{-\frac{2}{3}}/2\right) \geq 0.$$

It will be convenient to use the following notation:

$$P = \nu(\tanh\alpha - \alpha), \quad Q = \frac{\nu}{3}\tanh^3\alpha.$$

The formula 8.43(4) in [43] reads

(4.7)
$$J_\nu(2r) = \frac{\tanh\alpha}{\pi\sqrt{3}}e^{P+Q}K_{\frac{1}{3}}(Q) + \frac{3\gamma_1}{\nu}e^P$$

where $|\gamma_1| < 1$. We have the following estimates as $r \to +\infty$:

$$\alpha = x^{\frac{1}{2}}r^{-\frac{1}{3}} + O(r^{-1}),$$

$$\tanh\alpha = \alpha + O(\alpha^3) = x^{\frac{1}{2}}r^{-\frac{1}{3}} + O(r^{-1}),$$

$$P + Q = \nu \cdot O(\alpha^5) = O(r^{-\frac{2}{3}}), \quad e^{P+Q} = 1 + O(r^{-\frac{2}{3}}),$$

$$Q = \frac{1}{3}\left(2r + xr^{\frac{1}{3}}\right)\left(x^{\frac{3}{2}}r^{-1} + O(r^{-\frac{4}{3}})\right) = \frac{2x^{\frac{3}{2}}}{3} + O(r^{-\frac{1}{3}}),$$

$$K_{\frac{1}{3}}(Q) = K_{\frac{1}{3}}\left(\frac{2x^{\frac{3}{2}}}{3}\right) + O(r^{-\frac{1}{3}}),$$

$$P \leq 0, \quad \frac{3\gamma_1}{\nu}e^P = O(r^{-1}).$$

Substituting this into (4.7), we obtain the claim (4.6) for $x \geq 0$.

Assume now that $x \leq 0$. Denote

$$\nu = 2r + xr^{\frac{1}{3}}, \quad \beta = \arccos\left(1 + xr^{-\frac{2}{3}}/2\right) \geq 0, \quad y = |x|.$$

Introduce the notation

$$\widetilde{P} = \nu(\tan\beta - \beta), \quad \widetilde{Q} = \frac{\nu}{3}\tan^3\beta.$$

The formula 8.43 (5) in [43] reads

$$(4.8) \quad J_\nu(r) = \frac{1}{3} \tan \beta \cos\left(\widetilde{P} - \widetilde{Q}\right) \left[J_{-\frac{1}{3}}\left(\widetilde{Q}\right) + J_{\frac{1}{3}}\left(\widetilde{Q}\right) \right]$$
$$+ \frac{1}{\sqrt{3}} \tan \beta \sin\left(\widetilde{P} - \widetilde{Q}\right) \left[J_{-\frac{1}{3}}\left(\widetilde{Q}\right) - J_{\frac{1}{3}}\left(\widetilde{Q}\right) \right] + \frac{24\gamma_2}{\nu}$$

where $|\gamma_2| < 1$. Again we have the estimates as $r \to +\infty$

$$\beta = y^{\frac{1}{2}} r^{-\frac{1}{3}} + O(r^{-1}),$$
$$\tan \beta = \beta + O(\beta^3) = y^{\frac{1}{2}} r^{-\frac{1}{3}} + O(r^{-1}),$$
$$\widetilde{P} - \widetilde{Q} = \nu \cdot O(\beta^5) = O(r^{-\frac{2}{3}}),$$
$$\cos\left(\widetilde{P} - \widetilde{Q}\right) = 1 + O(r^{-\frac{4}{3}}), \quad \sin\left(\widetilde{P} - \widetilde{Q}\right) = O(r^{-\frac{2}{3}}),$$
$$\widetilde{Q} = \frac{1}{3}\left(2r - yr^{\frac{1}{3}}\right)\left(y^{\frac{3}{2}} r^{-1} + O(r^{-\frac{4}{3}})\right) = \frac{2y^{\frac{3}{2}}}{3} + O(r^{-\frac{1}{3}}),$$
$$J_{\pm\frac{1}{3}}\left(\widetilde{Q}\right) = J_{\pm\frac{1}{3}}\left(\frac{2y^{\frac{3}{2}}}{3}\right) + O(r^{-\frac{1}{3}}).$$

These estimates after substituting into (4.8) produce (4.6) for $x \le 0$. $\qquad \square$

Lemma 4.5. *There exist $C_1, C_2, C_3, \varepsilon > 0$ such that for any $A > 0$ and $s > 0$ we have*

$$(4.9) \qquad \left| J_{r+Ar^{\frac{1}{3}}+s}(r) \right| \le C_1\, r^{-\frac{1}{3}} \exp\left(-C_2\left(A^{\frac{3}{2}} + sA^{\frac{1}{2}} r^{-\frac{1}{3}}\right)\right), \quad s \le \varepsilon r,$$

$$(4.10) \qquad \left| J_{r+Ar^{\frac{1}{3}}+s}(r) \right| \le \exp\left(-C_3(r+s)\right), \qquad\qquad\quad s \ge \varepsilon r,$$

for all $r \gg 0$.

Proof. First suppose that $s \le \varepsilon r$. Set $\nu = r + Ar^{\frac{1}{3}} + s$. We shall use (4.7) with $\alpha = \operatorname{arccosh}(\nu/r)$. Provided ε is chosen small enough and r is sufficiently large, α will be close to 0 and we will be able to use Taylor expansions. For $r \gg 0$ we have

$$\alpha = \operatorname{arccosh}(1 + Ar^{-\frac{2}{3}} + sr^{-1}) \ge \operatorname{const}\,(Ar^{-\frac{2}{3}} + sr^{-1})^{\frac{1}{2}},$$

and, similarly,

$$-P = \nu(\alpha - \tanh \alpha) \ge \operatorname{const}\,(A + sr^{-\frac{1}{3}})^{\frac{3}{2}}.$$

Since the function $x^{\frac{3}{2}}$ is concave, we have

$$-P \ge \operatorname{const}\,(A^{\frac{3}{2}} + sA^{\frac{1}{2}} r^{-\frac{1}{3}}).$$

The constant here is strictly positive.

Since $K_{\frac{1}{3}}(x) \le \operatorname{const} x^{-\frac{1}{2}} e^{-x}$ (see, for example, the formula 7.23 (1) in [43]) we obtain

$$\tanh \alpha\, e^{P+Q} K_{\frac{1}{3}}(Q) \le \operatorname{const} \frac{e^P}{\sqrt{\nu \tanh \alpha}}$$
$$\le \frac{\operatorname{const}}{r^{\frac{1}{3}}} \exp\left(-\operatorname{const}\left(A^{\frac{3}{2}} + sA^{\frac{1}{2}} r^{-\frac{1}{3}}\right)\right),$$

where we used that $\tanh \alpha \geq \text{const } r^{-\frac{1}{3}}$. Finally, we note that

$$\frac{e^P}{\nu} \leq \frac{1}{r} \exp\left(-\text{const}\left(A^{\frac{3}{2}} + sA^{\frac{1}{2}}r^{-\frac{1}{3}}\right)\right),$$

and this completes the proof of (4.9).

The estimate (4.10) follows directly from the formulas 8.5 (9), (4), (5) in [43]. □

Lemma 4.6. *For any* $\delta > 0$ *there exists* $M > 0$ *such that for all* $x, y > M$ *and large enough* r

$$\left| \mathsf{J}\left(2r + xr^{\frac{1}{3}}, 2r + yr^{\frac{1}{3}}, r^2\right) \right| < \delta r^{-\frac{1}{3}}.$$

Proof. From (2.15) we have

$$(4.11) \qquad \mathsf{J}\left(2r + xr^{\frac{1}{3}}, 2r + yr^{\frac{1}{3}}, r^2\right) = \sum_{s=1}^{\infty} J_{2r+xr^{\frac{1}{3}}+s}(2r) \, J_{2r+yr^{\frac{1}{3}}+s}(2r).$$

Let us split the sum in (4.11) into two parts,

$$\sum\nolimits_1 = \sum_{l \leq \varepsilon r}, \quad \sum\nolimits_2 = \sum_{l > \varepsilon r},$$

that is, one sum for $l \leq \varepsilon r$ and the other for $l > \varepsilon r$, and apply Lemma 4.5 to these two sums. Note that $2r$ here corresponds to r in Lemma 4.5; this produces factors of $2^{\frac{1}{3}}$ and does not affect the estimate.

Let the c_i's stand for some positive constants not depending on M. From (4.9) we obtain the following estimate for the first sum:

$$\sum\nolimits_1 \leq c_1 \, r^{-\frac{2}{3}} \exp\left(-c_2 \, M^{\frac{3}{2}}\right) \sum_{s=1}^{[\varepsilon r]} q^s$$

where

$$q = \exp\left(-c_2 M^{\frac{1}{2}} r^{-\frac{1}{3}}\right), \quad 0 < q < 1.$$

Therefore,

$$\sum\nolimits_1 \leq \frac{c_1 \, r^{-\frac{2}{3}} \exp\left(-c_2 \, M^{\frac{3}{2}}\right)}{1 - q} \leq r^{-\frac{1}{3}} \cdot c_3 \exp(-c_2 M^{\frac{3}{2}}) M^{-\frac{1}{2}}.$$

We can choose M so that $c_3 \exp(-c_2 M^{\frac{3}{2}}) M^{-\frac{1}{2}} < \delta/2$.

For the second sum we use (4.10) and obtain

$$\sum\nolimits_2 \leq \sum_{s \geq \varepsilon r} \exp(-c_4(r+s)) \leq c_5 \exp(-c_4 r).$$

Clearly, this is less than $\delta r^{-\frac{1}{3}}/2$ for $r \gg 0$. □

Proof of Proposition 4.1. As shown in [10, 36], the Airy kernel has the following integral representation:

$$(4.12) \qquad \mathsf{A}(x, y) = \int_0^{\infty} A(x+t)A(y+t)dt.$$

The formula (4.11) implies that for any integer $N > 0$

$$(4.13) \quad \mathsf{J}\left(2r + xr^{\frac{1}{3}}, 2r + yr^{\frac{1}{3}}, r^2\right) = \sum_{s=1}^{N} J_{2r+xr^{\frac{1}{3}}+s}(2r)\, J_{2r+yr^{\frac{1}{3}}+s}(2r)$$
$$+ \mathsf{J}\left(2r + xr^{\frac{1}{3}} + N, 2r + yr^{\frac{1}{3}} + N, r^2\right).$$

Let us fix $\delta > 0$ and pick $M > 0$ according to Lemma 4.6. Since, by assumption, x and y lie in a compact set of $\mathbb{R}$, we can fix m such that $x, y \geq m$. Set

$$N = \left[(M - m + 1)\, r^{\frac{1}{3}}\right].$$

Then the inequalities

$$x + Nr^{-\frac{1}{3}} > M, \quad y + Nr^{-\frac{1}{3}} > M$$

are satisfied for all x, y in our compact set and Lemma 4.4 applies to the sum in (4.13). We obtain

$$\left| r^{\frac{2}{3}} \sum_{s=1}^{N} J_{2r+xr^{\frac{1}{3}}+s}(2r)\, J_{2r+yr^{\frac{1}{3}}+s}(2r) - \sum_{s=1}^{N} A(x + sr^{-\frac{1}{3}})A(y + sr^{-\frac{1}{3}}) \right| = O(1)$$

because the number of summands is $N = O(r^{\frac{1}{3}})$ and $A(x)$ is bounded on subsets of $\mathbb{R}$ which are bounded from below. Note that

$$r^{-\frac{1}{3}} \sum_{s=1}^{N} A(x + sr^{-\frac{1}{3}})A(x + sr^{-\frac{1}{3}})$$

is a Riemann integral sum for the integral

$$\int_{0}^{M-m+1} A(x + t)A(y + t)\, dt,$$

and it converges to this integral as $r \to +\infty$. Since the absolute value of the second term in the right-hand side of (4.13) does not exceed $\delta r^{-\frac{1}{3}}$ by the choice of N, we get

$$\left| r^{\frac{1}{3}} \mathsf{J}\left(2r + xr^{\frac{1}{3}}, 2r + yr^{\frac{1}{3}}, r^2\right) - \int_{0}^{M-m+1} A(x + t)A(y + t)dt \right| \leq \delta + o(1)$$

as $r \to +\infty$, and this estimate is uniform on compact sets. Now let $\delta \to 0$ and $M \to +\infty$. By (4.5) the integral (4.12) converges uniformly in x and y on compact sets and we obtain the claim of the proposition. $\qquad\square$

Proof of Proposition 4.3. It is clear that Proposition 4.1 implies the convergence of $[\mathsf{J}_r]_a$ to $[\mathsf{A}]_a$ in the weak operator topology. Therefore, by Proposition A.9, it remains to prove that $\mathrm{tr}[\mathsf{J}_r]_a \to \mathrm{tr}[\mathsf{A}]_a$ as $r \to +\infty$. We have

$$\mathrm{tr}[\mathsf{J}_r]_a = \sum_{k=[2r+ar^{\frac{1}{3}}]}^{\infty} \mathsf{J}(k, k; r^2) + o(1),$$

where the $o(1)$ correction comes from the fact that a may not be a number of the form $\frac{k-2r}{r^{1/3}}$, $k \in \mathbb{Z}$. By (4.11) we have

$$(4.14) \qquad \sum_{k=[2r+ar^{\frac{1}{3}}]}^{\infty} \mathsf{J}(k,k;r^2) = \sum_{l=1}^{\infty} l \left(J_{[2r+ar^{\frac{1}{3}}]+l}(2r) \right)^2 .$$

Similarly,

$$(4.15) \qquad \mathrm{tr}[\mathsf{A}]_a = \int_a^{\infty} \mathsf{A}(s,s)ds = \int_0^{\infty} t(A(a+t))^2 dt .$$

Since we already established the uniform convergence of kernels on compact sets, it is enough to show that both (4.14) and (4.15) go to zero as $a \to +\infty$ and $r \to +\infty$. For the Airy kernel this is clear from (4.5). For the kernel J_r it is equivalent to the following statement: for any $\delta > 0$ there exists $M_0 > 0$ such that for all $M > M_0$ and large enough r we have

$$(4.16) \qquad \left| \sum_{l=1}^{\infty} l \, J^2_{2r+Mr^{\frac{1}{3}}+l}(2r) \right| < \delta .$$

We shall employ Lemma 4.5 for $A = M$. Again, we split the sum in (4.16) into two parts:

$$\sum_1 = \sum_{l \leq \varepsilon r} , \quad \sum_2 = \sum_{l > \varepsilon r} .$$

For the first sum Lemma 4.5 gives

$$\sum_1 \leq c_1 r^{-\frac{2}{3}} \exp\left(-c_2 M^{\frac{3}{2}}\right) \sum_{l \leq [\varepsilon r]} l \, q^l ,$$

where

$$q = \exp\left(-c_2 M^{\frac{1}{2}} r^{-\frac{1}{3}}\right) , \quad 0 < q < 1 ,$$

and the c_i's are some positive constants that do not depend on M. Since $\sum l \, q^l = q(1-q)^{-2}$ we obtain

$$\sum_1 \leq c_1 r^{-\frac{2}{3}} \exp\left(-c_2 M^{\frac{3}{2}}\right) \frac{q}{(1-q)^2} \leq c_3 \frac{\exp\left(-c_2 M^{\frac{3}{2}}\right)}{M} .$$

This can be made arbitrarily small by taking M sufficiently large.

For the other part of the sum we have the estimate

$$\sum_2 \leq \sum_{l > \varepsilon r} l \exp(-c_4(r+l))$$

which, evidently, goes to zero as $r \to +\infty$. $\qquad\qquad\square$

4.3. Depoissonization and proof of Theorem 4.

Fix some $m = 1, 2, \dots$ and denote by F_n the distribution function of $\lambda_1, \dots, \lambda_m$ under the Plancherel measure M_n,

$$F_n(x_1, \dots, x_m) = M_n \left(\{ \lambda \mid \lambda_i < x_i , \ 1 \leq i \leq m \} \right) .$$

Also, set

$$F(\theta, x) = e^{-\theta} \sum_{k=0}^{\infty} \frac{\theta^k}{k!} F_k(x).$$

This is the distribution function corresponding to the measure M^{θ}.

The measures M_n can be obtained as distribution at time n of a certain random growth process of a Young diagram; see e.g. [42]. This implies that

$$F_{n+1}(x) \leq F_n(x), \quad x \in \mathbb{R}^m.$$

Also, by construction, F_n is monotone in x and similarly

(4.17) $$F(\theta, x) \leq F(\theta, y), \quad x_i \leq y_i, \quad i = 1, \ldots, m.$$

We shall use these monotonicity properties together with the following lemma.

Lemma 4.7 (Johansson, [16]). *There exist constants $C > 0$ and $n_0 > 0$ such that for any nonincreasing sequence $\{b_n\}_{n=0}^{\infty} \subset [0, 1]$,*

$$1 \geq b_0 \geq b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0,$$

and its exponential generating function

$$B(\theta) = e^{-\theta} \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \cdot b_k$$

we have for all $n > n_0$ the following inequalities:

$$B(n + 4\sqrt{n \ln n}) - \frac{C}{n^2} \leq b_n \leq B(n - 4\sqrt{n \ln n}) + \frac{C}{n^2}.$$

This lemma implies that for all $x \in \mathbb{R}^m$

(4.18) $$F(n + 4\sqrt{n \ln n}, x) - \frac{C}{n^2} \leq F_n(x) \leq F(n - 4\sqrt{n \ln n}, x) + \frac{C}{n^2}.$$

Set

$$\bar{1} = (1, \ldots, 1).$$

Theorem 5 asserts that

(4.19) $$F\left(\theta, 2\theta^{\frac{1}{2}} \bar{1} + \theta^{\frac{1}{6}} x\right) \to F(x), \quad \theta \to +\infty, \quad x \in \mathbb{R}^m,$$

where $F(x)$ is the corresponding distribution function for the Airy ensemble. Note that $F(x)$ is continuous.

Denote $n_{\pm} = n \pm 4\sqrt{n \ln n}$. Then for $i = 1, \ldots, m$

$$2n_{\pm}^{\frac{1}{2}} + n_{\pm}^{\frac{1}{6}} x_i = 2n^{\frac{1}{2}} + n^{\frac{1}{6}} x_i + O((\ln n)^{1/2}).$$

Hence, for any $\varepsilon > 0$ and all sufficiently large n we have

$$2n_{+}^{\frac{1}{2}} + n_{+}^{\frac{1}{6}} (x_i - \varepsilon) \leq 2n^{\frac{1}{2}} + n^{\frac{1}{6}} x_i \leq 2n_{-}^{\frac{1}{2}} + n_{-}^{\frac{1}{6}} (x_i + \varepsilon),$$

for $i = 1, \ldots, m$. By (4.17) this implies that

$$F\left(n_+, 2n^{\frac{1}{2}} \bar{1} + n^{\frac{1}{6}} x\right) \geq F\left(n_+, 2n_+^{\frac{1}{2}} \bar{1} + n_+^{\frac{1}{6}} (x - \varepsilon \bar{1})\right),$$

$$F\left(n_-, 2n^{\frac{1}{2}} \bar{1} + n^{\frac{1}{6}} x\right) \leq F\left(n_-, 2n_-^{\frac{1}{2}} \bar{1} + n_-^{\frac{1}{6}} (x + \varepsilon \bar{1})\right).$$

From this and (4.18) we obtain

$$F\left(n_+, 2n_+^{\frac{1}{2}}\,\bar{1} + n_+^{\frac{1}{6}}\,(x - \varepsilon\,\bar{1})\right) - \frac{C}{n^2}$$
$$\leq F_n\left(2n^{\frac{1}{2}}\,\bar{1} + n^{\frac{1}{6}}\,x\right)$$
$$\leq F\left(n_-, 2n_-^{\frac{1}{2}}\,\bar{1} + n_-^{\frac{1}{6}}\,(x + \varepsilon\,\bar{1})\right) + \frac{C}{n^2}\,.$$

From this and (4.19) we conclude that

$$F\left(x - \varepsilon\,\bar{1}\right) + o(1) \leq F_n\left(2n^{\frac{1}{2}}\,\bar{1} + n^{\frac{1}{6}}\,x\right) \leq F\left(x + \varepsilon\,\bar{1}\right) + o(1)$$

as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary and $F(x)$ is continuous we obtain

$$F_n\left(2n^{\frac{1}{2}}\,\bar{1} + n^{\frac{1}{6}}\,x\right) \to F(x), \quad n \to \infty, \quad x \in \mathbb{R}^m,$$

which is the statement of Theorem 4.

APPENDIX A. GENERAL PROPERTIES OF DETERMINANTAL POINT PROCESSES

In this Appendix, we collect some necessary facts about determinantal point processes, their correlation functions, Fredholm determinants, and convergence of trace class operators.

Let $\mathfrak{X}$ be a countable set, let $\mathrm{Conf}(\mathfrak{X}) = 2^{\mathfrak{X}}$ be the set of subsets of $\mathfrak{X}$ and denote by $\mathrm{Conf}(\mathfrak{X})_0 \subset \mathrm{Conf}(\mathfrak{X})$ the set of finite subsets of $\mathfrak{X}$. We call elements of $\mathrm{Conf}(\mathfrak{X})$ configurations. Let L be a kernel on $\mathfrak{X}$, that is, a function on $\mathfrak{X} \times \mathfrak{X}$ also viewed as a matrix of an operator in $H = \ell^2(\mathfrak{X})$.

By a *determinantal point process* on $\mathfrak{X}$ we mean a probability measure on $\mathrm{Conf}(\mathfrak{X})_0$ such that

$$\mathrm{Prob}(X) = \frac{\det\left[L(x_i, x_j)\right]_{x_i \in X}}{\det(1 + L)}, \quad X \in \mathrm{Conf}(\mathfrak{X})_0\,.$$

Here the determinant in the numerator is the usual determinant of linear algebra, whereas the determinant in the denominator is, in general, a Fredholm determinant. Some sufficient conditions under which $\det(1 + L)$ makes sense are described in the following subsection.

A.1. **Fredholm determinants and determinantal processes.** Let H be a complex Hilbert space, $\mathcal{L}(H)$ the algebra of bounded operators in H, and $\mathcal{L}_1(H)$, $\mathcal{L}_2(H)$ the ideals of trace class and Hilbert–Schmidt operators, respectively.

Assume we are given a splitting $H = H_+ \oplus H_-$. According to this splitting, write operators $A \in \mathcal{L}(H)$ in block form, $A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}$, where

$$A_{++} : H_+ \to H_+, \quad A_{+-} : H_- \to H_+,$$
$$A_{-+} : H_+ \to H_-, \quad A_{--} : H_- \to H_-\,.$$

The algebra $\mathcal{L}(H)$ is equipped with a natural $\mathbb{Z}_2$-grading. Specifically, given A, its even part A_{even} and odd part A_{odd} are defined as follows:

$$A_{\mathrm{even}} = \begin{bmatrix} A_{++} & 0 \\ 0 & A_{--} \end{bmatrix}, \qquad A_{\mathrm{odd}} = \begin{bmatrix} 0 & A_{+-} \\ A_{-+} & 0 \end{bmatrix}\,.$$

Denote by $\mathcal{L}_{1|2}(H)$ the set of operators $A \in \mathcal{L}(H)$ such that A_{even} is in the trace class $\mathcal{L}_1(H)$ while A_{odd} is in the Hilbert–Schmidt class $\mathcal{L}_2(H)$. It is readily seen

that $\mathcal{L}_{1|2}(H)$ is an algebra. We endow it with the topology induced by the trace norm on the even part and the Hilbert–Schmidt norm on the odd part.

It is well known that the determinant $\det(1 + A)$ makes sense if $A \in \mathcal{L}_1(H)$. It can be characterized as the only function which is continuous in A with respect to the trace norm $\|A\|_1 = \operatorname{tr}\sqrt{AA^*}$ and which coincides with the usual determinant when A is a finite–dimensional operator. See, e.g., [33].

Proposition A.1. *The function $A \mapsto \det(1 + A)$ admits a unique extension to $\mathcal{L}_{1|2}(H)$, which is continuous in the topology of that algebra.*

Proof. For $A \in \mathcal{L}_{1|2}(H)$, set

$$(A.1) \qquad \det(1 + A) = \det((1 + A)e^{-A}) \cdot e^{\operatorname{tr} A_{\mathrm{even}}}.$$

As is well known (e.g., [33]),

$$A \mapsto (1 + A)e^{-A} - 1$$

is a continuous map from $\mathcal{L}_2(H)$ to $\mathcal{L}_1(H)$. Next, $A \mapsto \operatorname{tr} A_{\mathrm{even}}$ evidently is a continuous function on $\mathcal{L}_{1|2}(H)$. Consequently, (A.1) is well defined and is a continuous function on $\mathcal{L}_{1|2}(H)$. When $A \in \mathcal{L}_1(H)$, (A.1) agrees with the conventional definition, because then

$$\det((1 + A)e^{-A}) \cdot e^{\operatorname{tr} A_{\mathrm{even}}} = \det(1 + A)e^{-\operatorname{tr} A + \operatorname{tr} A_{\mathrm{even}}} = \det(1 + A).$$

This concludes the proof.   □

Corollary A.2. *If $\{P_n\}$ is an ascending sequence of even projection operators in H such that $P_n \to 1$ strongly, then*

$$\det(1 + A) = \lim_{n \to \infty} \det(1 + P_n A P_n).$$

Proof. Indeed, $P_n A P_n$ approximates A in the topology of $\mathcal{L}_{1|2}(H)$.   □

Corollary A.3. *If $A, B \in \mathcal{L}_{1|2}(H)$, then*

$$\det(1 + A)\det(1 + B) = \det((1 + A)(1 + B)).$$

Proof. Indeed, this is true for finite–dimensional A, B, and then we use the continuity argument.   □

In our particular case, the splitting of $H = \ell^2(\mathfrak{X})$ will come from a splitting of $\mathfrak{X} = \mathfrak{X}_+ \sqcup \mathfrak{X}_-$ into two complementary subsets as follows:

$$H_\pm = \ell^2(\mathfrak{X}_\pm).$$

An operator L in H will be viewed as an infinite matrix whose rows and columns are indexed by elements of $\mathfrak{X}$. Given $X \subset \mathfrak{X}$, we denote by L_X the corresponding finite submatrix in L.

Proposition A.4. *If $L \in \mathcal{L}_{1|2}(H)$, then*

$$(A.2) \qquad \sum_X \det L_X = \det(1 + L),$$

where summation is taken over all finite subsets $X \subset \mathfrak{X}$ including the empty set with the understanding that $\det L_\varnothing = 1$.

The exact meaning of the sum in the left-hand side is explained in the proof.

Proof. Given a finite subset $Y \subset \mathfrak{X}$, we assign to it, in the natural way, a projection operator P_Y. Note that P_Y is even. By elementary linear algebra, we have

$$\sum_{X \subseteq Y} \det L_X = \det(1 + P_Y L P_Y).$$

Assume Y becomes larger and larger, so that in the limit it covers the whole $\mathfrak{X}$. Then the left-hand side tends to the left-hand side of (A.2). More precisely, this is evident if all the minors $\det L_X$ are nonnegative. In general, instead of proving that the sum in the left-hand side of (A.2) is absolutely convergent we simply *define* it as

$$\sum_X \det L_X = \lim_{\{Y\}} \sum_{X \subseteq Y} \det L_X \, .$$

On the other hand, the right-hand side tends to $\det(1 + L)$ by Corollary A.2.   □

Remark A.5. Suppose that $L = \begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}$, where A is of Hilbert–Schmidt class. Then $L \in \mathcal{L}_{1|2}(H)$. It is readily seen that $\det L_X \geq 0$ for all X, and it is worth noting that $\det L_X = 0$ unless $|X_+| = |X_-|$. By Proposition A.4, we can define a probability measure on finite subsets X of $\mathfrak{X}$ by

$$\mathrm{Prob}(X) = \frac{\det L_X}{\det(1 + L)} \, , \quad X \in \mathrm{Conf}(\mathfrak{X})_0 \, .$$

A.2. **Correlation functions of determinantal processes.** Given $X \in \mathrm{Conf}(\mathfrak{X})_0$, let $\rho(X)$ be the probability that a random configuration contains X, that is,

$$\rho(X) = \mathrm{Prob} \left(\{ Y \in \mathrm{Conf}(\mathfrak{X})_0, X \subset Y \} \right) \, .$$

We call $\rho(X)$ the *correlation functions*. The fundamental fact about determinantal point processes is that their correlation functions again have a determinantal form.

Proposition A.6. *Let L be as above and set $K = L(1 + L)^{-1}$. Then $\rho(X) = \det K_X$.*

Proof. We follow the argument in [11], Exercise 5.4.7. Let $f(x)$ be an arbitrary function on $\mathfrak{X}$ such that $f(x) = 1$ for all but a finite number of x's. Form the probability generating functional:

$$\Phi(f) = \sum_X \prod_{x \in X} f(x) \cdot \mathrm{Prob}(X).$$

Then, viewing f as a diagonal matrix, we get

$$\Phi(f) = \frac{\sum_X \det(fL)}{\det(1 + L)} = \frac{\det(1 + fL)}{\det(1 + L)} \, ,$$

where the last equality is justified by Proposition A.4 applied to the operator fL.

Now, set $g(x) = f(x) - 1$, so that $g(x) = 0$ for all but finitely many x's. Then we can rewrite this relation as

$$\Phi(f) = \frac{\det(1 + fL)}{\det(1 + L)} = \frac{\det(1 + L + gL)}{\det(1 + L)} = \det(1 + gK),$$

where the last equality follows by Corollary A.3.

Next, as gK is in $\mathcal{L}_{1|2}(H)$ (it is even finite–dimensional), this can be rewritten as

$$\Phi(f) = \sum_X \det((gK)_X) = \sum_X \prod_{x \in X} g(x) \cdot \det K_X.$$

On the other hand, by the very definition of $\Phi(f)$,

$$\Phi(f) = \sum_X \prod_{x \in X} g(x) \cdot \rho(X).$$

This implies $\rho(X) = \det K_X$, as desired. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark A.7. If $L = \begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}$, then

$$K = \begin{bmatrix} AA^*(1 + AA^*)^{-1} & (1 + AA^*)^{-1}A \\ -(1 + A^*A)^{-1}A^* & A^*A(1 + A^*A)^{-1} \end{bmatrix}.$$

In the recent survey [35], the determinantal formula $\rho(X) = \det K_X$ for the correlation functions is taken as a definition. The paper [35] contains a more general and detailed discussion of the basics of the theory of determinantal processes which in [35] are called determinantal random point fields.

A.3. **Complementation principle.** In this section we discuss a simple but useful observation which was communicated to us by S. Kerov. Consider an arbitrary probability measure on $\mathrm{Conf}(\mathfrak{X})$ such that its correlation functions

$$\rho(X) = \mathrm{Prob}\left(\{Y \in \mathrm{Conf}(\mathfrak{X}), X \subset Y\}\right), \quad X \in \mathrm{Conf}(\mathfrak{X})_0,$$

have a determinantal form

$$\rho(X) = \det \left[K(x_i, x_j)\right]_{x_i \in X}$$

for some kernel K.

Let $Z \subset \mathfrak{X}$ be an arbitrary subset of $\mathfrak{X}$. Consider the symmetric difference mapping

$$\triangle_Z : \mathrm{Conf}(\mathfrak{X}) \to \mathrm{Conf}(\mathfrak{X}), \quad Y \mapsto Y \triangle Z,$$

which is an involution in $\mathrm{Conf}(\mathfrak{X})$. Let $\mathrm{Prob}^\triangle = (\triangle_Z)_* \mathrm{Prob}$ be the image of our probability measure under $\triangle_Z$ and let $\rho^\triangle(X)$ be the correlation functions of the measure $\mathrm{Prob}^\triangle$. Define a new kernel $K^\triangle$ as follows. Let $Z' = \mathfrak{X} \setminus Z$ be the complement of Z and write the matrix K in block form with respect to the decomposition $\mathfrak{X} = Z' \sqcup Z$:

$$K_{Z' \sqcup Z} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

By definition, set

$$K^\triangle_{Z' \sqcup Z} = \begin{bmatrix} A & B \\ -C & 1 - D \end{bmatrix}.$$

We have the following

Proposition A.8. $\rho^\triangle(X) = \det \left[K^\triangle(x_i, x_j)\right]_{x_i \in X}.$

Proof. Set $X_1 = X \setminus Z$, $X_2 = Z \setminus X$. By the inclusion-exclusion principle we have

$$\rho^\triangle(X) = \text{Prob}\left(\{Y \in \text{Conf}(\mathfrak{X}), X_1 \subset Y, X_2 \cap Y = \emptyset\}\right)$$
$$= \sum_{S \subset X_2} (-1)^{|S|} \rho(X_1 \cup S).$$

This alternating sum is easily seen to be identical by linearity to the expansion of $\det\left[K^\triangle(x_i, x_j)\right]_{x_i \in X}$ using

$$\begin{bmatrix} A & B \\ -C & 1-D \end{bmatrix} = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\square$

A.4. **Convergence of trace class operators.** Let $K_1, K_2, \ldots$ and K be Hermitian nonnegative operators in $\mathcal{L}_1(H)$. The following proposition is a special case of Theorem 2.20 in the book [34] (we are grateful to P. Deift for this reference). For the reader's convenience we give a proof here.

Proposition A.9. *The following conditions are equivalent:*
  (i) $\|K_n - K\|_1 \to 0$;
  (ii) $\text{tr}\, K_n \to \text{tr}\, K$ *and* $K_n \to K$ *in the weak operator topology.*

First, we prove a lemma:

Lemma A.10. *Let* $X = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ *be a nonnegative operator* 2×2 *matrix. Then* $\|B\|_1 \leq \sqrt{\text{tr}\, A \cdot \text{tr}\, D}$.

Proof of Lemma A.10. Without loss of generality one can assume that the block B is a nonnegative diagonal matrix, $B = \text{diag}(b_1, b_2, \ldots)$. Write the blocks A and D as matrices, too, and let a_i and d_i be their diagonal entries. Since $X \geq 0$, we have $b_i^2 \leq a_i d_i$ and therefore

$$\|B\|_1 = \sum b_i \leq \sum \sqrt{a_i d_i} \leq \sqrt{\sum a_i \cdot \sum d_i} \leq \sqrt{\text{tr}\, A \cdot \text{tr}\, D}.$$

$\square$

Proof of Proposition A.9. Clearly, (i) implies (ii). To check the converse claim, write K in block form,

$$K = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix},$$

where A is of finite size and $\text{tr}\, D$ is small. Write all the K_n's in block form with respect to the same decomposition of the Hilbert space,

$$K_n = \begin{bmatrix} A_n & B_n \\ B_n^* & D_n \end{bmatrix}.$$

Since $K_n \to K$ weakly, we have convergence of finite blocks, $A_n \to A$, which implies $\text{tr}\, A_n \to \text{tr}\, A$. Since $\text{tr}\, K_n \to \text{tr}\, K$, we get $\text{tr}\, D_n \to \text{tr}\, D$, so that all the traces $\text{tr}\, D_n$ are small together with $\text{tr}\, D$ provided that n is large enough.

Write

$$K' = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

and similarly for K_n. Then

$$\|K_n - K\|_1 \le \|K_n - K'_n\|_1 + \|K'_n - K'\|_1 + \|K' - K\|_1.$$

In the right-hand side, the first and the third summands are small because of the lemma, while the second summand is small because it is equal to $\|A_n - A\|_1$. $\qquad \square$

Proposition A.11. *The map $(A_1, \ldots, A_n) \mapsto \det(I + \lambda_1 A_1 + \cdots + \lambda_n A_n)$ defines a continuous map from $(\mathcal{L}_1(H))^n$ to the algebra of entire functions in n variables with the topology of uniform convergence on compact sets.*

Proof. The fact that $\det(I + \lambda_1 A_1 + \cdots + \lambda_n A_n)$ is holomorphic in $\{\lambda_i\}$ for any trace class operators $A_1, \ldots, A_n$ is proved in [33]. The continuity of the map follows from the inequality

$$|\det(I + B) - \det(I + C)| \le \|B - C\|_1 \exp(\|B\|_1 + \|C\|_1 + 1)$$

which holds for any $B, C \in \mathcal{L}_1(H)$; see [30, 34]. $\qquad \square$

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Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104–6395 and Dobrushin Mathematics Laboratory, Institute for Problems of Information Transmission, Bolshoy Karetny 19, 101447, Moscow, Russia
E-mail address: `borodine@math.upenn.edu`

University of Chicago, Department of Mathematics, 5734 University Ave., Chicago, Illinois 60637
Current address: Department of Mathematics, University of California at Berkeley, Evans Hall, Berkeley, California 94720-3840
E-mail address: `okounkov@math.berkeley.edu`

Dobrushin Mathematics Laboratory, Institute for Problems of Information Transmission, Bolshoy Karetny 19, 101447, Moscow, Russia
E-mail address: `olsh@glasnet.ru`