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Asymptotics of supremum distribution of a Gaussian process over a Weibullian time

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Let $\{X(t): t \in [0, \infty)\}$ be a centered Gaussian process with stationary increments and variance function $\sigma_X^2(t)$. We study the exact asymptotics of $\mathbb{P}(\sup_{t\in[0,T]} X(t) > u)$ as $u \to \infty$, where T is an independent of $\{X(t)\}$ non-negative Weibullian random variable. As an illustration, we work out the asymptotics of the supremum distribution of fractional Laplace motion.

Keywords: exact asymptotics; fractional Laplace motion; Gaussian process

1. Introduction

The problem of analyzing the asymptotic properties of

$$\mathbb{P}\Big(\sup_{t\in[0,T]} X(t) > u\Big) \qquad \text{as } u \to \infty \tag{1}$$

for a centered Gaussian process with stationary increments $\{X(t)\}$ and deterministic $T > 0$ plays an important role in many fields of applied and theoretical probability.

One of the seminal results in this area is the exact asymptotic

$$\mathbb{P}\Big(\sup_{t\in[0,T]} X(t) > u\Big) = \mathbb{P}\big(X(T) > u\big)\big(1 + \mathrm{o}(1)\big) \tag{2}$$

as $u \to \infty$, which holds for a wide class of centered Gaussian processes (see [8,10] and [9] for extensions of this result).

Some recently studied problems in, for example, queueing theory (dual risk theory) or hydrodynamics, motivate the analysis of (1) for T being a non-negative random variable independent of $\{X(t)\}$. In particular, the tail asymptotics of the *steady-state buffer content* for a *hybrid fluid queue* with the input modeled by a superposition of an integrated *on-off process* and a Gaussian process with stationary increments can be reduced (under some assumptions) to the analysis of (1) for some suitably chosen random T (see, e.g., [11] and references therein). Additionally, the analysis of the supremum distribution of subordinated Gaussian processes is strongly related to (1) over random T. For example, the asymptotics of the supremum of a *fractional Laplace motion* (used in hydrodynamic models – see, e.g., [6,7]) over a deterministic interval can be reduced to (1) with $X(t)$ being a fractional Brownian motion and T having Weibull distribution. We refer to Section 5 for details.

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We note that the additional variability of T may influence the form of the asymptotics of (1), leading to structures qualitatively different from (2). This was observed in [4], under the scenario that T has a regularly varying tail distribution (see also [1]).

Motivated by the above applications, in this paper, we focus on the exact asymptotics of (1) when T is a random variable, independent of $\{X(t)\}$, with asymptotically Weibullian tail distribution. In Theorem 3.1, we find the structural form of the asymptotics that holds for a wide class of Gaussian processes with stationary increments and convex variance function (see assumptions (A1)–(A3) in Section 2). Complementing this, in Corollary 3.2, we obtain an explicit form for the asymptotics, which appear to be Weibullian.

Additionally, for $\{X(t)\}$ being a fractional Brownian motion, we provide the exact asymptotics of (1) for the whole range of Hurst parameters $H \in (0, 1]$. It appears that in the case of $H < 1/2$ (concave variance function), the exact asymptotics takes a form qualitatively different from (2).

Finally, in Section 5, we apply the obtained results to the analysis of extremal behavior of fractional Laplace motion; see [6,7].

2. Notation and preliminary results

Let $\{X(t) : t \in [0, \infty)\}$ be a centered Gaussian process with stationary increments, a.s. continuous sample paths, $X(0) = 0$ a.s. and variance function $\sigma_X^2(t) := \mathbb{V}\mathrm{ar}(X(t))$. We assume that:

(A1) $\sigma_X^2(\cdot) \in C^1([0, \infty))$ is convex;
(A2) $\sigma_X^2(\cdot)$ is regularly varying at ∞ with parameter $\alpha_\infty \in (1, 2)$;
(A3) there exists $D > 0$ such that $\sigma_X^2(t) \le D t^{\alpha_\infty}$ for each $t \ge 0$.

We introduce the following classes of Gaussian processes:

- **fBm**: $X(t) = B_H(t)$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1]$, that is, a centered Gaussian process with stationary increments and $\sigma_{B_H}^2(t) = t^{2H}$ (note that (A2) is satisfied for $H \in (1/2, 1)$);
- **IG**: $X(t) = \int_0^t Z(s)\,\mathrm{d}s$, where $\{Z(t) : t \ge 0\}$ is a centered stationary Gaussian process with covariance function $R(t) = \mathbb{C}\mathrm{ov}(Z(s), Z(s+t))$ which is regularly varying at ∞ with parameter $\alpha_\infty - 2$.

In this paper, we analyze the asymptotics of

$$\mathbb{P}\Big(\sup_{t \in [0,T]} X(t) > u\Big) \tag{3}$$

as $u \to \infty$, where T is a non-negative random variable, independent of $\{X(t)\}$, with asymptotically Weibullian tail distribution, that is,

$$\mathbb{P}(T > t) = C t^\gamma \exp(-\beta t^\alpha)\big(1 + \mathrm{o}(1)\big) \tag{4}$$

as $t \to \infty$, where $\alpha, \beta, C > 0$, $\gamma \in \mathbb{R}$. We write $T \in \mathcal{W}(\alpha, \beta, \gamma, C)$ if T satisfies (4).

Let us introduce some notation. For given $H \in (0,1]$, by $\mathcal{H}_H$, we denote the *Pickands's constant* defined by the limit

$$\mathcal{H}_H = \lim_{T\to\infty} \frac{\mathcal{H}_H(T)}{T},$$

where $\mathcal{H}_H(T) := \mathbb{E}\exp(\sup_{t\in[0,T]} \sqrt{2}B_H(t) - t^{2H})$. Moreover, let $\Psi(u) := \mathbb{P}(\mathcal{N} > u)$, where $\mathcal{N}$ denotes the standard normal random variable. $\dot{\sigma}_X(t)$ denotes the first derivative of $\sigma_X(t)$ and $\dot{\sigma}_X^2(t) = 2\sigma_X(t)\dot{\sigma}_X(t)$ the first derivative of $\sigma_X^2(t)$.

Finally, we present a useful lemma, which is also of independent interest.

Lemma 2.1. *Let $X \in \mathcal{W}(\alpha_1,\beta_1,\gamma_1,C_1)$, $Y \in \mathcal{W}(\alpha_2,\beta_2,\gamma_2,C_2)$ be independent non-negative random variables. Then $X \cdot Y \in \mathcal{W}(\alpha,\beta,\gamma,C)$ with*

$$\alpha = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2},$$

$$\beta = \beta_1^{\alpha_2/(\alpha_1+\alpha_2)}\beta_2^{\alpha_1/(\alpha_1+\alpha_2)}\left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2/(\alpha_1+\alpha_2)} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_1/(\alpha_1+\alpha_2)}\right],$$

$$\gamma = \frac{\alpha_1\alpha_2 + 2\alpha_1\gamma_2 + 2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)},$$

$$C = \sqrt{2\pi}C_1C_2\frac{1}{\sqrt{\alpha_1+\alpha_2}}(\alpha_1\beta_1)^{(\alpha_2-2\gamma_1+2\gamma_2)/(2(\alpha_1+\alpha_2))}(\alpha_2\beta_2)^{(\alpha_1-2\gamma_2+2\gamma_1)/(2(\alpha_1+\alpha_2))}.$$

The proof of Lemma 2.1 is presented in Section 6.1.

3. Main results

In this section, we present the main results of the paper. We begin with the structural form of the analyzed asymptotics (Theorem 3.1), then we present an explicit asymptotic expansion (Corollary 3.2).

Theorem 3.1. *Let $X(t)$ be a centered Gaussian process with stationary increments and variance function that satisfies* (A1)–(A3) *and $T \in \mathcal{W}(\alpha,\beta,\gamma,C)$ be a non-negative random variable, independent of $\{X(t)\}$. Then, as $u \to \infty$,*

$$\mathbb{P}\Big(\sup_{s\in[0,T]} X(s) > u\Big) = \mathbb{P}\big(X(T) > u\big)\big(1+\mathrm{o}(1)\big) = \mathbb{P}\big(\sigma_X(T)\cdot\mathcal{N} > u\big)\big(1+\mathrm{o}(1)\big).$$

The proof of Theorem 3.1 is presented in Section 6.2.

Remark 3.1. It is tempting to ask to what extent (1) behaves as $\mathbb{P}(X(T) > u)$ for other (than Weibullian) distributions of T. Some limitations on the heaviness of the tail distribution of T can be inferred from [4], Theorem 2.1, which states that

$$\mathbb{P}\Big(\sup_{s\in[0,T]} X(s) > u\Big) = \mathrm{Const}\,\mathbb{P}\big(T > \sigma_X^{-1}(u)\big) \qquad \text{as } u \to \infty, \tag{5}$$

if T has regularly varying tail distribution at ∞. Thus, the asymptotics of (5) are qualitatively different from those observed in Theorem 3.1. We conjecture that an analog of Theorem 3.1 is also true for lighter-than-Weibullian tail distributions of T.

If the variance function of $\{X(t)\}$ is regular enough (in such a way that $\sigma_X(T)$ is asymptotically Weibullian), then the combination of Theorem 3.1 with Lemma 2.1 enables us to obtain the exact form of the asymptotics.

Corollary 3.2. *Let $X(t)$ be a centered Gaussian process with stationary increments and variance function that satisfies* (A1) *and $\sigma_X^2(t) = Dt^{\alpha_\infty} + o(t^{\alpha_\infty-\alpha})$ as $t \to \infty$ for $\alpha_\infty \in (1,2)$ and $D > 0$. If $T \in \mathcal{W}(\alpha,\beta,\gamma,C)$ is a non-negative random variable independent of $\{X(t)\}$, then*

$$\sup_{s\in[0,T]} X(s) \in \mathcal{W}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{C})$$

with

$$\widetilde{\alpha} = \frac{2\alpha}{\alpha+\alpha_\infty}, \qquad \widetilde{\beta} = \beta^{\alpha_\infty/(\alpha+\alpha_\infty)}\left(\frac{D}{2}\right)^{\alpha/(\alpha+\alpha_\infty)}\left(\left(\frac{\alpha}{\alpha_\infty}\right)^{\alpha_\infty/(\alpha+\alpha_\infty)} + \left(\frac{\alpha_\infty}{\alpha}\right)^{\alpha/(\alpha+\alpha_\infty)}\right),$$

$$\widetilde{\gamma} = \frac{2\gamma}{\alpha+\alpha_\infty}, \qquad \widetilde{C} = CD^{-1/\alpha_\infty}\sqrt{\frac{\alpha_\infty}{2(\alpha+\alpha_\infty)}}\left(\frac{\alpha_\infty}{2\alpha\beta}D^{\alpha_\infty/\alpha}\right)^{\gamma/(\alpha+\alpha_\infty)}.$$

The proof of Corollary 3.2 is given in Section 6.3.

Below, we apply the obtained asymptotics to IG processes. The family of fBm is analyzed separately in Section 4. Due to the self-similarity of fBm, we are able to give a proof (independent of Theorem 3.1) that covers the whole range of Hurst parameters $H \in (0,1]$.

Example 3.1. Let $T \in \mathcal{W}(\alpha,\beta,\gamma,C)$ and $X(t) = \int_0^t Z(s)\,\mathrm{d}s$, where $\{Z(s): s \ge 0\}$ is a centered stationary Gaussian process with continuous covariance function $R(t)$ such that $R(t) = Dt^{\alpha_\infty-2} + o(t^{\alpha_\infty-2-\alpha})$ as $t \to \infty$ with $\alpha_\infty \in (1,2)$. Following Karamata's theorem (see, e.g., [3], Proposition 1.5.8),

$$\sigma_X^2(t) = 2\int_0^t \mathrm{d}s \int_0^s R(v)\,\mathrm{d}v = \frac{2D}{\alpha_\infty(\alpha_\infty-1)}t^{\alpha_\infty} + o(t^{\alpha_\infty-\alpha})$$

as $t \to \infty$. Hence, by Corollary 3.2, we have

$$\sup_{t\in[0,T]} X(t) \in \mathcal{W}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{C})$$

with

$$\widetilde{\alpha} = \frac{2\alpha}{\alpha + \alpha_\infty},$$

$$\widetilde{\beta} = \beta^{\alpha_\infty/(\alpha+\alpha_\infty)} \left(\frac{D}{\alpha_\infty(\alpha_\infty - 1)} \right)^{\alpha/(\alpha+\alpha_\infty)} \left(\left(\frac{\alpha}{\alpha_\infty} \right)^{\alpha_\infty/(\alpha+\alpha_\infty)} + \left(\frac{\alpha_\infty}{\alpha} \right)^{\alpha/(\alpha+\alpha_\infty)} \right),$$

$$\widetilde{\gamma} = \frac{2\gamma}{\alpha + \alpha_\infty},$$

$$\widetilde{C} = C \left(\frac{2D}{\alpha_\infty(\alpha_\infty - 1)} \right)^{-1/\alpha_\infty} \sqrt{\frac{\alpha_\infty}{2(\alpha + \alpha_\infty)}} \left(\frac{\alpha_\infty}{2\alpha\beta} \left(\frac{2D}{\alpha_\infty(\alpha_\infty - 1)} \right)^{\alpha_\infty/\alpha} \right)^{\gamma/(\alpha+\alpha_\infty)}.$$

4. The case of fBm

In this section, we focus on the exact asymptotics of (3) for $\{X(t)\}$ being an fBm. The self-similarity of fBm, combined with Lemma 2.1, enables us to provide the following theorem.

Theorem 4.1. *Let $\{B_H(s) : s \geq 0\}$ be an fBm with Hurst parameter $H \in (0, 1]$ and $T \in \mathcal{W}(\alpha, \beta, \gamma, C)$ be a non-negative random variable independent of $\{B_H(s) : s \geq 0\}$. If:*

(i) $H \in (0, 1/2)$, *then*

$$\sup_{s\in[0,T]} B_H(s) \in \mathcal{W}\left(\frac{2\alpha}{2H + \alpha}, \beta_1, \frac{2\alpha - 3\alpha H + 2\gamma}{\alpha + 2H}, C_1 \right);$$

(ii) $H = 1/2$, *then*

$$\sup_{s\in[0,T]} B_H(s) \in \mathcal{W}\left(\frac{2\alpha}{2H + \alpha}, \beta_1, \frac{2\gamma}{\alpha + 2H}, 2C_2 \right);$$

(iii) $H \in (1/2, 1]$, *then*

$$\sup_{s\in[0,T]} B_H(s) \in \mathcal{W}\left(\frac{2\alpha}{2H + \alpha}, \beta_1, \frac{2\gamma}{\alpha + 2H}, C_2 \right),$$

where

$$\beta_1 = \beta^{2H/(2H+\alpha)} \left(\frac{1}{2} \left(\frac{\alpha}{H} \right)^{2H/(2H+\alpha)} + \left(\frac{\alpha}{H} \right)^{-\alpha/(2H+\alpha)} \right)$$

$$C_1 = \mathcal{H}_H \left(\frac{1}{2} \right)^{1/(2H)} \frac{C}{\sqrt{2H + \alpha}} H^{(\alpha+6H+2\gamma-2)/(2\alpha+4H)} (\alpha\beta)^{(1-2H-\gamma)/(\alpha+2H)},$$

$$C_2 = \frac{C\sqrt{H}}{\sqrt{\alpha + 2H}} \left(\frac{H}{\alpha\beta} \right)^{\gamma/(\alpha+2H)}.$$

The following lemma plays an important role in the proof of Theorem 4.1.

Lemma 4.2. *Let $B_H(\cdot)$ be an fBm with Hurst parameter $H \in (0,1]$. If*:

(i) $H \in (0, 1/2)$, *then*

$$\sup_{t\in[0,1]} B_H(t) \in \mathcal{W}\left(2, \frac{1}{2}, \frac{1}{H} - 3, \frac{1}{H\sqrt{\pi}} 2^{-(H+1)/(2H)}\right);$$

(ii) $H = 1/2$, *then*

$$\sup_{t\in[0,1]} B_H(t) \in \mathcal{W}\left(2, \frac{1}{2}, -1, \frac{2}{\sqrt{2\pi}}\right);$$

(iii) $H \in (1/2, 1]$, *then*

$$\sup_{t\in[0,1]} B_H(t) \in \mathcal{W}\left(2, \frac{1}{2}, -1, \frac{1}{\sqrt{2\pi}}\right).$$

The proof of Lemma 4.2 follows by a straightforward application of [9], Theorem D.3.

Proof of Theorem 4.1. Using the self-similarity of fBm, we have

$$\mathbb{P}\Big(\sup_{s\in[0,T]} B_H(s) > u\Big) = \mathbb{P}\Big(T^H \sup_{s\in[0,1]} B_H(s) > u\Big).$$

Note that $T^H \in \mathcal{W}(\frac{\alpha}{H}, \beta, \frac{\gamma}{H}, C)$ and (due to Lemma 4.2) $\sup_{s\in[0,1]} B_H(s)$ is asymptotically Weibullian.

Thus, all of the cases (i), (ii) and (iii) follow by a straightforward application of Lemma 2.1. □

Remark 4.1. Note that if $\mathbb{P}(T > t) = \exp(-At)$, then for a standard Brownian motion case, some straightforward calculations give

$$\mathbb{P}\Big(\sup_{t\in[0,T]} B_{1/2}(t) > u\Big) = \exp\big(-\sqrt{2A}u\big)$$

for each $u \geq 0$.

5. Application to extremes of fractional Laplace motion

In this section, we apply Theorem 4.1 to the analysis of the asymptotics of the supremum distribution of fractional Laplace motion over a deterministic interval.

Following [7], we recall the definition of fractional Laplace motion.

Let $\{\Gamma_t; t \geq 0\}$ be a gamma process with parameter $\nu > 0$, that is, a Lévy process such that the increments $\Gamma_{t+s} - \Gamma_t$ have gamma distributions $\mathcal{G}(s/\nu, 1)$ with density

$$f(x) = \frac{1}{\Gamma(s/\nu)} x^{s/\nu - 1} \exp(-x),$$

where $\Gamma(\cdot)$ denotes the gamma function.

Then, by fractional Laplace motion *fLm*(σ, ν), we denote the process $\{L_H(t); t \geq 0\}$ defined as follows:

$$\{L_H(t); t \geq 0\} \stackrel{d}{=} \{\sigma B_H(\Gamma_t); t \geq 0\}.$$

A standard fractional Laplace motion corresponds to $\sigma = \nu = 1$ and is denoted by fLm. We refer to Kozubowski *et al.* [6,7] for motivations of interest in the analysis of this class of stochastic processes.

Before we present the asymptotics of $\mathbb{P}(\sup_{s\in[0,S]} L_H(s) > u)$, let us observe that for given $S > 0$, we have $\Gamma_S \in \mathcal{W}(1, 1, S-1, \frac{1}{\Gamma(S)})$. Indeed, applying Karamata's theorem (see, e.g., [3], Proposition 1.5.10), we have

$$\mathbb{P}(\Gamma_S > u) = \frac{1}{\Gamma(S)} \int_u^\infty x^{S-1} \mathrm{e}^{-x}\,\mathrm{d}x = \frac{1}{\Gamma(S)} \int_{\mathrm{e}^u}^\infty (\log y)^{S-1} y^{-2}\,\mathrm{d}y = \frac{1}{\Gamma(S)} u^{S-1} \mathrm{e}^{-u}\big(1 + \mathrm{o}(1)\big)$$

as $u \to \infty$.

In the following proposition, we give the exact asymptotics of the supremum of fLm for $H > 1/2$. Let

$$m_H = \left(\frac{1}{2}\right)^{1/(2H+1)} \left[\left(\frac{1}{2H}\right)^{2H/(2H+1)} + \left(\frac{1}{2H}\right)^{1/(2H+1)}\right].$$

Proposition 5.1. *Let L_H be a standard fLm. If $H > 1/2$, then*

$$\sup_{s\in[0,S]} L_H(s) \in \mathcal{W}\left(\frac{2}{2H+1}, m_H, \frac{2S-2}{1+2H}, \frac{H^{(S+2H)/(2+4H)}}{\Gamma(S)\sqrt{1+2H}}\right).$$

Proof. First, we consider the lower bound. We observe that

$$\mathbb{P}\Big(\sup_{s\in[0,S]} B_H(\Gamma_s) > u\Big) \geq \mathbb{P}\big(B_H(\Gamma_S) > u\big) = \mathbb{P}\big((\Gamma_S)^H \mathcal{N} > u\big).$$

Combining the above with the facts that $(\Gamma_S)^H \in \mathcal{W}(\frac{1}{H}, 1, \frac{S-1}{H}, \frac{1}{\Gamma(S)})$ and $\mathcal{N} \in \mathcal{W}(2, \frac{1}{2}, -1, \frac{1}{\sqrt{2\pi}})$, together with Lemma 2.1, we obtain a tight asymptotic lower bound.

We now focus on the upper bound. Using the fact that sample paths of a gamma process are non-decreasing, we get

$$\mathbb{P}\Big(\sup_{s\in[0,S]} B_H(\Gamma_s) > u\Big) \leq \mathbb{P}\Big(\sup_{s\in[0,\Gamma_S]} B_H(s) > u\Big).$$

In order to complete the proof, it suffices to apply (iii) of Theorem 4.1. $\square$

Remark 5.1. The case $H \leq 1/2$ should be handled with care. Applying the argument presented in the proof of Proposition 5.1 gives

$$\mathbb{P}\Big(\sup_{s\in[0,S]} L_H(s) > u\Big) \geq \frac{1}{\Gamma(S)\sqrt{1+2H}} H^{(S+2H)/(2+4H)} u^{(2S-2)/(1+2H)} \exp\big(-m_H u^{2/(2H+1)}\big)\big(1+\mathrm{o}(1)\big)$$

as $u \to \infty$, and

$$\mathbb{P}\Big(\sup_{s\in[0,S]} L_H(s) > u\Big) \leq \frac{1}{\Gamma(S)} 2^{-1/(2H)} H^{-(2H+S+4)/(4H+2)} \mathcal{H}_H u^{(2SH-4H+1)/(H(2H+1))} \times \exp\big(-m_H u^{2/(2H+1)}\big)\big(1+\mathrm{o}(1)\big)$$

as $u \to \infty$. The above leads to the following logarithmic asymptotics for $H \in (0, \frac{1}{2}]$:

$$\frac{\log(\mathbb{P}\sup_{s\in[0,S]} L_H(s) > u)}{u^{2/(2H+1)}} = -m_H\big(1+\mathrm{o}(1)\big)$$

as $u \to \infty$.

In the case $H = \frac{1}{2}$, $S = 1$, due to Remark 4.1, we have

$$\frac{1}{2}\exp\big(-\sqrt{2}u\big) \leq P\Big(\sup_{s\in[0,1]} L_{1/2}(s) > u\Big) \leq \exp\big(-\sqrt{2}u\big)$$

for each $u \geq 0$. We conjecture that the exact asymptotics for $H \leq 1/2$ are influenced by the distribution of jumps of the gamma process.

6. Proofs

In this section, we present detailed proofs of Lemma 2.1, Theorem 3.1 and Corollary 3.2.

6.1. Proof of Lemma 2.1

We begin by considering the asymptotic

$$\int_{U(x_0(u))} f(x,u)\exp[S(x,u)]\,\mathrm{d}x$$

as $u \to \infty$ for particular forms of $f(x,u)$ and $S(x,u)$, where $x_0(u)$ denotes the point at which the function $S(x,u)$ of x achieves its maximum over $[0,\infty)$ and

$$U(x_0(u)) = \{x : |x - x_0(u)| \leq q(u)|S''_{x,x}(x_0(u),u)|^{-1/2}\}$$

for some suitable chosen function $q(u)$.

The following theorem can be found in, for example, [5], Theorem 2.2.

Lemma 6.1 (Fedoryouk). *Suppose that there exists a function $q(u) \to \infty$ as $u \to \infty$ such that*

$$S''_{x,x}(x,u) = S''_{x,x}(x_0(u),u)[1+\mathrm{o}(1)] \tag{6}$$

and

$$f(x,u) = f(x_0(u),u)[1+\mathrm{o}(1)] \tag{7}$$

as $u \to \infty$ uniformly for $x \in U(x_0(u))$. Then

$$\int_{U(x_0(u))} f(x,u)\exp[S(x,u)]\,\mathrm{d}x = \sqrt{-\frac{2\pi}{S''_{x,x}(x_0(u),u)}}\, f(x_0(u),u)\exp[S(x_0(u),u)]\big(1+\mathrm{o}(1)\big)$$

as $u \to \infty$.

Lemma 6.1 enables us to get the following exact asymptotics, which will play an important role in further analysis.

Lemma 6.2. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $\gamma \in \mathbb{R}$ and $a(u) = u^{\alpha_1/(2(\alpha_1+\alpha_2))}$, $A(u) = u^{2\alpha_1/(\alpha_1+\alpha_2)}$. Then*

$$\int_{a(u)}^{A(u)} x^{\gamma} \exp\left(-\frac{\beta_1 u^{\alpha_1}}{x^{\alpha_1}} - \beta_2 x^{\alpha_2}\right)\mathrm{d}x = C u^{\delta} \exp[-\beta_3 u^{\alpha_3}]\big(1+\mathrm{o}(1)\big)$$

as $u \to \infty$, where

$$\alpha_3 = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \qquad \beta_3 = \beta_1^{\alpha_2/(\alpha_1+\alpha_2)} \beta_2^{\alpha_1/(\alpha_1+\alpha_2)} \left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2/(\alpha_1+\alpha_2)} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_1/(\alpha_1+\alpha_2)}\right],$$

$$\delta = \frac{\alpha_1(-\alpha_2+2\gamma+2)}{2(\alpha_1+\alpha_2)},$$

$$C = \sqrt{2\pi}\frac{1}{\sqrt{\alpha_1+\alpha_2}} (\alpha_1\beta_1)^{(-\alpha_2+2\gamma+2)/(2(\alpha_1+\alpha_2))} (\alpha_2\beta_2)^{(-\alpha_1-2\gamma-2)/(2(\alpha_1+\alpha_2))}.$$

Proof. Let $x_0(u) = (\frac{\alpha_1\beta_1}{\alpha_2\beta_2})^{1/(\alpha_1+\alpha_2)} u^{\alpha_1/(\alpha_1+\alpha_2)}$, $r(u) = u^{(1-\varepsilon)\alpha_1/(\alpha_1+\alpha_2)}$ for some $\varepsilon \in (0, \min(\alpha_2/2, 1))$ and $\alpha_3, \beta_3, \delta, C$ be as in Lemma 6.2. It is convenient to decompose the analyzed integral in the following way:

$$\int_{a(u)}^{A(u)} x^{\gamma} \exp\left(-\frac{\beta_1 u^{\alpha_1}}{x^{\alpha_1}} - \beta_2 x^{\alpha_2}\right)\mathrm{d}x$$
$$= \int_{a(u)}^{x_0(u)-r(u)} + \int_{x_0(u)-r(u)}^{x_0(u)+r(u)} + \int_{x_0(u)+r(u)}^{A(u)} = I_1 + I_2 + I_3.$$

Applying Lemma 6.1, we have, as $u \to \infty$,

$$I_2 = \int_{x_0(u)-r(u)}^{x_0(u)+r(u)} x^\gamma \exp\biggl(-\frac{\beta_1 u^{\alpha_1}}{x^{\alpha_1}} - \beta_2 x^{\alpha_2}\biggr)\,\mathrm{d}x = Cu^\delta \exp[-\beta_3 u^{\alpha_3}]\bigl(1+\mathrm{o}(1)\bigr). \tag{8}$$

In order to complete the proof, it suffices to show that $I_1, I_3 = \mathrm{o}(I_2)$ as $u \to \infty$. Since proofs for I_1 and I_3 are similar, we focus on the argument that shows $I_1 = \mathrm{o}(I_2)$ as $u \to \infty$. Without loss of generality, we assume that $\gamma > 0$. Then

$$I_1 \le \bigl(x_0(u) - a(u)\bigr)^\gamma \bigl(x_0(u) - r(u) - a(u)\bigr) \exp\biggl(-\frac{\beta_1 u^{\alpha_1}}{(x_0(u)-r(u))^{\alpha_1}} - \beta_2\bigl(x_0(u)-r(u)\bigr)^{\alpha_2}\biggr),$$

which, combined with the fact that (using a Taylor expansion)

$$\begin{aligned} &-\frac{\beta_1 u^{\alpha_1}}{(x_0(u)-r(u))^{\alpha_1}} - \beta_2\bigl(x_0(u)-r(u)\bigr)^{\alpha_2} \\ &\quad = -\beta_3 u^{\alpha_3} - \frac{1}{2}(\alpha_1+\alpha_2)(\alpha_1\beta_1)^{(\alpha_2-2)/(\alpha_1+\alpha_2)}(\alpha_2\beta_2)^{(\alpha_1+2)/(\alpha_1+\alpha_2)} \\ &\qquad \times u^{\alpha_1(\alpha_2-2\varepsilon)/(\alpha_1+\alpha_2)}\bigl(1+\mathrm{o}(1)\bigr) \end{aligned} \tag{9}$$

as $u \to \infty$, straightforwardly implies that $I_1 = \mathrm{o}(I_3)$ as $u \to \infty$ (since $\varepsilon < \alpha_2/2$). This completes the proof. □

Proof of Lemma 2.1. Let $X \in \mathcal{W}(\alpha_1, \beta_1, \gamma_1, C_1)$ and $Y \in \mathcal{W}(\alpha_2, \beta_2, \gamma_2, C_2)$ be independent non-negative random variables. Define $a(u) = u^{\alpha_1/(2(\alpha_1+\alpha_2))}$, $A(u) = u^{2\alpha_1/(\alpha_1+\alpha_2)}$ and consider the decomposition

$$\begin{aligned} \mathbb{P}(XY > u) &= \int_0^\infty \mathbb{P}\biggl(X > \frac{u}{y}\biggr)\,\mathrm{d}F_Y(y) \\ &= \int_0^{a(u)} \mathbb{P}\biggl(X > \frac{u}{y}\biggr)\,\mathrm{d}F_Y(y) + \int_{a(u)}^{A(u)} \mathbb{P}\biggl(X > \frac{u}{y}\biggr)\,\mathrm{d}F_Y(y) \\ &\quad + \int_{A(u)}^\infty \mathbb{P}\biggl(X > \frac{u}{y}\biggr)\,\mathrm{d}F_Y(y) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We analyze each of the integrals I_1, I_2, I_3 separately. In order to simplify notation, we introduce $h_1(u) = C_1 u^{\gamma_1}\exp(-\beta_1 u^{\alpha_1})$ and $h_2(u) = C_2 u^{\gamma_2}\exp(-\beta_2 u^{\alpha_2})$.

Integral I_1. Since $X \in \mathcal{W}(\alpha_1, \beta_1, \gamma_1, C_1)$, for given $\varepsilon > 0$ and u large enough, we have

$$\begin{aligned} I_1 &\le (1+\varepsilon) h_1\biggl(\frac{u}{a(u)}\biggr) \\ &= (1+\varepsilon) C_1 u^{(\alpha_1+2\alpha_2)/(2(\alpha_1+\alpha_2))\gamma_1} \exp\Bigl(-\beta_1 u^{\alpha_1\alpha_2/(\alpha_1+\alpha_2)+\alpha_1^2/(2(\alpha_1+\alpha_2))}\Bigr). \end{aligned}$$

Integral I_3. For u sufficiently large, we have, as $u \to \infty$,

$$I_3 \le \mathbb{P}\big(Y > A(u)\big) = C_2 u^{2\alpha_1\gamma_2/(\alpha_1+\alpha_2)} \exp\big(-\beta_2 u^{2\alpha_1\alpha_2/(\alpha_1+\alpha_2)}\big)\big(1+\mathrm{o}(1)\big).$$

Integral I_2. We find upper and lower bounds of I_2 separately. Using the fact that X, Y are asymptotically Weibullian, we get, for sufficiently large u,

$$\begin{aligned}
&\int_{a(u)}^{A(u)} \mathbb{P}\left(X > \frac{u}{y}\right) \mathrm{d}F_Y(y) \ge (1-\varepsilon) \int_{a(u)}^{A(u)} h_1\left(\frac{u}{y}\right) \mathrm{d}F_Y(y) \\
&\ge (1-\varepsilon) \int_{a(u)}^{A(u)} \frac{\partial}{\partial y}\left[h_1\left(\frac{u}{y}\right)\right] \mathbb{P}(Y > y)\,\mathrm{d}y + (1-\varepsilon) h_1\left(\frac{u}{a(u)}\right) \mathbb{P}\big(Y > (a(u))\big) \\
&\quad - (1-\varepsilon) h_1\left(\frac{u}{A(u)}\right) \mathbb{P}\big(Y > A(u)\big) \\
&\ge (1-\varepsilon)^2 \int_{a(u)}^{A(u)} \frac{\partial}{\partial y}\left[h_1\left(\frac{u}{y}\right)\right] h_2(y)\,\mathrm{d}y + (1-\varepsilon)^2 h_1\left(\frac{u}{a(u)}\right) h_2(a(u)) \\
&\quad - (1-\varepsilon^2) h_1\left(\frac{u}{A(u)}\right) h_2(A(u)) \\
&= (1-\varepsilon)^2 I_4 + (1-\varepsilon)^2 R_1 - (1-\varepsilon^2) R_2.
\end{aligned}$$

Analogously, for sufficiently large u, we have the upper bound

$$I_2 \le (1+\varepsilon)^2 I_4 + (1+\varepsilon)^2 R_1 - (1-\varepsilon^2) R_2.$$

Additionally,

$$\begin{aligned}
R_1 &= h_1\left(\frac{u}{a(u)}\right) h_2(a(u)) \le h_1\left(\frac{u}{a(u)}\right) \\
&= C_1 u^{(\alpha_1\gamma_1+2\alpha_2\gamma_1)/(2(\alpha_1+\alpha_2))} \exp\big(-\beta_1 u^{\alpha_1\alpha_2/(\alpha_1+\alpha_2)+\alpha_1^2/(2(\alpha_1+\alpha_1))}\big)
\end{aligned}$$

and

$$R_2 = h_1\left(\frac{u}{A(u)}\right) h_2(A(u)) \le h_2(A(u)) = C_2 u^{2\alpha_1\gamma_2/(\alpha_1+\alpha_2)} \exp\big(-\beta_2 u^{2\alpha_1\alpha_2/(\alpha_1+\alpha_2)}\big).$$

Finally, applying Lemma 6.2, we find the asymptotics of integral I_4:

$$I_4 = C_3 u^{\gamma_3} \exp(-\beta_3 u^{\alpha_3})\big(1+\mathrm{o}(1)\big)$$

as $u \to \infty$, with

$$\alpha_3 = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \qquad \beta_3 = \beta_1^{\alpha_2/(\alpha_1+\alpha_2)} \beta_2^{\alpha_1/(\alpha_1+\alpha_2)} \left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2/(\alpha_1+\alpha_2)} + \left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_1/(\alpha_1+\alpha_2)}\right],$$

$$\gamma_3 = \frac{\alpha_1\alpha_2 + 2\alpha_1\gamma_2 + 2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)},$$

$$C_3 = \sqrt{2\pi}C_1C_2\frac{1}{\sqrt{\alpha_1+\alpha_2}}(\alpha_1\beta_1)^{(\alpha_2-2\gamma_1+2\gamma_2)/(2(\alpha_1+\alpha_2))}(\alpha_2\beta_2)^{(\alpha_1-2\gamma_2+2\gamma_1)/(2(\alpha_1+\alpha_2))}.$$

Since $I_1, I_2, R_1, R_2 = \mathrm{o}((C_3u^{\gamma_3}\exp(-\beta_3u^{\alpha_3}))$ as $u\to\infty$, we have

$$\mathbb{P}(X\cdot Y>u) = I_4\big(1+\mathrm{o}(1)\big) = C_3u^{\gamma_3}\exp(-\beta_3u^{\alpha_3})\big(1+\mathrm{o}(1)\big)$$

as $u\to\infty$. This completes the proof. □

6.2. Proof of Theorem 3.1

Let $\tau_1 = \frac{2}{\alpha_\infty+2\alpha}$, $\tau_2 = \frac{4}{2\alpha_\infty+\alpha}$ and $\delta = \delta(u) = \frac{\sigma_X^3(t)}{\dot\sigma_X(t)}2u^{-2}\log^2 u$. Additionally, let $\{Z(s): s\geq 0\}$ be a centered stationary Gaussian process with covariance function $\mathbb{C}\mathrm{ov}(Z(s), Z(s+t)) = \mathrm{e}^{-t^{\alpha_\infty}}$. The existence of such a process is guaranteed by the fact that $\alpha_\infty < 2$, which implies that the covariance of $Z(\cdot)$ is positively defined; see, for example, proof of [8], Theorem D.3.

The proof of Theorem 3.1 is based on the following two lemmas.

Lemma 6.3. *Let $X(t)$ be a centered Gaussian process with stationary increments such that conditions* (A1)–(A3) *are satisfied. Then, for sufficiently large u,*

$$\mathbb{P}\Big(\sup_{s\in[0,t-\delta]} X(s) > u\Big) \leq \Psi\bigg(\frac{u}{\sigma_X(t)}\bigg)\exp\big(-\log^2(u)/2\big)$$

uniformly for $t := t(u)\in[u^{\tau_1}, u^{\tau_2}]$.

Proof. Let $t := t(u)\in[u^{\tau_1}, u^{\tau_2}]$. Observe that $\sigma_X^2(t)2u^{-2}\log^2(u)\to 0$ uniformly for $t\in[u^{\tau_1}, u^{\tau_2}]$ as $u\to\infty$ and $\lim_{t\to\infty}\frac{\sigma_X(t)}{t\dot\sigma_X(t)} = \lim_{t\to\infty}\frac{2\sigma_X^2(t)}{t\dot\sigma_X^2(t)} = \frac{2}{\alpha_\infty}$ (due to [3], formula (1.11.1)). Hence,

$$\delta(u) = \mathrm{o}(t) \qquad \text{as } u\to\infty. \tag{10}$$

Now, for sufficiently large u, we consider the following decomposition:

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{s\in[0,t-\delta]} X(s) > u\Big)\\
&\quad\leq \mathbb{P}\Big(\sup_{s\in[0,1]} X(s) > u\Big) \qquad (11)\\
&\qquad + \sum_{k=0}^{(D/\sigma_X^2(1))^{1/\alpha_\infty}[t-\delta]} \mathbb{P}\bigg(\sup_{s\in[1+(\sigma_X^2(1)/D)^{1/\alpha_\infty}k,\,1+(\sigma_X^2(1)/D)^{1/\alpha_\infty}(k+1)]} \frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(t-\delta)}\bigg).
\end{aligned}$$

According to the Borell inequality (see, e.g., [2], Theorem 2.1), the first term is bounded by

$$\mathbb{P}\Big(\sup_{s\in[0,1]} X(s) > u\Big) \le \exp\bigg(-\frac{(u-\mathbb{E}\sup_{s\in[0,1]} X(s))^2}{2}\bigg)$$

as $u \to \infty$.

Due to (A1), (A3), for each $v, w \ge 1$ such that $|v-w| \le (\frac{\sigma_X^2(1)}{D})^{1/\alpha_\infty}$,

$$\mathbb{C}\mathrm{ov}\bigg(\frac{X(v)}{\sigma_X(v)}, \frac{X(w)}{\sigma_X(w)}\bigg) \ge \mathbb{C}\mathrm{ov}\bigg(Z\bigg(\bigg(\frac{D}{\sigma_X^2(1)}\bigg)^{1/\alpha_\infty} v\bigg), Z\bigg(\bigg(\frac{D}{\sigma_X^2(1)}\bigg)^{1/\alpha_\infty} w\bigg)\bigg).$$

Thus, Slepian's inequality (see, e.g., [9], Theorem C.1) combined with [9], Theorem D.2, straightforwardly leads to the following upper bound of (11):

$$\begin{aligned}
&\bigg(\frac{D}{\sigma_X^2(1)}\bigg)^{1/\alpha_\infty}[t-\delta]\mathbb{P}\bigg(\sup_{s\in[0,(\sigma_X^2(1)/D)^{1/\alpha_\infty}]} Z(s) > \frac{u}{\sigma_X(t-\delta)}\bigg)\\
&\quad = \mathcal{H}_{\alpha_\infty} t \bigg(\frac{u}{\sigma_X(t)}\bigg)^{2/\alpha_\infty} \Psi\bigg(\frac{u}{\sigma_X(t-\delta)}\bigg)\big(1+\mathrm{o}(1)\big)
\end{aligned} \tag{12}$$

as $u \to \infty$. Hence, in order to complete the proof, it suffices to note that

$$\begin{aligned}
\Psi\bigg(\frac{u}{\sigma_X(t-\delta)}\bigg) &\le 4\Psi\bigg(\frac{u}{\sigma_X(t)}\bigg)\exp\bigg(-\frac{u^2(\sigma_X^2(t)-\sigma_X^2(t-\delta))}{2\sigma_X^2(t)\sigma_X^2(t-\delta)}\bigg)\\
&\le 4\Psi\bigg(\frac{u}{\sigma_X(t)}\bigg)\exp\bigg(-\frac{u^2\delta 2\sigma_X(t-\theta\delta)\dot{\sigma}_X(t-\theta\delta)}{2\sigma_X^4(t)}\bigg)\\
&\le 4\Psi\bigg(\frac{u}{\sigma_X(t)}\bigg)\exp\bigg(-\frac{u^2\delta\sigma_X(t)\dot{\sigma}_X(t)}{2\sigma^4(t)}\bigg) \qquad (13)\\
&= 4\Psi\bigg(\frac{u}{\sigma_X(t)}\bigg)\exp(-\log^2(u)), \qquad (14)
\end{aligned}$$

where $\theta \in (0,1)$, and (13) is a consequence of (10) and of the fact that, by condition (A1), $\dot{\sigma}_X^2 = 2\sigma_X(t)\dot{\sigma}_X(t)$ is monotone and (in view of [3], formula (1.11.1)) regularly varying at ∞.

Thus, combining (11) with (12) and (14), for sufficiently large u,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{s\in[0,t-\delta]} X(s) > u\Big) &\le 4\mathcal{H}_{\alpha_\infty} t \cdot \bigg(\frac{u}{\sigma_X(t)}\bigg)^{2/\alpha_\infty}\Psi\bigg(\frac{u}{\sigma_X(t)}\bigg)\exp\big(-\log^2(u)\big)\big(1+\mathrm{o}(1)\big)\\
&\le \Psi\bigg(\frac{u}{\sigma_X(t)}\bigg)\exp\big(-\log^2(u)/2\big),
\end{aligned}$$

uniformly for $t \in [u^{\tau_1}, u^{\tau_2}]$. This completes the proof. □

Lemma 6.4. *Let $X(t)$ be a centered Gaussian process with stationary increments such that conditions* (A1)–(A3) *are satisfied. Then, for sufficiently large u,*

$$\mathbb{P}\Big(\sup_{s\in[t-\delta,t]} X(s) > u\Big) \le (1+\varepsilon)\Psi\bigg(\frac{u}{\sigma_X(t)}\bigg)$$

uniformly for $t := t(u) \in [u^{\tau_1}, u^{\tau_2}]$.

Proof. Let $\varepsilon > 0$. Then

$$\mathbb{P}\Big(\sup_{s\in[t-\delta,t]} X(s) > u\Big) \le \mathbb{P}\bigg(\sup_{s\in[t-\delta,t]} \frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(t)}\bigg).$$

Using the fact that for each $v, w \in [t-\delta, t]$,

$$\mathbb{C}\mathrm{ov}\bigg(\frac{X(v)}{\sigma_X(v)}, \frac{X(w)}{\sigma_X(w)}\bigg) \ge \mathbb{C}\mathrm{ov}\bigg(Z\bigg(\bigg(\frac{2D}{\sigma_X^2(t)}\bigg)^{1/\alpha_\infty} v\bigg), Z\bigg(\bigg(\frac{2D}{\sigma_X^2(t)}\bigg)^{1/\alpha_\infty} w\bigg)\bigg),$$

Slepian's inequality gives

$$\begin{aligned}
&\mathbb{P}\bigg(\sup_{s\in[t-\delta,t]} \frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(t)}\bigg)\\
&\quad\le \mathbb{P}\bigg(\sup_{s\in[t-\delta,t]} Z\bigg(\bigg(\frac{2D}{\sigma_X^2(t)}\bigg)^{1/\alpha_\infty} s\bigg) > \frac{u}{\sigma_X(t)}\bigg) \qquad (15)\\
&\quad= \mathbb{P}\bigg(\sup_{s\in[0,(2D)^{1/\alpha_\infty}u^{2/\alpha_\infty}\delta(\sigma_X(t))^{-4/\alpha_\infty}(u/\sigma_X(t))^{-2/\alpha_\infty}]} Z(s) > \frac{u}{\sigma_X(t)}\bigg).
\end{aligned}$$

Observe that for each $\varepsilon_1 > 0$, there exists u large enough such that $(2D)^{1/\alpha_\infty}u^{2/\alpha_\infty} \times \delta(\sigma_X(t))^{-4/\alpha_\infty} \le \varepsilon_1$ uniformly for $t \in [u^{\tau_1}, u^{\tau_2}]$, which, combined with [9], Theorem D.1, implies the following upper bound for (15):

$$\mathbb{P}\bigg(\sup_{s\in[0,\varepsilon_1(u/\sigma_X(t))^{-2/\alpha_\infty}]} Z(s) > \frac{u}{\sigma_X(t)}\bigg) \le (1+\varepsilon_1)\mathcal{H}_{\alpha_\infty}(\varepsilon_1)\Psi\bigg(\frac{u}{\sigma_X(t)}\bigg) \le (1+\varepsilon)\Psi\bigg(\frac{u}{\sigma_X(t)}\bigg),$$

where the last inequality is due to the fact that $\mathcal{H}_{\alpha_\infty}(t) \to 1$ as $t \to 0$.

This completes the proof. □

Proof of Theorem 3.1. Since the lower bound is immediate, we focus on the analysis of the upper bound. We have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{s\in[0,T]} X(s) > u\Big)\\
&\quad\le \int_0^{u^{\tau_1}} \mathbb{P}\Big(\sup_{s\in[0,t]} X(s) > u\Big)\,\mathrm{d}F_T(t) + \int_{u^{\tau_1}}^{u^{\tau_2}} \mathbb{P}\Big(\sup_{s\in[0,t-\delta]} X(s) > u\Big)\,\mathrm{d}F_T(t)
\end{aligned}$$

$$+\int_{u^{\tau_1}}^{u^{\tau_2}} \mathbb{P}\Big(\sup_{s\in[t-\delta,t]} X(s) > u\Big)\,\mathrm{d}F_T(t) + \int_{u^{\tau_2}}^{\infty} \mathbb{P}\Big(\sup_{s\in[0,t]} X(s) > u\Big)\,\mathrm{d}F_T(t)$$
$$= I_1 + I_2 + I_3 + I_4.$$

We now investigate the asymptotic behavior of each of the integrals.

Integral I_1.

$$I_1 \le \mathbb{P}\Big(\sup_{s\in[0,1]} X(s) > u\Big) + \mathbb{P}\Big(\sup_{s\in[1,u^{\tau_1}]} X(s) > u\Big).$$

Following an argument analogous to that given in the proof of Lemma 6.3, we obtain the following asymptotic upper bound for the above sum:

$$\mathrm{Const}\, u^{\tau_1}\left(\frac{u}{\sigma_X(u^{\tau_1})}\right)^{2/\alpha_\infty} \Psi\left(\frac{u}{\sigma(u^{\tau_1})}\right) \le \exp\big(-u^{2\alpha/(\alpha+\alpha_\infty)+\varepsilon}\big)\big(1+\mathrm{o}(1)\big) \tag{16}$$

as $u \to \infty$, for some $\varepsilon > 0$.

Integral I_2. According to Lemma 6.3, for all $t \in [u^{\tau_1}, u^{\tau_2}]$ and for u large enough,

$$\mathbb{P}\Big(\sup_{s\in[0,t-\delta]} X(s) > u\Big) \le \Psi\left(\frac{u}{\sigma_X(t)}\right)\exp\big(-\log^2(u)/2\big).$$

Hence,

$$\begin{aligned} I_2 &\le \exp\big(-\log^2(u)/2\big)\int_0^\infty \Psi\left(\frac{u}{\sigma_X(t)}\right)\mathrm{d}F_T(t) \\ &= \exp\big(-\log^2(u)/2\big)\mathbb{P}\big(X(T) > u\big) = \mathrm{o}\big(\mathbb{P}\big(X(T) > u\big)\big). \end{aligned} \tag{17}$$

Integral I_3. Due to Lemma 6.4, for each $\varepsilon > 0$ and u large enough,

$$I_3 \le (1+\varepsilon)\int_0^\infty \psi\left(\frac{u}{\sigma_X(t)}\right)\mathrm{d}F_T(t) = (1+\varepsilon)\mathbb{P}\big(X(T) > u\big). \tag{18}$$

Integral I_4.

$$I_4 = \mathbb{P}(T > u^{\tau_2}) \le \exp\big(-u^{2\alpha/(\alpha+\alpha_\infty)+\varepsilon}\big)\big(1+\mathrm{o}(1)\big)$$

as $u \to \infty$, for some $\varepsilon > 0$.

Observe that for each $\epsilon > 0$ and sufficiently large u,

$$\begin{aligned} \mathbb{P}\big(X(T) > u\big) &= \mathbb{P}\big(\sigma_X(T)\mathcal{N} > u\big) \ge \mathbb{P}\big(\sigma_X(T) > u^{\alpha_\infty/(\alpha+\alpha_\infty)}\big)\mathbb{P}\big(\mathcal{N} > u^{\alpha/(\alpha+\alpha_\infty)}\big) \\ &\ge \exp\big(-u^{2\alpha/(\alpha+\alpha_\infty)+\epsilon}\big). \end{aligned}$$

Thus $I_1, I_2, I_4 = \mathrm{o}(I_3)$ as $u \to \infty$, which, in view of (18), implies that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} X(t) > u\Big) \le (1+\varepsilon)\mathbb{P}\big(X(T) > u\big)$$

for each $\varepsilon > 0$ and sufficiently large u. This completes the proof. □

6.3. Proof of Corollary 3.2

By a straightforward inspection, we observe that $\sigma_X^2(t)$ satisfies (A1)–(A3). Thus, in view of Theorem 3.1, we have

$$\mathbb{P}\Big(\sup_{s\in[0,T]} X(s) > u\Big) = \mathbb{P}\big(\sigma_X(T)\cdot\mathcal{N} > u\big)\big(1+\mathrm{o}(1)\big)$$

as $u\to\infty$. Since $\mathcal{N}\in\mathcal{W}(2,1/2,-1,1/\sqrt{2\pi})$, due to Lemma 2.1, in order to complete the proof, it suffices to show that $\sigma_X(T)\in\mathcal{W}(\frac{2\alpha}{\alpha_\infty},\beta D^{\alpha/\alpha_\infty},\frac{2\gamma}{\alpha_\infty},CD^{-1/\alpha_\infty})$. In view of

$$\mathbb{P}\big(\sigma_X(T) > u\big) = C((\sigma_X)^{-1}(u))^\gamma \exp(-\beta((\sigma_X)^{-1}(u))^\alpha)\big(1+\mathrm{o}(1)\big)$$

and the fact that $(\sigma_X)^{-1}(u) = D^{-1/\alpha_\infty}u^{2/\alpha_\infty}(1+\mathrm{o}(1))$, this reduces to

$$\exp(-\beta((\sigma_X)^{-1}(u))^\alpha) = \exp\bigg(-\frac{\beta}{D^{\alpha/\alpha_\infty}}u^{2\alpha/\alpha_\infty}\bigg)\big(1+\mathrm{o}(1)\big)$$

as $u\to\infty$, which follows by inspection.

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