ASYMPTOTICS OF THE FAST DIFFUSION EQUATION VIA ENTROPY ESTIMATES

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Abstract. We consider non-negative solutions of the fast diffusion equation $u_t = \Delta u^m$ with $m \in (0,1)$, in the Euclidean space $\mathbb{R}^d$, $d \geq 3$, and study the asymptotic behavior of a natural class of solutions, in the limit corresponding to $t \to \infty$ for $m \geq m_c = (d-2)/d$, or as t approaches the extinction time when $m < m_c$. For a class of initial data we prove that the solution converges with a polynomial rate to a self-similar solution, for t large enough if $m \geq m_c$, or close enough to the extinction time if $m < m_c$. Such results are new in the range $m \leq m_c$ where previous approaches fail. In the range $m_c < m < 1$ we improve on known results.

1. Introduction

We study the Cauchy problem for the fast diffusion equation posed in the whole Euclidean space, that is, we consider the solutions $u(\tau, y)$ of

$$\begin{cases} \partial_\tau u = \Delta u^m \\ u(0,\cdot) = u_0 \,, \end{cases} \tag{1.1}$$

where $m \in (0,1)$ (which means fast diffusion) and $(\tau, y) \in (0,T) \times \mathbb{R}^d$ for some $T > 0$. We consider non-negative initial data and solutions. Existence and uniqueness of weak solutions of this problem with initial data in $\mathrm{L1}_{\mathrm{loc}}(\mathbb{R}^d)$ was first proved by M.A. Herrero and M. Pierre in [30]. In the whole space, the behavior of the solutions is quite different in the parameter ranges $m_c < m < 1$ and $0 < m < m_c$, the critical exponent being defined as

$$m_c := \frac{d-2}{d} \,.$$

Note that $m_c > 0$ only if $d \geq 3$, so that the lower range does not exist for $d = 1$, 2. For $m > m_c$ the mass $\int_{\mathbb{R}^d} u(y,t)\,\mathrm{d}y$ is preserved in time if the initial datum u_0 is integrable in $\mathbb{R}^d$. Besides, non-negative solutions are positive and smooth for all $x \in \mathbb{R}^d$ and $t > 0$. On the contrary, solutions may extinguish in finite time in the lower range $m < m_c$, for instance when the initial data is in $\mathrm{L}^{p_*}(\mathbb{R}^d)$ with $p_* = d\,(1-m)/2$: then there exists a time $T > 0$ such that

$$\lim_{\tau \nearrow T} u(\tau, y) = 0 \,.$$

Many computations are however similar in both ranges, from an algebraic point of view. We refer to the monograph [47] for a detailed discussion of the existence theory and references to the subject. The extension to exponents $m \leq 0$ is also treated, and it is natural but it will not be the focus of this paper.

In the last two decades, special attention has been given to the study of large time asymptotics of these equations, starting with the pioneering work of A. Friedman and S. Kamin [28] and completed in [45], when m is in the range (m_c, ∞). In those studies the class of non-negative,

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finite mass solutions are considered. Asymptotic stabilization towards *self-similar asymptotic solutions* known as Barenblatt solutions is shown. For $m_c < m < 1$, such solutions take the form:

$$U_{D,T}(\tau, y) := \frac{1}{R(\tau)^d} \left(D + \frac{1-m}{2\,m} \left| \frac{y}{R(\tau)} \right|^2 \right)^{-\frac{1}{1-m}} \tag{1.2}$$

with $R(\tau) := \left[d\,(m - m_c)\,(\tau + T) \right]^{\frac{1}{d\,(m-m_c)}}$. Here $D, T \geq 0$ are free parameters. While the second parameter means a time displacement and does not play much role in the asymptotic behavior, the first does and can be computed from the mass of the solution. The value m_c is the critical exponent below which the Barenblatt solutions cease to exist in this standard form.

Here, we are mainly interested in addressing the question of the asymptotic behavior of (1.1) when $0 < m < m_c$. We consider a wide class of solutions which vanish in finite time T and describe their behavior as τ goes to T. We point out that our methods allow to treat simultaneously the ranges $0 < m < m_c$ and $m_c \leq m < 1$, in which one is interested in the behavior of the solutions as τ goes to infinity. For this purpose, we extend the Barenblatt solutions to the range $0 < m < m_c$ with the same expression (1.2), but a different form for R, that is

$$R(\tau) := \left[d\,(m_c - m)\,(T - \tau) \right]^{-\frac{1}{d\,(m_c-m)}} .$$

The parameter T now denotes the extinction time. Following [47], we shall call such solutions the *pseudo-Barenblatt solutions*. Notice that Barenblatt and pseudo-Barenblatt solutions $U_{D,T}$, with D, $T > 0$, are such that $U_{D,T}^p$ is integrable if and only if $p > p_*$ (p_* is defined above, and $p_* > 1$ means $m < m_c$). Consistently with the above choices, for $m = m_c$, one has to choose $R(\tau) := e^{\tau+T}$ with free parameter T, see [47], in order to obtain pseudo-Barenblatt solutions; then, $p_* = 1$.

The family of Barenblatt (respectively pseudo-Barenblatt) solutions represents the asymptotic patterns to which many other solutions converge for large times if $m > m_c$ (respectively as t goes to T if $0 < m < m_c$). We are interested in the class of solutions for which such a convergence takes place and in the rates of convergence. Both questions strongly depend on m. Let us emphasize for instance that the Barenblatt solution $U_{D,T}$ is integrable in y for $m > m_c$, while the pseudo-Barenblatt solution corresponding to $m \leq m_c$ is not integrable. Since much is known in the case $m > m_c$, see for instance [16, 25] and [10, 11, 13, 14, 15, 27, 36, 45] for more complete results, the main novelty of our paper is concerned with the lower range $m \leq m_c$, which has several interesting new features. For instance, in the analysis in high space dimensions, that is $d > 4$, another critical exponent appears,

$$m_* := \frac{d-4}{d-2} < m_c .$$

A key property of m_* is that the difference of two pseudo-Barenblatt solutions is integrable for $m \in (m_*, m_c)$, while it is not integrable for $m \in (0, m_*]$.

The convergence towards Barenblatt and pseudo-Barenblatt solutions is subtle since the solutions converge to zero everywhere. To capture the asymptotic profiles, it is therefore convenient to rescale the solutions and replace the study of intermediate asymptotics by the study of the convergence to stationary solutions in *rescaled variables*,

$$t := \log \left(\frac{R(\tau)}{R(0)} \right) \quad \text{and} \quad x := \frac{y}{R(\tau)} , \tag{1.3}$$

with R as above. In these new variables, if u is a solution to (1.1), the function

$$v(t, x) := R(\tau)^d\, u(\tau, y)$$

solves a nonlinear *Fokker-Planck type equation,*

$$\begin{cases} \partial_t v(t,x) = \Delta v^m(t,x) + \nabla\cdot(x\,v(t,x)) & (t,x)\in(0,+\infty)\times\mathbb{R}^d\,, \\ v(0,x) = v_0(x) & x\in\mathbb{R}^d\,. \end{cases} \tag{1.4}$$

The initial data for (1.1) and for the rescaled equation (1.4) are related by

$$u_0(y) = R(0)^{-d}\, v_0(y/R(0))\;,$$

where $R(0) = \big[d\,|m-m_c|\,T\big]^{\frac{1}{d\,(m-m_c)}}$ only depends on T. In this formulation, the Barenblatt and pseudo-Barenblatt solutions are transformed into stationary solutions given by

$$V_D(x) := \left(D + \frac{1-m}{2\,m}\,|x|^2\right)^{-\frac{1}{1-m}} \tag{1.5}$$

where $0<m<1$ and $D>0$ is a free parameter. With a straightforward abuse of language, we say that V_D is a *Barenblatt profile,* including the case $m\le m_c$. The value $D=0$ can also be admitted as a limit case, but the corresponding solution is singular at $x=0$. See [47] for more details. The parameter T has disappeared from the new problem, but it enters in the change of variables. Note that in all cases, t runs from 0 to infinity in these rescaled variables.

Assumptions and main results. We can write the assumptions on the *initial conditions* in terms of either u_0 or v_0. We assume that

(H1) u_0 is a non-negative function in $\mathrm{L1}_{\mathrm{loc}}(\mathbb{R}^d)$ and that there exist positive constants T and $D_0>D_1$ such that

$$U_{D_0,T}(0,y)\le u_0(y)\le U_{D_1,T}(0,y)\quad\forall\,y\in\mathbb{R}^d\,.$$

(H2) There exist $D_*\in[D_1,D_0]$ and $f\in\mathrm{L1}(\mathbb{R}^d)$ such that

$$u_0(y) = U_{D_*,T}(0,y) + f(y)\quad\forall\,y\in\mathbb{R}^d\,.$$

Note that by the Comparison Principle, see Lemma 2.2 below, in the case $m<m_c$, (H1) implies that the extinction occurs at time T. When $m>m_*$, (H2) follows from (H1) since the difference of two Barenblatt solutions is always integrable. For $m\le m_*$, (H2) is an additional restriction. We shall assume throughout this paper that $d\ge3$ and observe that (H2) has to be taken into account only if $m_*>0$, that is, $d\ge5$.

In terms of v_0, with f replaced by $R(0)^{-d}\,f(y/R(0))$, conditions (H1) and (H2) can be rewritten as follows. To avoid more notations, we keep using f in (H2') although it is not the same function as in (H2).

(H1') v_0 is a non-negative function in $\mathrm{L1}_{\mathrm{loc}}(\mathbb{R}^d)$ and there exist positive constants $D_0>D_1$ such that

$$V_{D_0}(x)\le v_0(x)\le V_{D_1}(x)\quad\forall\,x\in\mathbb{R}^d\,.$$

(H2') There exist $D_*\in[D_1,D_0]$ and $f\in\mathrm{L1}(\mathbb{R}^d)$ such that

$$v_0(x) = V_{D_*}(x) + f(x)\quad\forall\,x\in\mathbb{R}^d\,.$$

If $m\in(m_*,1)$, the map $D\mapsto\int_{\mathbb{R}^d}(v_0-V_D)\,\mathrm{d}x$ is continuous, monotone increasing. Hence we can also define a unique $D_*\in[D_1,D_0]$ such that

$$\int_{\mathbb{R}^d}(v_0-V_{D_*})\,\mathrm{d}x = 0\;.$$

Before stating any result, one more exponent is needed. We define $p(m)$ as the infimum of all positive real numbers p for which two Barenblatt profiles V_{D_1} and V_{D_2} are such that $|V_{D_1}-V_{D_2}|$ belongs to $\mathrm{L}^p(\mathbb{R}^d)$:

$$p(m) := \frac{d\,(1-m)}{2\,(2-m)}\;.$$

We see that $p(m) > 1$ if $m \in (0, m_*)$, $p(m_*) = 1$, and $p(m) < 1$ if $m > m_*$.

We can now state the convergence of $v(t)$ towards a unique Barenblatt profile. For simplicity, we will write $v(t)$ instead of $x \mapsto v(t,x)$ whenever we want to emphasize the dependence in t.

Theorem 1.1 (Convergence to the asymptotic profile)**.** *Let $d \geq 3$, $m \in (0,1)$. Consider the solution v of* (1.4) *with initial data satisfying* (H1')-(H2').

(i) *For any $m > m_*$, there exists a unique $D_* \in [D_1, D_0]$ such that $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,\mathrm{d}x = 0$ for any $t > 0$. Moreover, for any $p \in (p(m), \infty]$, $\lim_{t\to\infty} \int_{\mathbb{R}^d} |v(t) - V_{D_*}|^p\,\mathrm{d}x = 0$.*
(ii) *For $m \leq m_*$, $v(t) - V_{D_*}$ is integrable, $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,\mathrm{d}x = \int_{\mathbb{R}^d} f\,\mathrm{d}x$ and $v(t)$ converges to V_{D_*} in $\mathrm{L}^p(\mathbb{R}^d)$ as $t \to \infty$, for any $p \in (1, \infty]$.*
(iii) (Convergence in Relative Error) *For any $p \in (d/2, \infty]$,*

$$\lim_{t\to\infty} \| \, v(t)/\, V_{D_*} - 1 \, \|_p = \; 0 \; .$$

In case $m > m_*$, the value of D_* can be computed at $t = 0$ as a consequence of the mass balance law $\int_{\mathbb{R}^d} (v_0 - V_{D_*})\,\mathrm{d}x = 0$, and then the conservation result holds for all $t > 0$, see Proposition 2.3 below. On the other hand, in the case $m \leq m_*$ the mass balance does not make sense, but D_* is determined by Assumption (H2). In this case, the presence of a perturbation f of V_{D_*} with nonzero mass, does not affect the asymptotic behavior of the solution at first order.

In a recent paper [22], P. Daskalopoulos and N. Sesum prove some of the results of Theorem 1.1 under similar hypotheses (see [22, Theorem 1.4]). Actually they only prove the L^∞ convergence in case (ii) and the $\mathrm{L}1 \cap \mathrm{L}^\infty$ convergence in case (i). Our proof was obtained independently and announced in [7]. It is based on entropy estimates and paves the way to the sharper results on convergence with rates, which are the main purpose of the present paper. Assertion (iii) says that the convergence of (i)-(ii) can be improved into a convergence in relative error, in the sense of [45]. Such a strong convergence may look surprising at first sight, but it is a consequence of Assumption (H1'): the tails of v_0 and V_{D_*} have the same behavior as $|x| \to \infty$.

We can now state our main asymptotic result, on rates of convergence. To state this second result, we need yet another exponent,

$$q_* := \frac{2\,d\,(1-m)}{2\,(2-m) + d\,(1-m)}$$

and note that $q_* > 1$ if $m < m_*$, $q_* = 1$ if $m = m_*$, and $q_* < 1$ if $m > m_*$. For any $q > q_*$, the function V_{D_*} is in $\mathrm{L}^{(2-m)q/(2-q)}(\mathbb{R}^d)$, which allows us to use convenient Hölder interpolation inequalities. We define the C^j semi-norm by

$$\|f\|_{C^j(\mathbb{R}^d)} := \max_{|\eta|=j} \; \sup_{x\in\mathbb{R}^d} \left|\partial^\eta f(x)\,\right| ,$$

where the standard multi-index notation is used: $|\eta| = \eta_1 + \ldots + \eta_d$ is the length of the multi-index $\eta = (\eta_1, \ldots, \eta_d) \in \mathbb{Z}^d$. The last ingredient is a Hardy-Poincaré constant, which is defined as follows. For any $m \in (0, m_*) \cup (m_*, 1)$, let

$$\lambda_{m,d} \;:=\; m \;\inf_h \; \frac{\int_{\mathbb{R}^d} |\nabla h|^2\, V_{D_*}\,\mathrm{d}x}{\int_{\mathbb{R}^d} |h - \bar{h}|^2\, V_{D_*}^{2-m}\,\mathrm{d}x} \; , \tag{1.6}$$

where the infimum is taken over the set of smooth functions h such that $\mathrm{supp}(h) \subset \mathbb{R}^d \setminus \{0\}$ and $\bar{h} = 0$ if $m < m_*$, while for $m > m_*$, $\bar{h} := \int_{\mathbb{R}^d} h\, V_{D_*}^{2-m}\,\mathrm{d}x / \int_{\mathbb{R}^d} V_{D_*}^{2-m}\,\mathrm{d}x$, *cf.* Theorem A.1 in the Appendix for more details. We shall prove that $\lambda_{m,d}$ is positive and independent of D_*. In the next result, the time decay rate is formulated in terms of the spectral gap $\lambda_{m,d}$. Analyzing the relationship between the optimal constant $\mathcal{C}_{m,d} = m/\lambda_{m,d}$ in the corresponding functional inequality and the asymptotic rates of the fast diffusion equation is the leitmotiv of this paper. The case $m = m_*$ has to be excluded for reasons which are deeply related to Hardy's inequality, see [7].

Theorem 1.2 (Convergence with rate)**.** *Under the assumptions of Theorem* 1.1*, with* $\lambda_{m,d}$ *given by* (1.6)*, if* $m \neq m_*$*, there exists* $t_0 \geq 0$ *such that the following properties hold:*

(i) *For any* $q \in (q_*, \infty]$*, there exists a positive constant* C_q *such that*

$$\|v(t) - V_{D_*}\|_q \leq C_q \, \mathrm{e}^{-\lambda_{m,d}\, t} \quad \forall\, t \geq t_0 \; .$$

(ii) *For any* $\vartheta \in [0, (2-m)/(1-m))$*, there exists a positive constant* K_ϑ *such that*

$$\big\| \, |x|^\vartheta (v(t) - V_{D_*}) \big\|_2 \leq K_\vartheta \, \mathrm{e}^{-\lambda_{m,d}\, t} \quad \forall\, t \geq t_0 \; .$$

(iii) *For any* $j \in \mathbb{N}$*, there exists a positive constant* H_j *such that*

$$\|v(t) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \leq H_j \, \mathrm{e}^{-\frac{2\,\lambda_{m,d}}{d+2(j+1)}\, t} \quad \forall\, t \geq t_0 \; .$$

The constants C_q, K_ϑ and H_j depend on t_0, m, d, v_0, D_0, D_1, and q, ϑ and j; t_0 also depends on D_0 and D_1. It is remarkable that the decay rate of the nonlinear problem is given exactly by $\lambda_{m,d}$ (see Section 6.3). Using (1.3), the results of Theorem 1.2 for the solution $v(t)$ of (1.4) can be translated into results for the solution $u(\tau)$ of (1.1) as follows.

Corollary 1.3 (Intermediate asymptotics)**.** *Let* $d \geq 3$*,* $m \in (0,1)$*,* $m \neq m_*$*. Consider a solution* u *of* (1.1) *with initial data satisfying* (H1)-(H2)*. For* τ *large enough, for any* $q \in (q_*, \infty]$*, there exists a positive constant* C *such that*

$$\|u(\tau) - U_{D_*}(\tau)\|_q \leq C \, R(\tau)^{-\alpha} \; ,$$

where $\alpha = \lambda_{m,d} + d\,(q-1)/q$ *with* $\lambda_{m,d}$ *given by* (1.6)*, and* large *means* $T - \tau > 0$*, small, if* $m < m_c$*, and* $\tau \to \infty$ *if* $m \geq m_c$*.*

We also obtain a convergence result in relative error. For any $p \in (d/2, \infty]$, define

$$\lambda(p) := \frac{(2\,p - d)\,(1-m)}{p\,(d+2)\,(2-m)}\,\lambda_{m,d} \; .$$

Theorem 1.4 (Exponential Decay of Relative Error)**.** *Under the assumptions of Theorem* 1.2*, if* $m \neq m_*$*, for any* $p \in (d/2, \infty]$*, there exists a positive constant* $\mathcal{C}$ *and* $\lambda \in (0, \lambda(p))$ *such that*

$$\big\| \, v(t)/\,V_{D_*} - 1 \, \big\|_p \leq \mathcal{C}\, \mathrm{e}^{-\lambda\, t} \quad \forall\, t \geq 0 \; .$$

Let us list a few observations on the above results.

(a) In the range $m_c < m < 1$, convergence with rates has been obtained under various assumptions, *cf.* [25, 13, 15] (optimal rates) for $m \in [m_1, 1)$, $m_1 = (d-1)/d$, and [14, 16] for $m \in (m_c, m_1)$, which are weaker than the ones of Theorem 1.2. See [27] for the detailed analysis of the spectrum of the linearized operator in the range $m > m_c$. A stronger convergence has also been proved in the sense of relative error under very mild assumptions, *cf.* [45].

(b) In the range $m_c < m < 1$, Assumption (H1) is less restrictive than one could think. By the global Harnack principle of [11], see Theorem 2.5 below, any solution with non-negative initial data $u_0 \in \mathrm{L1}_{\mathrm{loc}}(\mathbb{R}^d)$ that decays at infinity like $u_0(y) = O(|y|^{2/(1-m)})$, is indeed trapped for all $t > 0$ between two Barenblatt solutions if $m_c < m < 1$. The restrictions on the class of initial data are therefore not so essential as far as the asymptotic behavior is concerned, and can therefore be relaxed.

(c) In the range $0 < m \leq m_c$, the pseudo-Barenblatt solutions are not integrable. For $m < m_c$ many solutions vanish in finite time and have various asymptotic behaviors depending on the initial data. Solutions with bounded and integrable initial data are described by self-similar solutions with so-called anomalous exponents, see [34, 41] and [47, Chapter 7]. Even for solutions with initial data not so far from a pseudo-Barenblatt solution, the asymptotic behavior may significantly differ from the behavior of a pseudo-Barenblatt solution: in [22, Theorem 1.4] a solution of (1.1) is found, which is such that $\lim_{|x|\to\infty} u_0(x)/U_{D,T}(0,x) = 1$ and which, after rescaling, does not converge to V_D as $t \to \infty$, that is for $\tau \to T$. Assumption (H1)-(H2) are therefore more restrictive than for $m > m_c$.

(d) Proofs are constructive and the values of the various constants are explicit although not so easy to write. The interested reader will be able to recover their expressions by carefully reading the proofs, where all details are given. See Appendix A for more details on the constant $\lambda_{m,d}$ that controls the rate of convergence. The rate given by this exponent is sharp in the linear case, and a deeper analysis should prove that it is sharp also in the nonlinear case. Obtaining the optimal value of $\lambda_{m,d}$ is still an open question in the range $m \in (m_*, m_c]$. Our method gives convergence with rates even in the limit case $m = m_c$, which is new.

Further comments. After stating our main results, let us come back a little bit on the motivations of this paper, on the main tools and the originality of our results with respect to the existing literature.

During the last few years, asymptotic rates of convergence for the solutions of nonlinear diffusion equations have attracted lots of attention, usually in connection with time-dependent scalings and entropy methods. This has been first done in the range of exponents corresponding to the porous medium equation, with $1 < m < 2$, and in the range where standard Gagliardo-Nirenberg inequalities apply, $m_1 \le m < 1$, see [25, 13, 15]. The class of non-negative, finite mass solutions has to be narrowed to the smaller set of functions with finite free energy, or to be precise, with finite entropy and finite potential energy. In the rescaled variables, asymptotic stabilization to the Barenblatt profiles holds at an exponential rate, while in the original time variable τ, the convergence of the difference with the Barenblatt solutions holds at a power-law rate, which is shown to be optimal.

The next question was to understand what happens for $m < m_1$. After the linearized analysis of [14], the proof of convergence with rates was done in [16] in the range $m_c < m < m_1$ for which global existence of finite mass solutions still holds. The basin of attraction is narrowed to the class of solutions with finite relative entropy with respect to some Barenblatt solution.

A dramatic change occurs for $m < m_c$, since a large class of solutions vanish in finite time. As a consequence, mass is not conserved, and a key estimate for higher values of m is lost. It is however natural to investigate the basin of attraction of the pseudo-Barenblatt solutions for $m \le m_c$ using relative entropy techniques and to study the convergence rates. This can be done in a weighted space using functional inequalities, which can still be related to some spectral properties of a differential operator obtained by an appropriate linearization.

The *generalized entropy functional,* or *free energy functional,* is defined as

$$\mathcal{E}[v] := \int_{\mathbb{R}^d} \left[\varphi(v) + \frac{1}{2}\,|x|^2\,v\right] \mathrm{d}x \quad \text{where} \quad \varphi(v) := \frac{v^m}{m-1}\;.$$

It is then observed that the free energy of the Barenblatt profiles, *cf.* [15, 25], becomes infinite if $m \le m_0$, where $m_0 := d/(d+2) \in (m_c, m_1)$. In order to avoid this difficulty, it is convenient to work with the *relative entropy* of v with respect to V_D defined as follows:

$$\mathcal{E}[v|V_D] := \int_{\mathbb{R}^d} \Big[\varphi(v) - \varphi(V_D) - \varphi'(V_D)\,(v - V_D)\Big]\,\mathrm{d}x\;.$$

The *relative entropy* is the key tool of our analysis. It is such that $\mathcal{E}[v|V_D] := \mathcal{E}[v] - \mathcal{E}[V_D]$ if $m \in (m_0, 1)$ and $\int_{\mathbb{R}^d} v\,\mathrm{d}x = \int_{\mathbb{R}^d} V_D\,\mathrm{d}x$, that is for $D = D_*$, where D_* is as in Theorem 1.1. The functional $\mathcal{E}[v|V_{D_*}]$ can also be defined for $m \le m_0$. By homogeneity of φ, we can indeed rewrite it as

$$\mathcal{E}[v|V_{D_*}] := \int_{\mathbb{R}^d} \Big[\varphi(w) - \varphi(1) - \varphi'(1)\,(w-1)\Big]\,V_{D_*}^m\,\mathrm{d}x \quad \text{with} \quad w = \frac{v}{V_{D_*}}\;.$$

This makes clear why it is well defined at least for w close enough to 1 as $|x| \to \infty$. The functional $v \mapsto \mathcal{E}[v|V_{D_*}]$ is convex and achieves its minimum, 0, for $v = V_{D_*}$. If v is a solution of (1.4), the *entropy production term* takes the form

$$-\frac{d}{dt}\mathcal{E}[v(t)|V_{D_*}] = \mathcal{I}[v(t)|V_{D_*}] ,$$

where the functional

$$v \mapsto \mathcal{I}[v|V_D] := \int_{\mathbb{R}^d} v \left|\nabla\varphi'(v) - \nabla\varphi'(V_D)\right|^2 \mathrm{d}x$$

will be called *the relative Fisher information*. See Proposition 2.6 for more details. For any $m \in [m_1, 1)$, $\mathcal{E}[v|V_{D_*}] \leq \frac{1}{2}\mathcal{I}[v|V_{D_*}]$ holds for any smooth function v and the inequality is nothing else than the optimal Gagliardo-Nirenberg inequality, for which equality is achieved precisely by the Barenblatt profiles, see [25]. In such a case,

$$\mathcal{E}[v(t)|V_{D_*}] \leq \mathcal{E}[v_0|V_{D_*}]\, e^{-2t} \quad \forall\, t \geq 0 .$$

The limit case $m = m_1$ corresponds to the critical Sobolev inequality whose optimal form was established by T. Aubin and G. Talenti in [1, 44], while in the limit $m \to 1$ one recovers Gross' logarithmic Sobolev inequality, see [29, 25]. For $m \in [m_1, 1)$, F. Otto in [40] noticed that (1.4) can be interpreted as the gradient flow of the free energy with respect to the Wasserstein distance. The exponent $m = m_1$ is the limit case for which the *displacement convexity* property holds true.

Pushing the method to the case $0 < m < m_1$ requires the use of the relative entropy in place of the free energy. The method applies only to a class of initial data which have a finite relative entropy with respect to some Barenblatt profile V_{D_*} and satisfy convenient bounds. Mass can be finite in the case $m \in (m_c, m_1)$, which was the framework of some earlier studies, see [14, 16], or infinite if $m \in (0, m_c)$. Two Barenblatt profiles V_{D_0} and V_{D_1} have finite relative entropy, *i.e.* $\mathcal{E}[V_{D_1}|V_{D_0}] < \infty$ if and only if either $d \leq 4$, or $d \geq 5$ and $m > m_*$, $m_* = (d-4)/(d-2)$. Hence, for $d \geq 5$, $m = m_*$ is a threshold not only for defining the relative mass of two pseudo-Barenblatt solutions, but also for defining their relative entropies or for the integrability of $V_{D_*}^{2-m}$. Note that $m_* < m_c$ for all $d \geq 5$. The proof of Theorem 1.2 amounts to prove that the relative entropy $\mathcal{E}[v|V_{D_*}]$ decays in time and converges to 0 at an exponential rate when $t \to \infty$. For $m > \min\{0, m_*\}$, $\mathcal{E}[v|V_{D_*}]$ is well defined under condition (H1'). For $m < m_*$, an additional restriction is required, which is precisely the purpose of (H2').

Our approach of course covers the case $m \geq m_c$ and we recover some of the results found in [14, 16]. Some of our results can also be extended to the range $m < 0$, but additional technical complications arise, which are still to be studied. In this paper, we leave apart several interesting questions, like the precise study of the case of $m = m_*$ or the equation $u_t = \Delta \log u$ in dimension $d \geq 2$, see *e.g.* [20, 21, 42, 48], which is the natural limiting equation to study in the limit $m \to 0$. Also see [31, 32, 33] for results which seem closely related to ours, and [26] in the case $m = (d-2)/(d+2)$. In particular we do not use the Bakry-Emery method introduced in [2], on which the results of [15, 13, 36, 14, 16] are based. We prove a conservation of relative mass, which allows us to remove the limitation $m > m_c$. Neither mass transportation techniques nor Wasserstein distance are needed, although the approach of Section A.3 is not unrelated, see [8, 3, 4, 38].

The paper is organized as follows. In Section 2, we extend the property of mass conservation, which holds only for $m > m_c$, to a property of conservation of relative mass, see Proposition 2.3. This selects a unique Barenblatt profile, which governs the asymptotic behavior. We also establish regularity properties of the solutions. From there on we work with the quotient of the solution by the Barenblatt profile. In Section 3, we prove Theorem 1.1 and establish several properties which are used in the sequel. Sections 4 and 5 are respectively devoted to introducting a suitable linearization and to the derivation of entropy - entropy production estimates in the nonlinear case, from the corresponding spectral gap estimates for the linearized problem. The proof of Theorem 1.2 is given in Section 6.

Appendix A is devoted to the proofs of spectral gap estimates, that is, of weighted Poincaré-Hardy inequalities, which have already been studied in [7], and in [14, 27] in the special case $m > m_c$. We consider the family of weights of the form V_D or V_D^{2-m}, $D > 0$, that are obtained from the linearization of the relative entropy. In the limit $D \to 0$, they yield the case corresponding to the weighted L2 norm of the Caffarelli-Kohn-Nirenberg inequalities, *cf.* [12, 17].

A final section, Appendix B, explains how to extend the results of this paper to the fast diffusion with exponents $m \leq 0$. Note that the equation needs to be properly modified. The conclusion is that the results still hold and the proofs need only minor modifications that are indicated.

2. Basic Estimates

We establish in this section the main properties of the solutions that will be used in the sequel.

2.1. L1-contraction and Maximum Principle.

Lemma 2.1 (L1-contraction)**.** *For any two non-negative solutions u_1 and u_2 of* (1.1) *defined on a time interval $[0,T)$, with initial data in $\mathrm{L1}_{\mathrm{loc}}(\mathbb{R}^d)$, and any two times t_1 and t_2 such that $0 \leq t_1 \leq t_2 < T$, we have*

$$\int_{\mathbb{R}^d} |u_1(t_2) - u_2(t_2)| \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |u_1(t_1) - u_2(t_1)| \, \mathrm{d}x \, .$$

The above result is well-known to be true for solutions with L1 data, *cf.* [46, Proposition 9.1], even in the stronger form

$$\int_{\mathbb{R}^d} \big[u_1(t_2) - u_2(t_2)\big]_+ \, \mathrm{d}x \leq \int_{\mathbb{R}^d} \big[u_1(t_1) - u_2(t_1)\big]_+ \, \mathrm{d}x \, ,$$

where $[u]_+$ denotes the positive part of u. The result for data in $\mathrm{L1}_{\mathrm{loc}}(\mathbb{R}^d)$ is obtained by approximation, using the uniqueness of solutions to the Cauchy Problem, which has been established in [30]. Note that when the right-hand side is infinite the result applies but there is nothing to prove. As a consequence, we also obtain the following.

Lemma 2.2 (Comparison Principle)**.** *For any two non-negative solutions u_1 and u_2 of* (1.1) *on $[0,T)$, $T > 0$, with initial data satisfying $u_{01} \leq u_{02}$ a.e, $u_{02} \in \mathrm{L1}_{\mathrm{loc}}(\mathbb{R}^d)$, then we have $u_1(t) \leq u_2(t)$ for almost every $t \in [0,T)$.*

We will see below that under Assumption (H1)-(H2) the solutions are smooth functions, hence the comparison in the previous result holds everywhere in $[0,T) \times \mathbb{R}^d$.

2.2. Conservation of relative mass.

Mass conservation is used in the range $m > m_c$ to determine the parameter D which characterizes the Barenblatt profile V_D. In the range $m \leq m_c$, we can still prove that $\int_{\mathbb{R}^d} (v(t) - V_D) \, \mathrm{d}x$ is conserved for any $t > 0$, even if $V_D \notin \mathrm{L1}(\mathbb{R}^d)$.

Proposition 2.3. *Let $m \in (0,1)$. Consider a solution u of* (1.1) *with initial data u_0 satisfying* (H1)-(H2)*. If for some $D > 0$, $\int_{\mathbb{R}^d} (u_0 - U_{D,T}(0,\cdot)) \, \mathrm{d}x$ is finite, then*

$$\int_{\mathbb{R}^d} \big[u(\tau,x) - U_{D,T}(\tau,x)\big] \, \mathrm{d}x = \int_{\mathbb{R}^d} \big[u_0(x) - U_{D,T}(0,x)\big] \, \mathrm{d}x \quad \forall \, \tau \in (0,T) \, .$$

Proof. In the range $m > m_c$, u_0 and $U_{D,T}(0,\cdot)$ are integrable and mass conservation of the solutions of (1.1) is well-known.

Assume next that $m < m_c$ and let χ be a $C2$ function on $\mathbb{R}^+$ such that $\chi \equiv 1$ on $[0,1]$, $\chi \equiv 0$ on $[2,\infty)$, and $0 \le \chi \le 1$ on $[1,2]$. For any $\lambda > 0$, take $\phi_\lambda(x) := \chi(|x|/\lambda)$ as a test function and denote by B_λ the ball $B(0,\lambda)$. Then,

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^d}[u(\tau)-U_{D,T}(\tau)]\;\phi_\lambda\;\mathrm{d}y\right| &= \left|\int_{\mathbb{R}^d}\left[u^m(\tau)-U_{D,T}^m(\tau)\right]\;\Delta\phi_\lambda\;\mathrm{d}y\right|\\ &= \left|\int_{B_{2\lambda}\setminus B_\lambda}\left[u^m(\tau)-U_{D,T}^m(\tau)\right]\;\Delta\phi_\lambda\;\mathrm{d}y\right|\\ &\le C\int_{B_{2\lambda}\setminus B_\lambda}|u(\tau)-U_{D,T}(\tau)|\;U_{D_0,T}^{m-1}(\tau)\;|\Delta\phi_\lambda|\;\mathrm{d}y\;,\end{aligned}$$

where C is a numerical constant depending on D, D_0, D_1. As $\lambda \to \infty$, we observe that in $B_{2\lambda} \setminus B_\lambda$, $U_{D_0,T}^{m-1}$ and $|\Delta\phi_\lambda|$ behave like $\lambda2$ and λ^{-2}, so that $U_{D_0,T}^{m-1}(\tau)\,\Delta\phi$ is bounded uniformly with respect to λ. The right hand side is therefore bounded by $\int_{B_{2\lambda}\setminus B_\lambda}|u(\tau)-U_{D,T}(\tau)|\;\mathrm{d}y$. For any τ_1, $\tau_2 \in [0,T)$, we write

$$\left|\int_{\mathbb{R}^d}[u(\tau_2)-U_{D,T}(\tau_2)]\;\phi_\lambda\;\mathrm{d}y-\int_{\mathbb{R}^d}[u(\tau_1)-U_{D,T}(\tau_1)]\;\phi_\lambda\;\mathrm{d}y\right| \le \int_{\tau_1}^{\tau_2}\int_{B_{2\lambda}\setminus B_\lambda}|u-U_{D,T}|\;\mathrm{d}y\;\mathrm{d}\tau\;.$$

By the L1-contraction principle, see Lemma 2.1, we also know that $|u(\tau,y)-U_{D,T}(\tau,y)|$ is integrable in y, uniformly for all $\tau > 0$. The integrability condition implies that the right-hand side goes to zero in the limit $\lambda \to +\infty$.

The case $m = m_c$ is similar, except that there is no extinction time. □

In the rescaled variables given by (1.3), relative mass is also conserved. Consider a solution v of (1.4) with initial data v_0 satisfying (H1')-(H2'). If for some $D > 0$, $\int_{\mathbb{R}^d}(v_0(x)-V_D(t,x))\,\mathrm{d}x$ is finite, then

$$\int_{\mathbb{R}^d}\left[v(t,x)-V_D(t,x)\right]\,\mathrm{d}x = \int_{\mathbb{R}^d}\left[v_0(x)-V_D(t,x)\right]\,\mathrm{d}x\quad\forall\,t>0\;.$$

Whenever $m \le m_c$, we recall that pseudo-Barenblatt solutions have infinite mass, that is $\int_{\mathbb{R}^d} V_{D_*}\,\mathrm{d}y = \infty$, but we observe that the difference of two pseudo-Barenblatt solutions is integrable if $m > m_*$. In such a case, the parameter $D \in [D_1, D_0]$ in Proposition 2.3 can be arbitrary. In the proof, we can moreover estimate $\int_{B_{2\lambda}\setminus B_\lambda}|u-U_{D,T}|\;\mathrm{d}y$ by

$$\max\left\{\int_{B_{2\lambda/R(t)}\setminus B_{\lambda/R(t)}}|V_{D_0}-V_D|\;\mathrm{d}y\;,\quad\int_{B_{2\lambda/R(t)}\setminus B_{\lambda/R(t)}}|V_{D_1}-V_D|\;\mathrm{d}y\right\}\;,$$

which converges to 0 as $\lambda \to \infty$. As already quoted in the introduction, the map $D \mapsto \int_{\mathbb{R}^d}(v_0 - V_D)\,\mathrm{d}x$ is continuous, monotone increasing, and we can define a unique $D_* \in [D_1, D_0]$ such that $\int_{\mathbb{R}^d}(v_0 - V_{D_*})\,\mathrm{d}x = 0$. Under the assumptions of Proposition 2.3,

$$\int_{\mathbb{R}^d}[v(t,x)-V_{D_*}(x)]\,\mathrm{d}x = 0\quad\forall\,t>0\;.$$

This fact is used in the statement and proof of Theorem 1.1 for $m > m_*$. On the contrary, if $m \in (0, m_*]$, integrals are infinite unless $D = D_*$ in Proposition 2.3, and then, with the notations of Assumption (H2'),

$$\int_{\mathbb{R}^d}[v(t,x)-V_{D_*}(x)]\,\mathrm{d}x = \int_{\mathbb{R}^d} f\;\mathrm{d}x\quad\forall\,t>0\;.$$

In both cases, that is for any $m \in (0,1)$, we shall summarize the fact that $\frac{d}{dt}\int_{\mathbb{R}^d}[v - V_{D_*}]\,\mathrm{d}x = 0$ by saying that the *relative mass* is conserved.

2.3. **Passing to the quotient.** Consider a solution v of (1.4). As in [45, 11, 10], define

$$w(t,x) := \frac{v(t,x)}{V_{D_*}(x)} \quad \forall\,(t,x) \in (0,\infty)\times\mathbb{R}^d\,. \tag{2.1}$$

Next, we rewrite Problem (1.4) in terms of w:

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}}\,\nabla\cdot\left[w\,V_{D_*}\nabla\left(\dfrac{m}{m-1}\,(w^{m-1}-1)\,V_{D_*}^{m-1}\right)\right] & \text{in } (0,+\infty)\times\mathbb{R}^d\,,\\[2mm] w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d\,. \end{cases} \tag{2.2}$$

Define

$$W_0 := \inf_{x\in\mathbb{R}^d}\frac{V_{D_0}}{V_{D_*}} \le \sup_{x\in\mathbb{R}^d}\frac{V_{D_1}}{V_{D_*}} := W_1\,.$$

A straightforward calculation gives

$$W_0 = \left(\frac{D_*}{D_0}\right)^{\frac{1}{1-m}} \quad < \quad 1 \quad < \quad \left(\frac{D_*}{D_1}\right)^{\frac{1}{1-m}} := W_1\,.$$

In terms of w_0, assumptions (H1') and (H2') can be rewritten as follows:

(H1") w_0 is a non-negative function in $\mathrm{L1}_{\mathrm{loc}}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$0 < W_0 \le \frac{V_{D_0}(x)}{V_{D_*}(x)} \le w(x) \le \frac{V_{D_1}(x)}{V_{D_*}(x)} \le W_1 < +\infty \quad \forall\,x\in\mathbb{R}^d\,.$$

(H2") There exists $f \in \mathrm{L1}(\mathbb{R}^d)$ such that

$$w(x) = 1 + \frac{f(x)}{V_{D_*}(x)} \quad \forall\,x\in\mathbb{R}^d\,.$$

As a consequence of the Comparison Principle, see Lemma 2.2, (H1") is satisfied by a solution w of (2.2) if it is satisfied by w_0.

2.4. **Regularity estimates and Harnack principle.** We start by briefly recalling some well-known results for solutions of fast diffusion equations. A basic regularity result is due to DiBenedetto et al. , see [18, p. 270], and concerns local space-time Hölder regularity for Problem (1.1), with some Hölder exponent $\alpha\in(0,1)$; it holds for locally bounded initial data, possibly with sign changes. In the present situation of locally bounded and positive initial data, it is known that the solutions are C^∞ as long as they do not vanish identically because we avoid any degeneracy or singularity and the standard parabolic theory applies. In the sequel we are in particular interested in some sort of uniform $C1$ regularity under the assumption (H1"). We find that it is preferable to work with the function w introduced in (2.1) since it is uniformly bounded from above, and from below away from zero.–

Theorem 2.4 (Uniform C^k regularity)**.** *Let $m\in(0,1)$ and $w\in\mathrm{L}^\infty_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ be a solution of (2.2). Then for any $k\in\mathbb{N}$, for any $t_0\in(0,T)$,*

$$\sup_{t\ge t_0}\|w(t)\|_{C^k(\mathbb{R}^d)} < +\infty\,.$$

Proof. Take $t\ge t_0>0$. For a given $\lambda>0$, the equation for w is uniformly parabolic in B_λ, so we conclude that the regularity estimates hold on $B_{\lambda/2}$ for any $t\ge t_0$. Let v be the solution of (1.4) corresponding to w. In order to cover the large values of $|x|$, we consider the scaling

$$v_\lambda(x,t) = \lambda^{2/(1-m)}\,v(\lambda\,x,t)$$

with $\lambda \to \infty$. Then v_λ is again a solution of (1.4), but the region $\Omega_\lambda = \{x \in \mathbb{R}^d : \lambda \le |x| \le 2\,\lambda\}$ gets mapped into the region $\Omega_1 = \{x \in \mathbb{R}^d : 1 \le |x| \le 2\}$, for all λ. Note also that this scaling transforms the Barenblatt profiles according to

$$(V_D)_\lambda = V_{D/\lambda 2}\;,$$

so that, on Ω_1, $(V_D)_\lambda$ is uniformly bounded from above and from below in Ω_1 as $\lambda \to \infty$, and converges to V_0.

Next, we pass to the functions $w_\lambda(x,t) = v_\lambda(x,t)/(V_{D_*})_\lambda(x)$ and observe that in Ω_1, $(w_\lambda)_{\lambda\ge 1}$ is uniformly bounded from above, and from below away from zero. Since w_λ satisfies (2.2) with V_{D_*} replaced by $(V_{D_*})_\lambda$, we conclude as in part (i) that

$$\|w_\lambda(t)\|_{C^k(\Omega_1)} \le C_k$$

uniformly in $t \ge t_0$ and $\lambda \ge 1$. Undoing the scaling, we find a constant C independent of λ such that, for any $t \ge t_0$,

$$\frac{|\nabla w(t,\lambda\,x)|}{w(t,\lambda\,x)} \le \frac{C}{\lambda}\;.$$

We conclude that the result holds for $k = 1$. The same argument applies for $k > 1$. □

In the range $m \in (m_c, 1)$, the regularization effects of the fast diffusion equation allow to get rid of assumption (H1). See [11] for details expressions of the constants entering the statement.

Theorem 2.5 (Global Harnack principle, [11, Theorem 1.2])**.** *Let u_0 be a non-negative function in* L1$(\mathbb{R}^d)$ *solution of* (1.1)*. Assume that for some $R > 0$, $\sup_{|y|>R} u_0(y)\,|y|^{2/(1-m)}$ is finite. Then, for any $\varepsilon > 0$, there exist positive constants T_0, T_1, D_0 and D_1, such that*

$$U_{D_0,T_0}(\tau,y) \le u(\tau,y) \le U_{D_1,T_1}(\tau,y) \quad \forall\;(\tau,y) \in (\varepsilon,\infty)\times\mathbb{R}^d\,.$$

2.5. **Relative entropy.** In terms of w, we define the *relative entropy*

$$\mathcal{F}[w] := \frac{1}{1-m}\int_{\mathbb{R}^d}\Big[\,(w-1) - \frac{1}{m}\,(w^m-1)\Big]\,V_{D_*}^m\,\mathrm{d}x \tag{2.3}$$

and the *relative Fisher information*

$$\mathcal{J}[w] := \frac{m}{(m-1)2}\int_{\mathbb{R}^d}\big|\,\nabla\,\big[\big(w^{m-1}-1\big)\,V_{D_*}^{m-1}\big]\,\big|^2\,w\,V_{D_*}\,\mathrm{d}x\;. \tag{2.4}$$

These definitions are consistent with the ones given in the introduction, in the sense that, for $w = v/\,V_{D_*}$,

$$\mathcal{F}[w] = \frac{1}{m}\,\mathcal{E}[v|V_D] \quad\text{and}\quad \mathcal{J}[w] = \frac{1}{m}\,\mathcal{I}[v|V_D]\;.$$

The $1/m$ factor simplifies the expressions of the linearized relative entropy and Fisher information, as we shall see below. It has no impact on the rates. Consistently with the passage to the quotient, the relative entropy and the relative Fisher information are related as follows.

Proposition 2.6. *Under Assumptions* (H1")-(H2")*, if w is a solution of* (2.2)*, then*

$$\frac{d}{dt}\mathcal{F}[w(t)] = -\,\mathcal{J}[w(t)]\;. \tag{2.5}$$

Proof. As in Section 2.2, consider a test function $\phi_\lambda(x) := \chi(|x|/\lambda)$ where χ is a smooth function on $\mathbb{R}^+$ such that $\chi \equiv 1$ on $[0,1]$, $\chi \equiv 0$ on $[2,\infty)$, and $0 \le \chi \le 1$ on $[1,2]$. Then, using (2.2), the equality $(w^{m-1}-1)\,V_{D_*}^{m-1}=v^{m-1}-V_{D_*}^{m-1}$ and integration by parts we obtain

$$
\begin{aligned}
&\frac{\mathrm d}{\mathrm dt}\frac{1}{1-m}\int_{\mathbb R^d}\big[\,(w-1)-\frac{1}{m}(w^m-1)\big]\phi_\lambda\,V_{D_*}^m\;\mathrm dx\\
&\quad=-\frac{m}{(m-1)2}\int_{\mathbb R^d}\nabla\big[(w^{m-1}-1)\,V_{D_*}^{m-1}\big]\;\cdot\;\nabla\big[(w^{m-1}-1)\,V_{D_*}^{m-1}\,\phi_\lambda\big]\;\;w\,V_{D_*}\;\mathrm dx\\
&\quad=-\frac{m}{(m-1)2}\int_{\mathbb R^d}\nabla(v^{m-1}-\;V_{D_*}^{m-1})\cdot\big[\phi_\lambda\nabla(v^{m-1}-\;V_{D_*}^{m-1})+\nabla\phi_\lambda(v^{m-1}-\;V_{D_*}^{m-1})\big]\,v\;\mathrm dx\\
&\quad=-\frac{m}{(m-1)2}\int_{\mathbb R^d}\left|\nabla(v^{m-1}-\;V_{D_*}^{m-1})\right|^2\phi_\lambda\,v\;\mathrm dx+\mathcal R(\lambda)\;,
\end{aligned}
$$

where the last integral can be computed as

$$
\begin{aligned}
\mathcal R(\lambda)\;&:=\;-\frac{m}{2\,(m-1)2}\int_{\mathbb R^d}\nabla\big[(v^{m-1}-\;V_{D_*}^{m-1})2\big]\;v\,\nabla\phi_\lambda\;\mathrm dx\\
&=\frac{m}{2\,(m-1)2}\int_{\mathbb R^d}\left|v^{m-1}-\;V_{D_*}^{m-1}\right|^2\big(\nabla v\cdot\nabla\phi_\lambda+v\cdot\Delta\phi_\lambda\big)\;\mathrm dx\;.
\end{aligned}
$$

In the region $\Omega_\lambda=\{x\in\mathbb R^d:\lambda\le|x|\le2\,\lambda\}$, $\lambda>0$, we get

$$
\begin{aligned}
|\,\mathcal R(\lambda)\,|\;&\le\;k_1\int_{\Omega_\lambda}|v-\;V_{D_*}|^2\;V_{D_0}^{2(m-2)}\;(\,|\nabla v|\,|\nabla\phi_\lambda|+v\,|\Delta\phi_\lambda|\,)\;\mathrm dx\\
&\le\;k_1\sup_{\Omega_\lambda}\Big[V_{D_0}^{2(m-2)}\;|v-\;V_{D_*}|\;(\,|\nabla v|\,|\nabla\phi_\lambda|+v\,|\Delta\phi_\lambda|\,)\Big]\int_{\Omega_\lambda}|v-\;V_{D_*}|\;\mathrm dx\;,
\end{aligned}
$$

where the positive constant k_1 depends on m, d, D_0 and D_1.

As in the proof of Theorem 2.4, consider a solution v of (1.4) and define

$$v_\lambda(x,t)=\lambda^{2/(1-m)}\,v(\lambda\,x,t)\;.$$

In what follows, c_i will denote positive constants which may depend on m, d, D_0, D_1 and on the maximum of $\nabla\phi_1$, but not on λ.

For any $\lambda>0$, v_λ is a solution of (1.4) but the region $\Omega_\lambda=\{x\in\mathbb R^d:\lambda\le|x|\le2\lambda\}$ gets mapped into the region $\Omega_1=\{x\in\mathbb R^d:1\le|x|\le2\}$. We already know that ∇v_λ is uniformly bounded on Ω_1, by Theorem 2.4: $\sup_{x\in\Omega_1}\left|\nabla v_\lambda(t,x)\right|\le c_0$. In terms of v, this gives the estimate

$$\lambda^{\frac{2}{1-m}}\sup_{y\in\Omega_\lambda}\left|\nabla_y v(t,y)\right|=\lambda^{\frac{2}{1-m}}\sup_{x\in\Omega_1}\left|\lambda^{-1}\nabla_x v(t,\lambda\,x)\right|=\lambda^{-1}\sup_{x\in\Omega_1}\left|\nabla_x v_\lambda(t,x)\right|\le\lambda^{-1}\,c_0$$

and proves that

$$\sup_{x\in\Omega_\lambda}\left|\nabla v(t,x)\right|\le c_0\,\lambda^{-\frac{2}{1-m}-1}\,.$$

By our choice of ϕ_λ, we see that

$$\sup_{\Omega_\lambda}|\nabla\phi_\lambda|\le\frac{c_1}{\lambda}\quad\text{and}\quad\sup_{\Omega_\lambda}|\Delta\phi_\lambda|\le\frac{c_2}{\lambda2}\;.$$

Putting together these two estimates, we get

$$|\nabla v|\,|\nabla\phi_\lambda|+v\,|\Delta\phi_\lambda|\le c_3\,\lambda^{-\frac{2}{1-m}-2}\,.$$

Next, we observe that

$$-\frac{\partial\,V_D}{\partial D}=\frac{1}{1-m}\left[D+\frac{1-m}{2\,m}\,|x|^2\right]^{-\frac{2-m}{1-m}}=\frac{1}{1-m}\,V_D^{2-m}\;.$$

Hence, for some constant c_4 depending on m, d, D_0 and D_1,

$$|V_{D_1}-\;V_{D_0}|\le c_4\,V_{D_0}^{2-m}$$

and

$$\sup_{\Omega_\lambda}\left[V_{D_0}^{2(m-2)}\,|v-\,V_{D_*}|\,\left(\,|\nabla v|\,|\nabla\phi_\lambda|+v\,|\Delta\phi_\lambda|\,\right)\right]\le\,c_4\,\sup_{\Omega_\lambda}\left[V_{D_0}^{m-2}\left(\,|\nabla v|\,|\nabla\phi_\lambda|+v\,|\Delta\phi_\lambda|\,\right)\right].$$

Taking into account the fact that, for any $\lambda>0$, $V_D\le\,c_5\,\lambda^{-2/(1-m)}$ on Ω_λ, we obtain

$$\sup_{\Omega_\lambda}\left[V_{D_0}^{2(m-2)}\,|v-\,V_{D_*}|\,\left(\,|\nabla v|\,|\nabla\phi_\lambda|+v\,|\Delta\phi_\lambda|\,\right)\right]\le\,c_6\,\lambda^{2\frac{2-m}{1-m}}\,\lambda^{-\frac{2}{1-m}-2}=c_6\;,$$

for some positive constant c_6 which is independent of λ. By assumptions (H1')-(H2') and the L1-contraction principle, the difference $v-V_{D_*}$ is in L1, and so, $\lim_{\lambda\to\infty}\int_{\Omega_\lambda}|v-\,V_{D_*}|\;\mathrm{d}x=0$. This proves that $\lim_{\lambda\to\infty}\mathcal{R}(\lambda)=0$ and we conclude by passing to the limit as $\lambda\to\infty$. □

3. Convergence without rate and in relative error

This section is mostly devoted to the proof of Theorem 1.1.

3.1. **Relative entropy.** Under Assumptions (H1")-(H2"), the relative entropy is well defined.

Lemma 3.1 (An equivalence result)**.** *Let $m\in(0,1)$. If w satisfies* (H1")-(H2")*, then*

$$\frac{1}{2}\,W_1^{m-2}\int_{\mathbb{R}^d}|w-1|^2\,V_{D_*}^m\;\mathrm{d}x\le\mathcal{F}[w]\le\frac{1}{2}\,W_0^{m-2}\int_{\mathbb{R}^d}|w-1|^2\,V_{D_*}^m\;\mathrm{d}x\,.$$

Proof. For $a>0$, let $\phi_a(w):=\frac{1}{1-m}\left[(w-1)-(w^m-1)/m\right]-a\,(w-1)\,2$. We compute $\phi_a'(w)=\frac{1}{1-m}\left[1-w^{m-1}\right]-2\,a\,(w-1)$ and $\phi_a''(w)=\,w^{m-2}-2\,a$, and note that $\phi_a(1)=\phi_a'(1)=0$. With $a=W_1^{m-2}/2$, ϕ_a'' is positive on (W_0,W_1), which proves the lower bound after multiplying by V_D^m and integrating over $\mathbb{R}^d$. With $a=\,W_0^{m-2}/2$, ϕ_a'' is negative on (W_0,W_1) which proves the upper bound. □

Lemma 3.2 (Boundedness of the free energy)**.** *Let $m\in(0,1)$. If w_0 satisfies* (H1")-(H2")*, then the free energy $\mathcal{F}[w(t)]$ is finite for any $t\ge0$.*

Proof. By virtue of Proposition 2.6, we have to prove the result only for $w=w_0$. Notice that for any D_0, $D_1>0$ there exist a positive constant C such that $|V_{D_0}-V_{D_1}|\le C\,|x|^{-2(\frac{2-m}{1-m})}$ as $|x|\to\infty$. Indeed,

$$|V_{D_0}-V_{D_1}|=\left(\frac{1-m}{2\,m}\right)^{-\frac{1}{1-m}}\frac{2\,m\,|D_0-D_1|}{(1-m)2}\,|x|^{-2\left(\frac{2-m}{1-m}\right)}\,(1+o(1))\quad\text{as}\quad|x|\to\infty\,.$$

By Lemma 3.1, for some positive constant c depending on D_0 and D_1, we have

$$\frac{2}{W_0^{m-2}}\,\mathcal{F}[w]\le\int_{\mathbb{R}^d}\left|\frac{v}{V_{D_*}}-1\right|^2V_{D_*}^m\;\mathrm{d}x\le\int_{\mathbb{R}^d}|v-V_{D_*}|^2\;V_{D_*}^{m-2}\;\mathrm{d}x\le c\int_{\mathbb{R}^d}|V_{D_0}-V_{D_1}|^2\;V_{D_*}^{m-2}\;\mathrm{d}x\,.$$

If $m\in(m_*,1)$, then $|V_{D_0}-V_{D_1}|^2\;V_{D_*}^{m-2}=O\big(|x|^{-2\,\frac{2-m}{1-m}}\big)$ is integrable as $|x|\to\infty$. Otherwise, if $m\in(0,m_*]$, $\mathcal{F}[w]$ is also integrable as $|x|\to\infty$ because

$$\frac{2}{W_0^{m-2}}\,\mathcal{F}[w]\le\frac{2}{W_0^{m-2}}\,\mathcal{F}[w_0]\le\int_{\mathbb{R}^d}|f|^2\;V_{D_*}^{m-2}\;\mathrm{d}x\le\int_{\mathbb{R}^d}|f|\,|V_{D_0}-V_{D_1}|\;V_{D_*}^{m-2}\;\mathrm{d}x\;,$$

f is integrable and $|V_{D_0}-V_{D_1}|\;V_{D_*}^{m-2}$ is bounded (we ask the reader to check this fact). □

3.2. **Pointwise convergence in relative error.**

Lemma 3.3. *Let $m \in (0,1)$. If w is a solution of* (2.2) *with initial data w_0 satisfying* (H1")-(H2")*, then $\lim_{t\to\infty} w(t,x) = 1$ for any $x \in \mathbb{R}^d$.*

Proof. Let $w_\tau(t,x) = w(t+\tau,x)$. By the uniform C^k regularity, see Theorem 2.4, the functions w_τ are uniformly $C1$ continuous. Hence, by the Ascoli-Arzelà theorem, there exists a sequence $\tau_n \to \infty$ such that w_{τ_n} converges to a function w_∞, locally uniformly in (t,x). We know by the Comparison Principle, see Lemma 2.2, that $w_\infty > W_0 > 0$. By interior regularity of the solutions, the derivatives also converge everywhere.

By Lemma 3.2, $\mathcal{F}[w]$ is finite. Since

$$\mathcal{F}[w(\tau_n)] - \mathcal{F}[w(\tau_n+1)] = \int_{\tau_n}^{\tau_n+1} \mathcal{J}[w(s)]\,\mathrm{d}s = \int_0^1 \mathcal{J}[w(s+\tau_n)]\,\mathrm{d}s\;,$$

as a function of t, $\mathcal{J}[w_{\tau_n}]$ is integrable on $[0,1]$ and converges to zero as $n \to \infty$,

$$\lim_{n\to\infty} \int_0^1 \int_{\mathbb{R}^d} \left|\nabla\left[\left(w_{\tau_n}^{m-1}(t,x) - 1\right) V_{D_*}^{m-1}(x)\right]\right|^2 w_{\tau_n}(t,x)\, V_{D_*}(x)\,\mathrm{d}x\,\mathrm{d}t = 0\;.$$

By Fatou's lemma, $w_\infty = \lim_{n\to\infty} w_{\tau_n}$ satisfies $\nabla\left[\left(w_\infty^{m-1} - 1\right) V_{D_*}^{m-1}\right] = 0$ a.e. in $(0,1)\times\mathbb{R}^d$. As a consequence of the conservation of relative mass, see Proposition 2.3, $w_\infty = 1$. Thus, we have proved the convergence a.e., and by equi-continuity, the pointwise convergence. Since the limit is unique, the whole family $\{w_\tau\}_\tau$ converges everywhere as $\tau \to \infty$. □

3.3. **Proof of Theorem 1.1.**

Proof of Theorem 1.1, (i) and (ii). By Lemma 3.3, $\lim_{t\to\infty} |v(t,x) - V_{D_*}(x)| = 0$ for any $x \in \mathbb{R}^d$. Moreover, we observe that

$$|v(t) - V_{D_*}| \le \max\left\{\,|V_{D_0} - V_{D_*}|\,,\ |V_{D_1} - V_{D_*}|\,\right\} = O\left(|x|^{-2(2-m)/(1-m)}\right)$$

as $|x| \to \infty$. By Lebesgue's dominated convergence theorem, $v(t)$ converges to V_{D_*} in $\mathrm{L}^p(\mathbb{R}^d)$, for any $p \in (p(m),\infty)$, where $p(m) := \frac{d\,(1-m)}{2\,(2-m)}$ is the infimum of all positive p such that the difference between two different Barenblatt profiles belongs to $\mathrm{L}^p(\mathbb{R}^d)$.

The uniform convergence is based on the following interpolation lemma, due to Nirenberg, *cf.* [39, p. 126]. Let λ, μ and ν be such that $-\infty < \lambda \le \mu \le \nu < \infty$. Then there exists a positive constant $\mathcal{C}_{\lambda,\mu,\nu}$ such that

$$\|f\|_{1/\mu}^{\nu-\lambda} \le \mathcal{C}_{\lambda,\mu,\nu}\,\|f\|_{1/\lambda}^{\nu-\mu}\,\|f\|_{1/\nu}^{\mu-\lambda} \quad \forall\, f \in \mathcal{C}(\mathbb{R}^d)\;, \tag{3.1}$$

where $\|\cdot\|_{1/\sigma}$ stands for the following quantities:

(i) If $\sigma > 0$, then $\|f\|_{1/\sigma} = \left(\int_{\mathbb{R}^d} |f|^{1/\sigma}\,\mathrm{d}x\right)^\sigma$.

(ii) If $\sigma < 0$, let k be the integer part of $(-\sigma\,d)$ and $\alpha = |\sigma|\,d - k$ be the fractional (positive) part of σ. Using the standard multi-index notation, where $|\eta| = \eta_1 + \ldots + \eta_d$ is the length of the multi-index $\eta = (\eta_1, \ldots \eta_d) \in \mathbb{Z}^d$, we define

$$\|f\|_{1/\sigma} = \begin{cases} \max\limits_{|\eta|=k} \left|\partial^\eta f\right|_\alpha = \max\limits_{|\eta|=k}\ \sup\limits_{x,\,y\in\mathbb{R}^d} \dfrac{\left|\partial^\eta f(x) - \partial^\eta f(y)\right|}{|x-y|^\alpha} =: \|f\|_{C^\alpha(\mathbb{R}^d)} & \text{if } \alpha > 0\;, \\ \max\limits_{|\eta|=k}\ \sup\limits_{z\in\mathbb{R}^d} \left|\partial^\eta f(z)\right| := \|f\|_{C^k(\mathbb{R}^d)} & \text{if } \alpha = 0\;. \end{cases}$$

As a special case, we observe that $\|f\|_{-d/j} = \|f\|_{C^j(\mathbb{R}^d)}$.

(iii) By convention, we note $\|f\|_{1/0} = \sup_{z\in\mathbb{R}^d} |f(z)| = \|f\|_{C0(\mathbb{R}^d)} = \|f\|_\infty$.

Let $j \in \mathbb{N}$ and $\lambda = -(j+1)/d \le \mu = -j/d \le \nu = 1/2$ so that $k = j+1$ and $\alpha = 0$. Inequality (3.1) becomes

$$\|f\|_{C^j(\mathbb{R}^d)} \le \mathcal{C}_{-(j+1)/d,\,-j/d,\,1/2}^{\frac{2d}{d+2(j+1)}} \|f\|_{C^{j+1}(\mathbb{R}^d)}^{\frac{d+2j}{d+2(j+1)}} \|f\|_2^{\frac{2}{d+2(j+1)}} \tag{3.2}$$

for any $j \in \mathbb{N}$. By applying this interpolation inequality $f = v(t) - V_{D_*}$ with $j = 0$, we obtain

$$\|v(t) - V_{D_*}\|_\infty \le \mathcal{C}_{-1/d,\,0,\,1/2}^{\frac{2d}{d+2}} \|v(t) - V_{D_*}\|_{C^1(\mathbb{R}^d)}^{\frac{d}{d+2}} \|v(t) - V_{D_*}\|_2^{\frac{2}{d+2}} . \tag{3.3}$$

By Theorem 2.4, the $C1$ norm is uniformly bounded. If $q(m) < 2$, that is, if $d \le 8$, or $d \ge 9$ and $m > (d-8)/(d-4)$, we already know that $\lim_{t\to\infty} \|v(t) - V_{D_*}\|_2 = 0$. Otherwise, we can interpolate $\|v(t)-V_{D_*}\|_2$ between $\|v(t)-V_{D_*}\|_1 \le \|v_0-V_{D_*}\|_1$ (see Lemma 2.1) and $\|v(t)-V_{D_*}\|_q$ for some $q > q(m)$. This proves that $\lim_{t\to\infty} \|v(t) - V_{D_*}\|_\infty = 0$. □

Proof of Theorem 1.1, *(iii).*

Corollary 3.4 (Uniform convergence of the relative error)**.** *Let* $m \in (0,1)$*. If* w *is a solution of* (2.2) *with initial data* w_0 *satisfying* (H1")-(H2")*, then*

$$\lim_{t\to\infty} \|w(t) - 1\|_\infty = 0 .$$

Proof. Because of the convergence of $v(t)$ to V_{D_*} in $L^\infty(\mathbb{R}^d)$, we know that $w(t)$ converges uniformly to 1 on any compact set of $\mathbb{R}^d$. By Assumption (H1'), $v(t)$ is sandwiched between two Barenblatt profiles that have the same asymptotic behavior when $|x|$ is large. In terms of w, this means that $|w(t,x) - 1|$ is small for $|x|$ large, uniformly in t. Global uniform convergence follows. □

The fact that $w(t)$ converges uniformly to 1 as $t \to \infty$ allows us to improve the lower and upper bounds W_0 and W_1 for the function $w(t)$, at the price of waiting some time. For any $\varepsilon > 0$ there exists a time $t_0 = t_0(\varepsilon) \ge 0$ such that

$$1 - \varepsilon \le w(t,x) \le 1 + \varepsilon \quad \forall\, (t,x) \in (t_0, \infty) \times \mathbb{R}^d .$$

Corollary 3.5 (L^p Convergence of the relative error)**.** *Let* $m \in (0,1)$*. If* w *is a solution of* (2.2) *with initial data* w_0 *satisfying* (H1")-(H2")*, then* $w(t)$ *converges to* 1 *in* $L^p(\mathbb{R}^d)$ *for any* $p \in (d/2, \infty]$.

Proof. By Assumptions (H1")-(H2"), there exists a positive constant c_0 such that $w_0 - 1$ is bounded and for $|x|$ large,

$$|w_0 - 1| = \left|\frac{v_0 - V_{D_*}}{V_{D_*}}\right| \le \frac{V_{D_1} - V_{D_0}}{V_{D_*}} \le \frac{c_0}{1+|x|^2} .$$

By Lemma 2.2, the same estimate holds for $w(t)$. Hence $w(t) - 1 \in L^q$ for any $q > d/2$. Let $\delta = (p - d/2)/2 > 0$. By Hölder's inequality,

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} |w(t) - 1|^p \, dx \le \lim_{t\to\infty} \|w(t) - 1\|_\infty^\delta \int_{\mathbb{R}^d} \left(\frac{c_0}{1+|x|^2}\right)^{\delta + d/2} dx = 0 .$$

□

Theorem 1.1, (iii), results from Corollaries 3.4 and 3.5.

3.4. **Uniform convergence and C^α regularity.**

Lemma 3.6. *Let $m\in(0,1)$. Consider a solution v of* (1.4) *with initial data v_0 satisfying* (H1')-(H2'). *There exists $t_0\geq 0$, $\alpha\in(0,1)$ and a positive constant $\mathcal{H}$ such that $h(t):=v(t)-V_{D_*}$ is in C^α and*

$$\|h(t)\|_{C^\alpha(\mathbb{R}^d)}\leq \mathcal{H}\,\|h(t)\|_\infty \quad \forall\, t\geq t_0\,. \tag{3.4}$$

Proof. Since both v and V_{D_*} are solutions to equation (1.4), h solves

$$h_t=\nabla\cdot\left[m(V_{D_*}+h)^{m-1}\nabla\,h+m\left((V_{D_*}+h)^{m-1}-V_{D_*}^{m-1}-V_{D_*}^{m-2}h\right)\nabla\,V_{D_*}\right]\,.$$

Let $\lambda>0$. By Theorem 1.1, we know that for some $t_0\geq 0$, for any $t\geq t_0$, $\|h(t)\|_\infty$ can be taken uniformly small and v uniformly positive on $B_{2\lambda}$. We apply the standard quasilinear parabolic theory, see *e.g.* [35, Theorem 1.1, p. 418], with structure functions $a_i(x,t,h,\xi)=A\,\xi+B\,h$ and $a=0$, where $A(x,t):=m\,v^{m-1}$, $B(x,t):=m\left[\left(\frac{v^{m-1}-V_{D_*}^{m-1}}{v-V_{D_*}}-V_{D_*}^{m-2}\right)\nabla\,V_{D_*}\right]$. Hence there exists a Hölder exponent $\alpha\in(0,1)$ and a constant $\mathcal{H}$ depending on the uniform bounds for the coefficients, and on λ, such that (3.4) is verified in $B_\lambda\times(t_0+1,\infty)$. To extend the estimate uniformly to the whole space, $x\in\mathbb{R}^d$, we use the same scaling argument as in the proof of Theorem 2.4. We leave the details to the reader. □

4. Linearization

In order to better understand the asymptotic behavior of the solutions of (2.2), we linearize the equation around the equilibrium, introducing a convenient weight. Let g be such that

$$w(t,x)=1+\varepsilon\,\frac{g(t,x)}{V_{D_*}^{m-1}(x)}\quad\forall\,t>0\,,\quad\forall\,x\in\mathbb{R}^d\,, \tag{4.1}$$

for some $\varepsilon>0$, small. Substituting this expression in Equation (2.2) and letting $\varepsilon\to 0$, we formally obtain a linear equation for g,

$$g_t=A_m\,g\quad\text{where}\quad A_m\,g:=m\,V_{D_*}^{m-2}(x)\,\nabla\cdot\left[V_{D_*}\,\nabla g\right]\,. \tag{4.2}$$

The linear operator $A_m:\mathrm{L2}(\mathbb{R}^d,V_{D_*}^{2-m}\,\mathrm{d}x)\to\mathrm{L2}(\mathbb{R}^d,V_{D_*}^{2-m}\,\mathrm{d}x)$ is the positive self-adjoint operator associated to the closure of the quadratic form defined for $\phi\in C_c^\infty(\mathbb{R}^d)$ by

$$\mathsf{I}[\phi]:=m\int_{\mathbb{R}^d}|\nabla g|^2\,V_{D_*}\,\mathrm{d}x\,. \tag{4.3}$$

See [23, Theorem 2.6] for more details.

With the same heuristics, we linearize the relative entropy $\mathcal{F}$ and the relative Fisher information $\mathcal{J}$, which provides the functionals F and I, where I is given by (4.3) and F is defined by

$$\mathsf{F}[g]:=\frac{1}{2}\int_{\mathbb{R}^d}|g|^2\,V_{D_*}^{2-m}\,\mathrm{d}x\,. \tag{4.4}$$

Note that $\mathsf{F}[g]$ is the $\mathrm{L2}(\mathbb{R}^d,V_{D_*}^{2-m}\,\mathrm{d}x)$-norm up to a factor 1/2. If g is a solution of (4.2), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{F}[g(t)]=-\,\mathsf{I}[g(t)]\,. \tag{4.5}$$

Still at a formal level, the conservation of relative mass amounts to require

$$\int_{\mathbb{R}^d}(v_0-V_{D_*})\,\mathrm{d}x=\int_{\mathbb{R}^d}(w-1)\,V_{D_*}\,\mathrm{d}x=\varepsilon\int_{\mathbb{R}^d}g\,V_{D_*}^{2-m}\,\mathrm{d}x$$

in the limit $\varepsilon \to 0$. Hence, it makes sense to require that $\int_{\mathbb{R}^d} g\, V_{D_*}^{2-m}\, \mathrm{d}x = 0$ and use the spectral gap estimate, see [7] and Theorem A.1. With $\mathcal{C}_{m,d} = m/\lambda_{m,d}$, we obtain

$$2\,\mathsf{F}[g] \le \frac{\mathcal{C}_{m,d}}{m}\,\mathsf{I}[g]\;, \tag{4.6}$$

which gives, for the solution of (4.1), an exponential decay of the relative entropy,

$$\mathsf{F}[g(t)] \le \mathrm{e}^{-\,2\,\lambda_{m,d}\,t}\,\mathsf{F}[g(0)] \quad \forall\, t \ge 0\;.$$

In Sections 5 and 6, we will compare the relative entropy estimates for the solutions of (2.2) with the ones of the linearized problem. This is the main ingredient of the proof of Theorem 1.2.

The connection with the Fokker-Planck equation is easy to understand at the level of the linearized problem. In the limit $m \to 1$, we observe that

$$\lim_{m\to 1^-} D_*^{1/(1-m)}\, V_{D_*} = (2\,\pi\,D_*)^{d/2}\,\mu \quad \text{with} \quad \mu(x) = \frac{\mathrm{e}^{-\frac{|x|^2}{2\,D_*}}}{(2\,\pi\,D_*)^{d/2}}\;.$$

Equation (4.2) formally converges to the Ornstein-Uhlenbeck equation,

$$g_t = \mu^{-1}\,\nabla\cdot\big(\,\mu\,\nabla g\,\big)\;.$$

The spectral gap inequality (4.6) corresponds in such a limit to the well-known Poincaré inequality with gaussian weight,

$$\int_{\mathbb{R}^d} |\phi|^2\,\mathrm{d}\mu \le \int_{\mathbb{R}^d} |\nabla\phi|^2\,\mathrm{d}\mu \quad \forall\,\phi \in C^\infty(\mathbb{R}^d) \text{ such that } \int_{\mathbb{R}^d} \phi\,\mathrm{d}\mu = 0\;,$$

where $\mathrm{d}\mu := \mu\,\mathrm{d}x$. Note that in the Gaussian case, a logarithmic Sobolev inequality holds, see [29],

$$\int_{\mathbb{R}^d} |\phi|^2\,\log\left(\frac{|\phi|^2}{\int_{\mathbb{R}^d}|\phi|^2\,\mathrm{d}\mu}\right)\,\mathrm{d}\mu \;\le\; 2\int_{\mathbb{R}^d} |\nabla\phi|^2\,\mathrm{d}\mu\;,$$

which is stronger than the Gaussian Poincaré inequality. This is not the case with the measure $V_{D_*}\,\mathrm{d}x$. Although the spectral gap inequality (4.6) holds true, there is no corresponding logarithmic Sobolev inequality.

5. More on the relative Fisher information

In this section, we relate the relative Fisher and linearized Fisher informations. This and Lemma 3.1 provide us with an estimate of the relative entropy in terms of the relative Fisher information, or *entropy - entropy production inequality,* for the nonlinear problem.

5.1. Fisher information and linearized Fisher information.

Lemma 5.1 (Upper bound on the Fisher information)**.** *Let $m \in (0,1)$. There exists two positive constants β_1 and β_2 (depending on W_0, W_1 and m) such that, for any w satisfying* (H1")-(H2")*,*

$$\mathsf{I}[g] \le \beta_1\,\mathcal{J}[w] + \beta_2\,\mathsf{F}[g] \quad \textit{with} \quad g := (w-1)\,V_{D_*}^{m-1}\;.$$

Moreover, if $\eta := \max\{1-W_0, W_1-1\}$, then $\lim_{\eta\to 0_+}\big(|1-\beta_1| + \beta_2\big) = 0$.

The constant β_1 and β_2 are explicitly given in (5.2) in terms of m, D_*, W_0 and W_1.

Proof. Define $h_k(w) := (w^{k-1}-1)/(k-1)$. Let $\alpha_0 := W_0^{2(2-m)}$, $\alpha_1 := W_1^{2(2-m)}$. Since $|h_2/h_m|$ is non-decreasing,

$$\alpha_0 \le \left|\frac{h_2'(W_0)}{h_m'(W_0)}\right|^2 \le \left|\frac{h_2(w)}{h_m(w)}\right|^2 \le \left|\frac{h_2(W_1)}{h_m(W_1)}\right|^2 \le \left|\frac{h_2'(W_1)}{h_m'(W_1)}\right|^2 = \alpha_1\;. \tag{5.1}$$

Note that $\alpha_0=\alpha_0(W_0)<1<\alpha_1=\alpha_1(W_1)$ and both converge to 1 as $W_0, W_1\to 1$.

Using the fact that $V_{D_*}^{m-1}=D_*+\frac{1-m}{2\,m}\,|x|^2$ and an integration by part, we get

$$\int_{\mathbb{R}^d}\left|\nabla\left[h_k(w)V_{D_*}^{m-1}\right]\right|^2\,V_{D_*}\,{\rm d}x$$
$$=\int_{\mathbb{R}^d}|h_k'(w)|^2\,|\nabla w|^2\,V_{D_*}^{2m-1}\,{\rm d}x+\frac{1-m}{m2}\int_{\mathbb{R}^d}|x|^2\,|h_k(w)|^2\,V_{D_*}\,{\rm d}x-d\,\frac{1-m}{m}\int_{\mathbb{R}^d}|h_k(w)|^2\,V_{D_*}^m\,{\rm d}x\;.$$

Let $g:=(w-1)\,V_{D_*}^{m-1}$. Applied with $k=2$ and $k=m$, the above identity gives

$$\begin{aligned}\mathsf{I}[g]=m\int_{\mathbb{R}^d}\left|\nabla\left[h_2(w)V_{D_*}^{m-1}\right]\right|^2\,V_{D_*}\,{\rm d}x&\\ \le m\,\alpha_1\int_{\mathbb{R}^d}|h_m'(w)|^2\,|\nabla w|^2\,V_{D_*}^{2m-1}\,{\rm d}x&\\ +\alpha_1\,\frac{1-m}{m}\int_{\mathbb{R}^d}|x|^2\,|h_m(w)|^2\,V_{D_*}\,{\rm d}x-d\,(1-m)\int_{\mathbb{R}^d}|h_2(w)|^2\,V_{D_*}^m\,{\rm d}x&\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d}|h_m'(w)|^2\,|\nabla w|^2\,V_{D_*}^{2m-1}\,{\rm d}x&\\ =\int_{\mathbb{R}^d}\left|\nabla\left[h_m(w)V_{D_*}^{m-1}\right]\right|^2\,V_{D_*}\,{\rm d}x&\\ -\frac{1-m}{m2}\int_{\mathbb{R}^d}|x|^2\,|h_m(w)|^2\,V_{D_*}\,{\rm d}x+d\,\frac{1-m}{m}\int_{\mathbb{R}^d}|h_m(w)|^2\,V_{D_*}^m\,{\rm d}x\;.&\end{aligned}$$

Collecting these estimates, we obtain

$$\mathsf{I}[g]\le m\,\alpha_1\int_{\mathbb{R}^d}\left|\nabla\left[h_m(w)V_{D_*}^{m-1}\right]\right|^2\,V_{D_*}\,{\rm d}x+d\,(1-m)\int_{\mathbb{R}^d}\left(\alpha_1\,|h_m(w)|^2-|h_2(w)|^2\right)V_{D_*}^m\,{\rm d}x\;.$$

Note that

$$m\int_{\mathbb{R}^d}\left|\nabla\left[h_m(w)V_{D_*}^{m-1}\right]\right|^2\,V_{D_*}\,{\rm d}x\le\;W_0^{-1}\,\mathcal{J}[w]\;.$$

Using $\mathsf{F}[g]=\frac12\int_{\mathbb{R}^d}|g|^2\,V_{D_*}^{2-m}\,{\rm d}x=\frac12\int_{\mathbb{R}^d}|h_2(w)|^2\,V_{D_*}^m\,{\rm d}x$ with $g:=(w-1)\,V_{D_*}^{m-1}$, we obtain

$$\mathsf{I}[g]\le\beta_1\,\mathcal{J}[w]+\beta_2\,\mathsf{F}[w]\;,$$

with

$$\beta_1:=\frac{\alpha_1}{W_0}=\frac{W_1^{2(2-m)}}{W_0}\quad\text{and}\quad\beta_2:=2\,d\,(1-m)\left(\frac{\alpha_1}{\alpha_0}-1\right)\;.\tag{5.2}$$

Note that $\alpha_0=\alpha_0(W_0)<1<\alpha_1=\alpha_1(W_1)$ and both tend to 1 as $W_0, W_1\to 1$. □

5.2. **Entropy - entropy production inequality.**

Theorem 5.2 (Entropy - entropy production inequality)**.** *Let $m\in(0,1)$, $m\neq m_*$. For any function $w=w(x)$ satisfying* (H1")-(H2")*, if $1-W_0>0$ and $W_1-1>0$ are small enough, then there exists a positive constant γ such that*

$$\gamma\,\mathcal{F}[w]\le\mathcal{J}[w]\;.$$

As we shall see in the proof, the constant γ can be estimated as follows:

$$\gamma\ge 2\,\frac{m-\mathcal{C}_{m,d}\,d\,(1-m)\left[\left(\frac{W_1}{W_0}\right)^{2(2-m)}-1\right]}{\mathcal{C}_{m,d}\,W_0^{m-3}\,W_1^{2(2-m)}}\;.\tag{5.3}$$

The condition that $1-W_0>0$ *and* $W_1-1>0$ *are small enough* in the statement of Theorem 5.2 can be replaced by a weaker condition which amounts to ask that the right hand side of the above estimate is positive, that is, with $\lambda_{m,d}=m/\mathcal{C}_{m,d}$,

$$\frac{W_1}{W_0}<\left(1+\frac{\lambda_{m,d}}{d\,(1-m)}\right)^{\frac{1}{2\,(2-m)}}\;.$$

Proof. Let $g:=(w-1)\,V_{D_*}^{m-1}$. By definition of D_*, $0=\int_{\mathbb{R}^d}(w-1)\,V_{D_*}\,\mathrm{d}x=\int_{\mathbb{R}^d}g\,V_{D_*}^{2-m}\,\mathrm{d}x$ if $m>m_*$. By the spectral gap estimate (4.6) (also see Theorem A.1),

$$2\,\mathsf{F}[g]\le\frac{\mathcal{C}_{m,d}}{m}\,\mathsf{I}[g]\;.$$

By Lemma 5.1,

$$2\,\mathsf{F}[g]\le\frac{\mathcal{C}_{m,d}}{m}\,\mathsf{I}[g]\le\frac{\mathcal{C}_{m,d}}{m}\,(\beta_1\,\mathcal{J}[w]+\beta_2\,\mathsf{F}[g])\;,$$

from which we deduce that

$$\mathsf{F}[g]\le\frac{\mathcal{C}_{m,d}\,\beta_1}{2\,m-\mathcal{C}_{m,d}\,\beta_2}\,\mathcal{J}[w]\;.$$

By Lemma 3.1, the conclusion holds with

$$\gamma=\frac{2\,m-\mathcal{C}_{m,d}\,\beta_2}{W_0^{m-2}\,\mathcal{C}_{m,d}\,\beta_1}$$

under the condition $2\,m-\mathcal{C}_{m,d}\,\beta_2>0$. According to the definition (5.2) of β_2, this amounts to the condition

$$\mathcal{C}_{m,d}\,\beta_2=2\,d\,(1-m)\,\mathcal{C}_{m,d}\left(\frac{\alpha_1}{\alpha_0}-1\right)<2\,m\;,$$

that is α_1/α_0 close enough to 1, which follows from the requirement that $1-W_0>0$ and $W_1-1>0$ are small enough. □

6. Convergence with rates

6.1. Exponential decay of the relative entropy.

Proposition 6.1. *Let $m\in(0,1)$, $m\neq m_*$. There exists a positive constant γ such that, for any solution w of* (2.2) *with initial data w_0 satisfying* (H1")-(H2")*, if $1-W_0>0$ and $W_1-1>0$ are small enough, then*

$$\mathcal{F}[w(t)]\le\mathcal{F}[w_0]\,\mathrm{e}^{-\gamma\,t}\;.$$

The value of γ can be estimated from below by (5.3).

Proof. We combine formula $\frac{d}{dt}\mathcal{F}[w(t)]=-\mathcal{J}[w(t)]$ with the estimate $\mathcal{J}[w(t)]\ge\gamma\,\mathcal{F}[w(t)]$ obtained in Theorem 5.2, and then integrate the resulting differential inequality. □

6.2. Moments, L^p and C^k estimates.

We recall that $q_*:=\dfrac{2\,d\,(1-m)}{d\,(1-m)+2\,(2-m)}$ and define

$$\gamma(q):=\frac{1}{2}\ \text{if}\ q\in(q_*,2]\;,\quad\gamma(q)=\frac{q+d}{q(d+2)}\ \text{if}\ q\in(2,\infty)\;,\quad\gamma(\infty):=\frac{1}{d+2}\;.$$

The following lemma helps to understand better the consequences of the convergence of the free energy $\mathcal{E}\,[v|V_{D_*}]$, in terms of L^p, moment and also C^k convergence.

Lemma 6.2. *Let $m\in(0,1)$ and consider a function v satisfying* (H1')-(H2')*. Then*

(i) *For any* $\vartheta \in [0, \frac{2-m}{1-m}]$*, there exists a positive constant* K_ϑ *such that*

$$\Big\| \, |x|^\vartheta (v - V_{D_*}) \, \Big\|_2 \leq K_\vartheta \; (\mathcal{E}\left[v|V_{D_*}\right])^{1/2} \; .$$

(ii) *For any* $q \in (q_*, 2]$*, there exists a positive constant* $K(q)$ *such that*

$$\| v - V_{D_*} \|_q \leq K(q) \; (\mathcal{E}\left[v|V_{D_*}\right])^{\gamma(q)} \; .$$

Consider now a solution v *of* (1.4) *such that* v_0 *satisfies* (H1')-(H2') *and fix some* $t_0 > 0$*. Then*

(iii) *For any* $j \in \mathbb{N}$ *and any* $t_0 > 0$*, there exists a positive constant* H_j *such that*

$$\| v(t) - V_{D_*} \|_{C^j(\mathbb{R}^d)} \leq H_j \; (\mathcal{E}\left[v(t)|V_{D_*}\right])^{\frac{1}{d+2(j+1)}} \quad \forall \, t \geq t_0 \; .$$

(iv) *For any* $q \in (2, \infty]$*, there exists a positive constant* $K(q)$ *such that*

$$\| v(t) - V_{D_*} \|_q \leq K(q) \; (\mathcal{E}\left[v(t)|V_{D_*}\right])^{\gamma(q)} \quad \forall \, t \geq t_0 \; .$$

Proof. (i) With $\kappa_\vartheta := 2 \, \sup_{r>0} r^{2\vartheta} \left(D_* + \frac{1-m}{2\,m} \, r2 \right)^{-(2-m)/(1-m)}$,

$$\| \, |x|^\vartheta (v - V_{D_*}) \, \|_2 2 \leq \frac{1}{2} \, \kappa_\vartheta \int_{\mathbb{R}^d} |v - V_{D_*}|^2 \, V_{D_*}^{m-2} \, \mathrm{d}x \leq \kappa_\vartheta \, \mathsf{F}[(w-1) V_{D_*}^{m-1}] \; ,$$

and the right hand side is equivalent to $\mathcal{E}\left[v|V_{D_*}\right]$ by Lemma 3.1.

(ii) Let $q \in (q_*, 2)$. By Hölder's inequality,

$$\int_{\mathbb{R}^d} |v - V_{D_*}|^q \, \mathrm{d}x = \int_{\mathbb{R}^d} V_{D_*}^{(2-m)q/2} \cdot \left(|v - V_{D_*}|^2 \, V_{D_*}^{m-2} \right)^{q/2} \, \mathrm{d}x \leq c(q) \left(\int_{\mathbb{R}^d} |v - V_{D_*}|^2 \, V_{D_*}^{m-2} \, \mathrm{d}x \right)^{q/2} ,$$

where $c(q)^{2/(2-q)} := \int_{\mathbb{R}^d} V_{D_*}^{(2-m)\frac{q}{2-q}} \, \mathrm{d}x$ is finite for any $q > q_*$. By Lemma 3.1, the estimate holds with $K(q) := c(q)^{1/q} \, (2 \, W_1^{2-m})^{1/2}$. In the limit case $q = 2$, the same method applies with $c(2) = \| V_{D_*}^{(2-m)} \|_\infty = D_*^{-(2-m)/(1-m)}$.

(iii) We apply the interpolation inequality (3.2) to $f = v(t) - V_{D_*}$ and bound $\| v(t) - V_{D_*} \|_{C^j(\mathbb{R}^d)}$ in terms of $\| v(t) \|_{C^{j+1}}$, which is uniformly bounded by Theorem 2.4 and $\| v(t) - V_{D_*} \|_2^{2/(d+2(j+1))}$, for which we apply the result of Part (i) with $\vartheta = 0$.

(iv) By Theorem 2.4, $v(t) \in C1(\mathbb{R}^d)$ and $v(t) - V_{D_*}$ is bounded in $C1$ uniformly for any $t \geq t_0 > 0$. By (3.3),

$$\| v(t) - V_{D_*} \|_\infty \; \leq \; \mathcal{C}_{-1/d,\,0,\,1/2}^{\frac{2d}{d+2}} \, \| v(t) - V_{D_*} \|_{C^1(\mathbb{R}^d)}^{\frac{d}{d+2}} \, \| v(t) - V_{D_*} \|_2^{\frac{2}{d+2}} \; .$$

We conclude by using Hölder's inequality, $\| v(t) - V_{D_*} \|_q \leq \| v(t) - V_{D_*} \|_\infty^{(q-2)/q} \, \| v(t) - V_{D_*} \|_2^{2/q}$. □

6.3. **Improvement of the convergence.**

Theorem 6.3. *Let* $d \geq 3$*,* $m \in (0,1)$ *with* $m \neq m^*$*. Consider a solution* w *of* (2.2) *with initial data satisfying* (H1")-(H2")*. There exist a positive constant* $\mathcal{K}$ *and a time* $t_0 \geq 0$ *such that*

$$\mathcal{F}[w(t)] \leq \; \mathcal{K} \, \mathrm{e}^{-2\,\lambda_{m,d}\,t} \quad \forall \, t \geq t_0 \; .$$

Moreover, for any $\lambda \in \left(0, \lambda_{m,d}\right)$*, there exist a positive constant* $\mathcal{C}_\infty$ *and a time* $t_0 \geq 0$ *such that*

$$\| w(t) - 1 \|_\infty \leq \; \mathcal{C}_\infty \, \mathrm{e}^{-2\,\frac{1-m}{2-m}\,\frac{\lambda}{d+2}\,t} \quad \forall \, t \geq t_0 \; .$$

Here $\lambda_{m,d} = m/\mathcal{C}_{m,d}$ where $\mathcal{C}_{m,d}$ is given in Theorem A.1. Hence the rate of decay obtained by spectral methods for the linearized equation exactly gives the rate of decay for the nonlinear problem, and the price to be paid is only on the constant $\mathcal{K}$. As a subproduct of the proof, for some positive constants η_0 and γ_∞ which are defined below, we obtain the following estimate

$$\mathcal{K} \le \mathcal{F}[w(t_0)]\, \mathrm{e}^{\frac{\eta_0\,(2-m)}{\gamma_\infty\,(1-m)}}\, \mathrm{e}^{2\,\lambda_{m,d}\,t_0}\;.$$

Proof. By Corollary 3.4, for any $\varepsilon > 0$, there exists $t_0 > 0$ such that $\tilde{w}(t) = w(t + t_0)$ satisfies Assumption (H2") at $t = 0$ with $0 < 1 - W_0 < \varepsilon$ and $0 < W_1 - 1 < \varepsilon$. With ε small enough, $\tilde{w}$ enters in the framework of Proposition 6.1, with γ as in Theorem 5.2. From now on, we assume that $t \ge t_0$ and simply write w instead of $\tilde{w}$.

On the one hand, by Lemma 6.2 and Proposition 6.1, we have

$$\|v(t) - V_{D_*}\|_\infty \le \sigma_0\, \mathrm{e}^{-\gamma_\infty\,(t-t_0)} \quad \text{with} \quad \sigma_0 := K(\infty)\, \mathcal{F}[w_0]^{\frac{1}{d+2}} \quad \text{and} \quad \gamma_\infty = \frac{\gamma}{d+2}\;,$$

which, in terms of w, gives the estimate

$$|w(t,x) - 1| \le \sigma_0\, \mathrm{e}^{-\gamma_\infty\,(t-t_0)} \left[D_* + \frac{1-m}{2\,m}\,|x|^2\right]^{\frac{1}{1-m}} \qquad \forall\, t \ge t_0\;, \quad \forall\, x \in \mathbb{R}^d\;.$$

On the other hand, let $h_\alpha(s) := (1+s)^\alpha$, $\alpha > 1$. For any $s \in [0, s_0]$,

$$\frac{h_\alpha(s) - 1}{s} \le \alpha + \frac{s_0}{2} \max_{s\in[0,s_0]} h''_\alpha(s) = \begin{cases} \alpha + \frac{s_0}{2}\,\alpha\,(\alpha-1)\,(1+s_0)^{\alpha-2} & \text{if} \quad \alpha \ge 2\;,\\[2mm] \alpha + \frac{s_0}{2}\,\alpha\,(\alpha-1) & \text{if} \quad \alpha \le 2\;. \end{cases}$$

Apply then this inequality with $\alpha = 1/(1-m)$, $s = (D_* - D_1)/\big(D_1 + \frac{1-m}{2\,m}\,|x|^2\big) \le s_0 = \frac{D_*}{D_1} - 1$ to get the existence of a positive constant $\mathcal{M}_1 = \mathcal{M}_1(m, D_*, D_1)$ such that

$$w(t,x) - 1 \le \frac{V_{D_1}}{V_{D_*}} - 1 = \left(1 + \frac{D_* - D_1}{D_1 + \frac{1-m}{2\,m}\,|x|^2}\right)^{\frac{1}{1-m}} - 1 \le \frac{\mathcal{M}_1}{D_1 + \frac{1-m}{2\,m}\,|x|^2} \quad \forall\, x \in \mathbb{R}^d\;.$$

Similarly, for any $s \in [-s_0, 0]$,

$$\frac{h_\alpha(s) - 1}{s} \ge \alpha - \frac{s_0}{2} \max_{s\in[-s_0,0]} h''_\alpha(s) = \begin{cases} \alpha - \frac{s_0}{2}\,\alpha\,(\alpha-1) & \text{if} \quad \alpha \ge 2\;,\\[2mm] \alpha - \frac{s_0}{2}\,\alpha\,(\alpha-1)\,(1-s_0)^{\alpha-2} & \text{if} \quad \alpha \le 2\;, \end{cases}$$

so that, with $\alpha = 1/(1-m)$, $s = -(D_0 - D_*)/\big(D_0 + \frac{1-m}{2\,m}\,|x|^2\big) \le -s_0 = \frac{D_*}{D_0} - 1$, we get the existence of a positive constant $\mathcal{M}_0 = \mathcal{M}_0(m, D_*, D_0)$ such that

$$w(t,x) - 1 \ge \frac{V_{D_0}}{V_{D_*}} - 1 = \left(1 - \frac{D_0 - D_*}{D_0 + \frac{1-m}{2\,m}\,|x|^2}\right)^{\frac{1}{1-m}} - 1 \ge \frac{\mathcal{M}_0}{D_0 + \frac{1-m}{2\,m}\,|x|^2} \quad \forall\, x \in \mathbb{R}^d\;.$$

Hence, there exists a positive constant $\mathcal{M}$ depending on $\max\{\mathcal{M}_0, \mathcal{M}_1\}$, D_0 and D_1, for which we obtain

$$|w(t,x) - 1| \le \min\left\{\frac{\sigma_0\,\mathrm{e}^{-\gamma_\infty\,(t-t_0)}}{V_{D_*}}\;,\; \mathcal{M}\,V_{D_*}^{1-m}\right\} = \sigma\,\mathrm{e}^{-\gamma_\infty\,\frac{1-m}{2-m}\,(t-t_0)}\;, \quad \sigma := \mathcal{M}^{\frac{1}{2-m}}\,\sigma_0^{\frac{1-m}{2-m}}\;.$$

As a consequence, for any $t \ge t_0$, we have improved bounds on w, with W_0 and W_1 replaced respectively by $\sigma_0(t) := 1 - \sigma\,\mathrm{e}^{-\gamma_\infty\,\frac{1-m}{2-m}\,(t-t_0)}$ and $\sigma_1(t) := 1 + \sigma\,\mathrm{e}^{-\gamma_\infty\,\frac{1-m}{2-m}\,(t-t_0)}$. As in Proposition 6.1, according to Theorem 5.2 and Inequality (5.3), $z(t) = \mathcal{F}[w(t)]$ satisfies

$$\frac{dz}{dt} \le -\gamma(t)\,z(t)$$

with

$$\gamma(t) := 2\,\frac{m\,\sigma_0(t)^{2(2-m)} - \mathcal{C}_{m,d}\,d\,(1-m)\,\left[\sigma_1(t)^{2(2-m)} - \sigma_0(t)^{2(2-m)}\right]}{\mathcal{C}_{m,d}\,\sigma_0(t)^{1-m}\,\sigma_1(t)^{2(2-m)}} = 2\,\lambda_{m,d} - \eta(t)\;,$$

$\lambda_{m,d} = m/\mathcal{C}_{m,d}$ and $\eta(t) \le \eta_0\,\mathrm{e}^{-\gamma_\infty \frac{1-m}{2-m}\,(t-t_0)}$ for some $\eta_0 > 0$. A Gronwall argument then shows that for any $t \ge t_0$,

$$\log\left(\frac{z(t)}{z(t_0)}\right) \le -\,2\,\lambda_{m,d}\,(t-t_0) + \frac{\eta_0}{\gamma_\infty\frac{1-m}{2-m}}\,\left[1-\mathrm{e}^{-\gamma_\infty\frac{1-m}{2-m}(t-t_0)}\right] \le -\,2\,\lambda_{m,d}\,(t-t_0) + \frac{\eta_0\,(2-m)}{\gamma_\infty\,(1-m)}\;.$$

which completes the estimate on $\mathcal{F}[w(t)]$. For t large enough, $\frac{1}{2}\,\gamma(t) \in (\lambda, \lambda_{m,d})$ and the L^∞ estimate follows. □

Proof of Theorem 1.4. As in the proof of Corollary 3.5, a Hölder interpolation inequality shows that, for any $\delta > 0$,

$$\int_{\mathbb{R}^d} |w(t)-1|^p\;\mathrm{d}x \le \|w(t)-1\|_\infty^{p-\delta-\frac{d}{2}}\;\int_{\mathbb{R}^d}\left(\frac{c_0}{1+|x|^2}\right)^{\delta+\frac{d}{2}}\mathrm{d}x\;.$$

□

6.4. **Proof of Theorem 1.2.** We first apply Theorem 6.3 and Lemma 6.2, (i), with $\vartheta = 0$ to obtain, for some $t_0 \ge 0$,

$$\|v(t) - V_{D_*}\|_2 \le K_{\vartheta=0}\,\left(\mathcal{E}\left[v(t)|V_{D_*}\right]\right)^{\frac{1}{2}} \le C_2\,\mathrm{e}^{-\lambda_{m,d}\,t} \quad \forall\,t \ge t_0\;,$$

for some positive constant C_2. By the interpolation inequality (3.1) with $\lambda = -\alpha\,d < 0 = \mu < 1/2 = \nu$, $C = \mathcal{C}_{-\alpha d,\,0,\,1/2}$, and Lemma 3.6, (3.4), we have

$$\|v(t) - V_{D_*}\|_\infty \le\, C\,\|v(t) - V_{D_*}\|_{C^\alpha}^\theta\;\|v(t) - V_{D_*}\|_2^{1-\theta} \le\, C\,\mathcal{H}^\theta\,\|v(t) - V_{D_*}\|_\infty^\theta\;\|v(t) - V_{D_*}\|_2^{1-\theta}$$

where $\theta = 1/(2+\alpha\,d)$. This implies

$$\|v(t) - V_{D_*}\|_\infty \le C^{1/(1-\theta)}\,\mathcal{H}^{\theta/(1-\theta)}\,\|v(t) - V_{D_*}\|_2 \quad \forall\,t \ge t_0\;.$$

From Hölder's inequality, $\|v(t) - V_{D_*}\|_q \le \|v(t) - V_{D_*}\|_\infty^{(q-2)/q}\,\|v(t) - V_{D_*}\|_2^{2/q}$, $q \in (2,\infty]$, we deduce that $\|v(t) - V_{D_*}\|_q$ decays with the same rate as $\|v(t) - V_{D_*}\|_2$. If $q \in (q_*, 2)$, we apply Lemma 6.2, (ii), and Theorem 6.3 to prove that for some positive constant C_q and for some $t_0 \ge 0$,

$$\|v(t) - V_{D_*}\|_q \le C_q\,\mathrm{e}^{-\lambda_{m,d}\,t} \quad \forall\,t \ge t_0\;.$$

Similarly, the estimate $\|v(t) - V_{D_*}\|_{C^j(\mathbb{R}^d)}$ follows from Lemma 6.2, (iii), and Theorem 6.3. This completes the proof of Theorem 1.2. □

Appendix A: Hardy-Poincaré inequalities

In this appendix, we state and prove a result on inequalities which we have already been partially studied in [7]. Here we give more details and a few improvements. We are especially interested in the explicit values of the constants which enter in the convergence rates of Theorems 1.2 and 1.4. This is why we take weights which are adapted to Equation (1.4) and define the measures

$$d\mu := V_D^{2-m}\,\mathrm{d}x \quad \text{and} \quad d\nu := V_D\,\mathrm{d}x\;,$$

where $V_D(x) = \left(D + \frac{1-m}{2\,m}\,|x|^2\right)^{-1/(1-m)}$. Incidentally we observe that $d\mu = V_D^{1-m}\,d\nu$. To a function $g \in \mathrm{L}1(d\mu)$, we associate its average $\overline{g} = \int_{\mathbb{R}^d} g(x)\,d\mu$. Recall that $m_* = (d-4)/(d-2)$.

A.1. **Statement and comments.**

Theorem A.1. *Let $d \geq 1$ and $D > 0$. If $m \in (0,1)$ and $1 \leq d \leq 4$, or $m \in (m_*, 1)$ and $d \geq 5$, then there exists a positive constant $\mathcal{C}_{m,d}$, which does not depend on D, such that*

$$\int_{\mathbb{R}^d} |g - \overline{g}|^2 \, d\mu \leq \mathcal{C}_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2 \, d\nu \quad \forall\, g \in \mathcal{D}(\mathbb{R}^d)\,, \quad \overline{g} = \int_{\mathbb{R}^d} g \, d\mu\,. \tag{A.1}$$

In case $d \geq 5$ and $m \in (0, m_*)$, we have

$$\int_{\mathbb{R}^d} g2 \, d\mu \leq \mathcal{C}_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2 \, d\nu \quad \forall\, g \in \mathcal{D}(\mathbb{R}^d) \tag{A.2}$$

and $\mathcal{C}_{m,d} = \frac{8\,m\,(1-m)}{[(d-2)\,(m-m_*)]2}$ is optimal.

Estimates of the optimal constant $\mathcal{C}_{m,d}$ when $m > m_*$ are given below in Proposition A.3. With $v_m(x) = (1+|x|^2)^{-1/(1-m)}$, a simple change of variables shows that $\lambda_{m,d} = m/\mathcal{C}_{m,d}$ is such that

$$\lambda_{m,d} \;=\; m \inf_h \frac{\int_{\mathbb{R}^d} |\nabla h|^2\, V_D \,\mathrm{d}x}{\int_{\mathbb{R}^d} |h - \bar{h}|^2\, V_D^{2-m} \,\mathrm{d}x} \;=\; \frac{1-m}{2} \inf_h \frac{\int_{\mathbb{R}^d} |\nabla h|^2\, v_m \,\mathrm{d}x}{\int_{\mathbb{R}^d} |h - \tilde{h}|^2\, v_m^{2-m} \,\mathrm{d}x}\,, \tag{A.3}$$

where the infima are taken over the set of smooth functions h such that
- either $m < m_*$ and $\mathrm{supp}(h) \subset \mathbb{R}^d \setminus \{0\}$ and $\bar{h} = 0$, $\tilde{h} = 0$,
- or $m > m_*$,

$$\bar{h} := \frac{\int_{\mathbb{R}^d} h\, V_D^{2-m} \,\mathrm{d}x}{\int_{\mathbb{R}^d} V_D^{2-m} \,\mathrm{d}x} \quad \text{and} \quad \tilde{h} := \frac{\int_{\mathbb{R}^d} h\, v_m^{2-m} \,\mathrm{d}x}{\int_{\mathbb{R}^d} v_m^{2-m} \,\mathrm{d}x}\,.$$

This already shows that $\lambda_{m,d}$ is independent of D.

We observe that as $|x| \to \infty$, $d\mu \sim d\nu/|x|^2$. Hence, if $m \in (0, m_*)$, Inequality (A.2) is of Hardy type. Otherwise, if $m \in (m_*, 1)$, Inequality (A.1) involves an average and is rather of Poincaré type. In such a case, we shall also say that it is a weighted Poincaré inequality, or that there is a spectral gap, since for the associated operator, the lowest eigenvalue, 0, is achieved by the constant functions, and the second eigenvalue corresponds to $\lambda_{m,d} = m/\mathcal{C}_{m,d}$ where $\mathcal{C}_{m,d}$ is the best constant in the inequality. See [7] for further considerations on these issues.

We also remark that Theorem A.1 provides an explicit example for which the weighted Poincaré inequality holds, while the corresponding weighted logarithmic Sobolev inequality does not hold, even in dimension $d = 1$, as shown by [3, Theorem 3].

The proofs of (A.1) and (A.2) are quite different and for this reason we treat the two cases separately. We start with the proof of (A.2) corresponding to the case $m < m_*$, $d \geq 5$.

A.2. **Case $m \in (0, m_*)$.** The proof follows the ideas of [7]. We reproduce it here for completeness. We compute

$$|\nabla V_D(x)|^2 = \frac{|x|^2}{m2}\, V_D(x)^{2(2-m)}$$

and

$$-\,2\,m2\, \frac{\Delta V_D(x)}{V_D(x)^{3-2m}} = 2\,d\,D\,m + (d-2)\,(m_* - m)|x|^2\,.$$

An integration by parts and the Cauchy-Schwarz inequality show that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} |g|^2\, \Delta V_D \,\mathrm{d}x\right| &\leq 2 \int_{\mathbb{R}^d} |g|\, |\nabla g|\, |\nabla V_D| \,\mathrm{d}x \\ &\leq 2 \left(\int_{\mathbb{R}^d} |g|^2\, |\Delta V_D| \,\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^d} |\nabla g|^2\, |\nabla V_D|^2\, |\Delta V_D|^{-1} \,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

As in [24], we remark that ΔV_D has a constant sign and get the estimate

$$\left|\int_{\mathbb{R}^d}|g|^2\,\Delta V_D\,\mathrm{d}x\right|=\int_{\mathbb{R}^d}|g|^2\,|\Delta V_D|\,\mathrm{d}x\le 4\int_{\mathbb{R}^d}|\nabla g|^2\,|\nabla V_D|^2\,|\Delta V_D|^{-1}\,\mathrm{d}x\;.$$

Weights can be estimated on both sides of the inequality:

$$\begin{aligned}\frac{|\Delta V_D|}{V_D^{2-m}}&=\frac{2\,d\,D\,m+(d-2)\,(m_*-m)\,|x|^2}{m\,(2\,D\,m+(1-m)\,|x|^2)}\ge\frac{(d-2)\,(m_*-m)}{m\,(1-m)}\;,\\ \frac{|\nabla V_D|^2}{|\Delta V_D|\,V_D}&\le\frac{2\,|x|^2}{2\,d\,D\,m+(d-2)\,(m_*-m)\,|x|^2}\le\frac{2}{(d-2)\,(m_*-m)}\;,\end{aligned}$$

which proves (A.2). See [7] for further details.

We now consider the limit $D\to 0^+$. With $\alpha:=1/(m-1)\in(1-d/2,-1)$, that is $m\in(0,m_*)$, and

$$\kappa_\alpha:=\frac{8\,m\,(1-m)}{[(d-2)\,(m-m_*)]2}\cdot\frac{1-m}{2\,m}=\frac{4\,(1-m)2}{[(d-4)-(d-2)\,m]2}\;,$$

Inequality (A.2) takes the form of a weighted Hardy inequality,

$$\int_{\mathbb{R}^d}\frac{|g|^2}{|x|^2}\,|x|^\alpha\,\mathrm{d}x\le\kappa_\alpha\int_{\mathbb{R}^d}|\nabla g|^2\,|x|^\alpha\,\mathrm{d}x\quad\forall\,g\in\mathcal{D}(\mathbb{R}^d)\;.$$

Such an inequality is easy to establish by the "completing the square method" as follows. Let $\alpha\in\mathbb{R}\setminus\{\alpha\}$ with $\alpha_*:=1-d/2$, and $g\in\mathcal{D}(\mathbb{R}^d)$. Then

$$\begin{aligned}0\;&\le\;\int_{\mathbb{R}^d}\left|\nabla g+\lambda\,\frac{x}{|x|^2}\,g\right|^2\,|x|^{2\alpha}\,\mathrm{d}x\\ &=\int_{\mathbb{R}^d}|\nabla g|^2\,|x|^{2\alpha}\,\mathrm{d}x+\Big[\lambda 2-\lambda\,(2\,\alpha+d-2)\Big]\int_{\mathbb{R}^d}\frac{|g|^2}{|x|^2}\,|x|^{2\alpha}\,\mathrm{d}x\;.\end{aligned}$$

An optimization of the right hand side with respect to λ results in choosing $\lambda=(2\,\alpha+d-2)/2$, that is

$$\frac{1}{\lambda 2}=\frac{4}{(d+2\,\alpha-2)2}=\frac{4\,(1-m)2}{[(d-4)-(d-2)\,m]2}=\kappa_\alpha\;.$$

The weighted Hardy inequality is optimal, with optimal constant κ_α, as follows by considering the test functions $g_\varepsilon(x):=\min\{\varepsilon^{-\lambda},(|x|^{-\lambda}-\varepsilon^\lambda)_+\}$ and letting $\varepsilon\to 0$. □

A closer inspection of the proof reveals that the constant κ_α in the weighted Hardy inequality also is optimal when $m>m_*$. Consider indeed the test functions $g_\varepsilon(x):=|x|^{1-\alpha-d/2+\varepsilon}$ for $|x|<1$ and $g_\varepsilon(x)=(2-|x|)_+$ for $|x|\ge 1$, and then let $\varepsilon\to 0$.

Proposition A.2 (Weighted Hardy inequality)**.** *With the above notations, for any $\alpha\in\mathbb{R}$, $\alpha\neq\alpha_*$,*

$$\int_{\mathbb{R}^d}\frac{|g|^2}{|x|^2}\,|x|^{2\alpha}\,\mathrm{d}x\le\kappa_\alpha\int_{\mathbb{R}^d}|\nabla g|^2\,|x|^{2\alpha}\,\mathrm{d}x\quad\forall\,g\in\mathcal{D}(\mathbb{R}^d)\;,$$

with the additional requirement that g is supported in $\mathbb{R}^d\setminus\{0\}$ if $\alpha<\alpha_*$, and κ_α is optimal.

The range $m\in(0,1)$ corresponds to $1/(m-1)=\alpha\in(-\infty,-1)$, so that $m=m_*$ is equivalent to $\alpha=\alpha_*$. Notice that the result holds without other restriction than $\alpha\neq\alpha_*$, but one has to be careful with integrability condition at $x=0$ if $\alpha<\alpha_*$.

A.3. **Case** $\max\{0,m_*\}<m<1$**.** Several partial results are known. In the range $m\in(m_c,1)$, see [14] for an estimate of $\mathcal{C}_{m,d}$ based on the Bakry-Emery method, [7] for other estimates, and [27] for the exact values of the optimal constant for a corresponding linear problem.

We now prove (A.1) with some explicit estimates of the constant $\mathcal{C}_{m,d}$ in the whole range $(\max\{0,m_*\},(d-2)/(d-1))\supset(m_*,m_c]$. Because of the change of variables (A.3), our task is now to characterize $\mathcal{C}_{m,d}$ as

$$\left(\frac{(1-m)\,\mathcal{C}_{m,d}}{2\,m}\right)^{-1}=\inf_h\frac{\int_{\mathbb{R}^d}|\nabla h|^2\,v_m\,\mathrm{d}x}{\int_{\mathbb{R}^d}|h-\tilde h|^2\,v_m^{2-m}\,\mathrm{d}x}\,.$$

On $\mathbb{R}^+$, consider the function $\mu(r):=r^{d-1}(1+|r|^2)^{(2-m)/(m-1)}$, and denote its median by η. Let $\nu(r):=r^{d-1}(1+r2)^{1/(m-1)}$ and define for all $\zeta>0$ the quantity

$$\mathsf{K}(\zeta):=\frac{2\,m}{1-m}\,\max\{\mathsf{A}(\zeta),\mathsf{B}(\zeta)\}$$

with

$$\mathsf{A}(\zeta):=\sup_{r<\zeta}\left[\int_0^r\mu(s)\,\mathrm{d}s\int_r^\zeta\frac{\mathrm{d}s}{\nu(s)}\right]\,,\quad \mathsf{B}(\zeta):=\sup_{r>\zeta}\left[\int_\zeta^r\frac{\mathrm{d}s}{\nu(s)}\int_r^{+\infty}\mu(s)\,\mathrm{d}s\right]\,.$$

By convention, we take $\mathsf{K}(0)=\frac{2\,m}{1-m}\,\mathsf{B}(0)$. The following result is inspired by [3, 8, 19, 38].

Proposition A.3. *Let $d\ge1$. For any $m\neq m_*$,*

$$\mathcal{C}_{m,d}\ge\frac{8\,m\,(1-m)}{[d-4-m\,(d-2)]\,2}\,.$$

If $m\in(m_*,1)$, then

$$\mathcal{C}_{m,1}\le\mathsf{K}(0)\quad\textit{and}\quad\mathcal{C}_{m,d}\le\max\left\{2\,\mathsf{K}(\eta)\,,\;\frac{4\,m}{(1-m)\,(d-1)}\right\}\quad\textit{if}\quad d\ge2\,,$$

where, for any $m\in(m_*,(d-2)/(d-1))$,

$$\mathsf{K}(\eta)\le\frac{m\,(2-m)\,2^{\frac{3-2m}{1-m}}\left(1+2^{\frac{2-m}{1-m}}\right)}{d\,[d-4-m\,(d-2)]\,2}\,.$$

The function v_m^{2-m} is integrable for any $m\in(m_*,1)$, so that $\mathsf{K}(\eta)$ is well defined in this range. The upper bound on $\mathcal{C}_{m,d}$ is equal to its exact value up to a factor which is at least $1/4$ (and at most 1). Such an interval is inherent to the method, see [38]. The case $m=m_c\le(d-2)/(d-1)$ is covered, showing in particular that $\mathcal{C}_{m_c,d}$ is positive, finite. The bounds diverge as $m\searrow m^*$ with same behavior at first order. Our approach can be extended easily to the case $((d-2)/(d-1),1)$, with slightly different estimates of $\mathsf{K}(\eta)$, but this case is already covered in [7, 14, 27] by other methods. The restriction $m<(d-2)/(d-1)$ is convenient from a technical point of view, and not essential at all.

Proof. The lower bound on $\mathcal{C}_{m,d}$ is achieved as in Section A.2 by taking the limit $D=0$, thus showing that $\mathcal{C}_{m,d}\ge\frac{1-m}{2\,m}\,\kappa_{1/(m-1)}$ and using Proposition A.2.

Let us prove the upper bounds. We introduce the standard change of variables from Cartesian to spherical coordinates, *i.e.* $r=|x|$, and $\vartheta=x/|x|$. In these coordinates, the gradient can be written as $(\partial_r,\frac1r\nabla_\theta)$ where $\partial_r=\frac{x}{r}\cdot\nabla$ is the partial derivative with respect to the radial variable r and ∇_θ is the derivative with respect to the angular variables. We shall denote by $\mathbb{S}^{d-1}\subset\mathbb{R}^d$ the unit sphere and parametrize it with the variable ϑ.

The radial density functions $r\mapsto\mu(r)$ and $r\mapsto\nu(r)$ are such that $v_m\,\mathrm{d}x=\mu(|x|)\,\mathrm{d}x$ and $v_m^{2-m}\,\mathrm{d}x=\nu(|x|)\,\mathrm{d}x$. We introduce the following normalization constants:

$$\omega_d=\int_{\mathbb{S}^{d-1}}\mathrm{d}\vartheta=\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\,,\qquad\widehat{\mathrm{d}\vartheta}=\omega_d^{-1}\,\mathrm{d}\vartheta\,,\qquad\int_{\mathbb{S}^{d-1}}\widehat{\mathrm{d}\vartheta}=1\,.$$

With these notations,

$$\mu(r)\,\mathrm{d}r\,\mathrm{d}\vartheta=\frac{v_m(x)}{1+|x|^2}\,\mathrm{d}x=v_m^{2-m}\,\mathrm{d}x\quad\text{and}\quad\nu(r)\,\mathrm{d}r\,\mathrm{d}\vartheta=v_m\,\mathrm{d}x\;.$$

We define a directional average of a function f by

$$\widetilde{f_\mu}(\vartheta):=\int_0^{+\infty}f(r,\vartheta)\,\hat\mu(r)\,\mathrm{d}r\quad\text{with}\quad\hat\mu(r):=\frac{\mu(r)}{\int_0^{+\infty}\mu(s)\,\mathrm{d}s}$$

and the global average of f by

$$\widetilde{f}:=\frac{\int_{\mathbb{R}^d}f\,v_m^{2-m}\,\mathrm{d}x}{\int_{\mathbb{R}^d}v_m^{2-m}\,\mathrm{d}x}=\iint_{(0,\infty)\times\mathbb{S}^{d-1}}f(r,\vartheta)\,\hat\mu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}=\int_{\mathbb{S}^{d-1}}\widetilde{f_\mu}(\vartheta)\,\widehat{\mathrm{d}\vartheta}\;.$$

In the case $d=1$, Theorem 2 of [4], also see [38], says that Inequality (A.1) holds with

$$\mathcal{C}_{m,1}\le\frac{2\,m}{1-m}\sup_{r>0}\left[\int_r^{+\infty}\mu(r)\,\mathrm{d}r\int_0^r\frac{\mathrm{d}r}{\nu(r)}\right]=\mathsf{K}(0)$$

in which case we also have the estimate $\mathcal{C}_{m,1}\le\mathsf{K}(0)\le4\,\mathcal{C}_{m,1}$.

In case of radial functions, Inequality (A.1) takes the form:

$$\int_0^{+\infty}\left|f(r)2-\widetilde{f}\right|^2\mu(r)\,\mathrm{d}r\le\frac{1-m}{2\,m}\,\mathcal{C}^{\rm rad}_{m,d}\int_0^{+\infty}\left|f'(r)\right|^2\nu(r)\,\mathrm{d}r\tag{A.4}$$

with

$$\mathcal{C}^{\rm rad}_{m,d}\le\frac{2\,m}{1-m}\,\max\left\{\sup_{r>\eta}\left[\int_r^{+\infty}\mu(r)\,\mathrm{d}r\int_\eta^r\frac{\mathrm{d}r}{\nu(r)}\right]\;,\;\sup_{r<\eta}\left[\int_0^r\mu(r)\,\mathrm{d}r\int_r^\eta\frac{\mathrm{d}r}{\nu(r)}\right]\right\}=\mathsf{K}(\eta)\;.$$

It is straightforward to show that $\mathsf{K}(\eta)$ is finite, with the present choices of μ and ν, for $m\in(m_*,(d-2)/(d-1))$, and as above, $\mathcal{C}^{\rm rad}_{m,d}\le\mathsf{K}(\eta)\le4\,\mathcal{C}^{\rm rad}_{m,d}$.

We now focus on the case of non radial functions, with $d\ge2$, and rewrite the left hand side of (A.1) in spherical coordinates.

$$\begin{aligned}\int_{\mathbb{R}^d}\left|f(x)-\widetilde{f}\right|^2v_m^{2-m}\,\mathrm{d}x&=\omega_d\iint_{(0,\infty)\times\mathbb{S}^{d-1}}\left|f(r,\vartheta)-\widetilde{f}\,\right|^2\mu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}\\&=\omega_d\iint_{(0,\infty)\times\mathbb{S}^{d-1}}\left|f(r,\vartheta)-\widetilde{f_\mu}(\vartheta)+\widetilde{f_\mu}(\vartheta)-\widetilde{f}\,\right|^2\mu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}\\&\le2\,\omega_d\left[(\mathrm{I})+(\mathrm{II})\right]\end{aligned}$$

with

$$\begin{aligned}(\mathrm{I})&=\iint_{(0,\infty)\times\mathbb{S}^{d-1}}\left|f(r,\vartheta)-\widetilde{f_\mu}(\vartheta)\right|^2\mu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}\;,\\(\mathrm{II})&=\iint_{(0,\infty)\times\mathbb{S}^{d-1}}\left|\widetilde{f_\mu}-\widetilde{f}\,\right|^2\mu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}=\int_{\mathbb{S}^{d-1}}\left|\widetilde{f_\mu}-\widetilde{f}\right|^2\widehat{\mathrm{d}\vartheta}\;.\end{aligned}$$

We estimate (I) by (A.4) and get

$$(\mathrm{I})=\iint_{(0,\infty)\times\mathbb{S}^{d-1}}\left|f(r)2-\widetilde{f_\mu}(\vartheta)\right|^2\mu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}\le\frac{1-m}{2\,m}\,\mathcal{C}^{\rm rad}_{m,d}\iint_{(0,\infty)\times\mathbb{S}^{d-1}}\left|\partial_r\,f(r,\vartheta)\right|^2\nu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}\;.$$

To estimate (II), we rely on the Poincaré inequality on the unit sphere $\mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}}|u-\hat u\,|^2\,\widehat{\mathrm{d}\vartheta}\le\frac{1}{d-1}\int_{\mathbb{S}^{d-1}}|\nabla_\vartheta\,u|^2\,\widehat{\mathrm{d}\vartheta}\quad\forall\,u\in H1(\mathbb{S}^{d-1})\;.$$

Here $\hat u:=\int_{\mathbb{S}^{d-1}}u\,\widehat{\mathrm{d}\vartheta}$. In the inequality, $1/(d-1)$ is the optimal constant, as can be checked using spherical harmonic functions. See for instance [5, 9, 43]. The inequality itself can be recovered by various methods. For example, using the inverse stereographic projection, see [37], the optimal Sobolev inequality on $\mathbb{R}^d$ becomes

$$\left(\int_{\mathbb{S}^{d-1}} |v|^p \,\widehat{\mathrm{d}\vartheta}\right)^{2/p} \le \int_{\mathbb{S}^{d-1}} |v|^2 \,\widehat{\mathrm{d}\vartheta} + \frac{p-2}{d-1}\int_{\mathbb{S}^{d-1}} |\nabla_\vartheta \, v|^2 \,\widehat{\mathrm{d}\vartheta}\;,$$

for any $u \in H1(\mathbb{S}^{d-1})$, with $p = 2\,d/(d-2)$, $d \ge 3$. The inequality also holds true for any $p \in (2, 2\,d/(d-2))$ if $d \ge 3$ and for any $p > 2$ if $d = 2$, see [6]. Hence we recover the Poincaré inequality on $\mathbb{S}^{d-1}$ by writing $v = 1 + \varepsilon\, u$ and keeping only the terms of order $\varepsilon 2$ as $\varepsilon \to 0$.

We apply the Poincaré inequality with $u = \widetilde{f_\mu}$.

$$\int_{\mathbb{S}^{d-1}} \left|\widetilde{f_\mu} - \widetilde{f}\right|^2 \widehat{\mathrm{d}\vartheta} \le \frac{1}{d-1}\int_{\mathbb{S}^{d-1}} \left|\nabla_\vartheta \,\widetilde{f_\mu}\right|^2 \widehat{\mathrm{d}\vartheta}$$

Recall that $|\nabla f|^2 = |\partial_r f(r,\vartheta)|^2 + \frac{1}{r2}\,|\nabla_\vartheta \, f(r,\vartheta)|^2$. Using the Cauchy-Schwarz inequality and the estimate $r2\,\mu(r)\,\mathrm{d}r \le \nu(r)\,\mathrm{d}r$, we get

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \left|\nabla_\vartheta \,\widetilde{f_\mu}\right|^2 \widehat{\mathrm{d}\vartheta} &\le \iint_{(0,\infty)\times\mathbb{S}^{d-1}} \left|\nabla_\vartheta \, f(r,\vartheta)\right|^2 \mu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}\\ &\le \iint_{(0,\infty)\times\mathbb{S}^{d-1}} \frac{1}{r2}\,|\nabla_\vartheta \, f(r,\vartheta)|^2\;\nu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}\;.\end{aligned}$$

This proves that

$$(\mathrm{II}) \le \frac{1}{d-1}\iint_{(0,\infty)\times\mathbb{S}^{d-1}} \frac{1}{r2}\,|\nabla_\vartheta \, f(r,\vartheta)|^2\;\nu(r)\,\mathrm{d}r\,\widehat{\mathrm{d}\vartheta}\;.$$

Summarizing, we have shown that

$$\int_{\mathbb{R}^d} \left|f(x) - \widetilde{f}\right|^2 \,v_m^{2-m}\,\mathrm{d}x \le 2\,\max\left\{\frac{1-m}{2\,m}\,\mathcal{C}^{\mathrm{rad}}_{m,d},\,\frac{1}{d-1}\right\}\int_{\mathbb{R}^d} |\nabla f|^2\,v_m\,\mathrm{d}x\;.$$

By undoing the change of coordinates as in (A.3), we get

$$\int_{\mathbb{R}^d} \left|f(x) - \overline{f}\right|^2 \mathrm{d}\mu \le \max\left\{2\,\mathcal{C}^{\mathrm{rad}}_{m,d},\,\frac{4\,m}{(1-m)\,(d-1)}\right\}\int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\nu\;.$$

The bounds on $\mathsf{K}(\eta)$ follow by quite long but straightforward calculations, omitted here. □

Appendix B: Extension to exponents $m \le 0$

The presence at several instances of factors of the form $1/m$ in the previous calculations may suggest that there is an essential divergence as $m \to 0$. In the present section we want to dispel that impression by introducing a normalization that is often used in the literature, consisting in a rescaling of the time variable of the form $\tau' = m\,\tau$ that modifies equation (1.1) into

$$\partial_{\tau'} u = \nabla\cdot(u^{m-1}\nabla u)\;. \tag{B.1}$$

One of the first consequences is that the new equation, that we will call *modified fast diffusion equation* for clarity, makes perfect sense as a nonlinear parabolic equation of singular type for all the range of exponents $m \in \mathbb{R}$ (including $m = 0$), in particular for all $m < 1$ that form the extended range of the fast diffusion. Such approach has been consistently used in [47] where it is shown that the effect on the self-similar solutions of Barenblatt type is just to eliminate the denominator m in the formulas (1.2), (1.5). Note the rescaling to obtain (B.1) from the standard fast diffusion equation (1.1) when $m > 0$ can also be done by changing the space variable in the form $x = \sqrt{m}\,x'$ and not changing time.

Since the general theory (existence, uniqueness, estimates, special solutions and extinction) has been developed to the measure we need it, we can follow the different stages of the present paper with due attention to chasing the m factors, and the results stated in Section 1 remain valid. For instance, the formula defining $\mathcal{F}[w]$ in Section 2.5 has to be replaced by

$$\mathcal{F}[w] = -\int_{\mathbb{R}^d} \left[\,\log w - (w-1)\right]\,\mathrm{d}x\;.$$

In the linearization of Section 4 there is no m factor in the definition of operator A_m and neither in the definition of $\mathsf{I}[g]$. Let us mention two other points of interest: the exponent m_* becomes negative for $d = 1, 2, 3$ and zero for $d = 4$, but it still plays the same role of an important critical exponent separating different behavior types. On the other hand, the constant $\lambda_{m,d}$ that gives the decay rate in our main result has a finite positive value as $m \to 0$, according to Theorem A.1. This constant determines the rates in all results concerning the asymptotic behavior of the solutions and is not affected by our m-rescaling.

We have refrained from treating the extension to $m \leq 0$ in the main body of the paper in order to avoid further distractions in an already very technical matter. Whole details will appear separately.

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