Asymptotics of weighted random sums

José Manuel Corcuera*, David Nualart†, Mark Podolskij‡

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Abstract

In this paper we study the asymptotic behaviour of weighted random sums when the sum process converges stably in law to a Brownian motion and the weight process has continuous trajectories, more regular than that of a Brownian motion. We show that these sums converge in law to the integral of the weight process with respect to the Brownian motion when the distance between observations goes to zero. The result is obtained with the help of fractional calculus showing the power of this technique. This study, though interesting by itself, is motivated by an error found in the proof of Theorem 4 in [2].

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Fix a time interval $[0,T]$ and consider a double sequence of random variables $\xi = \{\xi_{i,m}, m \in \mathbb{Z}_+, 1 \le i \le [mT]\}$. For any $m \ge 1$ we denote by $g_m(t)$ the stochastic process defined as the distribution function of the signed measure on $[0,T]$ which gives mass $\xi_{i,m}$ to the points $t_i = \frac{i}{m}$, for $1 \le i \le [mT]$, that is,

$$g_m(t) := \sum_{i=1}^{[mt]} \xi_{i,m}.$$

Notice that $\xi_{i,m} = g_m(t_i) - g_m(t_{i-1})$. Throughout this paper we assume the following hypotheses on the double sequence ξ:

(H1) The sequence of processes $(g_m(t))_{t\in[0,T]}$ satisfies

$$(g_m(t))_{t\in[0,T]} \overset{f.d.d.}{\underset{m\to\infty}{\to}} (w(t))_{t\in[0,T]} \qquad \mathcal{F}\text{-stably},$$

*University of Barcelona, Spain. E-mail: jmcorcuera@ub.edu
†University of Kansas, USA. E-mail: nualart@ku.edu
‡University of Heilderberg, Germany. E-mail:m.podolskij@uni-heidelberg.de

where $(w(t))_{t\in[0,T]}$ is a standard Brownian motion independent of $\mathcal{F}$, and the latter denotes convergence of finite dimensional distributions $\mathcal{F}$-stably in law (see the definition below).

(H2) The family of random variables ξ satisfies the tightness condition

$$\mathbb{E}\left(\left|\sum_{i=j+1}^{k}\xi_{i,m}\right|^4\right) \leq C\left(\frac{k-j}{m}\right)^2, \tag{1}$$

for any $1 \leq j < k \leq [mT]$.

Notice that these hypotheses imply that $(g_m(t))_{t\in[0,T]} \xrightarrow[m\to\infty]{\mathcal{L}} (w(t))_{t\in[0,T]}$, $\mathcal{F}$-stably in the Skorohod space $D[0,T]$ equipped with the uniform topology.

Under these assumptions, the purpose of this note is to establish the following result.

Theorem 1 *Let $(f(t))_{t\in[0,T]}$ be a α-Hölder continuous process with index $\alpha > 1/2$. Suppose that $\xi = \{\xi_{i,m}, m \in \mathbb{Z}_+, 1 \leq i \leq [mT]\}$ is a family of random variables satisfying hypotheses **(H1)** and **(H2)**. Set*

$$X_m(t) := \sum_{i=1}^{[mt]} f(t_i)\xi_{i,m} = \int_0^t f(s)\mathrm{d}g_m(s).$$

Then,

$$X_m(t) \xrightarrow[m\to\infty]{\mathcal{L}} \int_0^t f(s)\mathrm{d}w(s), \qquad \mathcal{F}\text{-stably}$$

in $D[0,T]$, where $(w(t))_{t\in[0,T]}$ is a standard Brownian motion independent of $\mathcal{F}$.

Recall that a sequence of random vectors or processes Y_n converges $\mathcal{F}$-stably in law to a random vector or process Y, where Y is defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability $(\Omega, \mathcal{F}, \mathbb{P})$, if $(Y_n, Z) \xrightarrow{\mathcal{L}} (Y, Z)$ for any $\mathcal{F}$-measurable random variable Z. If Y is $\mathcal{F}$-measurable, then we have convergence in probability. We refer to [5] and [1] for more details on stable convergence.

Remark 2 *The conclusion of Theorem 1 also holds for the forward-type Riemann sums*

$$\widetilde{X}_m(t) := \sum_{i=1}^{[mt]} f(t_{i-1})\xi_{i,m}.$$

Indeed, it suffices to put the random weights $\xi_{i,m}$ at the points t_{i-1}.

This theorem has been motivated by a mistake in the proof of Theorem 4 of the reference [2]. More precisely the fact that $\lim_{n\to\infty}\limsup_{m\to\infty}\mathbb{P}(\|B_t^{(n,m)}\|_\infty > \epsilon) = 0$ in page 724 of [2] is a particular case of the convergence (4). The rest of this note is devoted to the proof of Theorem 1. First we establish a basic decomposition, which reduces the proof of Theorem 1 to the proof of the convergence (4). In section 3 we discuss different attempts to prove this convergence using p-variation norms and martingale methods. Finally in Section 4 we provide a proof of (4) using techniques of fractional calculus.

2 The main decomposition

The basic idea of the proof of Theorem 1 is the classical Bernstein's big blocks/small blocks technique. For this purpose we set $u_j = \frac{j}{n}, n \le m$, and decompose the process $X_m(t)$ as follows

$$\begin{aligned} X_m(t) &= \sum_{i=1}^{[mt]} f(t_i)\xi_{i,m} \\ &= \sum_{j=1}^{[nt]+1} \sum_{i\in I_n(j)} \left(f(t_i) - f(u_{j-1})\right)\xi_{i,m} + \sum_{j=1}^{[nt]+1} f(u_{j-1}) \sum_{i\in I_n(j)} \xi_{i,m} \end{aligned} \tag{2}$$

with $I_n(j) := \{i : 1 \le i \le [mT], \frac{i}{m} \in [\frac{j-1}{n}, \frac{j}{n})\}$. According to our hypothesis it holds that

$$g_m \underset{m\to\infty}{\overset{\mathcal{L}}{\to}} w \qquad \mathcal{F}\text{-stably}$$

on $D[0,T]$ with $(w(t))_{t\in[0,T]}$ being a Brownian motion independent of $\mathcal{F}$. This implies, in particular, the $\mathcal{F}$-stable convergence

$$\sum_{j=1}^{[nt]+1} f(u_{j-1}) \sum_{i\in I_n(j)} \xi_{i,m} \underset{m\to\infty}{\overset{\mathcal{L}}{\to}} \sum_{j=1}^{[nt]+1} f(u_{j-1}) \left(w(u_j) - w(u_{j-1})\right).$$

We also have that

$$\sum_{j=1}^{[nt]+1} f(u_{j-1}) \left(w(u_j) - w(u_{j-1})\right) \underset{n\to\infty}{\overset{u.c.p.}{\to}} \int_0^t f(s)\mathrm{d}w(s).$$

where $u.c.p.$ stands for uniform convergence in probability.

Now we treat the first term of (2), but before we consider, separately, the last summand. We claim that

$$\mathbb{P}\text{-}\lim_{n\to\infty}\limsup_{m\to\infty}\sup_{t\in[0,T]}\left|\sum_{i\in I_n([nt]+1)}\Big(f(t_i) - f(u_{[nt]})\Big)\xi_{i,m}\right| = 0. \tag{3}$$

In fact, using the Hölder continuity of f we can write

$$\left| \sum_{i \in I_n([nt]+1)} \Big(f(t_i) - f(u_{[nt]}) \Big) \, \xi_{i,m} \right| \leq \|f\|_\alpha n^{-\alpha} \sum_{i \in I_n([nt]+1)} |\xi_{i,m}|,$$

where

$$||f||_\alpha := \sup_{|u-v| \leq T} \frac{|f(u) - f(v)|}{|u-v|^\alpha} < \infty.$$

Then, (3) follows from Hypothesis **(H2)** taking into account that the cardinality of the set $I_n([nt]+1)$ is bounded by $\frac{m}{n} + 1$ and $\alpha > \frac{1}{2}$.

Then, in order to finish the proof, we need to show that

$$\mathbb{P}\text{-} \lim_{n\to\infty} \limsup_{m\to\infty} \sup_{t\in[0,T]} \left| \sum_{j=1}^{[nt]} \sum_{i\in I_n(j)} \big(f(t_i) - f(u_{j-1}) \big) \, \xi_{i,m} \right| = 0. \tag{4}$$

In fact, this is a key step of the proof. In particular situations, such as e.g. in the martingale framework, there are various specific techniques of the proof. We will present some of them in the next section. However, proving convergence (4) in a general setting turns out to be not quite easy.

A first straightforward attempt is as follows. We set

$$R_{n,m}(t) := \sum_{j=1}^{[nt]} \sum_{i\in I_n(j)} \big(f(t_i) - f(u_{j-1}) \big) \, \xi_{i,m}.$$

Then we deduce

$$\sup_{t\in[0,T]} |R_{n,m}(t)| \leq n^{-\alpha} ||f||_\alpha \sum_{i=1}^{[mT]} |\xi_{i,m}| \, ,$$

but in general we have that

$$\lim_{m\to\infty} \sum_{i=1}^{[mT]} |\xi_{i,m}| = \infty$$

as the following simple example shows.

Example 3 *Consider the case where* $\xi_{i,m} = \frac{X_i}{\sqrt{m}}$, *where* $(X_i)_{i\geq 1}$ *are i.i.d with* $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. *Then*

$$g_m(t) = \sum_{i=1}^{[mt]} \frac{X_i}{\sqrt{m}},$$

and $g_m \xrightarrow[m\to\infty]{\mathcal{L}} w$ *on* $D[0,T]$ *and*

$$\sum_{i=1}^{[mT]} |\xi_{i,m}| \underset{m\to\infty}{\to} \infty.$$

3 Young's calculus

A more sophisticated approach for proving (4) is to use Young's integral. Consider the interval $I_j^n := \big[u_{j-1}, u_j\big)$. Then

$$\sum_{j=1}^{[nt]} \sum_{i\in I_n(j)} \big(f(t_i) - f(u_{j-1})\big)\, \xi_{i,m} = \sum_{j=1}^{[nt]} \int_{I_j^n} (f(s) - f(u_{j-1}))\mathrm{d}g_m(s).$$

By the Love-Young inequality and for $\beta > 1-\alpha$,

$$\left| \sum_{j=1}^{[nt]} \int_{I_j^n} (f(s) - f(u_{j-1}))\mathrm{d}g_m(s) \right| \le C_{\alpha,\beta} \sum_{j=1}^{[nt]} v^j_{\frac{1}{\alpha}}(f) v^j_{\frac{1}{\beta}}(g_m)$$

with

$$v^j_p(h) := \left(\sup_{\pi} \sum_{i=1}^{N} |h(s_i) - h(s_{i-1})|^p \right)^{1/p},$$

where the supremum runs over all partitions $\pi = \{s_0, \ldots, s_N\}$ of the interval $[u_{j-1}, u_j]$, and $C_{\alpha,\beta}$ is a positive constant; see [8]. Notice that $v^j_{\frac{1}{\alpha}}(f) \le n^{-\alpha}||f||_\alpha$. The problem is then to bound $v^j_{\frac{1}{\beta}}(g_m)$ under the hypothesis $g_m \xrightarrow[m\to\infty]{\mathcal{L}} w$ on $D[0,T]$. Unfortunately, the strong p-variation v_p is not a continuous functional on $D[0,T]$ equipped with the uniform topology. Nevertheless, we suppose for a moment that the convergence

$$v^j_{\frac{1}{\beta}}(g_m) \xrightarrow[m\to\infty]{\mathcal{L}} v^j_{\frac{1}{\beta}}(w),$$

holds. Then, recalling that $\alpha > \frac{1}{2}$, we can choose $\beta = \frac{1}{2} - \varepsilon$ with $\varepsilon < \alpha - \frac{1}{2}$ and we obtain that

$$v^j_{\frac{1}{\beta}}(w) \le ||w||_\beta n^{-\beta} < \infty.$$

Consequently,

$$v^j_{\frac{1}{\alpha}}(f) v^j_{\frac{1}{\beta}}(g_m) \xrightarrow[m\to\infty]{\mathcal{L}} v^j_{\frac{1}{\alpha}}(f) v^j_{\frac{1}{\beta}}(w) \le ||f||_\alpha ||w||_\beta n^{-\alpha-\beta}.$$

Thus, we deduce the desired convergence

$$\lim_{n\to\infty}\limsup_{m\to\infty}\mathbb{P}\left\{\sup_{t\in[0,T]}\left|\sum_{j=1}^{[nt]}\int_{I_j^n}\big(f(s)-f(u_{j-1})\big)\,dg_m(s)\right|>\varepsilon\right\}=0,$$

since $\alpha+\beta>1$. This would complete the proof of our central limit theorem.

Unfortunately, there are only few results about the asymptotic behaviour of $v_{\frac{1}{\beta}}(g_m)$ (the latter denotes strong $1/\beta$-variation on the interval $[0,1]$). Below, we shall mention some of them. Denote by $\mathcal{W}_p[0,1]$ the space of functions on $[0,1]$ such that $v_p<\infty$ $(p\geq 1)$ with the norm $||\cdot||_{[p]}:=\big(v_p(\cdot)\big)^{1/p}+||\cdot||_\infty$.

Proposition 4 *(Norvaiša-Račkauskas, 2008, [4]). Let $(X_i)_{i=1}^m$ be an iid sequence, set $S_n(t):=\sum_{i=1}^{[mt]}X_i$, $t\in[0,1]$. Then, for $p>2$,*

$$\frac{1}{\sqrt{m}}S_m\underset{m\to\infty}{\overset{\mathcal{L}}{\to}}w,$$

in $\mathcal{W}_p[0,1]$ iff $\mathbb{E}(X_1)=0$ and $\mathbb{E}(X_1^2)=1$.

As a consequence, if the random variables $\xi_{i,m}$ are iid with

$$\mathbb{E}\left(\xi_{i,m}\right)=0,\qquad \mathbb{E}\left(\xi_{i,m}^2\right)=\frac{1}{m},$$

we immediately deduce that $v_p(g_m)\underset{m\to\infty}{\overset{\mathcal{L}}{\to}}v_p(w)$.

Another result that can help is the following one.

Theorem 5 *(Lépingle 1976, [3]). If $p>2$ there exists a positive constant C depending on p, such that*

$$\mathbb{E}(v_p(M)^{1/p})\leq C\mathbb{E}(||M||_\infty)$$

for all martingales M.

Assume that g_m is a martingale for a fixed m, which is equivalent to say that $\{\xi_{i,m},1\leq i\leq[mT]\}$ is a martingale difference. Then we have

$$\mathbb{E}(v_p(g_m)^{1/p})\leq C\mathbb{E}(||g_m||_\infty),$$

and if

$$g_m\underset{m\to\infty}{\overset{\mathcal{L}}{\to}}w$$

in $D[0,T]$, we also have that

$$\|g_m\|_\infty \underset{m\to\infty}{\overset{\mathcal{L}}{\to}} \|w\|_\infty .$$

Using the tightness condition (1) on g_m and Doob's inequality, we obtain that

$$\mathbb{E}(||g_m||_\infty) < C.$$

Now, by the Skorohod representation theorem and dominated convergence theorem, we have that

$$\limsup_{m\to\infty} \mathbb{E}(v_p^j(g_m)^{1/p}) \leq Cn^{-\beta}||w||_\beta.$$

and we can obtain the central limit theorem.

So, in the both cases mentioned above we need additional conditions on the process g_m, in order to get the desired result. However, other interesting examples are not covered by above methods. For instance, consider a fractional Brownian motion $(B_t^H)_{t\in[0,T]}$ with Hurst parameter $H \in (0, 3/4)$ and define

$$g_m(t) = \frac{1}{\sqrt{m}} \sum_{i=1}^{[mt]} \left(m^{2H}(B_{t_i}^H - B_{t_{i-1}}^H)^2 - 1 \right).$$

It is well known that $g_m \overset{\mathcal{L}}{\to} w$ on $D[0,T]$, but, to the best of our knowledge, there is less known about the asymptotic behaviour of the strong p-variation of g_m. For this reason we will develop a new technique, which does not rely on p-variation concepts.

4 Proof of the convergence (4)

In this section we are going to prove (4) and, therefore, complete the proof of Theorem 1, using techniques of fractional calculus. We refer the reader to [6] for a detailed exposition of fractional calculus.

Proof. Fix $\gamma \in (0,1)$ such that $1/2 < \gamma < \alpha$. Throughout the proof all positive constants are denoted by C, although they may change from line to line. Denote by β_j the smallest integer greater or equal than mu_j. Then, an integer i belongs to $I_n(j)$ if and only if $u_{j-1} \leq \frac{i}{m} < \frac{\beta_j}{m}$. Let $J_j^{n,m}$ be the interval $\left[u_{j-1}, \frac{\beta_j}{m}\right)$. With this notation we can write

$$R_{n,m}(t) = \sum_{j=1}^{[nt]} \sum_{i\in I_n(j)} \left(f(t_i) - f(u_{j-1})\right) \xi_{i,m} = \sum_{j=1}^{[nt]} \int_{J_j^{n,m}} (f(s) - f(u_{j-1}))\mathrm{d}g_m(s).$$

Set

$$R_{n,m,j} := \int_{J_j^{n,m}} (f(s) - f(u_{j-1}))\mathrm{d}g_m(s).$$

We have that for any $0 \leq a < b \leq T$, the identity

$$\int_{[a,b)} (f(s) - f(a))\mathrm{d}g_m(s) = \int_a^b D_{a+}^{\gamma} f_a(s) D_{b-}^{1-\gamma}(g_m)_{b-}(s)\mathrm{d}s \tag{5}$$

holds, where $f_a(s) = f(s) - f(a)$, $(g_m)_{b-}(s) = g_m(s) - g_m(b-)$ for $s \in [a,b)$ and D_{a+}^{γ} and $D_{b-}^{1-\gamma}$ are the fractional derivative operators defined by

$$D_{a+}^{\gamma} f_a(s) = \frac{1}{\Gamma(1-\gamma)} \left(\frac{f(s) - f(a)}{(s-a)^{\gamma}} + \gamma \int_a^s \frac{f(s) - f(y)}{(s-y)^{\gamma+1}} \mathrm{d}y \right),$$

and

$$\begin{aligned} & D_{b-}^{1-\gamma}(g_m)_{b-}(s) \\ = \; & \frac{1}{\Gamma(\gamma)} \left(\frac{g_m(s) - g_m(b-)}{(b-s)^{1-\gamma}} + (1-\gamma) \int_s^b \frac{g_m(s) - g_m(y)}{(y-s)^{2-\gamma}} \mathrm{d}y \right). \end{aligned}$$

Notice that these operators are well defined by the α-Hölder continuity of f, with $\alpha > \gamma$ and since g_m is piecewise constant. The identity (5) can be found e.g. in [7, Theorem 3.1 (v)]. Now, if we take $a = u_{j-1}$, we can estimate $D_{a+}^{\gamma} f_a(s)$ by the Hölder norm of f and in this way we obtain that

$$|D_{a+}^{\gamma} f_a(s)| \leq C\|f\|_{\alpha} n^{-\alpha+\gamma} = Gn^{-\alpha+\gamma},$$

for some random variable G. As a consequence, if we put $a = u_{j-1}$ and $b = \frac{\beta_j}{m}$, we deduce the inequality

$$\begin{aligned} R_{n,m,j} \; : \; &= \left| \int_a^b D_{a+}^{\gamma} f_a(s) D_{b-}^{1-\gamma}(g_m)_{b-}(s)\mathrm{d}s \right| \\ &\leq \frac{Gn^{-\alpha+\gamma}}{\Gamma(1-\gamma)} \int_{u_{j-1}}^{\beta_j/m} |D_{b-}^{1-\gamma}(g_m)_{b-}(s)|\mathrm{d}s \\ &\leq \frac{Gn^{-\alpha+\gamma}}{\Gamma(1-\gamma)} \sum_{k=\beta_{j-1}}^{\beta_j} \int_{t_{k-1}}^{t_k} |D_{b-}^{1-\gamma}(g_m)_{b-}(s)|\mathrm{d}s, \end{aligned}$$

The last inequality follows from the inclusion $\left[u_{j-1}, \frac{\beta_j}{m}\right] \subset \left[t_{\beta_{j-1}-1}, t_{\beta_j}\right]$. Notice also that $g_m(b-) = g_m(t_{\beta_j-1})$.

Suppose that $t_{k-1} < s < t_k$. Then we obtain the identity,

$$\begin{aligned}
& D_{b-}^{1-\gamma}(g_m)_{b-}(s) \\
= \; & \frac{1}{\Gamma(\gamma)} \left(\frac{g_m(t_{k-1}) - g_m(t_{\beta_j-1})}{\left(t_{\beta_j} - s\right)^{1-\gamma}} + (1-\gamma) \int_s^{t_{\beta_j}} \frac{g_m(t_{k-1}) - g_m(y)}{(y-s)^{2-\gamma}} dy \right) \\
= \; & \frac{1}{\Gamma(\gamma)} \left(\frac{g_m(t_{k-1}) - g_m(t_{\beta_j-1})}{\left(t_{\beta_j} - s\right)^{1-\gamma}} + (1-\gamma) \sum_{l=k+1}^{\beta_j} \int_{t_{l-1}}^{t_l} \frac{g_m(t_{k-1}) - g_m(t_{l-1})}{(y-s)^{2-\gamma}} dy \right) \\
= \; & \frac{1}{\Gamma(\gamma)} \left(\frac{g_m(t_{k-1}) - g_m(t_{\beta_j-1})}{\left(t_{\beta_j} - s\right)^{1-\gamma}} \right. \\
& \left. - \sum_{l=k+1}^{\beta_j} (g_m(t_{k-1}) - g_m(t_{l-1}))[(t_l - s)^{\gamma-1} - (t_{l-1} - s)^{\gamma-1}] \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |D_{b-}^{1-\gamma}(g_m)_{b-}(s)| \\
\leq \; & \frac{1}{\Gamma(\gamma)} \left(\sum_{l=k+1}^{\beta_j} |g_m(t_{k-1}) - g_m(t_{l-1})| [(t_{l-1} - s)^{\gamma-1} - (t_l - s)^{\gamma-1}] \right. \\
& \left. + |g_m(t_k - 1) - g_m(t_{\beta_j-1})| \left(t_{\beta_j} - s\right)^{\gamma-1} \right).
\end{aligned}$$

Integrating in the variable s yields,

$$\begin{aligned}
& \int_{t_{k-1}}^{t_k} |D_{b-}^{1-\gamma}(g_m)_{b-}(s)| ds \\
\leq \; & C m^{-\gamma} \sum_{l=k+1}^{\beta_j} |g_m(t_{k-1}) - g_m(t_{l-1})| |(l-1-k)^\gamma - 2(l-k)^\gamma + (l-k+1)^\gamma| \\
& + C m^{-\gamma} |g_m(t_{k-1}) - g_m(t_{\beta_j-1})| [(\beta_j - k + 1)^\gamma - (\beta_j - k)^\gamma].
\end{aligned}$$

Then, for any constant $K > 0$, and $\varepsilon > 0$

$$\mathbb{P}(\sup_{t \in [0,T]} |R_{n,m}(t)| > \varepsilon) \leq \mathbb{P}(G > K) + \frac{1}{\varepsilon} \mathbb{E} \left(\sup_{t \in [0,T]} |R_{n,m}(t)| \mathbf{1}_{\{G \leq K\}} \right).$$

Now, we have

$$\mathbb{E}\left(\sup_{t\in[0,T]}|R_{n,m}(t)|\,\mathbf{1}_{\{G\leq K\}}\right)\leq\sum_{j=1}^{[nT]}\mathbb{E}\left(|R_{n,m,j}|\mathbf{1}_{\{G\leq K\}}\right)$$

$$\leq KCn^{-\alpha+\gamma}m^{-\gamma}\sum_{j=1}^{[nT]}\sum_{k=\beta_{j-1}}^{\beta_j}\sum_{l=k+1}^{\beta_j}\mathbb{E}\left(|g_m(t_{k-1})-g_m(t_{l-1})|\right)$$

$$\times|(l-1-k)^\gamma-2(l-k)^\gamma+(l-k+1)^\gamma|$$

$$+KCn^{-\alpha+\gamma}m^{-\gamma}\sum_{j=1}^{[nT]}\sum_{k=\beta_{j-1}}^{\beta_j}\mathbb{E}\left(|g_m(t_{k-1})-g_m(t_{\beta_j-1})|\right)[(\beta_j-k+1)^\gamma-(\beta_j-k)^\gamma].$$

Due to tightness condition (1) we obtain that

$$\mathbb{E}(|g_m(t_{k-1})-g_m(t_{l-1})|)\leq C|t_k-t_l|^{1/2},$$

hence,

$$\mathbb{E}\left(\sup_{t\in[0,T]}|R_{n,m}(t)|\,\mathbf{1}_{\{G\leq K\}}\right)\leq KCn^{-\alpha+\gamma}m^{-\gamma}\sum_{j=1}^{[nT]}\sum_{k=\beta_{j-1}}^{\beta_j}\sum_{l=k+1}^{\beta_j}\sqrt{l-k}$$

$$\times|(l-1-k)^\gamma-2(l-k)^\gamma+(l-k+1)^\gamma|$$

$$+KCn^{-\alpha+\gamma}m^{-\gamma}\sum_{j=1}^{[nT]}\sum_{k=\beta_{j-1}}^{\beta_j}\sqrt{\beta_j-k}[(\beta_j-k+1)^\gamma-(\beta_j-k)^\gamma]$$

$$\leq KCn^{-\alpha+\gamma}m^{-\gamma-\frac{1}{2}}\sum_{j=1}^{[nT]}\left((\beta_j-\beta_{j-1})\sum_{k=1}^{\beta_j-\beta_{j-1}}k^{\gamma-\frac{3}{2}}+\sum_{k=1}^{\beta_j-\beta_{j-1}}k^{\gamma-\frac{1}{2}}\right)$$

$$\leq KCn^{-\alpha+\gamma}m^{-\gamma-\frac{1}{2}}\sum_{j=1}^{[nT]}\left(\beta_j-\beta_{j-1}\right)^{\gamma+\frac{1}{2}}.$$

Then, since $\beta_j-\beta_{j-1}\leq\frac{m}{n}+1$, we conclude that

$$\mathbb{E}(\sup_{t\in[0,T]}|R_{n,m}(t)|\,\mathbf{1}_{\{G\leq K\}})\leq KCn^{1/2-\alpha}\to 0,$$

since $\alpha>1/2$. Therefore, by letting K goes to infinity, we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{P}(\sup_{t\in[0,T]}|R_{n,m}(t)|>\varepsilon)=0.$$

■

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