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ASYMPTOTICS TOWARD THE PLANAR RAREFACTION WAVE FOR VISCOUS CONSERVATION LAW IN TWO SPACE DIMENSIONS

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Abstract. This paper is concerned with the asymptotic behavior of the solution toward the planar rarefaction wave $r(\frac{x}{t})$ connecting u_+ and u_- for the scalar viscous conservation law in two space dimensions. We assume that the initial data $u_0(x,y)$ tends to constant states $u_\pm$ as $x \to \pm\infty$, respectively. Then, the convergence rate to $r(\frac{x}{t})$ of the solution $u(t,x,y)$ is investigated without the smallness conditions of $|u_+ - u_-|$ and the initial disturbance. The proof is given by elementary L^2-energy method.

1. Introduction

We consider the Cauchy problem for the scalar viscous conservation law in two space dimensions:

$$u_t + f(u)_x + g(u)_y = \mu\Delta u, \qquad (t,x,y) \in R_+ \times R^2, \tag{1.1}$$

$$u(0,x,y) = u_0(x,y), \tag{1.2}$$

where f and g are smooth functions, and μ is a positive constant. We assume that f is convex, i.e.,

$$f''(u) \geq \alpha > 0 \quad \text{for} \quad u \in R, \tag{1.3}$$

and that the initial data is asymptotically constant:

$$u_0(x,y) \to u_\pm \quad \text{as} \quad x \to \pm\infty \quad \text{for any fixed y} \in R, \tag{1.4}$$

where $u_\pm$ are constants satisfying $u_- < u_+$. The asymptotic behavior as $t \to \infty$ of the solution is closely related to that of the Riemann problem for the corresponding hyperbolic conservation law in one space dimension:

$$r_t + f(r)_x = 0, \qquad (t,x) \in (-1,\infty) \times R, \tag{1.5}$$

$$r(-1,x) = r_0^R(x) \equiv \begin{cases} u_- & \text{for} \quad x < 0, \\ u_+ & \text{for} \quad x > 0. \end{cases} \tag{1.6}$$

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The entropy solution $r(t,x)$ of (1.5), (1.6) is given by

$$r(t,x)=\begin{cases} u_- & \text{for } \ x<f'(u_-)(t+1),\\ (f')^{-1}(\dfrac{x}{t+1}) & \text{for } \ f'(u_-)(t+1)\le x\le f'(u_+)(t+1),\\ u_+ & \text{for } \ f'(u_+)(t+1)<x. \end{cases} \tag{1.7}$$

The function $(t,x,y)\to r(t,x)$ is called the planar rarefaction wave. In a one dimensional case, the asymptotic behaviors of solutions were originally investigated by Il'in and Oleinik [3]. Harabetian [1] obtained the convergence rate toward the rarefaction wave. Hattori and Nishihara [2] showed more precise behaviors of the solution for the Burgers equation, employing the Hopf-Cole transformation. See also [5], [6], [7], [8], [10].

In a two dimensional case, Xin [9] has first investigated the stability of the planar rarefaction wave. Ito [4] has recently shown the convergence rate toward the planar rarefaction wave. In both papers, the smallness of initial disturbance is essentially assumed. In [4], the rarefaction wave is also assumed to be weak.

Our main purpose in this paper is to show that the solution $u(t,x,y)$ asymptotically behaves as $r(t,x)$ with the same rate as that in [4] without smallness conditions, which improves their results.

Denote $R^2_+=\{(x,y)\in R^2; x>0\}, R^2_-=\{(x,y)\in R^2; x<0\}$ and $D=(\frac{\partial}{\partial x},\frac{\partial}{\partial y})$. Then, our main theorem is as follows.

Theorem 1. *Suppose that* $u_0(x,y)-u_\pm\in L^2(R^2_\pm)\cap L^1(R^2_\pm)$ *and* $D^\alpha u_0(x,y)\in H^1(R^2)$, $|\alpha|=1$. *Then the problem (1.1),(1.2) has a unique global solution* $u(t,x,y)$ *satisfying*

$$\sup_{y\in R}\|u(t,\cdot,y)-r(t,\cdot)\|_{L^2(R_x)}\le C(1+t)^{-\frac14}\log(2+t), \tag{1.8}$$

where C *is a positive constant depending on* u_0.

Our plan in this paper is as follows. In the next section, we construct a smooth rarefaction wave, which is different from that in [4], and reformulate our problem. In the last two sections, we give the proofs of theorems for the reformulated problems.

2. Smooth approximation and preliminaries

We first introduce the function $\tilde{w}(t,x)$ as a solution to the problem:

$$\tilde{w}_t+\tilde{w}\tilde{w}_x=\mu\tilde{w}_{xx},\quad (t,x)\in(-1,\infty)\times R, \tag{2.1}$$

$$\tilde{w}(-1,x)=\tilde{r}^R_0(x)\equiv f'(r^R_0(x)). \tag{2.2}$$

The Hopf-Cole transformation gives the information of the properties of $\tilde{w}$. Using $\tilde{w}(t,x)$, we define "the smooth rarefaction wave" $w(t,x)$ as

$$w(t,x)=(f')^{-1}(\tilde{w}(t,x))|_{t\ge0}. \tag{2.3}$$

According to (1.3), $w(t,x)$ satisfies

$$w_t+f(w)_x=\mu w_{xx}+\mu\frac{f'''(w)}{f''(w)}w_x^2,\quad (t,x)\in R_+\times R, \tag{2.4}$$

$$w(0,x)=w_0(x)\equiv f'(\tilde{w}(0,x)). \tag{2.5}$$

The properties of the smooth rarefaction wave $w(t,x)$ are stated in the following lemma. From now on, we denote several constants by C or c without confusion.

Lemma 1 (Hattori and Nishihara [2]). *The smooth rarefaction wave $w(t,x)$ given by (2.3) satisfies the following properties:*

(i) $|w(t,x) - u_\pm| \le C\exp(-c|x|^2)$,
(ii) $w_x(t,x) > 0$,
(iii) $\|w_x(t,\cdot)\|_{L^p(R)} \le (1+t)^{-1+\frac{1}{p}}$, $\|w_{xx}(t,\cdot)\|_{L^p(R)} \le (1+t)^{-1}$,
(iv) $\|w(t,\cdot) - r(t,\cdot)\|_{L^p(R)} \le C(1+t)^{-\frac{p-1}{2p}}$.

Since there is a "forcing term" $\frac{f'''(w)}{f''(w)}w_x^2$ in the equation (2.4), we further introduce the smooth rarefaction wave $U(t,x)$ approximate to w, which satisfies

$$U_t + f(U)_x = U_{xx}, \qquad (t,x) \in R_+ \times R, \tag{2.6}$$

$$U(0,x) = U_0(x) \equiv f'(\tilde{w}(0,x)). \tag{2.7}$$

The monotonicity in x of $U(t,x)$ was obtained by Xin [9], which is important in the a priori estimates in §4.

Lemma 2 (Xin [9]). *Suppose that $U_0(x)$ is monotonically increasing:*

$$\frac{d}{dt}U_0(x) > 0, \qquad x \in R. \tag{2.8}$$

Then, the solution $U(t,x)$ of (2.6),(2.7) satisfies

$$\frac{d}{dx}U(t,x) > 0, \qquad (t,x) \in R_+ \times R. \tag{2.9}$$

Thus, setting

$$\begin{aligned} u(t,x,y) - r(t,x) &= \{w(t,x) - r(t,x)\} + \{U(t,x) - w(t,x)\} + \{u(t,x,y) - U(t,x)\} \\ &\equiv \{w(t,x) - r(t,x)\} + v(t,x) + V(t,x,y), \end{aligned}$$

we have reached two reformulated problems:

$$v_t + \{f(w+v) - f(w)\}_x = \mu v_{xx} - \mu\frac{f'''(w)}{f''(w)}w_x^2, \qquad (t,x) \in R_+ \times R, \tag{2.10}$$

$$v(0,x) = U_0(x) - w(0,x) \equiv v_0(x), \tag{2.11}$$

and

$$V_t + \{f(U+V) - f(U)\}_x + g(U+V)_y = \mu\Delta V, \qquad (t,x,y) \in R_+ \times R^2, \tag{2.12}$$

$$V(0,x,y) = u_0(x,y) - U_0(x) \equiv V_0(x,y). \tag{2.13}$$

The perturbations v and V satisfy the following theorems, respectively.

Theorem 2 (Decay estimate). *Suppose that $v_0 \in H^2(R) \cap L^1(R)$. Then the problem (2.10),(2.11) has a unique global solution $v(t,x)$ satisfying*

$$v \in C^0([0,\infty); H^2(R)) \cap C^0([0,\infty); L^1(R)),$$

$$v_x \in L^2(0,T; H^2(R)),$$

and

$$\|v(t,\cdot)\|_{L^2(R)} \le C(1+t)^{-\frac{1}{4}}\log(2+t). \tag{2.14}$$

Theorem 3 (Decay estimate). *Suppose that $V_0 \in H^2(R^2) \cap L^1(R^2)$. Then, the problem (2.12),(2.13) has a unique global solution $V(t,x,y)$ satisfying*

$$V \in C^0([0,\infty); H^2(R)), \quad \nabla V \in L^2(0,\infty; H^2(R)),$$

and

$$\sup_{y\in R} \|V(t,\cdot,y)\|_{L^2(R_x)} \le C(1+t)^{-\frac{3}{4}}. \tag{2.15}$$

Theorem 2, Theorem 3 and Lemma 1 (iv) yield the desired estimate (1.8). In the next two sections, we devote ourselves to the proofs of Theorems 2 and 3, respectively.

3. Decay estimates for the perturbation v

We begin with the Cauchy problem

$$v_t + \{f(w+v) - f(w)\}_x = \mu v_{xx} - \mu \frac{f'''(w)}{f''(w)} w_x^2, \qquad (t,x) \in R_+ \times R, \tag{3.1}$$

$$v(0,x) = U(0,x) - w(0,x) \equiv v_0(x). \tag{3.2}$$

We shall show that the problem (3.1),(3.2) has a unique global solution in the solution space $X(0,\infty)$, where

$$X_M(0,T) = \left\{ \psi \,\middle|\, \begin{array}{ll} \psi \in C^0([0,T]; H^2(R)), & \psi_x \in L^2(0,T; H^2(R)) \\ \text{and} & \sup_{[0,T]} \|\psi(t,\cdot)\|_{H^2} \le M \end{array} \right\}.$$

In what follows, we often abbreviate the domain R of $H^2(R)$, etc.

Proposition 1 (Local existence). *Suppose that $v_0 \in H^2(R)$. For any $M > 0$, there exists a positive constant T_0 depending on M such that if $\|v_0\|_{H^2} \le M$, then the problem (3.1), (3.2) has a unique solution $v(t,x) \in X_{2M}(0,T_0)$.*

Proposition 1 can be proved in a standard way. So we omit the proof. Next, we show a priori estimates of v.

Proposition 2 (A priori estimate). *Suppose that v is a solution of (3.1),(3.2) in $X_M(0,T)$ for positive constants T and M. Then there exists a positive constant C_0 such that*

$$\|v(t)\|_{H^2}^2 + \int_0^t \int_R w_x v^2 dx d\tau + \int_0^t \|v_x(\tau)\|_{H^2}^2 d\tau \le C_0(\|v_0\|_{H^2}^2 + 1). \tag{3.3}$$

Proof. Multiplying (3.1) by v, we have

(3.4)
$$\frac{1}{2}\frac{d}{dt}\int_R v^2 dx + \int_R v\{f(w+v) - f(w)\}_x dx + \mu \int_R v_x^2 dx = -\int_R v \frac{f'''(w)}{f''(w)} w_x^2 dx.$$

The second term of (3.4) is estimated by the following:

$$\begin{aligned} &\int_R v\{f(w+v) - f(w)\}_x dx = -\int_R v_x\{f(w+v) - f(w)\} dx \\ &= \int_R \left[-\left(\int_w^{w+v} f(y) dy - f(w)v \right)_x + \{f(w+v) - f(w) - f'(w)v\} w_x \right] dx \qquad (3.5) \\ &\ge \frac{\alpha}{2} \int_R w_x v^2 dx. \end{aligned}$$

The right hand side is estimated as follows:

$$\left|\int_R v\frac{f'''(w)}{f''(w)}w_x^2dx\right| \le C\int_R w_x^2|v|dx \le \frac{\alpha}{4}\int_R w_x|v|^2dx + C\int_R w_x^3dx.$$

Integrating (3.4) over [0,t] and using Lemma 1 (iii), we get

$$\|v(t)\|^2 + \int_0^t\int_R w_x v^2dxd\tau + \int_0^t \|v_x(\tau)\|^2d\tau \le C_0(\|v_0\|^2+1). \tag{3.6}$$

Here and later, by $\|\cdot\|$ we denote the L^2-norm in R or R^2 without confusions. Next, we derive the higher order estimates. Multiplying (3.1) by $(-v_{xx})$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_R v_x^2dx - \int_R v_{xx}\{f(w+v)-f(w)\}_x dx + \mu\int_R v_{xx}^2dx\\ = \mu\int_R v_{xx}\frac{f'''(w)}{f''(w)}w_x^2dx.\end{aligned} \tag{3.7}$$

The right-hand side is estimated as

$$\left|\int_R v_{xx}\frac{f'''(w)}{f''(w)}w_x^2dx\right| \le \frac{1}{4}\|v_{xx}\|^2 + C\|w_x\|_{L^4}^4.$$

The second term of (3.7) is estimated as

$$\left|\int_R v_{xx}\{f(w+v)-f(w)\}_x dx\right| \le \frac{\mu}{4}\|v_{xx}\|^2 + C\left\{\|v_x(t)\|^2 + \int_R w_x v^2dx\right\}.$$

Here, the maximum principle for a parabolic equation has been employed. Hence, we have

$$\|v_x(t)\|^2 + \int_0^t \|v_{xx}(\tau)\|^2d\tau \le C_0(\|v_0\|_{H^1}^2+1). \tag{3.8}$$

Differentiating (3.1) twice in x, and multiplying it by v_{xx}, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_{xx}(t)\|^2 + \int_R v_{xx}\{f(w+v)-f(w)\}_{xxx}dx + \mu\|v_{xxx}(t)\|^2\\ = -\mu\int_R v_{xx}\left(\frac{f'''(w)}{f''(w)}w_x^2\right)_{xx}dx,\end{aligned}$$

which yields

$$\|v_{xx}(t)\|^2 + \int_0^t \|v_{xxx}(\tau)\|^2d\tau \le C_0(\|v_0\|_{H^2}^2+1). \tag{3.9}$$

Thus, the proof of Proposition 3 is complete. □

Combining Proposition 1 with Proposition 2, we obtain the global result.

Theorem 4 (Global existence)**.** *Suppose that $v_0(x) \in H^2(R)$. Then the problem (3.1),(3.2) has a unique global solution $v(t,x)$ satisfying*

$$v \in C^0([0,\infty);H^2(R)), \quad v_x \in L^2(0,\infty;H^2(R)),$$

and the estimate (3.3).

In order to obtain the decay order of v, we further assume that $v_0 \in L^1(R)$.

Lemma 3. *Suppose that $v_0 \in L^1(R) \cap H^2(R)$. Then the solution $v(t,x)$ also satisfies*

$$\|v(t)\|_{L^1} \le \|v_0\|_{L^1} + C_1 \log(1+t), \tag{3.10}$$

where C_1 is a constant depending on $|u_+ - u_-|$.

Proof. The L^1-estimate (3.10) of v can be proved by the same method as that in [4]. So we omit the proof. □

Theorem 5 (Decay estimate)**.** *Suppose that $v_0 \in H^2(R) \cap L^1(R)$. Then, for any $0 < \varepsilon < \frac{1}{2}$, the solution $v(t,x)$ of (3.1),(3.2) satisfies*

$$\begin{aligned} &(1+t)^{k+\frac{1}{2}+\varepsilon}\|\partial_x^k v(t)\|^2 \\ &\quad + \int_0^t (1+\tau)^{k+\frac{1}{2}+\varepsilon}\left(\int_R w_x|\partial_x^k v(\tau)|^2 dx + \|\partial_x^k v_x(\tau)\|^2\right) d\tau \\ &\quad \le C I_k (1+t)^{\varepsilon}\rho_k(t), \qquad k = 0, 1, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &(1+t)^{2+\varepsilon}\|\partial_x^2 v(t)\|^2 \\ &\quad + \int_0^t (1+\tau)^{2+\varepsilon}\left(\int_R w_x|\partial_x^2 v(\tau)|^2 dx + \|\partial_x^2 v_x(\tau)\|^2\right) d\tau \\ &\quad \le C I_2 (1+t)^{\varepsilon}\rho_2(t), \end{aligned} \tag{3.12}$$

where

$$I_k = (\|v_0\|_{L^1} + \|v_0\|_{H^k} + 1)^2, \quad k = 0, 1, \quad \rho_0 = \log^2(2+t), \quad \rho_1 = \log^{10}(2+t),$$

and

$$I_2 = (\|v_0\|_{L^1} + \|v_0\|_{H^2} + 1)^{\frac{70}{3}}, \quad \rho_2 = \log^6(2+t).$$

Remark. *The estimate* (3.11) *with $k = 0$ shows* (2.14) *in Theorem 2.*

Proof. The proof is similar to one in Ito [4]. However, the smooth rarefaction wave $w(t,x)$ in [4] is different from ours and its estimates are done for the linearized equation around $w(t,x)$. Hence, we give the outline of the proof.

First, we show (3.11) with $k = 0$. From (3.4) and Lemma 1 (iii), we have

$$\frac{d}{dt}\|v(t)\|^2 + \int_R w_x v^2 dx + \|v_x(t)\|^2 \le C(1+t)^{-2}. \tag{3.13}$$

Multiplying (3.13) by $(1+t)^{\frac{1}{2}+\varepsilon}$, we have

$$\begin{aligned} &\frac{d}{dt}\{(1+t)^{\frac{1}{2}+\varepsilon}\|v(t)\|^2\} + (1+t)^{\frac{1}{2}+\varepsilon}\left(\int_R w_x v^2 dx + \|v_x(t)\|^2\right) \\ &\quad \le C\{(1+t)^{-\frac{1}{2}+\varepsilon}\|v(t)\|^2 + (1+t)^{-\frac{3}{2}+\varepsilon}\}. \end{aligned} \tag{3.14}$$

By the Gagliardo-Nirenberg inequality

$$\|v(t)\|^2 \le C\|v(t)\|_{L^1(R)}^{\frac{4}{3}}\|v_x(t)\|^{\frac{2}{3}}, \tag{3.15}$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\{(1+t)^{\frac{1}{2}+\varepsilon}\|v(t)\|^2\} + (1+t)^{\frac{1}{2}+\varepsilon}\left(\int_R w_x v^2 dx + \|v_x(t)\|^2\right) \\
&\le C\{(1+t)^{-\frac{1}{2}+\varepsilon}\|v(t)\|_{L^1}^{\frac{4}{3}}\|v_x(t)\|^{\frac{2}{3}} + (1+t)^{-\frac{3}{2}+\varepsilon}\} \\
&\le \frac{1}{2}(1+t)^{\frac{1}{2}+\varepsilon}\|v_x(t)\|^2 + C\{(1+t)^{-1+\varepsilon}\|v(t)\|_{L^1}^2 + (1+t)^{-\frac{3}{2}+\varepsilon}\} \\
&\le \frac{1}{2}(1+t)^{\frac{1}{2}+\varepsilon}\|v_x(t)\|^2 + C\{(1+t)^{-1+\varepsilon}(\|v_0\|_{L^1}^2 + C_1\log^2(1+t)) + (1+t)^{-\frac{3}{2}+\varepsilon}\};
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{d}{dt}\{(1+t)^{\frac{1}{2}+\varepsilon}\|v(t)\|^2\} + (1+t)^{\frac{1}{2}+\varepsilon}\left(\int_R w_x v^2 dx + \|v_x(t)\|^2\right) \\
&\qquad \le C\{(1+t)^{-1+\varepsilon}(\|v_0\|_{L^1}^2 + C_1\log^2(1+t)) + (1+t)^{-\frac{3}{2}+\varepsilon}\}.
\end{aligned} \tag{3.16}$$

Integrating (3.16) over [0,t] in t, we obtain (3.11) with $k=0$.

Next, we derive (3.11) with $k=1$. From (3.7), we have

$$\frac{1}{2}\frac{d}{dt}\|v_x(t)\|^2 - \int_R v_{xx}\{f(w+v)-f(w)\}_x dx + \mu\|v_{xx}(t)\|^2 \le C(1+t)^{-3}. \tag{3.17}$$

Here

$$\begin{aligned}
&-\int_R v_{xx}\{f(w+v)-f(w)\}_x dx \\
&= \frac{1}{2}\int_R f''(w+v)w_x v_x^2 dx + \int_R\left[\frac{1}{2}v_x^3 - v_{xx}\{f'(w+v)-f'(w)\}w_x\right]dx.
\end{aligned}$$

Hence, due to (1.3), we have

$$\begin{aligned}
&\frac{d}{dt}\|v_x(t)\|^2 + \alpha\int_R w_x v_x^2 dx + \|v_{xx}(t)\|^2 \\
&\le C\left\{\int_R |v_{xx}||v||w_x| dx + \int_R |v_x|^3 dx + (1+t)^{-3}\right\} \\
&\le \frac{1}{2}\|v_{xx}(t)\|^2 + C\left\{\int_R w_x^2 v^2 dx + \|v_x(t)\|_{L^3}^3 + (1+t)^{-3}\right\}.
\end{aligned} \tag{3.18}$$

Multiplying (3.18) by $(1+t)^{\frac{3}{2}+\varepsilon}$, we have

$$\begin{aligned}
&\frac{d}{dt}\{(1+t)^{\frac{3}{2}+\varepsilon}\|v_x(t)\|^2\} + \alpha(1+t)^{\frac{3}{2}+\varepsilon}\int_R w_x v_x^2 dx + (1+t)^{\frac{3}{2}+\varepsilon}\|v_{xx}(t)\|^2 \\
&\le C\left\{(1+t)^{\frac{1}{2}+\varepsilon}\|v_x(t)\|^2 + (1+t)^{\frac{3}{2}+\varepsilon}\int_R w_x^2 v^2 dx\right. \\
&\qquad \left. + (1+t)^{\frac{3}{2}+\varepsilon}\|v_x(t)\|_{L^3}^3 + (1+t)^{-\frac{3}{2}+\varepsilon}\right\}.
\end{aligned} \tag{3.19}$$

Noting that

$$\begin{aligned}
(1+t)^{\frac{3}{2}+\varepsilon}\int_R w_x^2 v^2 dx &\le (1+t)^{\frac{3}{2}+\varepsilon}\|w_x(t)\|_{L^\infty}\int_R w_x v^2 dx \\
&\le C(1+t)^{\frac{1}{2}+\varepsilon}\int_R w_x v^2 dx,
\end{aligned}$$

and making use of (3.11) with $k=0$ and the Gagliardo-Nirenberg inequality

$$\|v_x(t)\|^3_{L^3(R)} \le C\|v_{xx}(t)\|^{\frac{7}{4}}_{L^2(R)}\|v(t)\|^{\frac{5}{4}}_{L^2(R)}, \tag{3.20}$$

we obtain

$$\begin{aligned}
&(1+t)^{\frac{3}{2}+\varepsilon}\|v_x^2(t)\|^2 + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\left(\alpha\int_R w_x v_x^2 dx + \|v_{xx}(\tau)\|^2\right)d\tau \\
&\le C\left\{I_0(1+t)^{\varepsilon}\rho_0 + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}\|v(\tau)\|^{10}_{L^2(R)}\,d\tau\right\} \\
&\le C\left\{I_0(1+t)^{\varepsilon}\rho_0 + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon}(I_0(1+\tau)^{-\frac{1}{2}}\rho_0)^5 d\tau\right\},
\end{aligned}$$

which yields (3.11) with $k=1$. Finally, multiply (3.9) by $(1+t)^{2+\varepsilon}$ and use (3.11). After several calculations, we can obtain the desired estimate (3.12). Though the details are omitted, we cannot multiply (3.9) by $(1+t)^{\frac{5}{2}+\varepsilon}$ in our method. Because we have the decay order $\|w_{xx}(t)\|^2 = O(t^{-2})$, not $O(t^{-\frac{5}{2}})$ (cf. Ito [4]).

Thus the proof is complete. □

4. Decay estimates for the perturbation V

In this section, we consider the Cauchy problem in two space dimension:

$$V_t + \{f(U+V) - f(U)\}_x + g(U+V)_y = \Delta V, \tag{4.1}$$

$$V(0,x,y) = V_0(x,y) \equiv u_0(x,y) - U_0(x). \tag{4.2}$$

The solution space is

$$\tilde{X}_M(0,T) = \left\{\psi \,\middle|\, \begin{array}{ll} \psi \in C^0([0,T];H^2(R^2)), & \nabla\psi \in L^2(0,T;H^2(R^2)) \\ \text{and} & \sup\limits_{[0,T]}\|\psi(t,\cdot,\cdot)\|_{H^2} \le M \end{array}\right\},$$

with $T>0$. Then we have

Proposition 3 (Local existence)**.** *Suppose that $V_0 \in H^2(R^2)$. For any $M>0$, there exists a positive constant T_0 depending on M such that if $\|V_0\|_{H^2} \le M$, then the problem (4.1), (4.2) has a unique solution $V(t,x,y) \in \tilde{X}_{2M}(0,T_0)$.*

Proposition 3 can be proved in a standard way. So we omit the proof. Next, we show a priori estimates of V.

Proposition 4 (A priori estimate)**.** *Suppose that V is a solution of (4.1),(4.2) in $\tilde{X}_M(0,T)$ for positive constants T and M. Then there exists a positive constant C_1 depending on V_0 such that*

$$\|V(t)\|^2_{H^2} + \int_0^t\int_{R^2} U_x V^2 dxdyd\tau + \int_0^t \|\nabla V(\tau)\|^2_{H^2}d\tau \le C_1\|V_0\|^2_{H^2}. \tag{4.3}$$

Proof. Multiplying (4.1) by V and integrating the resultant equation over R^2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V(t)\|^2 + \int_{R^2} V\{f(U+V)-f(U)\}_x dxdy \\
&\qquad + \int_{R^2} Vg(U+V)_y dxdy + \mu\|\nabla V(t)\|^2 = 0.
\end{aligned} \tag{4.4}$$

The second and third terms are, respectively, estimated as follows:

$$\begin{aligned}\int_{R^2} V\{f(U+V)-f(U)\}_x dxdy &= -\int_{R^2} V_x\{f(U+V)-f(U)\}dxdy \\ &= \int_{R^2}\Bigg[-\left(\int_U^{U+V} f(y)dy - f(U)V\right)_x \\ &\qquad + \{f(U+V)-f(U)-f'(U)V\}U_x\Bigg]dxdy \\ &\geq \frac{\alpha}{2}\int_{R^2} U_x V^2 dxdy.\end{aligned} \tag{4.5}$$

Since U is independent of y,

$$\begin{aligned}\int_{R^2} Vg(U+V)_y dxdy &= -\int_{R^2} V_y g(U+V)dxdy \\ &= -\int_{R^2}\partial_y\left(\int_U^{U+V} g(\xi)d\xi\right)dxdy = 0.\end{aligned} \tag{4.6}$$

Using (4.5) and (4.6), we have the basic estimate

$$\|V(t)\|^2 + \int_0^t\int_{R^2} U_x V^2 dxdyd\tau + \int_0^t \|\nabla V(\tau)\|^2 d\tau \leq C\|V_0\|^2. \tag{4.7}$$

The estimates of the derivatives in x, y of V can be obtained similarly to those in Proposition 2. We omit the details. □

The combination of Propositions 3 and 4 gives the global result.

Theorem 6 (Global existence). *Suppose that $V_0(x) \in H^2(R^2)$. Then the problem (4.1),(4.2) has a unique global solution $V(t,x,y)$ satisfying*

$$V \in C^0([0,\infty); H^2(R)), \quad \nabla V \in L^2(0,\infty; H^2(R)).$$

and the estimate (4.3).

We now show the decay estimates on V. As in Lemma 3, the following L^1-estimate plays an important roll.

Lemma 4 (Ito [4]). *Suppose further, in Theorem 6, that $V_0 \in L^1(R^2)$. Then the solution $V(t,x,y)$ also satisfies*

$$\|V(t)\|_{L^1(R^2)} \leq \|V_0\|_{L^1(R^2)}. \tag{4.8}$$

Applying Lemma 4, we have the following theorem.

Theorem 7 (Decay estimate). *Suppose that $V_0(x,y) \in H^2(R^2)\cap L^1(R^2)$ and let $V(t,x,y)$ be the solution of (4.1),(4.2). Then, for any $\varepsilon > 0$, there exists a constant $C > 0$ such that the following decay estimates hold:*

$$\begin{aligned}&(1+t)^{1+\varepsilon}\|V(t)\|^2 \\ &\quad + \int_0^t (1+\tau)^{1+\varepsilon}\left(\int_{R^2} U_x|V(\tau)|^2 dxdy + \|\nabla V(\tau)\|^2\right)d\tau \\ &\leq C(1+t)^{\varepsilon}(\|V_0\|_{L^1} + \|V_0\|)^2,\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&(1+t)^{\frac{15}{8}+\varepsilon}\|V_x(t)\|^2 \\
&\qquad + \int_0^t (1+\tau)^{\frac{15}{8}+\varepsilon}\left(\int_{R^2} U_x|V_x(\tau)|^2 dxdy + \|\nabla V_x(\tau)\|^2\right) d\tau \\
&\quad \le C(1+t)^{\varepsilon}\log^4(2+t)(\|V_0\|_{L^1}+\|V_0\|_{H^1})^2,
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
&(1+t)^{2+\varepsilon}\|V_y(t)\|^2 \\
&\qquad + \int_0^t (1+\tau)^{2+\varepsilon}\left(\int_{R^2} U_x|V_y(\tau)|^2 dxdy + \|\nabla V_y(\tau)\|^2\right) d\tau \\
&\quad \le C(1+t)^{\varepsilon}(\|V_0\|_{L^1}+\|V_0\|_{H^1})^2,
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
&(1+t)^{\frac{39}{16}+\varepsilon}\|V_{xx}(t)\|^2 \\
&\qquad + \int_0^t (1+\tau)^{\frac{39}{16}+\varepsilon}\left(\int_{R^2} U_x|V_{xx}(\tau)|^2 dxdy + \|\nabla V_{xx}(\tau)\|^2\right) d\tau \\
&\quad \le C(1+t)^{\varepsilon}\log^8(2+t)(\|V_0\|_{L^1}+\|V_0\|_{H^2})^2,
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
&(1+t)^{\frac{23}{8}+\varepsilon}\|V_{xy}(t)\|^2 \\
&\qquad + \int_0^t (1+\tau)^{\frac{23}{8}+\varepsilon}\left(\int_{R^2} U_x|V_{xy}(\tau)|^2 dxdy + \|\nabla V_{xy}(\tau)\|^2\right) d\tau \\
&\quad \le C(1+t)^{\varepsilon}\log^8(2+t)(\|V_0\|_{L^1}+\|V_0\|_{H^2})^2,
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
&(1+t)^{3+\varepsilon}\|V_{yy}(t)\|^2 \\
&\qquad + \int_0^t (1+\tau)^{3+\varepsilon}\left(\int_{R^2} U_x|V_{yy}(\tau)|^2 dxdy + \|\nabla V_{yy}(\tau)\|^2\right) d\tau \\
&\quad \le C(1+t)^{\varepsilon}(\|V_0\|_{L^1}+\|V_0\|_{H^2})^2.
\end{aligned} \tag{4.14}
$$

Remark. *From* (4.9) *and* (4.11)*, the estimate* (2.15) *in Theorem 3 is obtained as*

$$
\begin{aligned}
\sup_y \|V(t,\cdot,y)\|^2 &\le C\|V(t,\cdot,\cdot)\|\|V_y(t,\cdot,\cdot)\| \\
&\le C(1+t)^{-\frac{1}{2}-1} = C(1+t)^{-\frac{3}{2}}. \quad \square
\end{aligned}
$$

Proof. From (4.4)–(4.6), we get

$$
\frac{1}{2}\frac{d}{dt}\|V(t)\|^2 + \frac{1}{2}\alpha\int_{R^2} U_x V^2 dxdy + \mu\|\nabla V(t)\|^2 \le 0. \tag{4.15}
$$

Multiplying (4.15) by $2(1+t)^{1+\varepsilon}$, we have

$$
\begin{aligned}
\frac{d}{dt}\{(1+t)^{1+\varepsilon}\|V(t)\|^2\} + (1+t)^{1+\varepsilon}\left(\alpha\int_{R^2} U_x V^2 dxdy + 2\|\nabla V(t)\|^2\right)& \\
\le (1+\varepsilon)(1+t)^{\varepsilon}\|V(t)\|^2.&
\end{aligned} \tag{4.16}
$$

By the Gagliardo-Nirenberg inequality

$$
\|V(t)\|^2_{L^2(R^2)} \le C\|V(t)\|_{L^1(R^2)}\|\nabla V(t)\|_{L^2(R^2)},
$$

we obtain

$$
\begin{aligned}
&\frac{d}{dt}\{(1+t)^{1+\varepsilon}\|V(t)\|^2\} + (1+t)^{1+\varepsilon}\left(\alpha\int_{R^2} U_x V^2 dxdy + 2\|\nabla V(t)\|^2\right)\\
&\qquad \le C(1+t)^{\frac{1+\varepsilon}{2}}\|\nabla V(t)\|(1+t)^{\frac{\varepsilon-1}{2}}\|V(t)\|_{L^1(R^2)}\\
&\qquad \le (1+t)^{1+\varepsilon}\|\nabla V(t)\|^2 + C(1+t)^{\varepsilon-1}\|V(t)\|^2_{L^1(R^2)}\\
&\qquad \le (1+t)^{1+\varepsilon}\|\nabla V(t)\|^2 + C(1+t)^{\varepsilon-1}\|V_0\|^2_{L^1(R^2)}.
\end{aligned} \tag{4.17}
$$

Integrating (4.17) over $[0,t]$ in t, we obtain (4.9). Next, we estimate V_y and V_x. First, multiplying $\frac{\partial}{\partial y}(4.1)$ by V_y, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V_y(t)\|^2 + \int_{R^2} V_y\{f(U+V)-f(U)\}_{xy}dxdy\\
&\qquad + \int_{R^2} V_y g(U+V)_{yy}dxdy + \mu\|\nabla V_y(t)\|^2 = 0.
\end{aligned} \tag{4.18}
$$

The integration by parts gives:

The second and third terms of (4.18)

$$
= \frac{1}{2}\int_{R^2} f''(U+V)U_x V_y^2 dxdy + \frac{1}{2}\int_{R^2}\{f''(U+V)V_x V_y^2 + g''(U+V)V_y^3\}dxdy.
$$

Hence,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V_y(t)\|^2 + \frac{\alpha}{2}\int_{R^2} U_x V_y^2 dxdy + \mu\|\nabla V_y(t)\|^2\\
&\qquad \le C\int_{R^2}(|V_y|^3 + |V_x||V_y|^2)dxdy.
\end{aligned} \tag{4.19}
$$

Since

$$
\begin{aligned}
&C\int_{R^2}|V_x||V_y|^2 dxdy\\
&\le C\int_R \sup_{y\in R}|V_x(t,x,y)|\|V_y(t,x,\cdot)\|^2_{L^2(R_y)}dx\\
&\le C\int_R \|V_x(t,x,\cdot)\|^{\frac{1}{2}}_{L^2(R_y)}\|V_{xy}(t,x,\cdot)\|^{\frac{1}{2}}_{L^2(R_y)}\|V_y(t,x,\cdot)\|^2_{L^2(R_y)}dx\\
&\le \frac{\mu}{4}\|V_{xy}(t)\|^2_{L^2(R^2)} + C\int_R \|V_x(t,x,\cdot)\|^{\frac{2}{3}}_{L^2(R_y)}\|V_y(t,x,\cdot)\|^{\frac{8}{3}}_{L^2(R_y)}dx\\
&\le \frac{\mu}{4}\|V_{xy}(t)\|^2_{L^2(R^2)}\\
&\qquad + C\int_R \|V_x(t,x,\cdot)\|^{\frac{2}{3}}_{L^2(R_y)}\|V(t,x,\cdot)\|^{\frac{4}{3}}_{L^2(R_y)}\|V_{yy}(t,x,\cdot)\|^{\frac{4}{3}}_{L^2(R_y)}dx\\
&\le \frac{\mu}{4}\|\nabla V_y(t)\|^2_{L^2(R^2)} + C\int_R \|V_x(t,x,\cdot)\|^2_{L^2(R_y)}\|V(t,x,\cdot)\|^4_{L^2(R_y)}dx\\
&\le \frac{\mu}{4}\|\nabla V_y(t)\|^2_{L^2(R^2)} + C\sup_{x\in R}\|V(t,x,\cdot)\|^4_{L^2(R_y)}\|V_x(t)\|^2_{L^2(R^2)}\\
&\le \frac{\mu}{4}\|\nabla V_y(t)\|^2_{L^2(R^2)} + C\|V(t)\|^2_{L^2(R_y)}\|V_x(t)\|^4_{L^2(R^2)}
\end{aligned}
$$

and

$$C\int_{R^2}|V_y|^3dxdy \le \frac{\mu}{4}\|\nabla V_y(t)\|^2_{L^2(R^2)} + C\|V(t)\|^2_{L^2(R^2)}\|V_x(t)\|^2_{L^2(R^2)}\|V_y(t)\|^2_{L^2(R^2)},$$

we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|V_y(t)\|^2 + \frac{\alpha}{2}\int_{R^2}U_xV_y^2dxdy + \frac{\mu}{2}\|\nabla V_y(t)\|^2\\ &\le C\|V(t)\|^2_{L^2(R^2)}\|V_x(t)\|^2_{L^2(R^2)}(\|V_x(t)\|^2_{L^2(R^2)} + \|V_y(t)\|^2_{L^2(R^2)}).\end{aligned} \tag{4.20}$$

Noting that $\|V(t)\|^2_{L^2(R^2)} \le C(1+t)^{-1}$, we multiply (4.20) by $(1+t)^{2+\varepsilon}$ and integrating it over $[0,t]$ to obtain (4.11). Second, multiplying $\frac{\partial}{\partial x}(4.1)$ by V_x. Then, after similar calculations to the above, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|V_x(t)\|^2 + \frac{\alpha}{2}\int_{R^2}U_x|V_x|^2dxdy + \frac{\mu}{2}\|\nabla V_x(t)\|^2\\ &\le C\|V(t)\|^2_{L^2(R^2)}\|V_x(t)\|^2_{L^2(R^2)}(\|V_x(t)\|^2_{L^2(R^2)} + \|V_y(t)\|^2_{L^2(R^2)})\\ &\quad + C\|U_x(t)\|_{L^\infty}\int_{R^2}U_xV^2dxdy.\end{aligned} \tag{4.21}$$

Since $\|U_x(t)\|_{L^\infty(R)} \le \|w_x(t)\|_{L^\infty(R)} + \|v_x(t)\|_{L^\infty(R)} \le C(1+t)^{-\frac{7}{8}}\log^4(2+t)$ by virtue of Theorem 5, we can multiply (4.21) by $(1+t)^{\frac{15}{8}+\varepsilon}$, not $(1+t)^{2+\varepsilon}$, to obtain (4.10). The estimates (4.12)–(4.14) for the second derivatives of V are obtained by more complicated calculations than those for the first derivatives. We omit the details.

Thus the proof of Theorem 7 is complete. □

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