

Asynchronous Consensus in Continuous-Time Multi-Agent Systems With Switching Topology and Time-Varying Delays ¹

Feng Xiao, Long Wang

Intelligent Control Laboratory, Center for Systems and Control, Department of Industrial Engineering and Management, and Department of Mechanics and Space Technologies, College of Engineering, Peking University, Beijing 100871, China

Abstract

In this paper, we study asynchronous consensus problems of continuous-time multi-agent systems with discontinuous information transmission. The proposed consensus control strategy is implemented only based on the state information at some discrete times of each agent's neighbors. The asynchronization means that each agent's update times, at which the agent adjusts its dynamics, are independent of others'. Furthermore, it is assumed that the communication topology among agents is time-dependent and the information transmission is with bounded time-varying delays. If the union of the communication topology across any time interval with some given length contains a spanning tree, the consensus problem is shown to be solvable. The analysis tool developed in this paper is based on the nonnegative matrix theory and graph theory. The main contribution of this paper is to provide a valid distributed consensus algorithm that overcomes the difficulties caused by unreliable communication channels, such as intermittent information transmission, switching communication topology, and time-varying communication delays, and therefore has its obvious practical applications. Simulation examples are provided to demonstrate the effectiveness of our theoretical results.

Key words: Multi-agent systems, asynchronous consensus, switching topology, time-varying delays, coordination.

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Email addresses: fengxiao@pku.edu.cn (Feng Xiao), longwang@pku.edu.cn (Long Wang).

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1 Introduction

In recent years, decentralized coordination of multi-agent systems has become an active area of research and attracted the attention of multi-disciplinary researchers in a wide range including system control theory, statistical physics, biology, applied mathematics, and computer science. This is partly due to its broad applications in cooperative control of unmanned aerial vehicles, scheduling of automated highway systems, formation control of satellite clusters, distributed optimization of multiple mobile robotic systems, etc.

In cooperative control of multiple agents, in order to accomplish some complicated tasks or reach their common goals, groups of dynamic agents in multi-agent/multi-robot systems need to interact with each other and eventually reach an agreement on certain quantities of interest. Those problems are usually called consensus problems, which are one of fundamental research topics in decentralized control.

Consensus problems have been studied for a long time and the formal investigation of them can be traced back to 1960's in the field of management science and statistics (See DeGront [1] and references therein). In the field of systems and control theory, the pioneering work was done by Borkar and Varaiya[2] and Tsitsiklis and Athans[3], which is on asynchronous consensus problems with an application in distributed decision-making system. In [4], Vicsek et al. proposed a simple but interesting discrete-time model of multiple agents moving in the plane. Each agent's motion is updated using a local rule based on its own state and the states of its neighbors. The Vicsek model can be viewed as a special case of a computer model mimicking animal aggregation proposed in [5] for the computer animation industry. By using the graph theory and matrix theory, Jadbabaie et al. provided a theoretical explanation of the consensus property of the Vicsek model in [6], where each agent's set of neighbors changes with time as the system evolves. The typical continuous-time consensus model was presented by Olfati-Saber and Murray in [7], where the concepts of solvability of consensus problems and consensus protocols were first introduced. In [7], under the assumptions that the dynamics of each agent is a scalar continuous-time integrator and the communication among agents may be unidirectional, Olfati-Saber and Murray used a directed graph to model the communication topology and studied three agreement problems, namely, directed networks with fixed topology, directed networks with switching topology, and undirected networks with communication time-delays and fixed topology. And it was assumed that the directed topology is balanced and strongly connected. In [8], Ren and Beard extended the results of [6] and [7] and presented some improved conditions for state agreement under dynamically changing directed interaction topology, which is not necessarily balanced or strongly connected. In the past several years, consensus problems

of multi-agent systems have been developing very fast and several research topics have been addressed, such as agreement over random networks [9,10], asynchronous information consensus [11,12], dynamic consensus [13], consensus filters [14], networks with general communication structures [15], networks with nonlinear agreement protocols [16], and networks with switching topology and time-delays [6,7,8,16,17,18,19,20]. For more details, see the survey [21] and paper [22], where a theoretical framework for analysis of consensus algorithms for multi-agent networked systems was provided. In addition, it is necessary to emphasize that flocking of agents and swarms[23,24,25,26] and formation control of vehicles[27,28,29] are two active areas where many useful results obtained in consensus problems have been successfully applied. And in [23], the authors provided the first proof of convergence of Reynolds' rules on the basis of the convergence of consensus algorithms in [7].

In this paper, we propose a distributed asynchronous consensus control strategy that is only based on the state information of each agent's neighbors at some discrete times. This is partly motivated by the work of Olfati-Saber and Murray [7] and the difficulties encountered in the implementation of the typical continuous-time protocols proposed in [7]. In the realistic networks of agents, we may face the following problems:

- (1) Communication topology is always changing;
- (2) The received information is often with time-delays, and furthermore, the delays may be (randomly) time-varying and unknown;
- (3) Due to long distance transmission, unreliable information channels, and limited bandwidth of networks, the continuity of state information of each agent's neighbors cannot be ensured.

Here, we summarize some important closely related works. First, in continuous-time systems, none of the existing results guarantee the stability of the consensus protocols proposed by Olfati-Saber and Murray, in the presence of switching topology and time-varying delays. It was often assumed that the available topologies are finite[6,7,8] and the delays are constant[7,16,17]. In fact, the stability analysis of the protocols in [7] in the presence of time-varying delays is challenging and even impossible in theory, and is unnecessary in applications.

In the study of discrete-time systems, Tanner and Christodoulakis in [20] studied a discrete-time model with fixed undirected topology and assumed that all agents transmit their state information in turn. Consequently, outdated information may be used and the equivalent augmented system becomes a periodically switched system, which can be viewed as a multi-agent system with switching topology. In [11], Fang and Antsaklis studied the case with switching topology and time-dependent delays by an asynchronous system with fixed topology. However since the possible topologies are generated by the fixed topology of the asynchronous system, the "switching" topology is

not really switching. By matrix theory, Xiao and Wang derived some sufficient conditions for the solvability of consensus problems of discrete-time systems with switching topology and time-varying delays in [19], but it was assumed that all available topologies are finite.

In addition, all the proposed continuous-time consensus algorithms depend on continuous state signals. In engineering applications, continuous signals require large bandwidth of networks, and furthermore, in many cases, are not available. Therefore, we need to devise some continuous-time consensus algorithms that do not continuously depend on the external state information.

Our proposed consensus control strategy is built upon very general assumptions. The communication topology is switching among an infinite set of weighted directed graphs, communication delays are time-varying, and state information transmission is allowed to be intermittent. Moreover, our control strategy is also an asynchronous one, which means that each agent's update actions are independent of others'. Each agent adjusts its dynamics independently, which is inherent in distributed control systems. It is important to mention that asynchronous consensus problems were also studied by [2,3] and [11], where several asynchronous consensus algorithms were given. By using nonnegative matrix theory and graph theory, especially the properties of scrambling matrices, we provide some sufficient (and necessary) conditions for the convergence of our consensus control strategy.

This paper is organized as follows. Section II presents some basic definitions and results in matrix theory and graph theory. Section III formulates the problem to be studied. Convergence analysis and the technical proof are performed in Sections IV and V, respectively. In Section VI, simulation examples are presented. Finally, concluding remarks are stated in Section VII.

2 Preliminaries

This section presents some definitions and results in matrix theory and graph theory that will be used in this paper [30,31].

Let $\mathcal{I}_n = \{1, 2, \dots, n\}$, \mathbb{Z}_+ be the set of nonnegative integers, and $\mathbf{1} = [1, 1, \dots, 1]^T$ with compatible dimensions. Given $A = [a_{ij}] \in \mathbb{R}^{n \times r}$, A is said to be *nonnegative*, $A \geq 0$, if all its entries a_{ij} are nonnegative. A is said to be *positive*, $A > 0$, if all its entries a_{ij} are positive. Let $B \in \mathbb{R}^{n \times r}$. We write $A \geq B$ if $A - B \geq 0$, and $A > B$ if $A - B > 0$. A nonnegative square matrix A with the property that all its row sums are +1 is said to be a *stochastic matrix*. Throughout this paper, we let $\prod_{i=1}^k A_i = A_k A_{k-1} \cdots A_1$ denote the left product of matrices. A $n \times n$ stochastic matrix A is called indecomposable and

aperiodic (SIA) (or *ergodic*) if there exists $f \in \mathbb{R}^n$ such that $\lim_{k \rightarrow \infty} A^k = \mathbf{1}f^T$.

Directed graphs will be used to model the communication topologies among agents. A *directed graph* \mathcal{G} consists of a vertex set $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$ and an edge set $\mathcal{E}(\mathcal{G}) \subset \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(\mathcal{G})\}$, where an edge is an ordered pair of vertices in $\mathcal{V}(\mathcal{G})$ (Here, we allow for self-loops, namely, the edges with the same vertices). The set of *neighbors* of vertex v_i in \mathcal{G} is denoted by $\mathcal{N}(\mathcal{G}, v_i) = \{v_j : (v_j, v_i) \in \mathcal{E}(\mathcal{G}), j \neq i\}$. The associated index set of the neighbors is denoted by $\mathcal{N}(\mathcal{G}, i) = \{j : v_j \in \mathcal{N}(\mathcal{G}, v_i)\}$. If (v_i, v_j) is an edge of \mathcal{G} , v_i and v_j are defined as the parent and child vertices, respectively. A *subgraph* \mathcal{G}_s of a directed graph \mathcal{G} is a directed graph such that the vertex set $\mathcal{V}(\mathcal{G}_s) \subset \mathcal{V}(\mathcal{G})$ and the edge set $\mathcal{E}(\mathcal{G}_s) \subset \mathcal{E}(\mathcal{G})$. If $\mathcal{V}(\mathcal{G}_s) = \mathcal{V}(\mathcal{G})$, we call \mathcal{G}_s a *spanning subgraph* of \mathcal{G} . For any $v_i, v_j \in \mathcal{V}(\mathcal{G}_s)$, if $(v_i, v_j) \in \mathcal{E}(\mathcal{G}_s)$ if and only if $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$, \mathcal{G}_s is called an *induced subgraph*. In this case, \mathcal{G}_s is also said to be induced by $\mathcal{V}(\mathcal{G}_s)$. A *path* in a directed graph \mathcal{G} is a sequence v_{i_1}, \dots, v_{i_k} of vertices such that $(v_{i_s}, v_{i_{s+1}}) \in \mathcal{E}(\mathcal{G})$ for $s = 1, \dots, k-1$. A directed graph \mathcal{G} is *strongly connected* if between every pair of distinct vertices v_i, v_j in \mathcal{G} , there is a path that begins at v_i and ends at v_j (that is, from v_i to v_j). A *directed tree* is a directed graph, where every vertex, except one special vertex without any parent, which is called the *root vertex*, has exactly one parent, and the root vertex can be connected to any other vertices through paths. A *spanning tree* of \mathcal{G} is a directed tree that is a spanning subgraph of \mathcal{G} . We say that a graph has (or contains) a spanning tree if a subset of the edges forms a spanning tree. A *weighted directed graph* $\mathcal{G}(A)$ is a directed graph \mathcal{G} plus a nonnegative *weight matrix* $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ such that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}) \iff a_{ji} > 0$. And a_{ji} is called the *weight* of edge (v_i, v_j) .

3 Problem Formulation

We suppose that the system studied in this paper consists of n autonomous agents, e.g., birds, robots, etc., labeled 1 through n . All these agents share a common state space \mathbb{R} . Each agent adjusts its current state based upon the information received from other agents that are defined as neighbors of this agent. We use a weighted directed graph $\mathcal{G}(\mathcal{A}(t))$ to represent the communication topology or information flow, where $\mathcal{A}(t) = [\mathbf{a}_{ij}(t)] \in \mathbb{R}^{n \times n}$ is a nonnegative matrix. The appearance of parameter t implies that the communication topology may be dynamically changing. Agent i is represented by vertex v_i . Edge $(v_j, v_i) \in \mathcal{E}(\mathcal{G}(\mathcal{A}(t)))$ corresponds an available information channel from agent j to agent i . If agent i receives information from agent j at time t , then there exists an edge from vertex v_j to vertex v_i , i.e., $(v_j, v_i) \in \mathcal{E}(\mathcal{G}(\mathcal{A}(t)))$. And the *neighbors* of agent i are those agents whose information is received by agent i at time t . The associated index set of the neighbors is denoted by $\mathcal{N}(t, i)$. Notice that because of the existence of communication time-delays,

the index set $\mathcal{N}(\mathcal{G}(\mathcal{A}(t)), i)$ may not be equal to $\mathcal{N}(t, i)$. We will discuss them latter.

Let $x_i \in \mathbb{R}$ denote the state of agent i and let $x = [x_1, x_2, \dots, x_n]^T$. Then the whole system can be generally represented by the continuous-time model $\dot{x}(t) = f(t, u(t))$ or by the discrete-time model $x(t+1) = f(t, u(t))$, where $u(t)$ is a state feedback. If for any initial state, $x(t)$ converges to some equilibrium point x^* (dependent on the initial state) such that $x_i^* = x_j^*$ for all $i, j \in \mathcal{I}_n$, as $t \rightarrow \infty$, then we say that this system *solves a consensus problem*[7] (or *has the consensus property*). Let $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function of n variables x_1, x_2, \dots, x_n . For any initial state $x(0)$,² if $x^* = \mathbf{1}\chi(x(0))$, then we say that this system *solves the χ -consensus problem*, and the function χ is called the *consensus function*. The common value of x_i^* is called the *group decision value*.

3.1 The Model

For agent i , $i \in \mathcal{I}_n$, we assume that it receives or detects its neighbors' states at *update times* $t_0^i, t_1^i, \dots, t_k^i, \dots$, which can be seen as a real number sequence and are denoted by $\{t_k^i\}$. We assume that $\{t_k^i\}$ satisfies the following assumptions:

(A1) For any $k \in \mathbb{Z}_+$, $0 < \check{\tau}_u \leq t_{k+1}^i - t_k^i \leq \hat{\tau}_u$, where $\check{\tau}_u, \hat{\tau}_u \in \mathbb{R}$;

(A2) $t_0^i = 0$.

The simple reason for the calling of “update times” is that the neighbors' information known by agent i or the dynamics of agent i is updated at those times. The existence of lower bound $\check{\tau}_u$ of time intervals between any two consecutive update times is just to guarantee the validity of our consensus protocols (1) and (2). If there does not exist lower bound, it will be hard to analyze protocols (1) and (2) in theory and it is also unnecessary in applications. If there does not exist an upper bound $\hat{\tau}_u$ of $t_{k+1}^i - t_k^i$, it will be difficult for the states of agents to reach consensus (See Example 1). For Assumption (A2), we make it solely for the convenience of our theoretical analysis; otherwise, the main results of our paper is still obtainable.

If agent i receives the state information of its neighbors at t_k^i , then agent i is

² In this paper, consensus functions are only related to the systems without time-delays.

assumed to take the following dynamics in time interval $[t_k^i, t_{k+1}^i)$

$$\dot{x}_i(t) = \begin{cases} \sum_{j \in \mathcal{N}(t_k^i, i)} \frac{\mathbf{a}_{ij}(t_k^i)}{\sum_{j \in \mathcal{N}(t_k^i, i)} \mathbf{a}_{ij}(t_k^i)} (x_j(t_k^i) - x_i(t)), & \text{if } \mathcal{N}(t_k^i, i) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Remark 1 *The continuous-time model studied in [7] is that $\dot{x}_i(t) = \sum_{j \in \mathcal{N}(t, i)} \mathbf{a}_{ij}(t) (x_j(t) - x_i(t))$. In our discrete-communication framework, this system will turn into $\dot{x}_i(t) = \sum_{j \in \mathcal{N}(t, i)} \mathbf{a}_{ij}(t_k^i) (x_j(t_k^i) - x_i(t))$ accordingly, where $t \in [t_k^i, t_{k+1}^i)$. Seemingly, our model (1) is a special case of this one. But if system (1) solves a consensus problem, then the afore-mentioned system will also have the consensus property. That will be clear in our analysis process.*

It is well known that communication time-delays exist extensively in networks. Therefore, it is reasonable to assume that there exist communication time-delays in the information transmission. For agent i , $i \in \mathcal{I}_n$, we assume that the information received by agent i from agent j is with time-delay τ_{ij}^k at update time t_k^i , and then the considered system turns into

$$\dot{x}_i(t) = \begin{cases} \sum_{j \in \mathcal{N}(t_k^i, i)} \frac{\mathbf{a}_{ij}(t_k^i)}{\sum_{j \in \mathcal{N}(t_k^i, i)} \mathbf{a}_{ij}(t_k^i)} (x_j(t_k^i - \tau_{ij}^k) - x_i(t)), & \text{if } \mathcal{N}(t_k^i, i) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

where $t \in [t_k^i, t_{k+1}^i)$.

We make the following assumption about system (2) together with Assumption (A1, A2):

$$(A3) \quad 0 \leq \tau_{ij}^k \leq \tau_d = K\bar{\tau}_u, \text{ where } i \in \mathcal{I}_n, j \in \mathcal{N}(t_k^i, i), k, K \in \mathbb{Z}_+;$$

Remark 2 *Consider system (2). Generally, the time-delays may be unknown for each agent when they receive information. If the data is time stamped, the delays will be detectable. In addition, if communication time-delays only satisfy Assumption (A3), for any $i \in \mathcal{I}_n$, $j \in \mathcal{N}(t_k^i, i)$, the state information of agent j , $x_j(t_k^i - \tau_{ij}^k)$, received by agent i at update time t_k^i may be outdated compared with state information $x_j(t_{k'}^i - \tau_{ij}^{k'})$ received previously, i.e., $t_k^i - \tau_{ij}^k < t_{k'}^i - \tau_{ij}^{k'}$, where $k' < k$. If the delays are detectable, we can suppose that agent i always uses the most recent data of its neighbors, that is, if there exists $k' < k$ such that $t_k^i - \tau_{ij}^k < t_{k'}^i - \tau_{ij}^{k'}$, then agent i replaces the state information $x_j(t_k^i - \tau_{ij}^k)$ by $x_j(t_{k^*}^i - \tau_{ij}^{k^*})$ in time interval $[t_k^i, t_{k+1}^i)$, where $k^* = \arg \max_{k' < k} (t_{k'}^i - \tau_{ij}^{k'})$. We call this control strategy the-most-recent-data strategy. This control strategy can get better convergence rate (See Example 3).*

According to different properties of update times, we classify systems (1) and (2) as synchronous or as asynchronous systems.

Definition 1 (Synchronous and asynchronous systems) *We say that system (1) (or (2)) is synchronous if for any $i, j \in \mathcal{I}_n$, $\{t_k^i\} = \{t_k^j\}$, i.e., for any $k \in \mathbb{Z}_+$, $t_k^i = t_k^j$. We say that system (1) (or (2)) is asynchronous if for any $i, j \in \mathcal{I}_n$, $i \neq j$, $\{t_k^i\}$ is independent of $\{t_k^j\}$, i.e., agents may not adjust their dynamics at the same time (See Fig. 1).*

In realistic networks, it is difficult for all agents to be synchronous on update actions and therefore we mainly discuss the asynchronous consensus property of systems (1) and (2).

3.2 Communication Topology

Since our consensus protocols (1) and (2) only depend on discrete state information, we are not concerned with the actual communication topology $\mathcal{G}(\mathcal{A}(t))$ outside those update times. And we give the following definition of $\mathcal{G}^0(t)$, which is different from the actual communication topology and is also called communication topology.

Definition 2 (Communication topology) *Suppose that $\mathcal{G}^0(t)$ is with the same vertex set as $\mathcal{G}(\mathcal{A}(t))$. For any $i \in \mathcal{I}_n$, $k \in \mathbb{Z}_+$, if agent i receives the state information of agent j at time t_k^i , $j \neq i$, $j \in \mathcal{I}_n$, then $(v_j, v_i) \in \mathcal{G}^0(t)$, and if not, $(v_j, v_i) \notin \mathcal{G}^0(t)$, where $t \in [t_k^i, t_{k+1}^i)$. In addition, there are no self-loops in $\mathcal{G}^0(t)$.*

In Definition 2, $\mathcal{G}^0(t)$ is only compliant with the actual topology $\mathcal{G}(\mathcal{A}(t))$ on those edges corresponding to $\mathbf{a}_{ij}(t_k^i)$, $i \in \mathcal{I}_n$, $j \in \mathcal{N}(t_k^i, i)$. This way can facilitate our theoretical analysis and does not affect the dynamics of agents.

Proposition 1 $\mathcal{N}(t_k^i, i) = \mathcal{N}(\mathcal{G}^0(t_k^i), i) \subset \mathcal{N}(\mathcal{G}(\mathcal{A}(t_k^i)), i)$.

The above proposition follows from the fact that the available channel (v_j, v_i) at t_k^i can not ensure that agent i can receive the state information of agent j at time t_k^i if there exists time-delay, and agent i cannot receive any information from agent j if (v_j, v_i) does not exist.

Remark 3 *The reason for the introduction of $\mathcal{G}^0(t)$ is that $\mathcal{G}^0(t)$ is more important than $\mathcal{G}(\mathcal{A}(t))$ for our control strategies. We can see that $\mathcal{G}^0(t)$ is directly connected with the successful information transmission in the networks. $\mathcal{E}(\mathcal{G}(\mathcal{A}(t)))$ represents all available communication channels, while $\mathcal{E}(\mathcal{G}^0(t))$ stands for the communication channels through which the information has been successfully received.*

Here, for simplicity of presentation, we introduce another matrix $\mathbf{A}(t) = [\mathbf{a}_{ij}(t)]$, where $\mathbf{a}_{ij}(t) = \begin{cases} \frac{\mathbf{a}_{ij}(t_k^i)}{\sum_{j \in \mathcal{N}(t_k^i, i)} \mathbf{a}_{ij}(t_k^i)}, & j \neq i \\ 0, & j = i \end{cases}$ if $\mathcal{N}(t_k^i, i) \neq \phi$, and $\mathbf{a}_{ij}(t) = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}$ if $\mathcal{N}(t_k^i, i) = \phi$, $t \in [t_k^i, t_{k+1}^i)$. Obviously, $\mathbf{A}(t)$ is stochastic.

Proposition 2 *For any $t \geq 0$, $\mathcal{N}(\mathcal{G}(\mathbf{A}(t)), i) = \mathcal{N}(\mathcal{G}^0(t), i)$. And if we ignore the weight of each edge in $\mathcal{G}(\mathbf{A}(t))$ and self-loops in $\mathcal{G}(\mathbf{A}(t))$, then $\mathcal{G}(\mathbf{A}(t))$ and $\mathcal{G}^0(t)$ represent the same graph.*

Since $\mathcal{G}(\mathbf{A}(t))$ may be dynamically changing, we should investigate all possible directed graphs. Because of the finite number of vertices, there are at most $2^{n \times n}$ different kinds of directed graphs. Let $\Gamma_{\mathcal{G}}$ denote the set of all those directed graphs. Assume the existence of real number $\hat{\mathbf{a}} \geq \check{\mathbf{a}} > 0$, such that $\check{\mathbf{a}} \leq \mathbf{a}_{ij}(t) \leq \hat{\mathbf{a}}$ if $\mathbf{a}_{ij}(t) \neq 0$. Consequently all possible $\mathbf{A}(t)$ constitute a compact set³, denoted by $\Gamma_{\mathbf{A}}$, and if $\mathbf{a}_{ij}(t) \neq 0$, then $\frac{\check{\mathbf{a}}}{(n-1)\hat{\mathbf{a}}} < \mathbf{a}_{ij}(t) \leq 1$. It is important to note that all possible communication topologies are infinite if we take the weight of each edge into account.

Finally, we present the notion of the union of graphs that will be used in the remainder of this paper.

Definition 3 (Union of graphs) *The union of graph $\mathcal{G}^0(t)$ across time interval $[t^0, t^0 + T]$ is a directed graph with the same vertex set as $\mathcal{G}^0(t)$ and the edge set $\bigcup_{t' \in [t^0, t^0 + T]} \mathcal{E}(\mathcal{G}^0(t'))$. The union of graph $\mathcal{G}^0(t)$ on the time set $\{s_1, s_2, \dots, s_k\}$ is a directed graph with the same vertex set as $\mathcal{G}^0(t)$ and the edge set $\bigcup_{i=1}^k \mathcal{E}(\mathcal{G}^0(s_i))$.*

4 Convergence Results

This section presents the main result of this paper. As a preparation for the study of the general case, we first investigate two relatively simple cases: the synchronous system with fixed topology in the absence of time-delays and the asynchronous system without time-delays. Our approach is to transpose the continuous-time systems into their discrete-time counterparts, which possess the same consensus property as the original systems. The obtained discrete-time systems have some special structures and the proof of their consensus property is postponed to the next section.

³ The set of all $r \times s$ matrices can be viewed as the metric space \mathbb{R}^{rs} and compact sets are equivalent to bounded closed sets.

Some notations are used in this section. If the studied systems are free of time-delays, we let $\{t_k\} = \{t_k^i, i \in \mathcal{I}_n, k \in \mathbb{Z}_+\}$, and if the systems are with time-delays, let $\{t_k\} = \{t_k^i \text{ or } t_k^i - \tau_{ij}^k, i \in \mathcal{I}_n, k \in \mathbb{Z}_+, j \in \mathcal{N}(t_k^i, i)\}$, such that $t_0 = 0$ and $t_{k+1} > t_k$. Let $\tau_k = t_{k+1} - t_k, k \in \mathbb{Z}_+$, and let $\Lambda(A) = \{B = [b_{ij}] \in \mathbb{R}^{n \times n} : \text{for any } i, j \in \mathcal{I}_n, b_{ij} = a_{ij}, \text{ or } b_{ij} = 0\}$, where $A = [a_{ij}] \in \mathbb{R}^{n \times n}$. Obviously $\Lambda(A)$ is a finite set. Let $\Pi(m, t)$ denote the set of matrices

$$\begin{bmatrix} e^{-h}I + (1 - e^{-h})A_1(t) & (1 - e^{-h})A_2(t) & \cdots & (1 - e^{-h})A_{m-1}(t) & (1 - e^{-h})A_m(t) \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}_{mn \times mn}, \quad (3)$$

where $0 < h \leq \hat{\tau}_u, A_1(t), \dots, A_m(t) \in \Lambda(\mathbf{A}(t))$, and $A_1(t) + \dots + A_m(t) = \mathbf{A}(t)$. We denote the above matrix (3) by $\pi(h, A_1(t), \dots, A_m(t))$.

4.1 The Synchronous Consensus

We first restrict our attention to the simplest case and we study the synchronous system with time-invariant topology in the absence of time-delays. Since there exist no communication delays, we assume that for any $i \in \mathcal{I}_n, j \in \mathcal{N}(\mathcal{G}(\mathcal{A}(t_k^i)), i)$, agent i can receive state information from agent j at time t_k^i . Therefore $\mathcal{N}(\mathcal{G}(\mathcal{A}(t)), i) = \mathcal{N}(\mathcal{G}(\mathbf{A}(t)), i), \mathcal{N}(t_k^i, i) = \mathcal{N}(\mathcal{G}(\mathcal{A}(t_k^i)), i)$, and $\mathbf{A}(t)$ is time-invariant. For simplicity, notations \mathcal{A} and \mathbf{A} are used instead of $\mathcal{A}(t)$ and $\mathbf{A}(t)$. Apparently, $\{t_k^i\} = \{t_k\}$. Rewriting system (1) yields that for any $i \in \mathcal{I}_n, t \in [t_k, t_{k+1})$

$$\dot{x}_i(t) = \begin{cases} -x_i(t) + \sum_{j \in \mathcal{N}(\mathcal{G}(\mathcal{A}), i)} \mathbf{a}_{ij} x_j(t_k), & \text{if } \mathcal{N}(\mathcal{G}(\mathcal{A}), i) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Then we have

$$\begin{aligned} x(t_k + h) &= (e^{-h}I + (1 - e^{-h})\mathbf{A})x(t_k) \\ &= ((1 - e^{-h})(\mathbf{A} - I) + I)x(t_k), \end{aligned} \quad (5)$$

where $0 \leq h \leq \tau_k$ and I is the identity matrix with compatible dimensions.

The following theorem characterizes the consensus property of synchronous system (4).

Theorem 1 *System (4) solves a consensus problem if and only if $\mathcal{G}(\mathcal{A})$ has*

a spanning tree. In addition, the group decision value is uniquely determined by $\mathcal{G}(\mathcal{A})$ and the initial state.

The sufficiency of the first part is a direct consequence of Theorem 3, and we only prove the necessity and the second statement.

Proof: The necessity is shown first. If $\mathcal{G}(\mathcal{A})$ has not any spanning tree, then there will be several subsystems, among which there will not be information communication, and thus system (4) will not solve any consensus problem. Next, we prove the second statement. Because \mathbf{A} is a stochastic matrix, there exists an $f \in \mathbb{R}^n$, $f \geq 0$, such that $f^T \mathbf{1} = 1$ and $f^T \mathbf{A} = f^T$ (Theorem 1, [32]). Let $\chi(x) = f^T x$.

$$\chi(x(t_k + h)) = f^T((1 - e^{-h})(A - I) + I)x(t_k) = f^T x(t_k) = \chi(x(t_k)).$$

Therefore, $\chi(x)$ is time-invariant.

Suppose that the final state is $\mathbf{1}a$, $a \in \mathbb{R}$. We have that $\lim_{t \rightarrow \infty} x(t) = \mathbf{1}a$ and $\lim_{t \rightarrow \infty} \chi(x(t)) = \chi(\mathbf{1}a) = f^T \mathbf{1}a = a$. It follows that the group decision value $a = \chi(x(0))$, which is uniquely determined by $\mathcal{G}(\mathcal{A})$ and the initial state, and the consensus function is $\chi(x)$.

Now, we give an example to show that the assumption of the existence of upper bound $\hat{\tau}_u$ of update intervals is necessary.

Example 1 (Counterexample) Consider the synchronous case and let $n = 2$ and $\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. If update intervals $\tau_0 = \ln 8$, $\tau_1 = \ln 16$, \dots , $\tau_k = \ln 2^{k+3}$, \dots , and the initial state $x(0) = [1, -1]^T$, then we obtain that

$$\lim_{k \rightarrow \infty} |t_{k+1} - t_k| = \lim_{k \rightarrow \infty} \ln 2^{k+3} = \infty,$$

and

$$\begin{aligned} x_1(t_0) - x_2(t_0) &= 2; \\ x_2(t_1) - x_1(t_1) &= 2\left(1 - \frac{1}{4}\right); \\ x_1(t_2) - x_2(t_2) &= 2\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right); \\ &\vdots \end{aligned}$$

For any $k \geq 1$, we have

$$\begin{aligned} |x_1(t_k) - x_2(t_k)| &= 2\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{8}\right)\cdots\left(1 - \frac{1}{2^{k+1}}\right) \\ &\geq 2\left(1 - \frac{1}{4} - \frac{1}{8} - \cdots - \frac{1}{2^{k+1}}\right). \end{aligned}$$

Therefore, $\lim_{k \rightarrow \infty} |x_1(t_k) - x_2(t_k)| \geq 2(1 - 0.5) = 1$. Consequently, the states of agent 1 and agent 2 will never reach consensus.

4.2 The Asynchronous Consensus

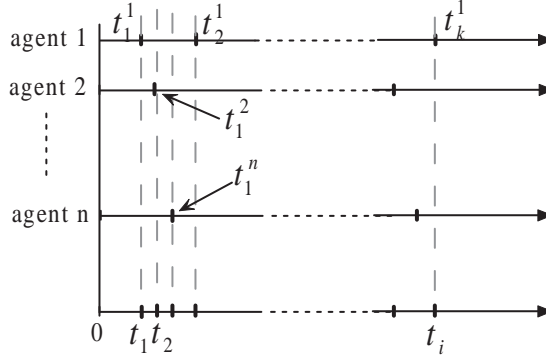


Fig. 1. Update times in asynchronous systems without time-delays.

4.2.1 The Case Without Time-Delays

Consider the asynchronous case of system (1). Lemma 1 gives a proper characterization of update times, which is important in the forthcoming analysis. From Assumption (A1), we have the following fact

Lemma 1 Consider system (1). For any $i \in \mathcal{I}_n$, the number of elements in set $\{t_j : t_j \in [t_k^i, t_{k+1}^i)\}$ is no greater than $(\lfloor \frac{\hat{\tau}_u}{\tau_u} \rfloor + 1)(n - 1) + 1$ denoted by \tilde{m} , where $\lfloor \frac{\hat{\tau}_u}{\tau_u} \rfloor$ is the maximum integer no greater than $\frac{\hat{\tau}_u}{\tau_u}$.

Proof: By Assumption (A1), we have that $t_{k+1}^i - t_k^i \leq \hat{\tau}_u$. For any $j \in \mathcal{I}_n, j \neq i$, agent j updates state information of its neighbors at most $\lfloor \frac{\hat{\tau}_u}{\tau_u} \rfloor + 1$ times in time interval (t_k^i, t_{k+1}^i) . And there are $n - 1$ possible js . Take t_k^i into account and therefore the number of elements in $\{t_j : t_j \in [t_k^i, t_{k+1}^i)\}$ is not greater than $(\lfloor \frac{\hat{\tau}_u}{\tau_u} \rfloor + 1)(n - 1) + 1$.

For agent $i, i \in \mathcal{I}_n$, and update time $t_k, k \geq \tilde{m} - 1$, there exists $s \in \mathbb{Z}_+$ such that $t_s^i \leq t_k < t_{k+1} \leq t_{s+1}^i$. By solving equation equation (1), we have that

$$x_i(t_{k+1}) = e^{-\tau_k} x_i(t_k) + (1 - e^{-\tau_k}) \sum_{j \in \mathcal{N}(t_s^i, i) \cup \{i\}} \mathbf{a}_{ij}(t_s^i) x_j(t_s^i),$$

where $\mathbf{a}_{ij}(t_s^i) = \mathbf{a}_{ij}(t_k)$.

By Lemma 1, $t_s^i \geq t_{k-\tilde{m}+1}$. Let $y(k) = [x(t_k)^T, x(t_{k-1})^T, \dots, x(t_{k-\tilde{m}+1})^T]^T$, where $k \geq \tilde{m} - 1$. From the above discussion, there exists a matrix $\pi(\tau_k, \mathbf{A}_1(t_k), \dots,$

$\mathbf{A}_{\tilde{m}}(t_k) \in \Pi(\tilde{m}, t_k)$ such that

$$y(k+1) = \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\tilde{m}}(t_k))y(k). \quad (6)$$

Proposition 3 *System (1) solves a consensus problem if and only if system (6) solves a consensus problem.*

Proof: The necessity follows from the definition of state variable $y(k)$. Assume that system (6) solves a consensus problem. Let $a \in \mathbb{R}$ such that $\lim_{k \rightarrow \infty} y(k) = \mathbf{1}a$. For any $i \in \mathcal{I}_n$, $x_i(t) - x_i(t_k^i) = (1 - e^{-(t-t_k^i)}) (\sum_{j=1}^n \mathbf{a}_{ij}(t_k^i) x_j(t_k^i) - x_i(t_k^i))$, where $t_k^i < t \leq t_{k+1}^i$. Since $t_k^i \rightarrow \infty$ as $t \rightarrow \infty$, $\lim_{t \rightarrow \infty} (\sum_{j=1}^n \mathbf{a}_{ij}(t_k^i) x_j(t_k^i) - x_i(t_k^i)) = \lim_{t \rightarrow \infty} \sum_{j=1}^n \mathbf{a}_{ij}(t_k^i) (x_j(t_k^i) - x_i(t_k^i)) = 0$ (for $\lim_{t \rightarrow \infty} (x_j(t_k^i) - x_i(t_k^i)) = a - a = 0$), that is, $\lim_{t \rightarrow \infty} (x_i(t) - x_i(t_k^i)) = 0$. Therefore, for any $i \in \mathcal{I}_n$, $\lim_{t \rightarrow \infty} x_i(t) = a$, and system (1) solves a consensus problem.

The following theorem is also a consequence of Theorem 3.

Theorem 2 *If there exists $T \geq 0$ such that for all $t^0 \geq 0$, the union of graph $\mathcal{G}^0(t)$ across interval $[t^0, t^0 + T]$ contains a spanning tree, then system (1) solves a consensus problem.*

4.2.2 The Case With Time-Delays

As a parallel with Lemma 1, we present the following lemma.

Lemma 2 *Consider the asynchronous case of system (2). For any $i \in \mathcal{I}_n, k \in \mathbb{Z}_+$, the number of elements in set $\{t_j : t_j \in [t_k^i, t_{k+1}^i]\}$ is not greater than $\tilde{m}n(K(n-1) + 1)$. We let $\tilde{m} = \tilde{m}n(K(n-1) + 1)$ and $\hat{m} = (K+1)\tilde{m}$;*

Proof: Obviously, there can not be infinite elements in $\{t_j : t_j \in [t_k^i, t_{k+1}^i]\}$ by Assumption (A1) and (A3). We work out one upper bound of the number of elements in this set.

For any i , let $\{t_k^{(i)}\} = \{t_k^i - \tau_{ij}^k, k \in \mathbb{Z}_+, j \in \mathcal{N}(t_k^i, i)\} \cup \{t_k^i\}$. Given $k \in \mathbb{Z}_+, i \in \mathcal{I}_n$, by Assumption (A3), all possible elements in $\{t_k^{(i)}\} \cap (t_k^i, t_{k+1}^i)$ are $t_{k+1}^i - \tau_{ij}^{k+1}, j \in \mathcal{N}(t_{k+1}^i, i), t_{k+2}^i - \tau_{ij}^{k+2}, j \in \mathcal{N}(t_{k+2}^i, i), \dots, t_{k+K}^i - \tau_{ij}^{k+K}, j \in \mathcal{N}(t_{k+K}^i, i)$. Because $|\mathcal{N}(t_{k'}^i, i)| \leq n-1$ for any k' , where $|\mathcal{N}(t_{k'}^i, i)|$ is the number of elements in set $\mathcal{N}(t_{k'}^i, i)$. Therefore

$$|\{t_k^{(i)}\} \cap [t_k^i, t_{k+1}^i]| \leq K(n-1) + 1. \quad (7)$$

Let $i \in \mathcal{I}_n, k \in \mathbb{Z}_+$ be given. By Lemma 1, $|\{t_{k'}^j : t_{k'}^j \in [t_k^i, t_{k+1}^i], k' \in \mathbb{Z}_+, j \in \mathcal{I}_n\}| \leq \tilde{m} + 1$. Let its elements be $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{m'+1}$ such that $\bar{t}_{k'} < \bar{t}_{k'+1}, k' = 1, 2, \dots, m'$, where $m' \leq \tilde{m}$. For any $k' \in \{1, 2, \dots, m'\}, j \in \mathcal{I}_n$, there

exists k'' such that $t_{k''}^j \leq \bar{t}_{k'} < \bar{t}_{k'+1} \leq t_{k''+1}^j$. From (7),

$$|[\bar{t}_{k'}, \bar{t}_{k'+1}] \cap \{t_k^{(j)}\}| \leq |[t_{k''}^j, t_{k''+1}^j] \cap \{t_k^{(j)}\}| \leq K(n+1) + 1.$$

Since $\{t_j : t_j \in [t_k^i, t_{k+1}^i]\} = \bigcup_{k'=1}^{m'} ([\bar{t}_{k'}, \bar{t}_{k'+1}] \cap (\bigcup_{j=1}^n \{t_k^{(j)}\}))$, $|\{t_j : t_j \in [t_k^i, t_{k+1}^i]\}| \leq \check{m}n(K(n-1) + 1)$.

For agent i and any $k \in \mathbb{Z}_+$, $k \geq \hat{m} - 1$, there exists s such that $t_s^i \leq t_k < t_{k+1} \leq t_{s+1}^i$, and agent i 's dynamics can be written

$$\dot{x}_i(t) = \begin{cases} \sum_{j \in \mathcal{N}(t_s^i, i)} \mathbf{a}_{ij}(t_s^i) (x_j(t_s^i - \tau_{ij}^s) - x_i(t)), & \text{if } \mathcal{N}(t_s^i, i) \neq \emptyset; \\ 0, & \text{otherwise,} \end{cases}$$

where $t \in [t_k, t_{k+1})$. From Lemma 2 and Assumption (A3), $t_{k-\hat{m}+1} \leq t_s^i - \tau_{ij}^s$. Let $\tau_{ii}^s = 0$. Solving the above equation gives

$$x_i(t_{k+1}) = e^{-\tau_k} x_i(t_k) + (1 - e^{-\tau_k}) \sum_{j \in \mathcal{N}(t_s^i, i) \cup \{i\}} \mathbf{a}_{ij}(t_k) x_j(t_s^i - \tau_{ij}^s).$$

Let $z(k) = [x(t_k)^T, x(t_{k-1})^T, \dots, x(t_{k-\hat{m}+1})^T]^T$. For any $k \geq \hat{m} - 1$, there exists a matrix $\pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}}(t_k)) \in \Pi(\hat{m}, t_k)$ such that

$$z(k+1) = \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}}(t_k)) z(k). \quad (8)$$

With the same arguments as Proposition 3, we have

Proposition 4 *System (2) solves a consensus problem if and only if system (8) solves a consensus problem.*

Now, we present the main result of this paper.

Theorem 3 *If there exists $T \geq 0$ such that for all $t^0 \geq 0$, the union of graph $\mathcal{G}^0(t)$ across interval $[t^0, t^0 + T]$ contains a spanning tree, then the time-delayed system (2) solves a consensus problem. Moreover, if $\mathcal{G}(\mathcal{A}(t))$ is time-invariant, then the solvability of the consensus problem of system (2) implies that $\mathcal{G}(\mathcal{A}(t))$ contains a spanning tree.*

Remark 4 *The second part of Theorem 3 is obvious. In order to prove Theorem 3, it suffices to prove the possession of consensus property of discrete-time system (8) under the hypothesis of Theorem 3. However, the proof of consensus property of system (8) is not easy. The following two properties of system (8) make all previous results, such as those in [8, 16], inapplicable to our system, and therefore we have to explore another efficient way.*

(1) $\Pi(\hat{m}, t_k)$, the set including all possible state matrices $\pi(\tau_k, \cdot)$, is not a

finite set, and furthermore is not compact because that $\Gamma_{\mathbf{A}}$ is an infinite set and $\tau_k \in (0, \hat{\tau}_u]$, which is not compact.

(2) The diagonal entries of any $\pi(\tau_k, \cdot)$ are not all non-zeros.

Since $\Pi(\hat{m}, t_k)$ is a infinite set, system (8) can be viewed as a discrete-time consensus model with the topology that switches among an infinite set of directed graphs. Therefore, previous results on discrete-time systems with finite available topologies, such as those in [6,7,8], are not applicable. The only general result touching on infinite available topologies was given in [16], while some of the basic pre-requisitions such as the assumption of the compactness of $e_k(\mathcal{A}(t))(x)$ and the assumption of the strict convexity (namely, (3) of Assumption 1 in [16]) are not satisfied by our model (See Assumption 1 in [16]).

The proof of Theorem 3, presented in the next section, is on the basis of the spacial structures of $\Pi(\hat{m}, t_k)$ and properties of $\{t_k\}$.

5 Technical Proof

This section presents a complete proof of Theorem 3. We first give an equivalent formulation of the condition in Theorem 3.

Lemma 3 *The existence of $T \geq 0$ such that for all $t^0 \geq 0$, the union of graph $\mathcal{G}^0(t)$ across interval $[t^0, t^0 + T]$ contains a spanning tree, is equivalent to the condition that there exists $e \in \mathbb{Z}_+$ and $\hat{\tau}_v > 0$ with the following property:*

For any $U_k = \{t_{ke+1}, t_{ke+2}, \dots, t_{(k+1)e}\}$, there exists a subset of U_k , denoted by V_k , such that the union of $\mathcal{G}^0(t)$ on V_k contains a spanning tree and for any $t_s \in V_k$, $t_{s+1} - t_s \geq \hat{\tau}_v$.

Proof: The sufficiency is rather straightforward and only the necessity is proved. By Lemma 2, there exists $p \in \mathbb{Z}_+$ such that for any $k \in \mathbb{Z}_+$, $t_{k+p} - t_k \geq T$, such as $p = \tilde{m}(\lfloor \frac{T}{\tilde{\tau}_u} \rfloor + 2)$ (For any given $i \in \mathcal{I}_n$, there exists k' such that $t_{k'}^i \leq t_k < t_{k'+1}^i$, and $t_{k'+\frac{p}{\tilde{m}}}^i - t_{k'+1}^i \geq \tilde{\tau}_u(\frac{p}{\tilde{m}} - 1) = \tilde{\tau}_u(\lfloor \frac{T}{\tilde{\tau}_u} \rfloor + 1) > T$. We claim that $t_{k+p} \geq t_{k'+\frac{p}{\tilde{m}}}^i$. If not, $|(t_{k'}^i, t_{k'+\frac{p}{\tilde{m}}}^i) \cap \{t_k\}| \geq |(t_k, t_{k+p}) \cap \{t_k\}| = p + 1$. But by Lemma 2, $|(t_{k'}^i, t_{k'+\frac{p}{\tilde{m}}}^i) \cap \{t_k\}| \leq \frac{p}{\tilde{m}}\tilde{m} = p$, which is a contradiction. Therefore $t_{k+p} - t_k \geq t_{k'+\frac{p}{\tilde{m}}}^i - t_k > t_{k'+\frac{p}{\tilde{m}}}^i - t_{k'+1}^i > T$). Let $e = p + 2\tilde{m}$. We consider U_k . Obviously $t_{ke+\tilde{m}+p} - t_{ke+\tilde{m}} \geq T$. Therefore the union graph $\mathcal{G}^0(t)$ on $\{t_{ke+\tilde{m}}, \dots, t_{ke+\tilde{m}+p}\}$ contains a spanning tree. Let the edge set of the spanning tree be \mathcal{E} . If $(v_j, v_i) \in \mathcal{E}$, there exists k' such that $ke+\tilde{m} \leq k' \leq ke+\tilde{m}+p$ and $(v_j, v_i) \in \mathcal{G}^0(t_{k'})$. For $t_{k'}$, there exists k'' such that $t_{k''}^i \leq t_{k'} < t_{k''+1}^i$, and thus (v_j, v_i) is an edge of the graph $\mathcal{G}^0(t)$, $t \in [t_{k''}^i, t_{k''+1}^i)$.

We claim that $t_{k''}^i \geq t_{k_{e+1}}$ and $t_{k''+1}^i \leq t_{(k+1)e}$. In fact, if $t_{k''}^i < t_{k_{e+1}}$, then $|(t_{k''}^i, t_{k''+1}^i) \cap \{t_k\}| > |[t_{k_{e+1}}, t_{k'}] \cap \{t_k\}| \geq |[t_{k_{e+1}}, t_{k_{e+\tilde{m}}}] \cap \{t_k\}| = \tilde{m}$, which contradicts Lemma 2. And if $t_{k''+1}^i > t_{(k+1)e}$, then $|(t_{k''}^i, t_{k''+1}^i) \cap \{t_k\}| \geq |[t_{k'}, t_{(k+1)e}] \cap \{t_k\}| \geq |[t_{k_{e+\tilde{m}+p}}, t_{(k+1)e}] \cap \{t_k\}| = \tilde{m} + 1$, which also contradicts Lemma 2.

Since $\{t_k\} \cap [t_{k''}^i, t_{k''+1}^i)$ has at most \tilde{m} elements, there exists $t_s \in \{t_k\} \cap [t_{k''}^i, t_{k''+1}^i) \subset U_k$ such that $t_{s+1} - t_s \geq \frac{t_{k''+1}^i - t_{k''}^i}{\tilde{m}} \geq \frac{\tilde{m}}{\tilde{m}}$. Let $\hat{\tau}_v = \frac{\tilde{m}}{\tilde{m}}$. If (v_j, v_i) takes very possible edge in \mathcal{E} , we obtain all possible t_s s. Let the set of them be V_k . Then V_k has the aforementioned property and the necessity is proved.

Let A, B be $r \times r$ stochastic matrices and let $\delta(A) = \max_j \max_{i_1, i_2} |a_{i_1 j} - a_{i_2 j}|$. Thus $\delta(A)$ measures how different the rows of A are. If the rows of A are identical, $\delta(A) = 0$ and conversely. We say that A, B are of the same type, $A \sim B$, if they have zero elements and positive elements in the same place. Let $\mathbf{n}(r)$ be the number of different types of all $r \times r$ SIA matrices. Define $\lambda(A) = 1 - \min_{i_1, i_2} \sum_j \min(a_{i_1, j}, a_{i_2, j})$. If $\lambda(A) < 1$ we call A a *scrambling matrix*.

Lemma 4 ([33], Lemma 2) *For any stochastic matrices A_1, A_2, \dots, A_k , $k > 0$,*

$$\delta(A_1 A_2 \cdots A_k) \leq \prod_{i=1}^k \lambda(A_i).$$

The next lemma generalizes the result of Lemma 4 in [33].

Lemma 5 *Let A_1, A_2, \dots, A_k (repetitions permitted) be $r \times r$ SIA matrices with the property that for any $1 \leq k_1 < k_2 \leq k$, $\prod_{i=k_1}^{k_2} A_i$ is SIA. If $k > \mathbf{n}(r)$, then $\prod_{i=1}^k A_i$ is a scrambling matrix.*

Proof: Since $k > \mathbf{n}(r)$, there exist $k_1 < k_2$, such that $\prod_{i=1}^{k_1} A_i \sim \prod_{i=1}^{k_2} A_i$. It follows from Lemma 10 and $\prod_{i=k_1+1}^{k_2} A_i$ being SIA that $\prod_{i=1}^{k_1} A_i$ is a scrambling matrix. Thus $\prod_{i=1}^k A_i$ is also a scrambling matrix by Lemma 9.

To investigate the properties of matrices in $\Pi(\hat{m}, t_k)$, we introduce some notations. Let $\mathcal{F}(A) = \sum_{i=1}^{\hat{m}} A_{1i}$, where $A = [A_{ij}]$ is an $\hat{m} \times \hat{m}$ block matrix and $A_{ij} \in \mathbb{R}^{n \times n}$. Let Γ_s denote the set of square matrices such that $A \in \Gamma_s$ if and only if $\mathcal{G}(A)$ contains a spanning tree with the property that the root vertex of the spanning tree has a self-loop in $\mathcal{G}(A)$.

Lemma 6 *Let A be a stochastic matrix. If $A \in \Gamma_s$, then A is SIA.*

Proof: Since A is stochastic, $A\mathbf{1} = \mathbf{1}$ and $\rho(A) = 1$. We assume that there exists a spanning tree with vertex v_{k_1} as its root and $(v_{k_1}, v_{k_1}) \in \mathcal{E}(\mathcal{G}(A))$. Suppose that subgraph \mathcal{G}_s induced by $v_{k_1}, v_{k_2}, \dots, v_{k_s}$ ($1 \leq s \leq n$) is the

maximal induced subgraph that is strongly connected. Let the vertices in $\mathcal{V}(\mathcal{G}(A)) \setminus \{v_{k_1}, v_{k_2}, \dots, v_{k_s}\}$ be $v_{k_{s+1}}, \dots, v_{k_n}$. Then there exists a permutation matrix T such that

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = T \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}.$$

Therefore,

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$\text{where } A_{11} = \begin{bmatrix} a_{k_1, k_1} & \cdots & a_{k_1, k_s} \\ \vdots & \ddots & \vdots \\ a_{k_s, k_1} & \cdots & a_{k_s, k_s} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{k_1, k_{s+1}} & \cdots & a_{k_1, k_n} \\ \vdots & \ddots & \vdots \\ a_{k_s, k_{s+1}} & \cdots & a_{k_s, k_n} \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} a_{k_{s+1}, k_1} & \cdots & a_{k_{s+1}, k_s} \\ \vdots & \ddots & \vdots \\ a_{k_n, k_1} & \cdots & a_{k_n, k_s} \end{bmatrix} \quad \text{and} \quad A_{22} = \begin{bmatrix} a_{k_{s+1}, k_{s+1}} & \cdots & a_{k_{s+1}, k_n} \\ \vdots & \ddots & \vdots \\ a_{k_n, k_{s+1}} & \cdots & a_{k_n, k_n} \end{bmatrix}.$$

By the assumption that \mathcal{G}_s is maximal, $A_{12} = 0$. Since \mathcal{G}_s is strongly connected, A_{11} is irreducible. And from $(v_{k_1}, v_{k_1}) \in \mathcal{E}(\mathcal{G}(A))$, $a_{k_1, k_1} > 0$. By Lemma 11, A_{11} is primitive (Its definition is provided in the Appendix) and thus 1 is the only eigenvalue of A_{11} with maximum modulus. Since 1 is an eigenvalue of A_{11} , by Lemma 13, 1 is not an eigenvalue of A_{22} . On the other hand, let $\rho(A_{22})$ denote the spectral radius of A_{22} . By Geršgorin disc theorem, $\rho(A_{22}) \leq 1$ and by Lemma 14, $\rho(A)$ is an eigenvalue of A . It follows that $\rho(A_{22}) < 1$. Consequently, 1 is the only eigenvalue of A with maximum modulus. By Lemma 13 and that $\rho(A) = 1$, it is easy to check that A satisfies the conditions of Lemma 12. Let $f^T A = f^T$ such that $f^T \mathbf{1} = 1$. Then $\lim_{k \rightarrow \infty} A^k = \mathbf{1} f^T$.

Lemma 7 Let $A_1, \dots, A_{\hat{m}}$ be $n \times n$ nonnegative matrices and let

$$\begin{aligned}
M_0 &= \begin{bmatrix} I & & & & & \\ I & & & & & \mathbf{0} \\ & I & & & & \\ & & \ddots & & & \\ \mathbf{0} & & & I & & 0 \end{bmatrix}_{\hat{m}n \times \hat{m}n}, M_1 = \begin{bmatrix} I + A_1 & A_2 & \cdots & A_{\hat{m}-1} & A_{\hat{m}} \\ & I & & & \\ & & I & & \mathbf{0} \\ & & & \ddots & \\ \mathbf{0} & & & & I \end{bmatrix}, \\
M_2 &= \begin{bmatrix} I + A_1 & \cdots & A_{m-2} & A_{\hat{m}-1} & A_{\hat{m}} \\ & I & & & \\ I & & & & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & & & I & \end{bmatrix}, \dots, M_{\hat{m}-1} = \begin{bmatrix} I + A_1 & A_2 & A_3 & \cdots & A_{\hat{m}} \\ & I & & & \\ & & I & & \mathbf{0} \\ & & & \vdots & \\ & & & & I \end{bmatrix}. \tag{9}
\end{aligned}$$

For any $i \in \{1, 2, \dots, \hat{m} - 1\}$, if $\mathcal{G}(\mathcal{F}(M_i))$ contains a spanning tree, then $M_i \in \Gamma_s$.

Proof: Let $N = M_1 - M_0$. Then we have $M_i = M_0^i + N$. Let $\mathcal{G}(M_i)$, $\mathcal{G}(M_0^i)$, and $\mathcal{G}(N)$ be with the same vertex set $\{u_1, u_2, \dots, u_{\hat{m}n}\}$ and let $\mathcal{E}(\mathcal{G}(\mathcal{F}(M_i))) = \{v_1, v_2, \dots, v_n\}$. Apparently $\mathcal{E}(\mathcal{G}(M_i)) = \mathcal{E}(\mathcal{G}(M_0^i)) \cup \mathcal{E}(\mathcal{G}(N))$. We first investigate the edge sets $\mathcal{E}(\mathcal{G}(M_0^i))$ and $\mathcal{E}(\mathcal{G}(N))$. For any $j \in \mathcal{I}_n$, $0 \leq i \leq \hat{m} - 1$,

$$\begin{aligned}
&\{(u_j, u_{n+j}), (u_j, u_{2n+j}), \dots, (u_j, u_{in+j}), \\
&(u_{n+j}, u_{(i+1)n+j}), (u_{2n+j}, u_{(i+2)n+j}), \dots, (u_{(\hat{m}-i-1)n+j}, u_{(\hat{m}-1)n+j})\} \subset \mathcal{E}(\mathcal{G}(M_0^i)).
\end{aligned}$$

Therefore for any $j \in \mathcal{I}_n$, there exist paths from u_j to $u_{n+j}, \dots, u_{(\hat{m}-1)n+j}$ in $\mathcal{E}(\mathcal{G}(M_0^i))$ (See Fig. 2). If there exists an edge $(v_j, v_k) \in \mathcal{E}(\mathcal{G}(\mathcal{F}(M_i)))$, then there exists $0 \leq s \leq \hat{m} - 1$ such that $(u_{j+sn}, u_k) \in \mathcal{E}(\mathcal{G}(N))$. Therefore there exists a path from u_j to u_k in $\mathcal{G}(M_i)$. It follows that if $\mathcal{G}(\mathcal{F}(M_i))$ contains a spanning tree with root vertex v_j , $\mathcal{G}(M_i)$ also contains a spanning tree with root vertex u_j . Since the entry in the j th row and the j th column of M_i is not less than 1, u_j has a self-loop in $\mathcal{G}(M_i)$. Consequently $\mathcal{G}(M_i) \in \Gamma_s$. Let $\Pi(\hat{m})$ be the set of matrices

$$\begin{bmatrix} e^{-h}I + (1 - e^{-h})A_1 & (1 - e^{-h})A_2 & \cdots & (1 - e^{-h})A_{m-1} & (1 - e^{-h})A_{\hat{m}} \\ & I & & 0 & 0 \\ & 0 & I & \cdots & 0 \\ & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & I & 0 \end{bmatrix}_{\hat{m}n \times \hat{m}n},$$

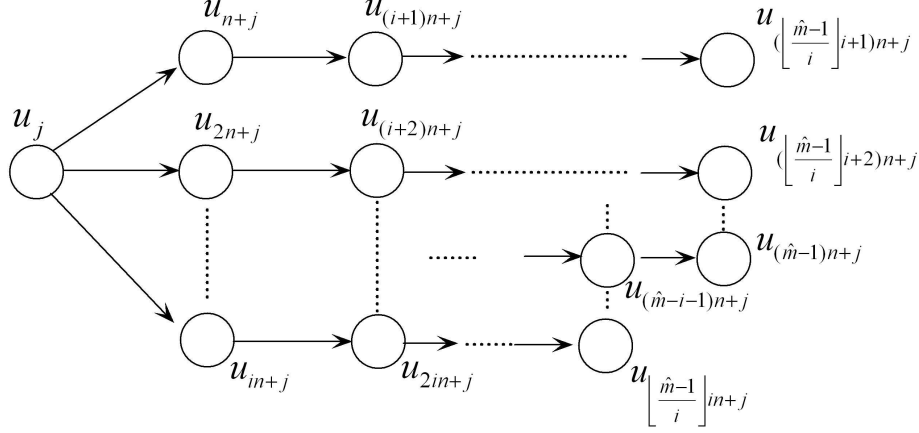


Fig. 2. Paths starting from u_j in $\mathcal{G}(M_0^i)$. If $\lfloor \frac{\hat{m}-1}{i} \rfloor i = \hat{m} - 1$, $u_{\lfloor \frac{\hat{m}-1}{i} \rfloor in+j}$ and $u_{(\hat{m}-1)n+j}$ are the same vertex.

where $0 \leq h \leq \hat{\tau}_u$, and there exists some $\mathbf{A}_0 \in \Gamma_{\mathbf{A}}$, such that $A_1, \dots, A_{\hat{m}} \in \Lambda(\mathbf{A}_0)$, and $A_1 + \dots + A_{\hat{m}} = \mathbf{A}_0$. Since $\Gamma_{\mathbf{A}}$ is compact and given any \mathbf{A}_0 , all possible choices of $A_1, \dots, A_{\hat{m}}$ are finite, $\Pi(\hat{m})$ is a compact set, and for any $k \in \mathbb{Z}_+$, $\Pi(\hat{m}, t_k) \subset \Pi(\hat{m})$.

Let $\Pi_0 = \{\prod_{i=1}^e \pi(h_i, A_{i1}, A_{i2}, \dots, A_{i\hat{m}}) : \pi(h_i, \cdot) \in \Pi(\hat{m}), \text{ and there exists a subset of } \{1, 2, \dots, e\}, \text{ denoted by } \mathcal{H}, \text{ such that for any } s \in \mathcal{H}, h_s \geq \hat{\tau}_v, \text{ and } \mathcal{G}(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij}) \text{ contains a spanning tree}\}$, where e and $\hat{\tau}_v$ are defined in Lemma 3.

Lemma 8 Π_0 is a compact set and for any $\pi \in \Pi_0$, π is SIA. And for any $k > \mathbf{n}(n\hat{m})$, if $\pi_1, \dots, \pi_k \in \Pi_0$, then $\prod_{i=1}^k \pi_i$ is a scrambling matrix and there exists a $\hat{\lambda}(k) \in [0, 1)$, such that $\lambda(\prod_{i=1}^k \pi_i) \leq \hat{\lambda}(k)$.

Proof: The compactness of Π_0 follows from the following facts

- (1) $\Pi(\hat{m})$ is a compact set;
- (2) All possible choices of \mathcal{H} are finite;
- (3) $h_s \in [\hat{\tau}_v, \hat{\tau}_u]$, which is a compact set;
- (4) All possible choices of the spanning tree are finite;
- (5) Given the spanning tree and \mathcal{H} , let $\Pi_1 = \{\prod_{i=1}^e \pi(h_i, A_{i1}, \dots, A_{i\hat{m}}) : \pi(h_i, \cdot) \in \Pi(\hat{m}), \text{ and for any } s \in \mathcal{H}, h_s \in [\hat{\tau}_v, \hat{\tau}_u], \text{ and } \mathcal{G}(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij}) \text{ contains the specified spanning tree}\}$ is compact.

We only prove the fact 5)

Let $|\mathcal{H}| = q$, $q \leq e$. Since the product of e matrices is continuous, it suffices to prove that $\Pi_2 = \{[\pi(h_1, A_{11}, \dots, A_{1\hat{m}}), \dots, \pi(h_e, A_{e1}, \dots, A_{e\hat{m}})] : \pi(h_i, \cdot) \in \Pi(\hat{m}), \text{ and for any } s \in \mathcal{H}, h_s \in [\hat{\tau}_v, \hat{\tau}_u], \text{ and } \mathcal{G}(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij}) \text{ contains the specified spanning tree}\}$ is compact. Since $\pi(h_i, \cdot) \in \Pi(\hat{m})$, which is compact,

it suffices to prove that $\Pi_3 = \{[\pi(h_1, A_{11}, \dots, A_{1\hat{m}}), \dots, \pi(h_q, A_q, \dots, A_{q\hat{m}})] : \pi(h_i, \cdot) \in \Pi(\hat{m}), h_i \in [\hat{\tau}_v, \hat{\tau}_u], i = 1, 2, \dots, q, \text{ and } \mathcal{G}(\sum_{i=1}^q \sum_{j=1}^{\hat{m}} A_{ij}) \text{ contains the specified spanning tree } \}$ is compact.

Let $B^{(p)} = [b_{ij}^{(p)}] \in \Pi_3, p = 1, 2, \dots$ be a sequence of matrices, and $\lim_{p \rightarrow \infty} B^{(p)} = B = [b_{ij}]$. Since Π_3 is a bounded set, it suffices to prove that $B \in \Pi_3$.

It is clear that $\lim_{p \rightarrow \infty} b_{ij}^{(p)} = b_{ij}$. For any $i, j \in \mathcal{I}_n$, if $b_{ij}^{(p)} \neq 0$, then by the definition of $\Gamma_{\mathbf{A}}, b_{ij}^{(p)} > (1 - e^{-\hat{\tau}_v}) \frac{\check{\mathbf{a}}}{(n-1)\check{\mathbf{a}}}$. Therefore if $b_{ij} > 0$, then $\lim_{p \rightarrow \infty} b_{ij}^{(p)} = b_{ij} \geq (1 - e^{-\hat{\tau}_v}) \frac{\check{\mathbf{a}}}{(n-1)\check{\mathbf{a}}} > 0$, and thus there exists $P_{ij} \in \mathbb{Z}_+$ such that for any $p > P_{ij}, b_{ij}^{(p)} > 0$. If $b_{ij} = 0$, from $\lim_{p \rightarrow \infty} b_{ij}^{(p)} = b_{ij} = 0$, there exists $P_{ij} \in \mathbb{Z}_+$ such that for any $p > P_{ij}, b_{ij}^{(p)} = 0$. Let $P = \max_{ij} P_{ij}$, and then for any $p > P, B^{(p)} \sim B$.

Let $B = [B_1, B_2, \dots, B_q], B_i \in \mathbb{R}^{\hat{m}n \times \hat{m}n}, 1 \leq i \leq q$. From the compactness of $\Pi(\hat{m}), B_i \in \Pi(\hat{m})$. If $B_i = \pi(h_{b_i}, B_{i1}, \dots, B_{i\hat{m}}), 1 \leq i \leq q$, then $h_{b_i} \in [\hat{\tau}_v, \hat{\tau}_u]$. And from $B^{(p)} \sim B, \mathcal{G}(\sum_{i=1}^q \sum_{j=1}^{\hat{m}} B_{ij})$ contains the spanning tree. To conclude, $B \in \Pi_3$.

Next, we prove that for any $\pi \in \Pi_0, \pi$ is SIA. Let $\pi = \sum_{i=1}^e \pi(h_i, A_{i1}, \dots, A_{i\hat{m}})$ and \mathcal{H} be the associated subset of $\{1, 2, \dots, e\}$ defined in the definition of Π_0 . Let M_0 be the same as in Lemma 7 and let

$$D_i = \begin{bmatrix} A_{i1} & A_{i2} & \cdots & A_{i\hat{m}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

$$\begin{aligned} \prod_{i=1}^e \pi(h_i, \cdot) &\geq \prod_{i=1}^e (e^{-h_i} M_0 + (1 - e^{-h_i}) D_i) \\ &\geq e^{-e\hat{\tau}_u} M_0^e + e^{-(e-1)\hat{\tau}_u} \sum_{i=1}^e (1 - e^{-h_i}) (M_0)^{i-1} D_i (M_0)^{e-i} \\ &\geq e^{-e\hat{\tau}_u} M_0^e + e^{-(e-1)\hat{\tau}_u} \sum_{i \in \mathcal{H}} (1 - e^{-h_i}) D_i M_0^{e-i} \\ &\geq e^{-e\hat{\tau}_u} M_0^e + e^{-(e-1)\hat{\tau}_u} (1 - e^{-\hat{\tau}_v}) \sum_{i \in \mathcal{H}} D_i M_0^{e-i} \\ &\geq \min\{e^{-e\hat{\tau}_u}, e^{-(e-1)\hat{\tau}_u} (1 - e^{-\hat{\tau}_v})\} (M_0^e + \sum_{i \in \mathcal{H}} D_i M_0^{e-i}). \end{aligned}$$

The second inequality follows from $0 < h_i \leq \hat{\tau}_u$, the third follows from $M_0^{i-1} D_i \geq D_i$, and the fourth follows from $h_i \geq \hat{\tau}_v, i \in \mathcal{H}$.

From $\mathcal{F}(D_i M_0^{e-i}) = \mathcal{F}(D_i)$, we have $\mathcal{F}(\sum_{i \in \mathcal{H}} D_i M_0^{e-i}) = \sum_{i \in \mathcal{H}} \mathcal{F}(D_i M_0^{e-i}) = \sum_{i \in \mathcal{H}} \mathcal{F}(D_i) = \sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij}$. Since $\mathcal{G}(\sum_{i \in \mathcal{H}} \sum_{j=1}^{\hat{m}} A_{ij})$ contains a spanning tree, $\mathcal{G}(\mathcal{F}(\sum_{i \in \mathcal{H}} D_i M_0^{e-i}))$ also contains a spanning tree. Let $N \in \mathbb{R}^{\hat{m}n \times \hat{m}n}$ be with the same first n rows as $\sum_{i \in \mathcal{H}} D_i M_0^{e-i}$ and all other rows are zeros. Then $\prod_{i=1}^e \pi(h_i, \cdot) \geq \min\{e^{-e\hat{\tau}_u}, e^{-(e-1)\hat{\tau}_u}(1 - e^{-\hat{\tau}_v})\}(M_0^e + N)$. By Lemma 7, $M_0^e + N \in \Gamma_s$, and thus $\pi \in \Gamma_s$. π is also stochastic, and therefore, by Lemma 6, π is SIA.

With the same arguments, we can conclude that for any $1 \leq k_1 < k_2 \leq k$, $\prod_{i=k_1}^{k_2} \pi_i$ is SIA (We only need to replace e by $(k_2 - k_1 + 1)e$ in the above arguments and \mathcal{H} be the index set associated to any π_i as defined in the definition of Π_0). By Lemma 5, for any $k > \mathbf{n}(n\hat{m})$, $\prod_{i=1}^k \pi_i$ is a scrambling matrix. Let

$$\hat{\lambda}(k) = \max_{\substack{\bar{\pi}_i \in \Pi_0 \\ 1 \leq i \leq k}} \lambda\left(\prod_{i=1}^k \bar{\pi}_i\right).$$

Since Π_0 is a compact set and $\lambda(\cdot)$ is continuous, $\hat{\lambda}(k)$ exists and $\hat{\lambda}(k) < 1$. Obviously $\lambda(\prod_{i=1}^k \pi_i) \leq \hat{\lambda}(k)$.

Proof of Theorem 3:

For any $k \in \mathbb{Z}_+$, let $\pi_k = \prod_{s=k}^{(k+1)e + \hat{m} - 2} \pi(\tau_s, \mathbf{A}_1(t_s), \mathbf{A}_2(t_s), \dots, \mathbf{A}_{\hat{m}}(t_s))$. By Lemma 3 and Proposition 2, $\pi_k \in \Pi_0$.

Let $p = \mathbf{n}(\hat{m}n) + 1$. For any $q > pe$, there exists $s \in \mathbb{Z}_+$ such that $q = spe + q'$, where $0 \leq q' < pe$. By Lemma 4,

$$\delta\left(\prod_{k=\hat{m}-1}^{q+\hat{m}-2} \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}}(t_k))\right) \leq \prod_{i=0}^{s-1} \lambda\left(\prod_{j=i\hat{p}}^{(i+1)\hat{p}-1} \pi_j\right) \leq (\hat{\lambda}(p))^s.$$

Therefore

$$\lim_{q \rightarrow \infty} \delta\left(\prod_{k=\hat{m}-1}^{q+\hat{m}-2} \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}}(t_k))\right) = 0,$$

which implies that there exists $a \in \mathbb{R}$ such that $\lim_{q \rightarrow \infty} \prod_{k=\hat{m}-1}^q \pi(\tau_k, \mathbf{A}_1(t_k), \dots, \mathbf{A}_{\hat{m}}(t_k)) = 1a$. And thus system (8) solves a consensus problem.

6 Simulations

In this section, we take some examples to illustrate the effectiveness of our results.

Example 2 (Fixed topology without time-delays) *Suppose that the sys-*

tem consists of 4 agents. Let $\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\check{\tau}_u = 0.2$ and $\hat{\tau}_u = 0.9$. And sup-

pose that there do not exist communication time-delays and each agent can get all its neighbors' states at its update times. For any $i \in \mathcal{I}_n$, $t_{k+1}^i - t_k^i$ is evenly distributed between 0.2 and 0.9. Since the communication topology has a spanning tree, the consensus is reachable under asynchronous consensus control strategy (1). Let initial state $x(0) = [5, 6, 7, 8]^T$. In the simulation experiment, the update times t_k^i , $i = 1, 2, 3, 4$, are randomly generated and independent of each other. The state trajectories of agents are shown in Fig. 4.

As a marked difference from the synchronous case, the final states of agents are dependent on the update times. We repeat the simulation experiment 100 times independently, and the final states of them are shown in Fig. 5.

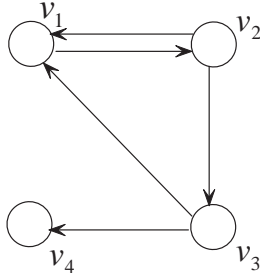


Fig. 3. $\mathcal{G}(\mathcal{A})$ in Example 2

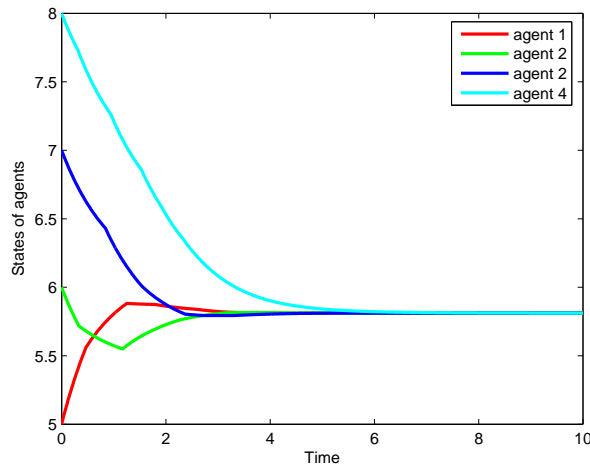


Fig. 4. State trajectories of agents in Example 2

Example 3 (Fixed topology with time-delays) We still consider the system in Example 2 and suppose that there exist communication time-delays bounded by τ_d and each agent can get all its neighbors' states at its update times. We let initial state $x(t) = [5, 6, 7, 8]$, $t \in [-\tau_d, 0]$. Fig. 6 and Fig. 7

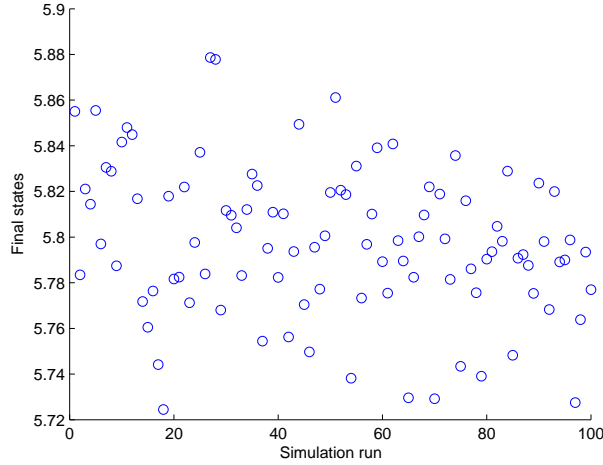


Fig. 5. Final states of different experiments in Example 2

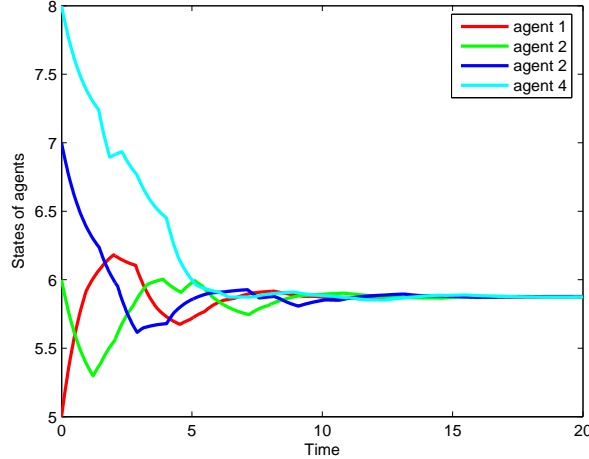


Fig. 6. State trajectories of agents with $\tau_d = 2$

show the state trajectories of agents with maximum communication time-delay $\tau_d = 2$ ($K = 10$) and $\tau_d = 10$ ($K = 50$) separately, where the time-delays are randomly generated. We can see that the system with $\tau_d = 2$ converges faster than the system with $\tau_d = 10$. If we adopt the-most-recent-data strategy, we can get better convergence rate. Fig. 8 and Fig. 9 show the state trajectories of agents under the-most-recent-data strategy with randomly generated time-delays bounded by $\tau_d = 2$ and $\tau_d = 10$ separately.

Example 4 (Switching topology with time-delays) We still consider the system consisting of 4 agents. Each agent's update intervals are evenly distributed between 0.2 and 0.9, and are randomly generated. Suppose that the maximum time-delay $\tau_d = 2$, and initial state $x(t) = [5, 6, 7, 8]$, $t \in [-\tau_d, 0]$.

We assume that the weight of each edge of the communication topology is 1

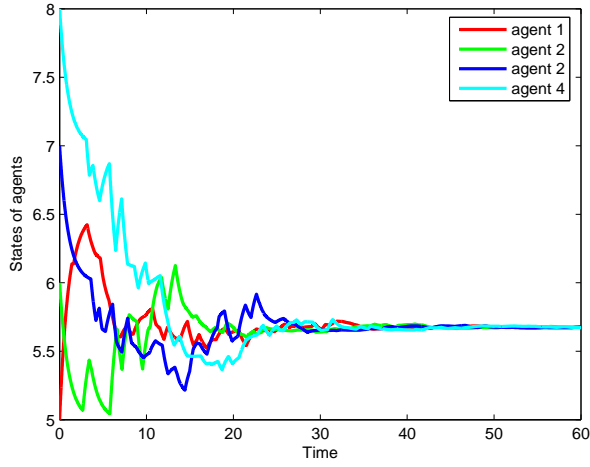


Fig. 7. State trajectories of agents with $\tau_d = 10$

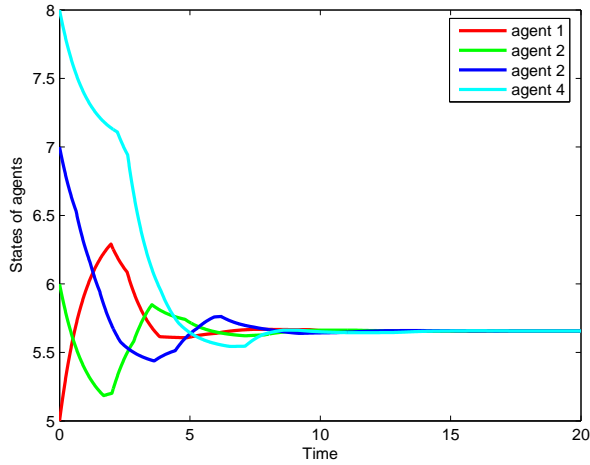


Fig. 8. State trajectories of agents by the-most-recent-data strategy with $\tau_d = 2$

and

- (1) agent 1 can get the state of agent 2 at update times t_{4k}^1 , $k \in \mathbb{Z}_+$, and can get the state of agent 3 at update times t_{4k+2}^1 , $k \in \mathbb{Z}_+$;
- (2) agent 2 can get the state of agent 1 at update times t_{4k+1}^2 , $k \in \mathbb{Z}_+$;
- (3) agent 3 can get the state of agent 2 at update times t_{4k+2}^3 , $k \in \mathbb{Z}_+$;
- (4) agent 4 can get the state of agent 3 at update times t_{4k+3}^4 , $k \in \mathbb{Z}_+$.

By Theorem 3, this system solves a consensus problem. The state trajectories of agents are shown in Fig. 10.

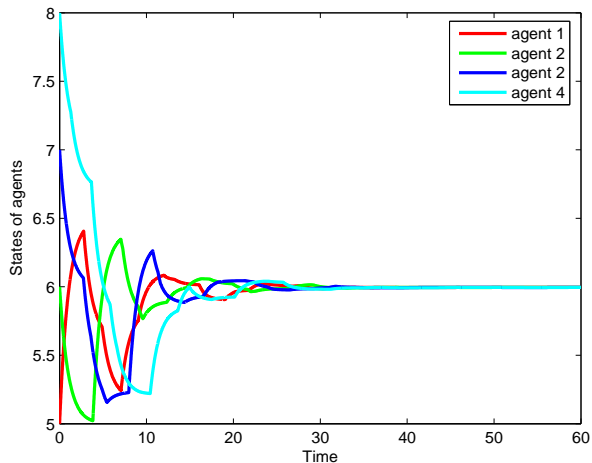


Fig. 9. State trajectories of agents by the-most-recent-data strategy with $\tau_d = 10$

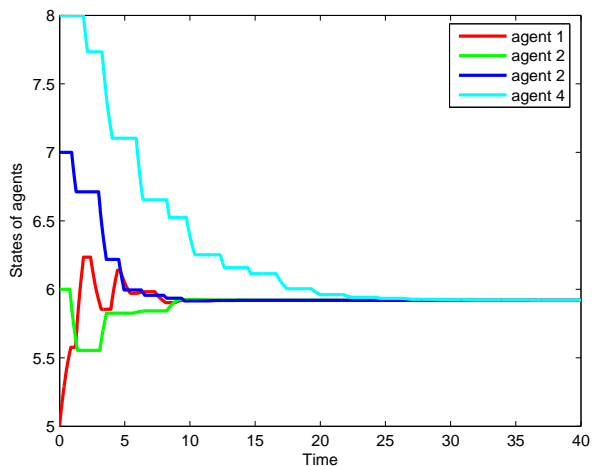


Fig. 10. State trajectories of agents in Example 4

7 Conclusion

We presented an asynchronous consensus control strategy, which is of obvious applications in realistic networks. By employing the tools from the nonnegative matrix theory and graph theory, we performed the convergence analysis of our consensus algorithm. The introduction of communication topology $\mathcal{G}^0(t)$ facilitated our analysis and it established a connection between the actual communication topology and our control strategy, and can be viewed as the estimation of the actual topology. Examples were provided to demonstrate the effectiveness of our theoretical results.

A Lemmas

Lemma 9 ([33], Lemma 1) *If one or more matrices in a product of stochastic matrices is scrambling, so is the product.*

Lemma 10 ([33], Lemma 3) *Let A_1, A_2 be stochastic matrices. If A_2 is an SIA matrix and $A_1 A_2 \sim A_1$, then A_1 is a scrambling matrix.*

Definition 4 *A nonnegative matrix $A \in \mathbb{R}^{n \times n}$ is said to be primitive if it is irreducible and has only one eigenvalue of maximum modulus.*

Lemma 11 ([30], pp.511, Corollary 8.4.8; pp.522, Problem 5) *Let $A \in \mathbb{R}^{n \times n}$ be nonnegative and irreducible. If at least one main diagonal entry is positive, then A is primitive.*

Lemma 12 ([30], pp.497, Lemma 8.2.7) *Let $A \in \mathbb{R}^{n \times n}$ be given, let $\lambda \in \mathbb{R}$ be given, and suppose ξ and ζ are vectors such that*

- (1) $A\xi = \lambda\xi$;
- (2) $A^T\zeta = \lambda\zeta$;
- (3) $\xi^T\zeta = 1$;
- (4) λ is an eigenvalue of A with geometric multiplicity 1;
- (5) $|\lambda| = \rho(A) > 0$; and
- (6) λ is the only eigenvalue of A with modulus $\rho(A)$,

where $\rho(A)$ is the spectral radius of A . Define $L = \xi\zeta^T$. Then $(\lambda^{-1}A)^k = L + (\lambda^{-1}A - L)^k \rightarrow L$ as $k \rightarrow \infty$.

Lemma 13 ([8], Lemma 3.4) *Let A be a stochastic matrix. $\mathcal{G}(A)$ has a spanning tree if and only if the eigenvalue 1 of A has algebraic multiplicity equal to one.*

Lemma 14 ([30], pp. 503, Theorem 8.3.1) *If $A \in \mathbb{R}^{n \times n}$ and $A \geq 0$, then $\rho(A)$ is an eigenvalue of A and there is a nonnegative vector $f \geq 0$, $f \neq 0$, such that $Af = \rho(A)f$.*

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