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Atangana-Baleanu derivative with fractional order applied to the model of groundwater within an unconfined aquifer

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Abstract

The power law has been used to construct the derivative with fractional order in Caputo and Riemann-Liouville sense, if we viewed them as a convolution. However, it is not always possible to find the power law behaviour in nature. In 2016 Abdon Atangana and Dumitru Baleanu proposed a derivative that is based upon the generalized Mittag-Leffler function, since the Mittag-Leffler function is more suitable in expressing nature than power function. In this paper, we applied their new finding to the model of groundwater flowing within an unconfined aquifer.

Keywords: Atangana-Baleanu derivatives, Laplace transforms, groundwater flow, unconfined aquifer.
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1. Introduction

Recently to step forward within the field of fractional calculus, Caputo and Fabrizio proposed a derivative for which the exponential decay law is used. Some benefits of this derivative over existing ones are that, the derivative can be used when observed the exponential law in nature; this derivative is free from singularity since the kernel is based on exponential function. However, there were some serious issues that were pointed out against this new finding. First the kernel used was not non-local; second the anti-derivative associated is just the average of the function and its integral [1, 2, 4, 8–11]. Therefore it was concluded that, the operator was rather a filter than a fractional derivative. However we should point out that many works were

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done around this new finding with great success [1, 2, 4, 10, 11]. This derivative was used in mechanical engineering, in groundwater studies, in thermal science, in diffusion model and other [1, 2, 4, 10]. Another important issue was pointed out by Atangana and Koca in their paper published in Chaos soliton and fractal were they said the solution of equation ${}_{0}^{CF}D_{t}^{\alpha}y\,(t)=y(t)$ was not a special function but an exponential function. To solve this problem, Atangana and Baleanu suggested a much better version of a derivative with no singular kernel that satisfy the issues pointed out against that of Caputo-Fabrizio [6]. Their derivative is based upon the well-known generalized Mittag-Leffler function. It can be recalled that, the Mittag-Leffler function has been introduced to provide response to conventional question of complex analysis, in particular to portray the procedure of the analytic continuation of power law series outside the disc of their convergence [3, 6]. Later on, as the investigation stepped a head, theoretical applications to the study of integral equations, as well as more practical applications to modelling non-standard processes have been established. The significance of the Mittag-Leffler function was re-discovered while its relationship to fractional calculus was completely comprehended. Due to the impact of this function to modelling complex problems, we apply the Atangana-Baleanu to the model of groundwater flowing within an unconfined aquifer. It is important to point out that, the study of groundwater is far more complex than it is studied nowadays, first the aquifer's properties within which the flow is taking place is not well known as it is complex. Therefore the model describing this system needs more than the local derivative. The model of groundwater flow based on the classical derivative is given as:

$$T\left[\frac{\partial^2 h(r,t)}{\partial r^2}+\frac{\partial h(r,t)}{r\partial r}\right]=S\frac{\partial h(r,t)}{\partial t}+DS_y\int_0^t\frac{\partial h(r,x)}{\partial x}exp\left[-D(t-x)\right]dx, \tag{1.1}$$

where T is the transmissivity of the aquifer, S the capacity of the aquifer to store water, h is the drawdown that can be expressed as hydraulic heads. In this paper, we will replace the time derivative with the Atangana-Baleanu time fractional derivative. First we present some useful properties of the new derivatives in the following section

1.1. Atangana-Baleanu derivatives with fractional order

In this section, we show the definitions of the new fractional derivatives with no singular and nonlocal kernel that were suggested by Atangana and Baleanu in their work published in press in thermal science journal [3, 6].

Definition 1.1. *Let $f\in H^1(a,\ b)\,,\ b>a,\ \alpha\in[0,\ 1]$ and not necessarily differentiable then, the definition of the new fractional derivative (Atangana-Baleanu fractional derivative in Riemann-Liouville sense) is given as* [3]:

$${}_{a}^{ABR}D_{t}^{\alpha}\left(f\left(t\right)\right)=\frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f\left(x\right)E_{\acute{\alpha}}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right]dx. \tag{1.2}$$

Definition 1.2 ([6])**.** *Let $f\in H^1(a,\ b)\,,\ b>a,\ \alpha\in[0,\ 1]$ then, the definition of the new fractional derivative (Atangana-Baleanu derivative in Caputo sense) is given as:*

$${}_{a}^{ABC}D_{t}^{\alpha}\left(f\left(t\right)\right)=\frac{B(\alpha)}{1-\alpha}\int_a^t f'\left(x\right)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right]dx. \tag{1.3}$$

They suggested that B has the same properties as in Caputo and Fabrizio case. The above definition will be helpful to real world problem and also will have great advantage when using Laplace transform to solve some physical problems.

Definition 1.3 ([3])**.** The fractional integral associated with the new fractional derivative with non-local kernel (Atangana-Baleanu fractional integral) is defined as:

$${}_{a}^{AB}I_{t}^{\alpha}\left[f\left(t\right)\right]=\frac{1-\alpha}{B(\alpha)}f\left(t\right)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f\left(j\right)(t-j)^{\alpha-1}dj. \tag{1.4}$$

They found that, when alpha was zero they recovered the initial function and if also alpha is 1, they obtained the ordinary integral. In addition to this, they computed the Laplace transform of both derivative and obtained the following [6]

$$\mathcal{L}\left\{{}_0^{ABR}D_t^{\alpha}\left(f\left(t\right)\right)\right\}(p)=\frac{B\left(\alpha\right)}{1-\alpha}\frac{p^{\alpha}F(p)}{p^{\alpha}+\frac{\alpha}{1-\acute{a}}} \tag{1.5}$$

and

$$\mathcal{L}\left\{{}_0^{ABC}D_t^{\alpha}\left(f\left(t\right)\right)\right\}(p)=\frac{B\left(\alpha\right)}{1-\alpha}\frac{F(p)p^{\alpha}-p^{\alpha-1}f(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{1.6}$$

2. Groundwater model in unconfined with Atangana-Baleanu derivative

When comparing the models based on power and local derivative with observed fact, the results let no doubt that the concept of power law and local derivative are unable to express accurately the movement of subsurface water within the geological formation known as unconfined aquifer. It is therefore important to note that, when the local derivative and the power law derivatives relax then Mittag-Leffler law can be used to further the investigation, it was proven that, the concept of Mittag-Leffler function was introduced to stipulate reaction to orthodox question of complex investigation, especially to represent the technique of the analytic extension of power law series outside the disc of their merging. To this end in this paper we replace the conventional time derivative in equation (1.1) by the Atangana-Baleanu time fractional derivative in Caputo sense. Therefore the new model of groundwater flow within subsurface geological formation called unconfined aquifer is give as [3, 5, 7, 12, 13]:

$$\begin{aligned} T\left[\frac{\partial^2 h(r,t)}{\partial r^2}+\frac{\partial h(r,t)}{r\partial r}\right]=&S\frac{AB\left(\alpha\right)}{1-\alpha}\int_0^t\frac{\partial h(r,x)}{\partial x}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]\\ &+DS_y\int_0^t\frac{\partial h(r,x)}{\partial x}exp\left[-D(t-x)\right]dx. \end{aligned} \tag{2.1}$$

In the next subsection, we present some analytical techniques to solve the above equation; some numerical approximation will also be used to present the solution numerically.

2.1. Analytical methods

In this subsection, two different approach will be used to solve equation (2.1). The method based on separation of variable and the method based on Laplace transform operator. Employing the method of separation of variable on equation (2.1) yields:

$$T\left[\frac{\partial^2 h_1(r)}{\partial r^2}+\frac{\partial h_1(r)}{r\partial r}\right]=-\lambda^2 h_1(r) \tag{2.2}$$

and

$$S\frac{AB\left(\alpha\right)}{1-\alpha}\int_0^t\frac{\partial h_2(x)}{\partial x}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]+DS_y\int_0^t\frac{\partial h_2(x)}{\partial x}exp\left[-D(t-x)\right]dx=-\lambda^2 h_2(t). \tag{2.3}$$

We will solve the above equations using the Laplace transform operator. First we apply the Laplace transform on both sides of the below equation

$$S\frac{AB\left(\alpha\right)}{1-\alpha}\int_0^t\frac{\partial h_2(x)}{\partial x}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-x)^{\alpha}\right]+DS_y\int_0^t\frac{\partial h_2(x)}{\partial x}exp\left[-D(t-x)\right]dx=-\lambda^2 h_2(t). \tag{2.4}$$

To obtain

$$H\left(p\right)=\left\{\frac{a\frac{p^{\alpha-1}}{p^{\alpha}+c}+\frac{b}{p+D}}{a\frac{p^{\alpha}}{p^{\alpha}+c}+\frac{bp}{p+D}+\lambda^2}\right\}h_1\left(0\right),\ a=S\frac{AB(\alpha)}{1-\alpha},\ b=S_yD,\ c=\frac{\alpha}{1-\alpha}. \tag{2.5}$$

The above can be reformulated as follows:

$$H\left(p\right)=\frac{(a+b)p^{\alpha}+aDp^{\alpha-1}+bc}{\left(a+b\right)p^{\alpha+1}+\left(aD+\lambda^{2}\right)p^{\alpha}+\left(Dc+c\lambda^{2}\right)p+cD\lambda^{2}}h\left(0\right). \tag{2.6}$$

To obtain the exact solution, we apply the inverse Laplace transform on both sides to obtain

$$h_{1}\left(t\lambda\right)=L^{-1}\left\{\frac{(a+b)p^{\alpha}+aDp^{\alpha-1}+bc}{\left(a+b\right)p^{\alpha+1}+\left(aD+\lambda^{2}\right)p^{\alpha}+\left(Dc+c\lambda^{2}\right)p+cD\lambda^{2}}h\left(0\right)\right\}. \tag{2.7}$$

The second equation is also solved via Laplace transform method. Thus applying the Laplace transform on both sides of the below equation,

$$T\left[\frac{\partial^{2}h_{2}(r)}{\partial r^{2}}+\frac{\partial h_{2}(r)}{r\partial r}\right]=-\lambda^{2}h_{2}(r). \tag{2.8}$$

The exact solution of the above equation is given as

$$h_{2}\left(r\right)=J_{0}(r\lambda)=\sum_{j=0}^{\infty}\frac{(-1)^{j}}{j!\Gamma\left(n+\alpha+1\right)}\left(\frac{r\lambda}{2}\right)^{2j+\alpha}. \tag{2.9}$$

Therefore the solution of the modified equation is given in series form as

$$h\left(r,t\right)=\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j}}{j!\Gamma\left(n+\alpha+1\right)}\left(\frac{r\lambda_{n}}{2}\right)^{2j+\alpha}h_{1}(\lambda_{n}t). \tag{2.10}$$

The next method is perhaps to use the Laplace transform directly. Thus applying the Laplace transform on both sides of (2.1) we obtain:

$$T\left[\frac{\partial^{2}h(r,p)}{\partial r^{2}}+\frac{\partial h(r,p)}{r\partial r}\right]=\left\{a\frac{p^{\alpha}}{p^{\alpha}+c}+\frac{bp}{p+D}\right\}h\left(r,p\right)+h(r,0)\left\{a\frac{p^{\alpha-1}}{p^{\alpha}+c}+\frac{b}{p+D}\right\}. \tag{2.11}$$

However according to the physical problem under study, when t = 0 the water level is considered to be zero therefore h(r,0)=0 then the above equation is reduced to

$$T\left[\frac{\partial^{2}h(r,p)}{\partial r^{2}}+\frac{\partial h(r,p)}{r\partial r}\right]=\left\{a\frac{p^{\alpha}}{p^{\alpha}+c}+\frac{bp}{p+D}\right\}h\left(r,p\right). \tag{2.12}$$

If we apply next the Fourier transform on both sides of the above equation, we obtain

$$T\left[-w^{2}H(w,p)+\int H(w,p)\right]=\left\{a\frac{p^{\alpha}}{p^{\alpha}+c}+\frac{bp}{p+D}\right\}H\left(w,p\right). \tag{2.13}$$

The above can be reformulated as follows:

$$\int_{w}^{\infty}H\left(w,p\right)=\left\{a\frac{p^{\alpha}}{p^{\alpha}+c}+\frac{bp}{p+D}+Tw^{2}\right\}H\left(w,p\right). \tag{2.14}$$

The exact solution of the above equation is given as

$$h\left(r,p\right)=J_{0}\left\{r\sqrt{\left\{a\frac{p^{\alpha}}{p^{\alpha}+c}+\frac{bp}{p+D}+T\right\}}\right\}. \tag{2.15}$$

Finally the solution is given as

$$h\left(r,t\right)=L^{-1}\left\{J_{0}\left\{r\sqrt{\left\{a\frac{p^{\alpha}}{p^{\alpha}+c}+\frac{bp}{p+D}+T\right\}}\right\}\right\}. \tag{2.16}$$

2.2. Numerical method

In this section, we solve numerically using the Crank-Nicholson scheme. To do this, we first present the numerical approximation of

$$\int_0^t \frac{\partial h(r,x)}{\partial x} exp\left[-D(t-x)\right] dx \tag{2.17}$$

and that of

$$\int_0^t \frac{\partial h(r,x)}{\partial x} E_\alpha \left[-\frac{\alpha}{1-\alpha}(t-x)^\alpha\right]. \tag{2.18}$$

Thus the numerical approximation of (2.17) is given as follows:

$$\int_0^{t_n} \frac{\partial h(r,x)}{\partial x} exp\left[-D(t_n-x)\right] dx = \sum_{j=1}^{n} \int_{(j-1)k}^{jk} \left(\frac{h_i^{j+1}-h_i^j}{k} + O(k)\right) exp\left[-\alpha\frac{t_n-x}{1-\alpha}\right] dx. \tag{2.19}$$

Before integration the above expression obtains the following expression [3]

$$\sum_{j=1}^{n} \left(\frac{h_i^{j+1}-h_i^j}{k} + O(k)\right) \int_{(j-1)k}^{jk} exp\left[-\alpha\frac{t_n-x}{1-\alpha}\right] dx. \tag{2.20}$$

$$\sum_{j=1}^{n} \left(\frac{h_i^{j+1}-h_i^j}{k} + O(k)\right) d_{j,k},$$

where

$$d_{j,k} = exp\left[-\alpha\frac{k}{1-\alpha}(n-j+1)\right] - exp\left[-\alpha\frac{k}{1-\alpha}(n-j)\right]. \tag{2.21}$$

The second order approximation of local derivative is given as follows:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= \frac{1}{2}\left(\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2(\Delta x)} + \frac{u_{i+1}^{j}-u_{i-1}^{j}}{2(\Delta x)}\right),\\ \frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{1}{2}\left(\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{2(\Delta x)^2} + \frac{u_{i+1}^{j}-2u_i^{j}+u_{i-1}^{j}}{2(\Delta x)^2}\right). \end{aligned} \tag{2.22}$$

Also the numerical approximation of (2.18) is given as follows:

$$\begin{aligned} &\int_0^{t_n} \frac{\partial h(r,x)}{\partial x} E_\alpha\left[-\frac{\alpha}{1-\alpha}(t_n-x)^\alpha\right] dx\\ &= \sum_{j=1}^{n} \int_{(j-1)k}^{jk} \left(\frac{h_i^{j+1}-h_i^j}{k} + O(k)\right) E_\alpha\left[-\frac{\alpha}{1-\alpha}(t_n-x)^\alpha\right] dx. \end{aligned} \tag{2.23}$$

Before integration the above expression obtains the following form [3]

$$\sum_{j=1}^{n} \left(\frac{h_i^{j+1}-h_i^j}{k} + O(k)\right) \int_{(j-1)k}^{jk} E_\alpha\left[-\frac{\alpha}{1-\alpha}(t_n-x)^\alpha\right] dx.$$

$$\sum_{j=1}^{n} \left(\frac{h_i^{j+1}-h_i^j}{k} + O(k)\right) \delta_{j,k},$$

where (2.24)

$$\delta_{j,k} = (t_n-t_{j+1}) E_\alpha\left[-\frac{\alpha}{1-\alpha}(t_n-t_{j+1})^\alpha\right] - (t_n-t_j) E_\alpha\left[-\frac{\alpha}{1-\alpha}(t_n-t_j)^\alpha\right]. \tag{2.24}$$

Replacing equations (2.24), (2.23) and (2.21) into equation

$$
\begin{aligned}
&T\left[\frac{h_{i+1}^{j+1}-2h_{i+1}^{j}+h_{i+1}^{j-1}}{2(\Delta x)^2}+\frac{h_{i}^{j+1}-2h_{i}^{j}+h_{i}^{j-1}}{2(\Delta x)^2}+\frac{1}{2}\left\{\frac{h_{i+1}^{j+1}-h_{i+1}^{j}}{2\Delta x}+\frac{h_{i}^{j+1}-h_{i}^{j}}{2\Delta x}\right\}\right]\\
&=S\frac{AB\left(\alpha\right)}{1-\alpha}\sum_{l=1}^{j}\left(\frac{h_{i}^{j+1}-h_{i}^{j}}{k}\right)\ d_{l,k}+DS_y\sum_{l=1}^{j}\left(\frac{h_{i}^{j+1}-h_{i}^{j}}{k}\right)\ \delta_{l,k}.
\end{aligned}
\tag{2.25}
$$

Rearranging we obtain

$$
\begin{aligned}
&\left\{\frac{T}{2(\Delta x)^2}+\frac{T}{4(\Delta x)}-S\frac{AB\left(\alpha\right)}{k\left(1-\alpha\right)}d_{j,n}-DS_y\delta_{j,n}\frac{1}{k}\right\}h_i^{j+1}\\
&=\left\{\frac{T}{(\Delta x)^2}+\frac{T}{2(\Delta x)}-S\frac{AB\left(\alpha\right)}{k\left(1-\alpha\right)}d_{j,n}-DS_y\delta_{j,n}\frac{1}{k}\right\}h_i^{j}\\
&-T\left[\frac{h_{i+1}^{j+1}+h_{i+1}^{j-1}}{2(\Delta x)^2}+\frac{h_{i}^{j-1}}{2(\Delta x)^2}+\frac{1}{2}\left\{\frac{h_{i+1}^{j+1}-h_{i+1}^{j}}{2\Delta x}\right\}\right]\\
&+\frac{AB\left(\alpha\right)}{1-\alpha}\sum_{j=1}^{l-1}\left(\frac{h_{i}^{j+1}-h_{i}^{j}}{k}\right)\ d_{l,k}+DS_y\sum_{l=1}^{j-1}\left(\frac{h_{i}^{j+1}-h_{i}^{j}}{k}\right)\ \delta_{l,k}.
\end{aligned}
\tag{2.26}
$$

When $j=1$ then we obtain the following iteration

$$
\begin{aligned}
&\left\{\frac{T}{2(\Delta x)^2}+\frac{T}{4(\Delta x)}-S\frac{AB\left(\alpha\right)}{k\left(1-\alpha\right)}d_{j,n}-DS_y\delta_{j,n}\frac{1}{k}\right\}h_i^{1}\\
&=\left\{\frac{T}{(\Delta x)^2}+\frac{T}{2(\Delta x)}-S\frac{AB\left(\alpha\right)}{k\left(1-\alpha\right)}d_{j,n}-DS_y\delta_{j,n}\frac{1}{k}\right\}h_i^{0}.
\end{aligned}
$$

Therefore

$$
\frac{1}{2}\geq\left\|\frac{\left\{\frac{T}{2(\Delta x)^2}+\frac{T}{4(\Delta x)}-S\frac{AB(\alpha)}{k(1-\alpha)}d_{j,n}-DS_y\delta_{j,n}\frac{1}{k}\right\}}{\left\{\frac{T}{(\Delta x)^2}+\frac{T}{2(\Delta x)}-S\frac{AB(\alpha)}{k(1-\alpha)}d_{j,n}-DS_y\delta_{j,n}\frac{1}{k}\right\}}\right\|\geq\left\|\frac{h_i^0}{h_i^1}\right\|.
\tag{2.27}
$$

3. The model with Caputo derivative

In this section, we present also the model of groundwater flowing with an unconfined aquifer with elastic method. It was proven by many researchers around the world that, the fractional derivative based upon the concept of power law is suitable and powerful mathematical tools to describe more accurately the flow within an elastic media. There exist two different derivatives based on power law commonly used in the literature, including the Caputo and Riemann-Liouville type. We shall recall that, the Caputo is much appreciated to describe real world problems. In this section, we also present the analysis of this model using the Caputo derivative.

$$
T\left[\frac{\partial^2 h(r,t)}{\partial r^2}+\frac{\partial h(r,t)}{r\partial r}\right]=S\ {}_0^C D_t^{\alpha}h(r,t)+DS_y\int_0^t\frac{\partial h(r,x)}{\partial x}exp\left[-D(t-x)\right]dx.
\tag{3.1}
$$

Again we employ the method of separation of variable, we obtain

$$
T\left[\frac{\partial^2 h_1(r)}{\partial r^2}+\frac{\partial h_1(r)}{r\partial r}\right]=-\lambda^2 h_1(r),
\tag{3.2}
$$

$$
S_0^C D_t^{\alpha}h_2(t)+DS_y\int_0^t\frac{\partial h_2(x)}{\partial x}exp\left[-D(t-x)\right]dx=-\lambda^2 h_2(t).
\tag{3.3}
$$

The solution of the first equation that provided in section below, we solve the second with Laplace transform operator to obtain

$$S\left\{p^{\alpha}h_2\left(p\right)-h_2(0)\right\}+\frac{DS_y\left\{h_2\left(p\right)-h_2(0)\right\}}{p+\alpha(1-p)}+\lambda^2 h_2\left(p\right)=0. \tag{3.4}$$

Rearranging the above we obtain

$$h_2\left(p\right)=\frac{pSh_2\left(0\right)\left(1-\alpha\right)+\alpha Sh_2\left(0\right)-h_2(0)}{p\lambda^2\left(1-\alpha\right)+\alpha\lambda^2+Sp^{\alpha+1}\left(1-\alpha\right)+\alpha p^{\alpha}}. \tag{3.5}$$

Then applying the inverse Laplace transform on both sides we obtain

$$h_2\left(t\right)=L^{-1}\left\{\frac{pSh_2\left(0\right)\left(1-\alpha\right)+\alpha Sh_2\left(0\right)-h_2(0)}{p\lambda^2\left(1-\alpha\right)+\alpha\lambda^2+Sp^{\alpha+1}\left(1-\alpha\right)+\alpha p^{\alpha}}\right\}.$$

Therefore the solution of the modified equation is given in series form as

$$h\left(r,t\right)=\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma\left(n+\alpha+1\right)}\left(\frac{r\lambda_n}{2}\right)^{2j+\alpha}h_2(\lambda_n t). \tag{3.6}$$

4. Numerical analysis

In this section, we present the numerical solution of the generalized equation using the Crank-Nicholson scheme. We shall recall that the numerical approximation of the Caputo derivative with fractional order is given as follows: Thus replacing the above in equation (3.1) we obtain

$$\begin{aligned}&T\left[\frac{h_{i+1}^{j+1}-2h_{i+1}^{j}+h_{i+1}^{j-1}}{2(\Delta x)^2}+\frac{h_{i}^{j+1}-2h_{i}^{j}+h_{i}^{j-1}}{2(\Delta x)^2}+\frac{1}{2}\left\{\frac{h_{i+1}^{j+1}-h_{i+1}^{j}}{2\Delta x}+\frac{h_{i}^{j+1}-h_{i}^{j}}{2\Delta x}\right\}\right]\\&\quad=S\frac{1}{\Gamma\left(\alpha\right)}\sum_{l=0}^{j}\frac{h_i^{j+1}-h_i^j}{\Delta t}\Omega_{l,j}+DS_y\sum_{l=1}^{j}\left(\frac{h_i^{j+1}-h_i^j}{k}\right)\ \delta_{l,k}.\end{aligned} \tag{4.1}$$

Rearranging we obtain

$$\begin{aligned}&\left\{\frac{T}{2(\Delta x)^2}+\frac{T}{4(\Delta x)}-S\frac{1}{\Gamma\left(\alpha\right)}\Omega_{j,l}-DS_y\delta_{j,n}\frac{1}{k}\right\}h_i^{j+1}\\&=\left\{\frac{T}{(\Delta x)^2}+\frac{T}{2(\Delta x)}-S\frac{1}{\Gamma\left(\alpha\right)}\Omega_{j,l}-DS_y\delta_{j,n}\frac{1}{k}\right\}h_i^{j}\\&\quad-T\left[\frac{h_{i+1}^{j+1}+h_{i+1}^{j-1}}{2(\Delta x)^2}+\frac{h_{i}^{j-1}}{2(\Delta x)^2}+\frac{1}{2}\left\{\frac{h_{i+1}^{j+1}-h_{i+1}^{j}}{2\Delta x}\right\}\right]\\&\quad+S\frac{1}{\Gamma\left(\alpha\right)}\sum_{l=0}^{j-1}\frac{h_i^{j+1}-h_i^j}{\Delta t}\Omega_{l,j}\\&\quad+DS_y\sum_{l=1}^{j-1}\left(\frac{h_i^{j+1}-h_i^j}{k}\right)\ \delta_{l,k}.\end{aligned} \tag{4.2}$$

When $j=1$ then we obtain the following iteration

$$\begin{aligned}&\left\{\frac{T}{2(\Delta x)^2}+\frac{T}{4(\Delta x)}-S\frac{1}{\Gamma\left(\alpha\right)}\Omega_{j,l}-DS_y\delta_{j,n}\frac{1}{k}\right\}h_i^{1}\\&=\left\{\frac{T}{(\Delta x)^2}+\frac{T}{2(\Delta x)}-S\frac{1}{\Gamma\left(\alpha\right)}\Omega_{j,l}-DS_y\delta_{j,n}\frac{1}{k}\right\}h_i^{0}.\end{aligned}$$

Therefore

$$\frac{1}{2}\geq\left\|\frac{\left\{\frac{T}{2(\Delta x)^2}+\frac{T}{4(\Delta x)}-S\frac{1}{\Gamma(\alpha)}\Omega_{j,l}-DS_y\delta_{j,n}\frac{1}{k}\right\}}{\left\{\frac{T}{(\Delta x)^2}+\frac{T}{2(\Delta x)}-S\frac{1}{\Gamma(\alpha)}\Omega_{j,l}-DS_y\delta_{j,n}\frac{1}{k}\right\}}\right\|\geq\left\|\frac{h_i^0}{h_i^1}\right\|. \tag{4.3}$$

5. Conclusion

In this paper we studied the model of groundwater flowing within the subsurface formation known as unconfined aquifer using the concept of derivative with fractional order with nonsingle and non-local kernel and also the fractional derivative based on the power law concept. Atangana and Baleanu proposed the derivative based on Mittag-Leffler function, which is more suitable to describe real world complex problems. The new model of groundwater flowing within unconfined aquifer based on the generalized Mittag-Leffler function was solved analytically via the Laplace transform operator and numerically via Crank-Nicholson scheme. This new model will surely lead to understanding new behaviour of the flow within this kind of geological formation.

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