Atmospheric and Oceanic Fluid Dynamics

Fundamentals and Large-Scale Circulation

Fluid dynamics is fundamental to our understanding of the atmosphere and oceans. Although many of the same principles of fluid dynamics apply to both the atmosphere and oceans, textbooks on the topic have tended to concentrate on either the atmosphere or the ocean, or on the theory of geophysical fluid dynamics (GFD). However, there is much to be said for a unified discussion, and this major new textbook provides a comprehensive, coherent treatment of all these topics. It is based on course notes that the author has developed over a number of years at Princeton and the University of California.

The first part of the book provides an introduction to the fundamentals of geophysical fluid dynamics, including discussions of rotation and stratification, the role of vorticity and potential vorticity, and scaling and approximations. The second part of the book discusses baroclinic and barotropic instabilities, wave–mean flow interactions and turbulence. The third and fourth parts discuss the general circulation of the atmosphere and ocean. Student problems and exercises, as well as bibliographic and historical notes, are included at the end of each chapter.

Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation will prove to be an invaluable graduate textbook on advanced courses in GFD, meteorology, atmospheric science, and oceanography, and will also be an excellent review volume for researchers. Additional resources are available at www.cambridge.org/9780521849692

GEOFFREY K. VALLIS is a senior scientist and professor in the Program in Atmospheric and Oceanic Sciences and NOAA's Geophysical Fluid Dynamics Laboratory at Princeton University. He is also an associate faculty member at the Program in Applied and Computational Mathematics, and a former professor at the University of California, Santa Cruz. Until recently he was editor of the *Journal of the Atmospheric Sciences*. His research interests include the general circulation of the ocean and atmosphere, turbulence theory, and climate dynamics. He has taught a wide range of topics at Princeton and the University of California, and he has published extensively in both the oceanographic and meteorological literature.

Pre-publication praise for *Atmospheric and Oceanic Fluid Dynamics*

"Geoff Vallis' *Atmosphere and Ocean Dynamics* will become the standard text on modern large-scale atmosphere and ocean dynamics. It covers the field from the equations of motion to modern developments such as wave–mean flow interaction theory and theories for the global-scale circulations of atmospheres and oceans. There is no book of comparable comprehensiveness, spanning the needs of beginning graduate students and researchers alike."
Tapio Schneider, California Institute of Technology

"This clearly written, self-contained new book is a modern treatment of atmospheric and oceanic dynamics. The book starts from classical concepts in fluid dynamics and thermodynamics and takes the reader to the frontier of current research. This is an accessible textbook for beginning students in meteorology, oceanography and climate sciences. Mature researchers will welcome this work as a stimulating resource. This is also the only textbook on geophysical fluid dynamics with a comprehensive collection of problems; these cement the material and expand it to a more advanced level. Highly recommended!"
Paola Cessi, Scripps Institution of Oceanography, University of California, San Diego

"Vallis provides a cohesive view of GFD that smoothly blends classic results with modern interpretations. The book strikes an ideal balance between mathematical rigor and physical intuition, and between atmosphere- and ocean-relevant applications. The use of a hierarchy of models is particularly welcome. Each physical phenomenon is modeled with the right degree of complexity, and the reader is introduced to the value of the hierarchy at an early stage. Well-designed homework problems spanning a broad range of difficulty make the book very appropriate for use in introductory courses in GFD."
Adam Sobel, Lamont-Doherty Earth Observatory, Columbia University

"I have adopted this text for my course in Atmosphere–Ocean Dynamics because the ideas are clearly presented and up-to-date. The text provides the flexibility for the instructor to choose among a variety of paths that take the student from the foundations of the subject to current research topics. For me as a researcher, the text is satisfying because it presents a unified view of the ideas that underlie the modern theory of large scale atmospheric and oceanic circulations."
Paul J. Kushner, University of Toronto

"The large-scale circulation in the atmosphere–ocean system is maintained by small-scale turbulent motions that interact with large-scale radiative processes. The first half of the book introduces the basic theories of large-scale atmosphere–ocean flows and of small-scale turbulent motions. In the second half, the two theories are brought together to explain how the interactions of motions on different scales maintain the global-scale climate. The emphasis on turbulent motions and their effect on larger scales makes this book a gem in the GFD literature. Finally, we have a textbook that is up to date with our current understanding of the climate system."
Raffaele Ferrari, Massachusetts Institute of Technology

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Fundamentals and Large-scale Circulation

GEOFFREY K. VALLIS
Princeton University, New Jersey

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To my parents, Jim and Doreen Vallis.

Contents

An asterisk indicates more advanced material that may be omitted on a first reading. A dagger indicates material that is still a topic of research or that is not settled.

We must be ignorant of much, if we would know anything.
Cardinal John Henry Newman (1801–1890).

Preface

THIS IS A BOOK on the fluid dynamics of the atmosphere and ocean, with an emphasis on the fundamentals and on the large-scale circulation, the latter meaning flows from the scale of the first deformation radius (a few tens of kilometres in the ocean, several hundred kilometres in the atmosphere) to the global scale. The book is primarily a textbook; it is designed to be accessible to students and could be used as a text for graduate courses. It may be also useful as an introduction to the field for scientists in other areas and as a reference for researchers in the field, and some aspects of the book have the flavour of a research monograph.

Atmospheric and oceanic fluid dynamics (AOFD) is fascinating field, and simultaneously both pure and applied. It is a pure field because it is intimately tied to some of the most fundamental and unsolved problems in fluid dynamics — problems in turbulence and wave–mean flow interaction, problems in chaos and predictability, and problems in the general circulation itself. Yet it is applied because the climate and weather so profoundly affect the human condition, and so a great deal of effort goes into making predictions — indeed the practice of weather forecasting is a remarkable example of a successful applied science, in spite of the natural limitations to predictability that are now reasonably well understood. The field is plainly important, for we live in the atmosphere and the ocean covers about two-thirds of the Earth. It is also very broad, encompassing such diverse topics as the general circulation, gyres, boundary layers, waves, convection and turbulence. My goal in this book is present a coherent selection of these topics, concentrating on the foundations but without shying away from the boundaries of active areas of research — for a book that limits itself to what is absolutely settled would, I think, be rather dry, a quality best reserved for martinis and humour.

AOFD is closely related to the field of *geophysical fluid dynamics* (GFD). The latter can be, depending on one's point of view, both a larger and a smaller field than the former. It is larger because GFD, in its broadest meaning, includes not just the fluid dynamics of the Earth's atmosphere and ocean, but also the fluid dynamics of such things as the Earth's interior, volcanoes, lava flows and planetary atmospheres; it is the fluid mechanics of all things geophysical. But at the same time the appellation 'GFD' implies a certain austerity, and the subject is often seen as the one that provides the fundamental principles and language for understanding geophysical flows without being suffocated by the overwhelming detail of the real world. In this book we are guided by the ascetic spirit of GFD, and my hope is that the reader will gain a solid grounding in the fundamentals, motivated by and with an appreciation for the problems of the real world.

The book is an outgrowth of various courses that I have taught over the years, mainly at Princeton University but also at the University of California and at summer schools or similar in Boulder and Kyoto. There are four parts to the book: fundamentals of geophysical fluid dynamics; instabilities, wave–mean flow interaction and turbulence; atmospheric circulation; and ocean circulation. Each corresponds, very roughly, to a one-term graduate course, although parts could also be used for undergraduates. Limitations enforced both by the need to keep the book coherent and focused, and my own expertise or (especially) lack thereof, naturally limit the choice of topics. In particular the chapters on the circulation focus on the steady and statistically steady large-scale circulation and perforce a number of important topics are omitted — tropical and equatorial dynamics, many of the effects of moisture on atmospheric circulation, the spin-up of the ocean circulation, atmospheric and oceanic tides, the quasi-biennial oscillation, and so on. I have however — and at no extra charge, mind you — discussed the large-scale circulation of both atmosphere and ocean. The similarities and differences between the two systems are, I believe, so instructive that even if one's interest is solely in one, there is much to be gained by studying the other. The references at the end of the book are representative and not exhaustive, and almost certainly disproportionately represent articles written in English and those with which I happen to be familiar. For the benefit of the reader interested in exploring the development of the subject I have included references to a number of historical articles, even when the presentation given does not draw from them. If there are other references that are particularly relevant I trust the reader will inform me.

I have tried to keep the overall treatment of topics as straightforward and as clear as I know how. In particular, I have tried to be as explicit as possible in my explanations, even at the risk of descending from pedagogy into pedantry. Relatedly, there is a certain amount of repetition between sections, and this serves both to emphasize the important things and to keep chapters reasonably self-contained. The chapters are of course intellectually linked, for example, heat transport in the atmosphere depends on baroclinic instability, but hopefully the reader already familiar with the latter will be able to read about the former without too much cross-referencing. The treatment generally is fairly physical and phenomenological, and rigour in the mathematical sense is absent; I treat the derivatives of integrals and of infinitesimal quantities rather informally, for example.

The figures (many in colour) may all be downloaded from the CUP web site associated with this book. An asterisk, *, next to a section heading means that the section may be omitted on first reading; although normally uncontroversial, it may contain advanced material that is not essential for subsequent sections. A dagger, †, next to a section heading means that the section discusses topics of research. Very roughly speaking, one might interpret as asterisk as indicating there is advanced manipulation of the equations, whereas a dagger might indicate there is approximation of the equations, or an interpretation that is not universally regarded as settled; *caveat emptor*. There is some arbitrariness in such markings, especially where the section deals with a well understood model of a poorly understood reality. Sections so-marked may be regarded as providing an introduction to the literature, rather than a complete or finished treatment, and the section may also require knowledge of material that appears later in the book. If the asterisk or dagger is applied to a section it applies to all the subsections within, and if a dagger or asterisk appears within a section that is already marked, the warning is even more emphatic. Reading the endnotes at the end of each chapter is not needed in order to follow the arguments in the main text. Problems marked with black diamonds may, like similarly marked ski-slopes, be difficult, and I do not know the solutions to all of them. Good answers to some of them may be publishable and I would appreciate hearing about any such work. I would also appreciate any comments on the material presented in the text. Qui docet discit.

Finally, I should say that this book owes its existence in part to my own hubris and selfishness: hubris to think that others might wish to read what I have written, and selfishness because the enjoyable task of writing such a book masquerades as work.

Summary of Contents

The chapters within each part, and the sections within each chapter, form *logical* units, and do not necessarily directly correspond to a single lecture or set number of lectures.

Part I. Fundamentals of geophysical fluid dynamics

Chapter 1 is a brief introduction to fluid dynamics and the basic equations of motion, assuming no prior knowledge of the field. Readers with prior knowledge of fluid dynamics might skim it lightly, concentrating on those aspects unique to the atmosphere or ocean.

Chapter 2 introduces the effects of stratification and rotation, these being the two main effects that most differentiate AOFD from other branches of fluid dynamics. Fundamental topics such as the primitive equations, the Boussinesq equations, and Ekman layers are introduced here, and these form the foundation for the rest of the book.

Chapter 3 focuses on the shallow water equations. Many of the principles of geophysical fluid dynamics have their simplest expression in the shallow water equations, because the effects of stratification are either eliminated or much simplified. The equations thus provide a relatively gentle introduction to the field.

Chapter 4 discusses vorticity and potential vorticity. Potential vorticity plays an especially important role in large-scale, rotating and stratified flows, and its conservation provides the basis for the equation sets of chapter 5.

Chapter 5 derives simplified equation sets for large-scale flows, in particular the quasi-geostrophic and planetary-geostrophic equation sets, and introduces a simple application, Rossby waves. Much of our theoretical understanding of the large-scale circulation has arisen through the use of these equations.

Part II. Instabilities, wave–mean flow interaction and turbulence

Chapter 6 covers barotropic and baroclinic instability, the latter being the instability that gives rise to weather — and therefore being, perhaps, the form of hydrodynamic instability that most affects the human condition.

Chapter 7 provides an introduction to the important topic, albeit one that is regarded as difficult, of wave–mean flow interaction. That is, how do the waves and instabilities affect the mean flow in which they propagate?

Chapter 8 and 9 are on the statistical theory of turbulence. Chapter 8 introduces the basic concepts of two- and three-dimensional turbulence, and chapter 9 applies similar ideas to geostrophic turbulence.

Chapter 10 discusses turbulent diffusion — a hoary subject, both used and abused, yet one that plays a central role in our thinking about the transport properties of eddies in the atmosphere and ocean.

Part III. Large-scale atmospheric circulation

Chapter 11 is mostly concerned with the dynamics of the Hadley Cell, and, rather descriptively, with the Ferrel Cell.

Chapter 12 addresses the mid-latitude circulation. The goal of this chapter is to provide the basis of an understanding of such topics as the surface westerly winds, the dynamics of the Ferrel Cell, the stratification of the atmosphere and the height of the tropopause. These are still active topics of research, so we cannot always be definitive, although many of the underlying principles are now established, and our discussion emphasizes the fundamentals that have, I think, permanent value.

Chapter 13 discusses stationary waves in the atmosphere, mainly produced by the interaction of the zonal wind with mountains and land–sea temperature contrasts. It also discusses the vertical propagation of Rossby waves, and how these induce a stratospheric circulation.

Part IV. Large-scale oceanic circulation

Chapter 14 discusses the wind-driven circulation, in particular the ocean gyres, either ignoring buoyancy effects or assuming that buoyancy forcing acts primarily to set up a stratification that can be taken as a given.

Chapter 15 discusses the buoyancy-driven circulation, largely neglecting the effects of wind forcing.

Chapter 16 addresses the combined effects of wind and buoyancy forcing in setting up the stratification and in driving both the predominantly horizontal gyral circulation and the overturning circulation. As with many sections of Part III, many of these topics are still being actively researched, although, again, many of the underlying principles have, I think, now been established and have permanent value.

Acknowledgements

This book would not have been possible without the input, criticism, encouragement and advice of a large number of people, from students to senior scientists. Students at Princeton University, New York University, Columbia University, California Institute of Technology, MIT and the University of Toronto have used earlier versions of the text in various courses, and I am grateful for the feedback received from the instructors and students about what works and what doesn't, as well as for numerous detailed comments. Parts of the first few chapters and many of the problems draw on notes prepared over the years for a graduate class at Princeton University taught by Steve Garner, Isaac Held, Yoshio Kurihara, Paul Kushner and me. Steve Garner has been notably generous with his time and ideas with respect to this material.

I would like to thank following individuals for providing comments on the text or for conversational input: Alistair Adcroft, Brian Arbic, Roger Berlind, Pavel Berloff, Thomas Birner, Paola Cessi, Sorin Codoban, Agatha de Boer, Roland de Szoeke, Raffaele Ferrari, Baylor Fox-Kemper, Dargan Frierson, Steve Griffies, Brian Hoskins, Huei-Ping Huang, Chris Hughes, Kosuke Ito and his fellow students in Kyoto, Laura Jackson, Martin Juckes, Allan Kaufman, Samar Khatiwhala, Andrew Kositsky, Paul Kushner, Joe LaCasce, Trevor McDougall, Paul O'Gorman, Tim Palmer, David Pearson, Lorenzo Polvani, Cathy Raphael, Adam Scaife, Urs Schaefer-Rolffs, John Scinocca, Rob Scott, Tiffany Shaw, Sabrina Speich, Robbie Toggweiler, Yue-Kin Tsang, Ross Tulloch, Eli Tziperman, Jacques Vanneste, Chris Walker, Ric Williams, Michal Ziemiánski and Pablo Zurita-Gotor. Some of the above provided detailed comments on several sections or chapters, and I am very grateful to them; a number of other people provided figures and these are acknowledged where they appear. Anders Persson and Roger Samelson provided a number of useful historical remarks and interpretations, and Michael McIntyre (see also McIntyre 1997) emphasized the importance of being as explicit as possible in my explanations. I am particularly indebted to Tapio Schneider, Adam Sobel, Jürgen Theiss and Andy White who read and gave detailed comments on many chapters of text, to Isaac Held for many conversations over the years on all aspects of atmospheric and oceanic fluid dynamics, to Shafer Smith for both comments and code, to Ed Gerber for several calculations and figures, and to Bill Young for very useful input on thermodynamics and the Boussinesq equations. I would also like to thank Matt Lloyd of CUP for his encouragement, my copy-editor, Louise Staples, for her light but careful touch, and Anna Valerio for efficient secretarial assistance. Needless to say, I take full responsibility for the errors, both scientific and presentational, that undoubtedly remain.

My first exposure to the field came as a graduate student in the fecund atmosphere of Atmospheric Physics Group at Imperial College in the late 1970s and early 1980s, and I'd like to thank everyone who was there at that time. You know who you are. I would also like to thank everyone at the Geophysical Fluid Dynamics Laboratory (GFDL) and in Princeton University's Atmospheric and Oceanic Science program for creating the pleasant and stimulating work environment that made it possible to write this book. I am grateful to the staff and scientists at the Plymouth Marine Laboratory and the U. K. Meteorological Office for their hospitality during a sabbatical visit. Finally, I would like to thank the National Science Foundation and the National Oceanic and Atmospheric Administration for providing financial support over the years, and Bess for support of a much greater kind.

Note on later printings

I have taken advantage the opportunity afforded by later printings of this book to correct a number of typographic errors and make a number of small corrections to the text throughout. The pagination and equation numbering are unaltered, although some microtypographic improvements have changed some of the line breaks. I am grateful to many readers who have sent in comments and corrections, and I would particularly like to acknowledge Roger Berlind for his exceptionally detailed and perceptive comments on the entire book. Additional resources, including downloadable figures and solutions to many of the end-of-chapter exercises, are available at www.vallisbook.org and www.cambridge.org/9780521849692.

NOTATION

The list below contains only the more important variables, or instances of non-obvious notation. Distinct meanings are separated with a semi-colon. Variables are normally set in italics, constants (e.g, π) in roman (i.e., upright), differential operators in roman, vectors in bold, and tensors in bold sans serif. Thus, vector variables are in bold italics, vector constants (e.g., unit vectors) in bold roman, and tensor variables are in bold slanting sans serif. Physical units are set in roman. A subscript denotes a derivative only if the subscript is a coordinate, such as x, y, z or t; a subscript 0 generally denotes a constant reference value (e.g., ρ_0). The components of a vector are denoted by superscripts.

Variable	Description
a	Radius of Earth.
b	Buoyancy, $-g\delta\rho/\rho_0$ or $g\delta\theta/\widetilde{\theta}$.
$\boldsymbol{c}_g$	Group velocity, (c_g^x, c_g^y, c_g^z).
c_p	Phase speed; heat capacity at constant pressure.
c_v	Heat capacity constant volume.
c_s	Sound speed.
f, f_0	Coriolis parameter, and its reference value.
$\boldsymbol{g}, g$	Vector acceleration due to gravity, magnitude of $\boldsymbol{g}$.
h	Layer thickness (in shallow water equations).
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Unit vectors in (x, y, z) directions.
i	An integer index.
i	Square root of -1.
$\boldsymbol{k}$	Wave vector, with components (k, l, m) or (k^x, k^y, k^z).
k_d	Wave number corresponding to deformation radius.
L_d	Deformation radius.
L, H	Horizontal length scale, vertical (height) scale.
m	Angular momentum about the Earth's axis of rotation.
M	Montgomery function, $M = c_pT + \Phi$.
N	Buoyancy, or Brunt–Väisälä, frequency.
p	Pressure.
Pr	Prandtl ratio, f_0/N.
q	Quasi-geostrophic potential vorticity.
Q	Potential vorticity (in particular Ertel PV).
$\dot{Q}$	Rate of heating.
Ra	Rayleigh number.
Re	Real part of expression.
Re	Reynolds number, UL/ν.
Ro	Rossby number, U/fL.
S	Salinity; source term on right-hand side of an evolution equation.
$S_o, \boldsymbol{S}_o$	Solenoidal term, solenoidal vector.
T	Temperature.
t	Time.
$\boldsymbol{u}$	Two-dimensional (horizontal) velocity, (u, v).
$\boldsymbol{v}$	Three-dimensional velocity, (u, v, z).
x, y, z	Cartesian coordinates, usually in zonal, meridional and vertical directions.
Z	Log-pressure, $-H \log p/p_R$. Usually, $H = 7.5$ km and $p_R = 10^5$ Pa.

Variable	Description
$\mathcal{A}$	Wave activity.
α	Inverse density, or specific volume; aspect ratio.
β	Rate of change of f with latitude, $\partial f/\partial y$.
β_T, β_S	Coefficient of expansion with respect to temperature, salinity.
ϵ	Generic small parameter (epsilon).
ε	Cascade or dissipation rate of energy (varepsilon).
η	Specific entropy; perturbation height; enstrophy cascade or dissipation rate.
$\boldsymbol{\mathcal{F}}$	Eliassen Palm flux, $(\mathcal{F}^y, \mathcal{F}^z)$.
γ	Vorticity gradient, $\beta - u_{yy}$; the ratio c_p/c_v.
Γ	Lapse rate.
κ	Diffusivity; the ratio R/c_p.
$\mathcal{K}$	Kolmogorov or Kolmogorov-like constant.
Λ	Shear, e.g., $\partial U/\partial z$.
μ	Viscosity.
ν	Kinematic viscosity, μ/ρ.
v	Meridional component of velocity.
ϕ	Pressure divided by density, p/ρ; passive tracer.
Φ	Geopotential, usually gz.
Π	Exner function, $\Pi = c_p T/\theta = c_p(p/p_R)^{R/c_p}$.
$\boldsymbol{\omega}$	Vorticity.
$\Omega, \boldsymbol{\Omega}$	Rotation rate of Earth and associated vector.
ψ	Streamfunction.
ρ	Density.
ρ_θ	Potential density.
σ	Layer thickness, $\partial z/\partial\theta$; Prandtl number ν/κ; measure of density, $\rho - 1000$.
$\boldsymbol{\tau}$	Stress vector, often wind stress.
$\tilde{\boldsymbol{\tau}}$	Kinematic stress, $\tilde{\boldsymbol{\tau}}/\rho$.
τ	Zonal component or magnitude of wind stress; eddy turnover time.
θ	Potential temperature.
ϑ, λ	Latitude, longitude.
ζ	Vertical component of vorticity.
$\left(\frac{\partial a}{\partial b}\right)_c$	Derivative of a with respect to b at constant c.
$\left.\frac{\partial a}{\partial b}\right\vert_{a=c}$	Derivative of a with respect to b evaluated at $a = c$.
∇_a	Gradient operator at constant value of coordinate a, e.g., $\nabla_z = \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y$.
$\nabla_a\cdot$	Divergence operator at constant value of coordinate a, e.g., $\nabla_z\cdot = (\mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y)\cdot$.
$\nabla^\perp$	Perpendicular gradient, $\nabla^\perp\phi \equiv \mathbf{k}\times\nabla\phi$.
curl_z	Vertical component of $\nabla\times$ operator, $\mathrm{curl}_z\boldsymbol{A} = \mathbf{k}\cdot\nabla\times\boldsymbol{A} = \partial_x A^y - \partial_y A^x$.
$\frac{\mathrm{D}}{\mathrm{D}t}$	Material derivative (generic).
$\frac{\mathrm{D}_g}{\mathrm{D}t}$	Material derivative using geostrophic velocity, for example $\partial/\partial t + \boldsymbol{u}_g\cdot\nabla$.
$\frac{\mathrm{D}_3}{\mathrm{D}t}, \frac{\mathrm{D}_2}{\mathrm{D}t}$	Material derivative in three dimensions and in two dimensions, for example $\partial/\partial t + \boldsymbol{v}\cdot\nabla$ and $\partial/\partial t + \boldsymbol{u}\cdot\nabla$ respectively.