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Atomic surfaces, tilings and coincidences II. Reducible case

ATOMIC SURFACES, TILINGS AND COINCIDENCES II. REDUCIBLE CASE

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ABSTRACT. — The atomic surfaces of unimodular Pisot substitutions of irreducible type have been studied by many authors. In this article, we study the atomic surfaces of Pisot substitutions of reducible type.

As an analogue of the irreducible case, we define the stepped-surface and the dual substitution over it. Using these notions, we give a simple proof to the fact that atomic surfaces form a self-similar tiling system. We show that the stepped-surface possesses the quasi-periodic property, which implies that a non-periodic covering by the atomic surfaces covers the space exactly k-times.

The atomic surfaces are originally designed by Rauzy to study the spectrum of the substitution dynamical system via a periodic tiling. However, we show that, since the stepped-surface is complicated in the reducible case, it is not clear whether the atomic surfaces can tile the space periodically or not. It seems that the geometry of the atomic surfaces can not applied directly to the spectral problem.

RÉSUMÉ. — Les surfaces atomiques des substitutions unimodulaires de type Pisot ont été étudiées par de nombreux auteurs. Dans cet article, nous étudions les surfaces atomiques des substitutions Pisot de type réductible.

Par analogie avec le cas irréductible, nous définissons la notion de surfaces plissées et de substitution duale agissant dessus. Grâce à ces notions, nous donnons une preuve simple du fait que les surfaces atomiques forment un système de pavage auto-similaire. Nous montrons que les surfaces atomiques sont quasi-périodiques, ce qui implique qu'un recouvrement non périodique par les surfaces atomiques recouvre l'espace exactement k fois.

Les surfaces atomiques ont été introduites à l'origine par Rauzy dans le but d'étudier le spectre des systèmes dynamiques substitutifs via un pavage périodique. Cependant, nous montrons qu'il n'est pas évident de savoir si les surfaces atomiques peuvent paver l'espace périodiquement ou non, en raison de la complexité du cas réductible. Il semble que la géométrie des surfaces atomiques ne peut pas être appliquée directement au problème spectral.

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1. Introduction

1.1. Atomic surfaces of Pisot substitution

A unimodular Pisot substitution determines a graph-directed iterated function system; the *atomic surfaces* (also called *Rauzy fractals*) are the invariant sets of the system.

Originally, the atomic surfaces are designed to give a geometrical realization of the *substitution dynamical systems* ([28, 5]). A domain-exchange transformation is defined on the atomic surface, and it is conjugate to the substitution dynamical system. The geometry of the Rauzy fractals may give information on the spectrum of the substitution dynamical system ([27, 17, 31]).

The atomic surfaces are also employed to construct nice Markov partitions for group automorphism of the torus([26, 30]). This construction has been applied to β-numeration system([22, 21]).

One major feature of atomic surfaces is that they form self-similar tiling systems([32]). The atomic surface tilings and the integral self-affine tilings ([7, 23, 24]) are the only two classes of self-similar tilings being studied systematically. As we have seen, the atomic surface tilings have rich dynamical properties.

Host([19]), Arnoux and Ito([5]) introduce the *coincidence condition* for Pisot substitution to the study of the atomic surfaces. (The notion of coincidence is originally introduced by Dekking [12] for substitutions of constant length.) Ito and Rao ([20]), Barge and Kwapisz ([9]) introduce a *super-coincidence condition* independently. (It is called *geometrical coincidence condition* in [9].) It is shown that the tiling and ergodic properties of the atomic surfaces are governed by the super-coincidence condition. Barge and Diamond show that a strong coincidence condition holds for Pisot substitution over two-letter alphabet, and it follows that the super-coincidence condition also holds in this case. Super-coincidence condition for many letters is an important and challenging open problem([10]).

However, all the previous discussions assume that the characteristic polynomial of the substitution is irreducible over $\mathbb{Q}$, which we call the *irreducible* case. Recently, the study of the reducible Pisot substitution has drawn the attentions of many people. One of the motivations comes form the so-called β-*tiling*, constructed by Thurston ([33]) and studied by Akiyama ([1, 2]). The β-tilings can be regarded as a special case of the atomic-surface tilings of Pisot substitutions, of irreducible or reducible case according to β([11]);

the (W)-property in [2, 3] becomes a special case of the super-coincidence condition.

The atomic surfaces of reducible Pisot substitutions were first studied by Ei and Ito [14]. Following this paper, several works are devoted to this topic, namely, Berthé and Siegel [11], Baker, Barge and Kwapisz [6] and the present paper. These works are carried out more or less independently, with emphasis on different aspects.

The paper [11] builds up the theory as a generalization of a construction of Thurston related to Pisot number. A numeration system due to Dumont and Thomas [13], plays an central role in [11], and the discussion is number theoretical. The paper [6] mainly concerns the spectral aspect and the super-coincidence condition.

In the present paper, we investigate the atomic surfaces of the reducible case in a geometrical point of view. Special attentions are paid to the stepped-surface, dual substitution, and periodic tiling of the atomic surfaces. These properties are dramatically different from the irreducible case. One of our main results is to show that atomic surfaces form a self-similar tiling system, where we give an elegant proof by using new results of Lagarias and Wang ([25]). Some results in this paper are already known in [11], but we keep them for self-containing and for alternative proofs.

1.2. Examples

Here we give two examples of Pisot substitutions of reducible case. Both examples come from β-numeration systems.

Example 1.1. — Let β be the dominant root of the polynomial $x^3 - Kx^2 - (K+1)x - 1$ where $K \geqslant 0$ is an integer. Then β is a Pisot unit (i.e., a Pisot number as well as an algebraic unit) and the expansion of 1 in base β is

$$1 = 0.(K+1)00K1.$$

The associated β-substitution σ and incidence matrix M_σ are

$$\sigma : \begin{cases} 1 \mapsto 1^{(K+1)}2 \\ 2 \mapsto 3 \\ 3 \mapsto 4 \\ 4 \mapsto 1^K 5 \\ 5 \mapsto 1 \end{cases}, \quad M_\sigma = \begin{pmatrix} K+1 & 0 & 0 & K & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of M_σ is

$$x^5 - (K+1)x^4 - Kx - 1 = (x^3 - Kx^2 - (K+1)x - 1)(x^2 - x + 1).$$

This substitution is called *Hokkaido substitution* by Akiyama, since its atomic surface looks like the map of Hokkaido, a northern island of Japan. The atomic surfaces of this substitution has been studied in detail in [14].
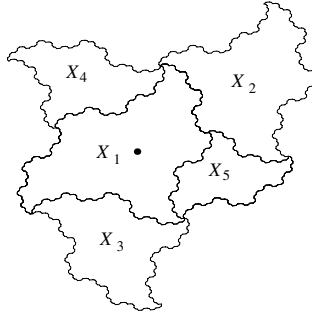

Figure 1.1. *Atomic surfaces $\bigcup_{i=1}^{5} X_i$ of Hokkaido substitution.*

Example 1.2. — Let β be the dominant root of the polynomial $x^3 - Kx^2 + Kx - 1$ where $K \geqslant 2$ is an integer. Then β is a Pisot number and the expansion of 1 in base β is

$$1 = 0.(K-1)(K-1)01.$$

The associated β-substitution τ and incidence matrix M_τ are

$$\tau : \begin{cases} 1 \mapsto 1^{(K-1)}2 \\ 2 \mapsto 1^{(K-1)}3 \\ 3 \mapsto 4 \\ 4 \mapsto 1 \end{cases} \quad (K \geqslant 2), \quad M_\tau = \begin{pmatrix} K-1 & K-1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of M_τ is

$$x^4 - (K-1)x^3 - (K-1)x^2 - 1 = (x^3 - Kx^2 + x - 1)(x+1).$$

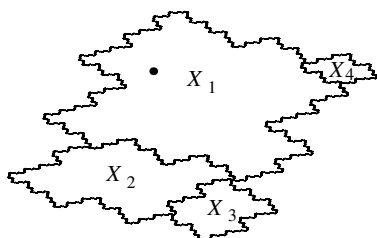In what follows, τ will always refer to this substitution.

Figure 1.2. Atomic surfaces $\bigcup_{i=1,2,3,4} X_i$ of substitution τ.

1.3. Main results

Let σ be an unimodular Pisot substitution over $\{1, 2, \ldots, d\}$. Let λ be the Perron-Frobenius eigenvalue of the incidence matrix $M = M_\sigma$, and let $g(x)$ be the minimal polynomial of λ. In this paper, we assume that $m = \deg g(x) < d$, that is, σ is reducible.

Projections. — Similar to the irreducible case, the atomic surfaces can be defined by projection method. Let $\lambda_1 = \lambda, \lambda_2, \cdots, \lambda_m$ be the roots of $g(x)$, then they are eigenvalues of M. Let P_1 be the sum of the eigenspaces of the eigenvalues λ_i, then P_1 is M-invariant and there is a unique M-invariant subspace V_1 satisfying $\mathbb{R}^d = P_1 \oplus V_1$. (See Proposition 2.2.)

Let V be the eigenspace corresponding to the eigenvalues λ_1, and let P be the direct sum of the eigenspaces corresponding to the eigenvalues $\lambda_2, \ldots, \lambda_m$. Then P is stable and V is unstable. According to the direct sum $\mathbb{R}^d = V_1 \oplus P \oplus V$, the following projections can be defined:

$$\phi : \mathbb{R}^d \mapsto P_1, \quad \pi : \mathbb{R}^d \mapsto P, \quad \pi' : \mathbb{R}^d \mapsto V.$$

We note that the projection ϕ is "rational", that is, the projection of $\mathbb{Z}^d$ is an m-dimensional lattice.

Atomic surface and self-similarity. — Let ω be a periodic point of σ. Then ω defines a broken line in $\mathbb{R}^d$. Projecting the integer points on the broken line to the stable plane P and taking a closure, we obtain the *atomic surface X*. The set X can be divided naturally into d parts $X_1, X_2, \ldots, X_d$, which are called *the partial atomic surfaces* of σ. The partial atomic surfaces $X_1, \ldots, X_d$ are compact subsets of the stable space P and they have a self-similar structure. This has also been shown in [11].

THEOREM 1.3. — *Let σ be an unimodular Pisot substitution. Then the partial atomic surfaces $\{X_i\}_{i=1}^d$ are compact and satisfy the following set equations*

$$(1.1) \qquad M^{-1}X_i = \bigcup_{j=1}^d \bigcup_{W_k^{(j)}=i} \left(X_j + M^{-1}\pi(f(P_k^{(j)})) \right), \quad 1 \leqslant i \leqslant d.$$

All the notations in the above formula will be precisely defined in the beginning of Section 3.

Stepped-surface and dual substitution. — The stepped-surface and dual substitution are powerful tools to study the set equations (1.1), as they do in the irreducible case. But the structure of the stepped-surface in the reducible case is more complicated.

The stepped-surface S is defined as a set of colored points on the lattice $\mathbb{L} = \phi(\mathbb{Z}^d)$, where these points are very close to the stable space P in some sense (see Section 4). We will use $[x, i^*]$ to denote an element of S, where x is a point in $\mathbb{L}$ and $i^* \in \{1^*, 2^*, \ldots, d^*\}$ indicates the color of x. Now the number of colors is larger than the rank of $\mathbb{L}$. This fact causes many difficulties. First, we do not have an obvious geometrical representation of the stepped-surface (we even do not know whether such a representation exists or not). Because of this, we do not know whether the atomic surfaces can tile the space P periodically or not.

Nevertheless, we could show the quasi-periodicity of this "abstract" stepped-surface.

THEOREM 1.4. — *The stepped-surface S of a Pisot substitution is quasi-periodic.*

According to the set equations (1.1), we define a dual substitution σ^*, a morphism on the stepped surface S. An important fact is that the dual substitution keeps the stepped surface S invariant.

THEOREM 1.5. — *Let σ be an unimodular Pisot substitution. Then the stepped surface is invariant under the action of the dual substitution σ^*. Precisely,*

    (i) $\sigma^*(S) = S$.

    (ii) $\sigma^*[x, i^*] \cap \sigma^*[y, j^*] = \emptyset$, *for distinct* $[x, i^*], [y, j^*] \in S$.

Self-similar tiling system. — Thanks to Theorem 1.5, now we can show that the partial atomic surfaces $X_1, \ldots, X_d$ form a self-similar tiling system. Precisely,

THEOREM 1.6. — *Let σ be an unimodular Pisot substitution. Then*

(i) *The interiors of the partial atomic surfaces X_i are not empty.*

(ii) *The right side of (1.1) consists of non-overlapping unions.*

(iii) *$X_i = \overline{X_i^\circ}$ and ∂X_i has Lebesgue measure 0, where ∂X_i denotes the boundary of X_i.*

In the irreducible case, this theorem is proved in several different ways ([5, 32]). Here we give a quick proof by using a result of Lagarias and Wang [25] on substitution Delone set. The idea of [25] is to construct a dual system of the set equations (1.1); if a family of Delone sets satisfies the dual system, then equation (1.1) gives a self-similar tiling system. It turns out that the projection of the stepped-surface gives a solution of the dual system.

A quasi-periodic tiling. — Combining the atomic surfaces and the stepped-surface together, we define the following collection

$$\mathcal{J} := \{\pi(x) + X_i; \ [x, i^*] \in S\}.$$

We will see that $\mathcal{J}$ is a *self-replicating collection* (see Section 5). From the quasi-periodicity of the stepped-surface and the self-replicating property of $\mathcal{J}$, one can show that there exists a constant k, such that almost every point of the space P is covered by k pieces of tiles in $\mathcal{J}$.

The problem whether $\mathcal{J}$ is always a tiling is an open problem. In a sequel paper [15], we will introduce the super-coincidence condition for reducible Pisot substitution and show that $\mathcal{J}$ is a tiling if and only if σ satisfies the super-coincidence condition. We also show there that for β-substitution, the super-coincidence condition is equivalent to a (W)-property introduced by Akiyama [2]. Several classes of Pisot numbers are proved to have (W)-property ([18, 3]) and hence the associated collections $\mathcal{J}$ are tilings.

Collection $\mathcal{J}_2$ and Markov partition. — As an analogue to the reducible case, we construct a family $\hat{X}_1, \ldots, \hat{X}_d$ which are subsets of P_1. Let $\hat{X} = \cup_{i=1}^d \hat{X}_i$. Set

$$\mathcal{J}_2 := \{\hat{X} + z : \ z \in \mathbb{L}\}.$$

It is shown that

THEOREM 1.7. — *The collection $\mathcal{J}_2$ is a tiling of P_1 if and only if $\mathcal{J}$ is a tiling of P.*

Since the matrix M is unimodular, it can be regarded as a group automorphism of the m-dimensional torus $P_1 \setminus \mathbb{L}$. It has been shown ([11]) that if $\mathcal{J}_2$ is a tiling, then the collection $\{\hat{X}_1, \ldots, \hat{X}_d\}$ is a Markov partition of group automorphism M on the torus $P_1 \setminus \mathbb{L}$.
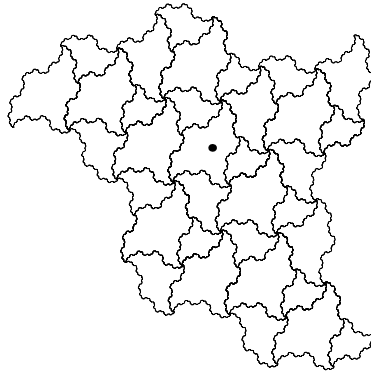
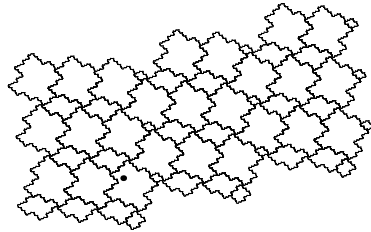Figure 1.3. Quasi-periodic tiling $\mathcal{J}$ of Hokkaido substitution.

Figure 1.4. Quasi-periodic tiling $\mathcal{J}$ of substitution τ.

Can X tile P periodically? — In the irreducible case, the atomic surface X can tile P periodically; actually this tiling is a lattice tiling. If σ satisfies the strong coincidence condition, then a *domain exchange transformation* is well-defined on X ([5]). Furthermore, if σ satisfies the super-coincidence condition, then X can be regarded as a $(d-1)$-dimensional torus and the domain exchange transformation is a rotation on the torus ([20]).

In the reducible case, the existence of geometrical representation of the stepped-surface is unclear. Hence in general, we do not know whether X can tile P periodically or not. This is a significant difference between the irreducible case and the reducible case.

Ei, Furukado and Ito have studied this problem for the substitutions in Example 1.1 and Example 1.2. By trial and error method, they found geometrical representation of the stepped-surface of these substitutions, consisting of polygons with strange shapes.

The paper is organized as follows. In Section 2, we give the basic notations and various projections are studied. The atomic surfaces are defined in Section 3 and the self-similarity is proved there (Theorem 1.3). In Section 4, we study the stepped-surface; quasi-periodicity is shown there (Theorem 1.4). Section 5 is devoted to dual substitution and Theorem 1.5 is proved. In Section 6, we show that the partial atomic surfaces form a self-similar tiling system (Theorem 1.6). Two tilings are studied in Section 7. In Section 8, some comments are given on the periodic tiling of atomic surfaces.

2. Projections

2.1. Pisot substitution

Let $\mathcal{A} = \{1, \cdots, d\}, d \geqslant 2$, be an alphabet. Let $\mathcal{A}^* = \cup_{n \geqslant 0} \mathcal{A}^n$ be the set of finite words over $\mathcal{A}$. A substitution σ is a function $\sigma : \mathcal{A} \mapsto \mathcal{A}^*$. The incidence matrix of σ is $M_\sigma = M = (m_{ij})_{1 \leqslant i,j \leqslant d}$, where m_{ij} is the number of occurrences of i in $\sigma(j)$. A substitution is said to be *primitive* if its incidence matrix M is primitive, i.e, M^N is a positive matrix for some N. By Perron-Frobenius Theorem, a non-negative primitive matrix has a simple real eigenvalue λ, which we call the *Perron-Frobenius eigenvalue*, that is larger than all the other eigenvalues in modulus.

The theory of atomic surfaces is related to a special type of substitutions, the so-called Pisot substitution.

DEFINITION 2.1 — *A substitution σ is a Pisot substitution if σ is primitive and the Perron-Frobenius eigenvalue of M_σ is a Pisot number.*

An algebraic integer is a *Pisot number* if all its algebraic conjugates have modulus strictly less than 1. A substitution σ is *unimodular* if $\det M = \pm 1$; it is *irreducible* (or of irreducible type) if the characteristic polynomial of M is irreducible over $\mathbb{Q}$.

2.2. Direct sum decomposition

Assume that $g(x)h(x)$ is the characteristic polynomial of M, and $g(x)$ is the minimal polynomial of the Pisot number λ. Then there is an integral $d \times d$ matrix Q with $\det Q = \pm 1$ such that

(2.1) $$Q^{-1}MQ = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1, A_2 are integral matrices with characteristic polynomials $g(x)$ and $h(x)$, respectively.

Let $\lambda_1 = \lambda, \lambda_2, \cdots, \lambda_m$ be the roots of $g(x)$, then they are eigenvalues of M as well as A_1. We arrange them in such an order: the real roots are ahead of the complex roots; for a complex root λ_i we put its complex conjugate $\bar{\lambda}_i$ next to it to make a pair. Let $v = v_1, v_2, \cdots, v_m$ be the eigenvectors of M corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$. To each real eigenvalue there is a 1-dimensional, M-invariant real eigenspace. To a pair of complex eigenvalues λ_i and $\lambda_{i+1} = \bar{\lambda}_i$, we choose $v_{i+1} = \overline{v_i}$; hence there is a 2-dimensional M-invariant real eigenspace

$$\{c_i v_i + c_{i+1} v_{i+1} : \ c_{i+1} = \bar{c}_i, \ c_i \in \mathbb{C}\}.$$

Let us denote by P_1 the direct sum of these eigenspaces,

$$(2.2) \quad P_1 = \{c_1 v_1 + \cdots + c_m v_m : c_i \in \mathbb{R} \text{ if } \lambda_i \in \mathbb{R}; c_{i+1}$$
$$= \overline{c_i} \in \mathbb{C} \text{ if } \lambda_{i+1} = \overline{\lambda_i}\}.$$

Hereafter we will use $v_1, \cdots, v_m$ as a basis of P_1 in the above sense. Using (2.1), it is easy to show that

PROPOSITION 2.2 — *There is a unique M-invariant subspace V_1 satisfying $\mathbb{R}^d = P_1 \oplus V_1$. Furthermore, if Q is a matrix satisfying (2.1), then*

$$V_1 = Q \begin{pmatrix} 0 \\ \mathbb{R}^{d-m} \end{pmatrix}.$$

We leave the easy proof to the reader. We decompose P_1 into $P_1 = P \oplus V$, where V is the eigenspace corresponding to λ, and P is the direct sum of the eigenspaces corresponding to $\lambda_2, \ldots, \lambda_m$.

2.3. Projections

According to the direct sum $\mathbb{R}^d = P_1 \oplus V_1$, a natural projection from $\mathbb{R}^d$ to P_1 is defined:

$$\phi: \ \mathbb{R}^d \mapsto P_1.$$

According to the direct sum $P_1 = P \oplus V$, we define two natural projections

$$(2.3) \qquad\qquad \pi_1: \ P_1 \mapsto P, \quad \pi_1': \ P_1 \mapsto V.$$

We define projections $\pi: \ \mathbb{R}^d \mapsto P$ and $\pi': \ \mathbb{R}^d \mapsto V$ by

$$\pi := \pi_1 \circ \phi, \quad \pi' := \pi_1' \circ \phi.$$

Since P_1, V_1, P and V are M-invariants, all the above projections commute with M.

LEMMA 2.3 — *The relation $\varphi \circ M = M \circ \varphi$ holds for*

$$\varphi \in \{\phi, \pi_1, \pi_1', \pi, \pi'\}.$$

Let $e_1, \cdots, e_d$ be the canonical basis of $\mathbb{R}^d$, let $\langle \cdot, \cdot \rangle$ be the inner product. Let ${}^t M$ denote the transpose of M.

Remark 2.4 — An explicit formula of the projections are given in [11]. Let v_1 be the eigenvectors of M for the eigenvalue $\lambda = \lambda_1$, we may assume that the entries of v_1 belong to the field $\mathbb{Q}(\lambda)$. Likewise, let u_1 be the eigenvectors of ${}^t M$ for the eigenvalue λ, with entries in $\mathbb{Q}(\lambda)$ and the inner product $\langle v_1, u_1 \rangle = 1$. Let v_i be the conjugates of v_1 and u_i be the conjugates of u_1. Then

$$\phi(x) = \langle x, u_1 \rangle v_1 + \langle x, u_2 \rangle v_2 + \cdots + \langle x, u_m \rangle v_m,$$
$$\pi(x) = \langle x, u_2 \rangle v_2 + \cdots + \langle x, u_m \rangle v_m,$$
$$\pi'(x) = \langle x, u_1 \rangle v_1.$$

Let v^* be a positive eigenvector of ${}^t M_\sigma$ with respect to the eigenvalue λ. Then

LEMMA 2.5 — *The following properties hold.*
  (i) $v^* \perp (P \oplus V_1)$.
  (ii) $\langle \phi(e_i), v^* \rangle > 0$ *holds for* $1 \leqslant i \leqslant d$.

Proof —
  (i) Since $\langle u_1, v_1 \rangle = 1$ and $\pi'(x) = \langle x, u_1 \rangle v_1$, for any vector $x \in \mathbb{R}^d$, we have ${}^t x v^* = {}^t x u_1^t v_1 v^*$, and hence $\langle x, v^* \rangle = \langle x, u_1 \rangle \langle v_1, v^* \rangle$. So that

$$\langle x - \pi'(x), v^* \rangle = 0,$$

  which is desired.
  (ii) Let $y = e_i - \phi(e_i) \in V_1$. Then $\langle \phi(e_i), v^* \rangle = \langle e_i - y, v^* \rangle = \langle e_i, v^* \rangle > 0$.
  $\square$

3. Atomic surfaces: definition and self-similarity

The following notations are used throughout this paper. Let $f : \mathcal{A}^* \to \mathbb{Z}^d$ be a map which sends a finite word to an integer in $\mathbb{R}^d$ as follows:

  (i) $f(\epsilon) = 0$, where ϵ denotes the empty word;
  (ii) $f(i) = e_i$ for $1 \leqslant i \leqslant d$ and $f(UV) = f(U) + f(V)$ for $U, V \in \mathcal{A}^*$.

Clearly $f \circ \sigma = M \circ f$.

We write $\sigma(i) = W_1^{(i)} \ldots W_{l_i}^{(i)}$, $i = 1, 2, \ldots, d$. Let $P_k^{(i)} = W_1^{(i)} \ldots W_{k-1}^{(i)}$ be the *prefix* of $\sigma(i)$ with length $k-1$, and denote by $S_k^{(i)} = W_{k+1}^{(i)} \ldots W_{l_i}^{(i)}$ the *suffix* of $\sigma(i)$. Then $\sigma(i)$ can be written as

$$\sigma(i) = P_k^{(i)} W_k^{(i)} S_k^{(i)}.$$

3.1. Inflation and substitution map

For $x \in \mathbb{Z}^d$, we denote by $(x, i) := \{x + \theta e_i; \; \theta \in [0, 1]\}$ the segment from x to $x + e_i$. Define $y + (x, i) := (x + y, i)$. Set

$$\mathcal{G} = \{(x, i) : \; x \in \mathbb{Z}^d, 1 \leqslant i \leqslant d\}.$$

The term *segment* will refer to any element of $\mathcal{G}$.

The *inflation and substitution map* F_σ is defined as follows on the set of subsets of $\mathcal{G}$ ([8]). Define

$$F_\sigma(0, i) := \bigcup_{k=1}^{l_i} \{(f(P_k^{(i)}), W_k^{(i)})\}, \quad 1 \leqslant i \leqslant d.$$

$$F_\sigma(x, i) := \{Mx + \mathbf{k} : \; \mathbf{k} \in F_\sigma(0, i)\},$$

and for $K \subseteq \mathcal{G}$,

$$F_\sigma(K) := \bigcup_{\mathbf{k} \in K} F_\sigma(\mathbf{k}).$$

Rigorously we should use $F_\sigma\{(x, i)\}$ instead of $F_\sigma(x, i)$. By the definitions of the incidence matrix and f, one has $M_{\sigma^n} = M_\sigma^n$ and $f(\sigma(U)) = F_\sigma(f(U))$ for every word U, and so that $F_{\sigma^n} = F_\sigma^n$.

3.2. Atomic surfaces

An infinite word over $\{1, 2, \ldots, d\}$ is a fixed point of σ if $\sigma(s) = s$, is a periodic point of σ if $\sigma^k(s) = s$ holds for some $k \geqslant 1$. A primitive substitution has at least one periodic point. For a finite or infinite word $s = s_1 \ldots s_n \ldots$, a broken line $\bar{s}$ starting from the origin is define as follows:

$$\bar{s} = \bigcup_{i \geqslant 1} \{(f(s_1 \ldots s_{i-1}), s_i)\} \subset \mathcal{G}.$$

Recall that $\pi = \pi_1 \circ \phi$ is a projection from $\mathbb{R}^d$ to the contractive subspace P. Let $s = s_1 s_2 \ldots$ be a periodic point of σ and $\bar{s}$ the broken line. Let

$Y = \{f(s_1 \ldots s_{k-1}) : \ k \geqslant 1\}$ be the set of integer points located on $\bar{s}$. Then the atomic surface of σ is the closure $X = \overline{\pi(Y)}$ of the projection of Y onto the contractive space P. Furthermore, let

$$Y_i = \{f(s_1 \ldots s_{k-1}) : \ s_k = i, \ k \geqslant 1\},$$

and $X_i = \overline{\pi(Y_i)}$, $1 \leqslant i \leqslant d$. We call the family $X_1, X_2, \ldots, X_d$ the partial atomic surfaces of σ.

As one expects, the partial atomic surfaces have a self-similar structure. (See also [11]).

THEOREM 1.3 — *Let σ be an unimodular Pisot substitution. Then the partial atomic surfaces $\{X_i\}_{i=1}^d$ are compact and satisfy the following set equation*

$$(3.1) \qquad M^{-1}X_i = \bigcup_{j=1}^{d} \bigcup_{W_k^{(j)}=i} \left(X_j + M^{-1}\pi(f(P_k^{(j)})) \right), \quad 1 \leqslant i \leqslant d.$$

The above set equation has exactly the same form as the irreducible case.

Proof —

    (i) By the definition of F_σ, we see that every element of Y can be expressed as $y = \sum_{k=0}^{N} M^k a_k$, where a_k belongs to the finite set $\{f(P_j^{(i)}) : \ 1 \leqslant i \leqslant d, \ 1 \leqslant j \leqslant l_i\}$. (If s is a periodic point satisfying $\sigma^k(s) = s$, we replace σ by σ^k.) By the Pisot assumption, we can choose a real θ such that $|\lambda_j| < \theta < 1$ holds for $2 \leqslant j \leqslant m$. Since $\pi(a_k)$ are points of the contractive eigenspace P, we have

$$|\pi(y)| = |\sum_{k=0}^{N} M^k \pi(a_k)| < C \sum_{k=0}^{\infty} \theta^k = \frac{C}{1-\theta},$$

        where $C > 0$ is a constant depends on the set $\{f(P_j^{(i)}) : \ 1 \leqslant i \leqslant d, \ 1 \leqslant j \leqslant l_i\}$. So X is bounded in the ball $B(0, C/1 - \theta) \subset P$ and the sets X_i are compact.

    (ii) First we consider the case that s is a fixed point of σ. In this case, we show that

$$(3.2) \qquad Y_i = \bigcup_{j=1}^{d} \bigcup_{W_k^{(j)}=i} (MY_j + f(P_k^{(j)})).$$

        Let us use the notation (Y_i, i) to denote the set $\{(y, i); \ y \in Y_i\}$.

Then the stair of the fixed point can be expressed as $\bar{s} = \bigcup_{i=1}^{d}(Y_i, i)$ and

$$(Y_i, i) = \{(y, i) : \ (y, i) \in \bar{s}\}$$

$$= \{(y, i) : \ (y, i) \in F_\sigma(\bigcup_{j=1}^{d}(Y_j, j))\}$$

$$= \bigcup_{j=1}^{d}\{(y, i) : \ (y, i) \in F_\sigma(Y_j, j)\}$$

$$= \bigcup_{j=1}^{d} \bigcup_{W_k^{(j)}=i} \{(My + f(P_k^{(j)}), i) : \ y \in Y_j\}.$$

Taking the starting points of the line segments in the above equation, we get (3.2). Formula (3.1) is obtained by taking the projection π and then taking a closure on both sides of (3.2).

(iii) Suppose s is a periodic point of σ, say $\sigma^k(s) = s$. Hence by (ii), $X_1, \ldots, X_d$ defined by s satisfy the set equation (3.1) for the substitution σ^k. Notice that the k-th iteration of the set equation (3.1) for σ coincides with the set equation (3.1) for σ^k, hence the set equations for σ and that for σ^k have the same invariant sets (see [16]). So $X_1, \ldots, X_d$ satisfy the set equation (3.1) for σ.

$\square$

Since the non-empty compact solution of (3.1) is unique ([16]), it is seen that the atomic surfaces do not depend on the choice of the periodic points.

4. Stepped-surface

In the reducible case, the stepped-surface and dual substitution are still powerful tools to study the atomic surfaces. In this section, we investigate the stepped-surface.

4.1. Definition of stepped-surface

Set $\mathbb{L} := \phi(\mathbb{Z}^d)$; it is seen that $\mathbb{L}$ is a full lattice (of rank m) in P_1 and $\{\phi(e_j)\}_{j=1}^{d}$ is a group of redundant generators of $\mathbb{L}$. Let

$$P^+ := \{x \in P_1 : \ \langle x, v^* \rangle \geqslant 0\} \quad \text{and} \quad P^- := \{x \in P_1 : \ \langle x, v^* \rangle < 0\}.$$

From its definition, P^+ denotes the half space of P_1 above the subspace P, according to the direction given by v^*.

We now intend to associate to any point on $\mathbb{L}$ a color described by a letter on $\mathcal{A}$. Formally, let us denote

$$\mathcal{G}^* := \{[x, i^*] : \; x \in \mathbb{L}, 1 \leqslant i \leqslant d\}$$

the set of such colored points. We define S, *the stepped-surface* of P, to be the set of *nearest* colored points $[x, i^*]$ above P, meaning that x belongs to P^+ whereas $x - \phi(e_i)$ does not.

(4.1) $\qquad S := \{[x, i^*] \in \mathcal{G}^* : \; x \in P^+ \text{ and } x - \phi(e_i) \in P^-\}.$

Notice that a point x in $\mathbb{L}$ may have several colors.

For $u \in \mathbb{R}^d$, denote by $P + u$ the hyperplane which is parallel to P and contains u. The area in P_1 between P and $P+u$, including P and excluding $P + u$, is denoted by $[P, P + u)$. As a consequence of the definition of S, $[x, i^*] \in S$ if and only if $x \in [P, P + \phi(e_i))$.

4.2. Geometrical representation of stepped-surface.

In the irreducible case, there is a natural geometrical representation of S. To any colored integer point $[x, i^*]$, one associates a face of a unit cube of $\mathbb{R}^d$ whose lowest vertex is in x and that is orthogonal to the direction e_i. We denote this face by $\overline{[x, i^*]}$, namely

$$\overline{[x, i^*]} := \{x + c_1 e_1 + \cdots + c_i e_i + \cdots + c_d e_d \in \mathbb{R}^d : \; c_i = 0, \; 0 \leqslant c_k \leqslant 1 \text{ if } k \neq i\}.$$

Then

$$\overline{S} := \bigcup_{[x, i^*] \in S} \overline{[x, i^*]}.$$

is an approximation of the space P, and sometimes it is also called the stepped-surface of P. Let us denote $\overline{[i^*]} = \overline{[0, i^*]}$ the faces of the unit cube placed in zero. Projecting the stepped-surface $\overline{S}$ to P, we obtain a polygonal tiling of P,

(4.2) $\qquad \mathcal{J}' := \{\pi \overline{[x, i^*]} : \; [x, i^*] \in S\},$

which uses d polygons $\pi\overline{[1^*]}, \ldots, \pi\overline{[d^*]}$ as prototiles.

Unfortunately all these nice properties are uncertain in the reducible case. This is the first big difference between the irreducible case and the reducible case. Here we present a possible definition of a geometrical stepped-surface. If there exist polyhedrons

$$\{T_1, T_2, \ldots, T_d\}$$

of P such that

(4.3) $\{\pi(x) + T_j;\ [x, j^*] \in S\}$

is a tiling of P, then we say this tiling is a *geometrical representation* of the stepped-surface S. It is very unclear whether there always exists a geometrical stepped-surface. However, Ei-Ito ([14]) found very strange geometrical representation for Hokkaido tiling. See Figure 5.
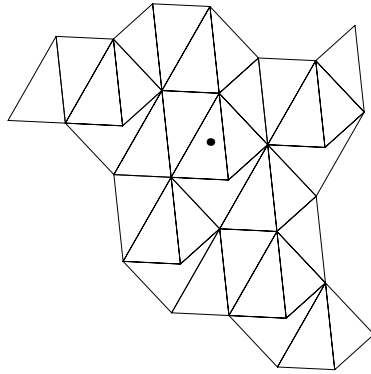

Figure 4.1. Geometrical representation of stepped-surface of Hokkaido substitution.
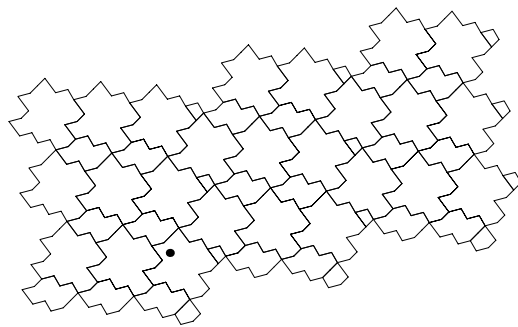

Figure 4.2. Geometrical representation of stepped-surface of substitution τ.

4.3. Quasi-periodicity of stepped-surface

Let Λ be a subset of $\mathbb{R}^d$. Λ is said to be *uniformly discrete* if there exists a positive real number r_0 such that, for any $x, y \in \Lambda$, $|x - y| > r_0$. Let E be a subset of $\mathbb{R}^d$. Λ is *relatively dense* in E if there is a positive real number R_0 such that every sphere with center in E and with radius greater than R_0 contains at least one point of Λ in its interior. A set Λ is said to be a *Delone set* in E, if it is uniformly discrete and it is relatively dense in E. (See for example [29]).

Let $\mathcal{J}$ be a translation tiling of $\mathbb{R}^d$ with finite prototiles. Denote by $B(x, r)$ the ball centered on x and with radius r. We call

$$\mathcal{P}(B(x, r)) = \{T : T \in \mathcal{J} \text{ and } T \cap B(x, r) \neq \emptyset\}$$

the *local arrangement* in $B(x, r)$.

If for any $r > 0$, there exists $R > 0$ such that for any $x, y \in \mathbb{R}^n$, the local arrangement of $B(x, r)$ appears (up to translation) in the ball $B(y, R)$, then we say that $\mathcal{J}$ is *quasi-periodic*.

Now we define the quasi-periodicity for a stepped-surface. Let

$$S_0 = \{x : [x, i^*] \in S\}$$

be the *base set* of S. For $z \in S_0$, we define the *local arrangement* of a ball $B(z, r)$ in S as

$$\mathcal{P}(B(z, r)) = \{[x, i^*] : [x, i^*] \in S \text{ and } x \in B(z, r)\}.$$

DEFINITION 4.1 — A stepped-surface S is said to be quasi-periodic provided that it satisfies the following property: for any $r > 0$, there exists $R > 0$ such that for any $x, y \in S_0$, the local arrangement of $B(x, r)$ appears (up to translation) in the local arrangement of the ball $B(y, R)$ in S.

THEOREM 1.4 — *The stepped-surface S of a Pisot substitution is quasi-periodic.*

In the irreducible case, this theorem is proved in [4] by associating the stepped-surface with a certain dynamical system on the real line. We do not know whether their method still works for the reducible case. Here we give a direct proof.

Proof — We use $D(A, B) = \inf\{|a - b|; \ a \in A, b \in B\}$ to denote the distance between two sets in $\mathbb{R}^d$. First, we rearrange these hyperplanes $P, P + \phi(e_1), \ldots, P + \phi(e_d)$ from below to above, and denote them by $P_0, P_1, \ldots, P_d$ after arrangement. The hyperplanes P_k, $0 \leqslant k \leqslant d$, divide the space into $d + 2$ parts $R_0, R_1, \ldots, R_{d+1}$ from below to above. Notice

that $R_0 = P^-$ and R_{d+1} is the space above (and including) the hyperplane P_d.

Since $[y, i^*] \in S$ if and only if $y \in [P, P + \phi(e_i))$, we infer that all points of $\mathbb{L}$ in a region R_k have the same color-configuration. Precisely, an integer point in R_0 and R_{d+1} has no color, an integer point in R_k, $1 \leqslant k \leqslant d$, has exact $d - k + 1$ colors.

For any $y \in \mathbb{L}$, there is a unique $j \in \{0, 1, \ldots, d+1\}$ such that $y \in R_j$; we set

$$\Delta(y) := \begin{cases} D(y, P_j) & \text{if } j < d+1 \\ +\infty & \text{if } j = d+1. \end{cases}$$

By the definition $\Delta(y) > 0$ always holds.
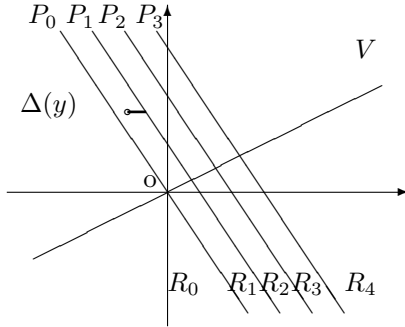
Let $z \in S_0$ and $r > 0$. We consider the distribution of the local arrangement of $\mathcal{P}(B(z, r))$ in the stepped-surface. Let

$$\epsilon(z, r) := \min\{\Delta(y) : \ y \in B(z, r) \cap \mathbb{L}\}.$$

Then $\epsilon(z, r) > 0$ since $B(z, r) \cap \mathbb{L}$ is a finite set. Suppose $P_d = P + \tilde{e}_{\max}$, i.e., $\tilde{e}_{\max}$ is the vector in $\phi(e_1), \ldots, \phi(e_d)$ such that $\pi'(\tilde{e}_{\max})$ attains the maximal value. Choose N large enough such that the distance between P and $P + \tilde{e}_{\max}/N$ is less than $\epsilon(z, r)$. We claim that $B(z, r)$ and $B(y, r)$ have the same local arrangements for $y \in [P + z, P + z + \tilde{e}_{\max}/N) \cap \mathbb{L}$.

Set $T(x) = x + (y - z)$; it is an one-one map from $B(z, r) \cap \mathbb{L}$ to $B(y, r) \cap \mathbb{L}$. Take an integer $x \in B(z, r)$. Let k be the unique number such that $x \in R_k$. If $k = d + 1$, then clearly $T(x)$ still belongs to the region R_k since $y - z$ is in the positive direction (according to v^*). If $k < d + 1$, then

$$D(P + x, P + T(x)) < D(P + x, P + x + \tilde{e}_{\max}/N)$$
$$< \epsilon(z, r) < D(P + x, P_k),$$

which means that $T(x)$ is between $P + x$ and P_k and so that $T(x)$ is still in the region R_k. Hence $x \in B(z, r)$ has the same color as $T(x) \in B(y, r)$. Our claim is proved.

It remains to show that $[P + z, P + z + \tilde{e}_{\max}/N) \cap \mathbb{L}$ is relatively dense in S_0. Since

$$S_0 = \cup_{k=0}^{N-1}[P + k\tilde{e}_{\max}/N, P + (k+1)\tilde{e}_{\max}/N)$$

is a finite union of translations of $[P, P + \tilde{e}_{\max}/N)$, we have $[P, P + \tilde{e}_{\max}/N)$ is relatively dense in S_0. Therefore $[P + z, P + z + \tilde{e}_{\max}/N) \cap \mathbb{L}$ is a Delone set, and this completes the proof. $\square$

5. Dual substitution

For $[y, i^*] \in \mathcal{G}^*$, we define $x + [y, i^*] := [x + y, i^*]$. According to set equation (3.1), we define the *dual substitution* σ^* on the set of subsets of $\mathcal{G}^*$ as follows.

$$\sigma^*[0, i^*] := \bigcup_{j=1}^{d} \bigcup_{W_k^{(j)} = i} \{[M^{-1}\phi \circ f(P_k^{(j)}), j^*]\}, \quad 1 \leqslant i \leqslant d,$$

$$\sigma^*[x, i^*] := \{M^{-1}x + \mathbf{k} : \mathbf{k} \in \sigma^*[0, i^*]\},$$

and for any $K \subseteq \mathcal{G}^*$,

$$\sigma^*(K) := \bigcup_{\mathbf{k} \in K} \sigma^*(\mathbf{k}).$$

For simplicity, we use $\sigma^*[x, i^*]$ to denote $\sigma^*\{[x, i^*]\}$. Using the dual substitution, formula (3.1) is equivalent to

$$(5.1) \qquad M^{-1}X_i = \bigcup_{[x, k^*] \in \sigma^*[0, i^*]} \pi(x) + X_k.$$

Arnoux and Ito ([5]) showed that the dual substitution keeps the stepped-surface invariant in the irreducible case. Here we generalize this remarkable property to the reducible case.

THEOREM 1.5 — *If σ is an unimodular Pisot substitution, then the stepped surface is invariant under the action of the dual substitution σ^*. Precisely,*

(i) $\sigma^*(S) = S$.

(ii) $\sigma^*[x, i^*] \cap \sigma^*[y, j^*] = \emptyset$, *for distinct* $[x, i^*], [y, j^*] \in S$.

The proof is very similar to the irreducible case ([5]).

Proof — We decompose the proof into the following three claims.

Claim 1. If $[x, j^*]$, $[x', j'^*] \in S$ *and* $[x, j^*] \neq [x', j'^*]$, *then* $\sigma^*[x, j^*] \cap \sigma^*[x', j'^*] = \emptyset$.

Suppose on the contrary that $[y, i^*]$ belongs to the intersection of $\sigma^*[x, j^*]$ and $\sigma^*[x', j'^*]$. Then, by the definition of σ^*, there exist k, k' such that

$$y = M^{-1}(x + \phi \circ f(P_k^{(i)})) = M^{-1}(x' + \phi \circ f(P_{k'}^{(i)})).$$

It follows that $x + \phi \circ f(P_k^{(i)}) = x' + \phi \circ f(P_{k'}^{(i)})$.

If $x = x'$, then $\phi \circ f(P_k^{(i)} - P_{k'}^{(i)}) = 0$, and so that $f(P_k^{(i)} - P_{k'}^{(i)}) = 0$ by Lemma 2.5 (ii). It follows that $k = k'$ and that $j = W_k^{(i)} = W_{k'}^{(i)} = j'$. Hence we get $[x, j^*] = [x', j'^*]$, which is a contradiction.

If $x \neq x'$, then $k \neq k'$ by the above discussion. Let us assume that $k < k'$ without loss of generality. Then

$$x - \phi \circ f(W_k^{(i)} \cdots W_{k'-1}^{(i)}) = x'.$$

Notice that $W_k^{(i)} = j$ and $f(W_k^{(i)}) = e_j$, so

$$\langle x', v^* \rangle = \langle x - \phi \circ f(W_k^{(i)} \cdots W_{k'-1}^{(i)}), v^* \rangle$$
$$= \langle x - \phi(e_j), v^* \rangle - \langle \phi(f(W_{k+1}^{(i)} \cdots W_{k'-1}^{(i)}), v^* \rangle$$

The first term of the last line is negative since $[x, j^*] \in S$, the second term is negative by Lemma 2.5 (ii). So $(x', v^*) < 0$ and $x' \in P^-$. Again we get a contradiction and our claim is proved.

Claim 2. $\sigma^*(S) \subseteq S$.

It suffices to show that if $[x, i^*] \in S$ and $W_k^{(j)} = i$, then $[M^{-1}(x + \phi \circ f(P_k^{(j)})), j^*] \in S$.

On one hand, $x \in P^+$ implies $x + \phi \circ f(P_k^{(j)}) \in P^+$ and so that $M^{-1}(x + \phi \circ f(P_k^{(j)})) \in P^+$. On the other hand, from $M^{-1}(f(\sigma(j))) = e_j$ we have

$$M^{-1}\left(x + \phi \circ f(P_k^{(j)})\right) - \phi(e_j)$$
$$= M^{-1}\left(x + \phi \circ f(P_k^{(j)}) - \phi \circ f(\sigma(j))\right)$$
$$= M^{-1}\left(x - \phi(e_i) - \phi \circ f(S_k^{(j)})\right).$$

Now $[x, i^*] \in S$ implies $x - \phi(e_i) \in P^-$, so the vector in the above formula belongs to P^-. Therefore $[M^{-1}(x + \phi \circ f(P_k^{(j)})), j^*] \in S$.

Claim 3. $S \subseteq \sigma^*(S)$.

Suppose $[x, i^*] \in S$. Then $x' = x - \phi(e_i) \in P^-$ and that $M(x') \in P^-$. Since $f(\sigma(i)) = Me_i$, we deduce that $M(x) - \{\phi \circ f(P_1^{(i)}), \cdots, \phi \circ f(P_{l_i}^{(i)}), \phi \circ$

$M(e_i)\}$ determines a path (on $\mathbb{L}$) from $M(x)$ to $M(x')$, i.e, from P^+ to P^-. This path must go through the hyperplane P. Suppose it intersects P at the k-th step, then we have $[Mx - \phi \circ f(P_k^{(i)}), j^*] \in S$ where $W_k^{(i)} = j$. Hence by the definition of σ^* we have

$$[x, i^*] \in \sigma^*[Mx - \phi \circ f(P_k^{(i)}), j^*] \subseteq \sigma^*(S).$$

$\square$

6. Self-similar tiling system

This section can be regard as the first application of the stepped-surface and the dual substitution. We prove that the partial atomic surfaces $X_1, \ldots, X_d$, as the unique invariant sets of (3.1), have non-empty interiors and the right sides of (3.1) consist in non-overlapping unions. Such system is called *self-similar tiling system*([35]).

6.1. Substitution Delone set

First, let us cite the definition of *inflation functional equation* of Lagarias and Wang [25].

Let A be an expanding $n \times n$ real matrix, i.e. all its eigenvalues $|\lambda| > 1$. Let

$$\{\mathcal{D}_{ij}; \ 1 \leqslant i, j \leqslant d\}$$

be finite sets in $\mathbb{R}^n$. These data define a graph-directed IFS,

$$(6.1) \qquad AX_i = \bigcup_{j=1}^{d} (X_j + \mathcal{D}_{ji})$$

for $1 \leqslant i \leqslant d$. There are unique non-empty compact sets $X_1, \ldots, X_d$, $X_i \subset \mathbb{R}^n$, satisfy the system (6.1) ([16]). The *subdivision matrix* associated to (6.1) is

$$B = [\sharp\mathcal{D}_{ji}]_{1 \leqslant i, j \leqslant d},$$

where $\sharp\mathcal{D}_{ij}$ denotes the cardinality of $\mathcal{D}_{ij}$.

The data A and $\mathcal{D}_{ij}$ define another system as follows.

$$(6.2) \qquad Z_i = \bigcup_{j=1}^{d} (A(Z_j) + \mathcal{D}_{ij}), \quad 1 \leqslant i \leqslant d.$$

This equation is called the *inflation functional equation* ([25]).

Actually in [25], the inflation functional equations are defined for a *multi-set* family $(Z_1, \ldots, Z_d)$. However, for our purpose, a normal family $(Z_1, \ldots, Z_d)$ is sufficient. A solution $(Z_1, \ldots, Z_d)$ of the inflation functional equation (6.2) is a *substitution Delone set family* if each set Z_i is a Delone set. It is seen that the subdivision matrix of (6.2) is ${}^tB$. Let us denote by $\rho(B)$ the absolute value of the maximal eigenvalue of B. The following theorem is a partial result of Theorem 5.1 in [25].

THEOREM 6.1 — *Let (6.2) be an inflation function equation that has a primitive subdivision matrix ${}^tB$ such that*

$$\rho({}^tB) = |\det A|.$$

If there exists a family of Delone sets $(Z_1, \ldots, Z_d)$ which is a solution of (6.2), then

  (i) *the unique non-empty compact solution $X_1, \ldots, X_d$ of (6.1) consists of sets X_i that have positive Lebesgue measure, $1 \leqslant i \leqslant d$.*
 (ii) *The system (6.1) satisfies an open set condition, i.e., the right side of (6.1) consists in non-overlapping union.*
(iii) *$X_i = \overline{X_i^\circ}$ and ∂X_i has Lebesgue measure 0, where ∂X_i denotes the boundary of X_i.*

6.2. Atomic surfaces

Now we apply Theorem 6.1 to the atomic surface. First, the action of M^{-1} on the subspace P is a linear map, and hence it is equivalent to a $(m-1) \times (m-1)$ real matrix. Let us denote this matrix by A. Since M is an unimodular Pisot matrix, we have that A is expanding and $|\det A| = \lambda$ is the Perron-Frobenius eigenvalue of M. Let us define

$$\mathcal{D}_{ji} := \{M^{-1}\pi(f(P_k^{(j)})); \ W_k^{(j)} = i\}, \quad 1 \leqslant i, j \leqslant d.$$

Using these data, system (6.1) coincides with the system (3.1), and hence the partial atomic surfaces $X_1, \ldots, X_d$ are the unique invariant sets of (6.1). It is seen that the subdivision matrix $B = {}^tM$.

A solution of Delone sets of (6.2) can be obtained by projecting the stepped-surface to the contractive space P. Precisely, let

$$Z_i := \{\pi(x); \ [x, i^*] \in S\},$$

then Theorem 1.5 amounts to say that $\{Z_1, \ldots, Z_d\}$ is a solution of (6.2). Therefore by Theorem 6.1, we have

THEOREM 1.6 — *Let σ be an unimodular Pisot substitution. Then*

(i) *The interiors of the partial atomic surfaces X_i are not empty.*

(ii) *The right side of (3.1) consists in non-overlapping unions.*

(iii) *$X_i = \overline{X_i^\circ}$ and ∂X_i has Lebesgue measure 0, where ∂X_i denotes the boundary of X_i.*

7. Two Tilings arising form atomic surfaces

7.1. A self-replicating collection $\mathcal{J}$

Combining the atomic surfaces and the stepped-surface together, we define the following collection

$$\mathcal{J} := \{\pi(x) + X_i; \ [x, i^*] \in S\}$$

$$= \bigcup_{i=1}^{d}\{z + X_i; \ z \in Z_i\}.$$

We say $\mathcal{J}$ is a *self-replicating collection* in the following sense: let us blow up the collection $\mathcal{J}$ by M^{-1}, then $M^{-1}\mathcal{J} := \{M^{-1}T : T \in \mathcal{J}\}$ is a collection with the prototiles $\{M^{-1}X_i\}_{i=1}^{d}$. Subdividing $M^{-1}X_i$ into small pieces according to the set equations (3.1), we obtain a new collection which coincides with $\mathcal{J}$ by Theorem 1.5. If $\mathcal{J}$ is a tiling of P, then it is a self-replicating tiling. The notion of self-replicating tiling is first introduced by Kenyon ([23]).

One may ask whether $\mathcal{J}$ is always a tiling. This problem is still open, in both the irreducible case and the reducible case. Similar to the irreducible case ([30, 34]), One can construct graphs to check whether the collection $\mathcal{J}$ is a tiling.

However, using the quasi-periodicity of the stepped-surface and the self-replicating property of $\mathcal{J}$, we could show that there exists a constant k, such that almost every point of the space P is covered by k pieces of tiles in $\mathcal{J}$. The proof is exactly the same as that of the irreducible case, which can be found in [20].

7.2. Collection $\mathcal{J}_2$ and Markov partition

In this subsection, we show how to construct Markov partition of group automorphism on torus by using atomic surfaces.

Recall π' is the projection from $\mathbb{R}^d$ to V defined in Section 2. Let

$$\hat{X}_i := \{X_i - \theta\pi'(e_i) : \ 0 \leqslant \theta < 1\}, \quad 1 \leqslant i \leqslant d,$$

and let $\hat{X} := \bigcup_{i=1}^d \hat{X}_i$. Set

$$\mathcal{J}_2 := \{z + \hat{X} : \ z \in \mathbb{L}\} = \hat{X} + \mathbb{L}.$$

THEOREM 1.7 — *The collection $\mathcal{J}_2$ is a tiling of P_1 if and only if $\mathcal{J}$ is a tiling of P.*

Proof — Suppose that $\mathcal{J}$ is a tiling of P. Let us consider the intersection of P and $\hat{X} + \mathbb{L}$. By the definition of $\hat{X}$, $x + \hat{X}$ intersects P if and only if x is on the stepped surface. Precisely, $x + \hat{X}$ does not intersect P if $[x, i^*] \notin S$ for any $1 \leqslant i \leqslant d$; $[x, i^*] \in S$ implies $|\pi'(x)| < |\pi'(e_i)|$, and so that $(\pi(x) + X_i) \subseteq P \cap (x + \hat{X})$. But $\cup_{[x,i^*]\in S}\pi(x) + X_i$ is a tiling of P by our assumption, so $P \subseteq \hat{X} + \mathbb{L}$ and in fact P is tiled by $\hat{X} + \mathbb{L}$ in the sense that almost every point of P is covered but only once by $\hat{X} + \mathbb{L}$.

Now it is ready to see for any integer point $z \in \mathbb{L}$, $P + z$ is covered and "tiled" by $\hat{X} + \mathbb{L}$. Since $\pi'(\mathbb{L})$ is dense in V, for any $x \in P_1$, $P + x$ is covered and "tiled" by $\hat{X} + \mathbb{L}$. Therefore $\hat{X} + \mathbb{L}$ is a tiling of P_1. In the same manner we can prove the inverse is also true. $\square$

Since the matrix M is unimodular, it can be regarded as a group automorphism of the d-dimensional torus $P_1 \setminus \mathbb{L}$. Set $I_i = \{\theta\pi'(e_i) : \ 0 \leqslant \theta \leqslant 1\}$, $1 \leqslant i \leqslant d$. Then I_i are subsets of the expanding eigenspace V. It is easy to see that $\{I_i\}$ satisfies the set equations

$$(7.1) \qquad\qquad MI_i = \bigcup_{k=1}^{l_i} I_{W_k^{(i)}} + f(P_k^{(i)})$$

Let us rewrite (7.1) as $MI_i = \bigcup_{k=1}^{l_i} I_{ik}$ and define

$$\hat{X}_{ik} = \{x - z : \ x \in X_i, z \in I_{ik}\}.$$

It has been shown that ([11])

THEOREM 7.1 — *If $\mathcal{J}_2$ is a tiling, then the collection $\{\hat{X}_{ik} : \ 1 \leqslant i \leqslant d, 1 \leqslant k \leqslant l_i\}$ is a Markov partition of group automorphism M on the torus $P_1 \setminus \mathbb{L}$.*

8. Can X tile P periodically?

In the irreducible case, the atomic surface X can tile P periodically. Actually this tiling is a lattice tiling, where the translation set Γ is a discrete

subgroup generated by the vectors $\pi(e_2) - \pi(e_1), \ldots, \pi(e_d) - \pi(e_1)$. If σ satisfies the strong coincidence condition, then a *domain exchange transformation* can be defined on X as:

$$(8.1) \qquad\qquad E(x) = x - \pi(e_i), \quad x \in X_i.$$

(See [5].) For a point x on the common boundaries of X_i, we can set x belongs to only one of X_i to avoid ambiguity. Furthermore, if σ satisfies the super-coincidence condition, then X can be regarded as a $(d-1)$-dimensional torus and the domain exchange transformation $E(x)$ is a rotation on the torus ([20]). In this case, the rotation on the torus is topologically conjugate to the substitution dynamical system associated to σ and hence the later one has a discrete spectrum. Unfortunately, the same argument does not work for the reducible case.

Domain-exchange transformation —   In the reducible case, we can still define a domain-exchange transformation E by (8.1), and it is metrically conjugate to the substitution dynamical system of σ. The trouble is that the atomic surface X is no longer a torus.

Periodic tiling —   To show that X can tile P periodically in the irreducible case, the geometrical representation of the stepped-surface plays a crucial role. As we has pointed out in Section 4, the existence of geometrical representation is very unclear in the reducible case. Hence in general, we do not know whether X can tile P periodically or not.

Ei, Furukado and Ito have studied this problem for the substitutions in Example 1.1 and Example 1.2. By trial and error method, they found geometrical representation of the stepped-surface of these substitutions.

For the substitution in Example 1.2, they shows that X can tile P by a lattice translation set. Hence X is a two-dimensional torus and a domain exchange transformation can be defined by formula (8.1). Actually the translation set is

$$\Gamma = \{m\pi(e_2 - e_1) + n\pi(e_4 - e_1): \ m, n \in \mathbb{Z}\}.$$

The situation is different for the Hokkaido substitution. Ito and Ei ([14]) show that X can not tile P by any lattice translation set; but there is a domain U consisting of X and a translation of reflection of X, such that U admits a lattice tiling of P. Precisely, set

$$U = X \cup (-X + \pi(e_1 + e_2 + e_3)),$$
$$\Gamma = \{m\pi(e_2 - e_4) + n\pi(e_1 + e_2 - 2e_3): \ m, n \in \mathbb{Z}\},$$

then $U + \Gamma$ is a tiling of P. They also managed to define a domain exchange on the set U.
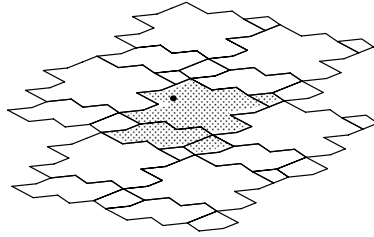

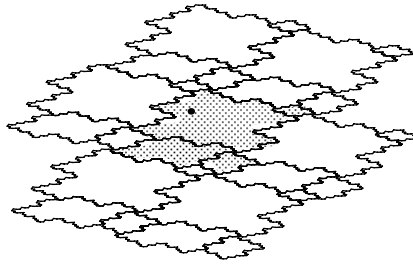Figure 8.1. *Periodic tiling by polygons of substitution τ.*

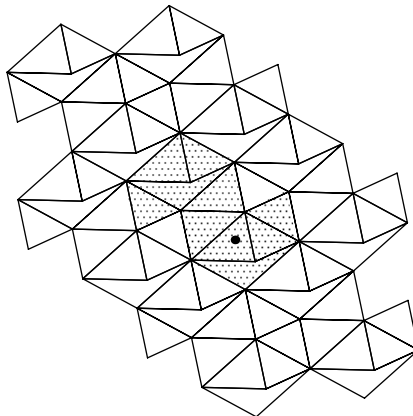Figure 8.2. *Periodic tiling by atomic surface X of substitution τ.*

Figure 8.3. *Periodic tiling by polygons of Hokkaido substitution.*

Figure 8.4. *Periodic tiling by atomic surface X of Hokkaido substitution.*

We conjecture that X can always tile P periodically if reflection and rotations are allowed.

Realization of the substitution dynamical system — Even if a periodical tiling exists, it is still a problem that how to define a rotation on the fundamental domain of the tiling to realize the substitution dynamical system. Remember that this is the original motivation of Rauzy's construction! More detailed study of this problem will carry out in [15].

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