MATH. SCAND. 31 (1972), 5–32

ATOMLESS PARTS OF SPACES

JOHN R. ISBELL

Introduction.

This paper concerns complete Brouwerian lattices regarded as a generalization of topologies (lattices of open sets). So regarding them is not a new idea. However, the previous work [1], [6], [15] seems to suppose that topology is embedded unchanged in the enlarged system. Of course one can adumbrate numerous changes of one sort or another. At least one is a considerable improvement: products of paracompact (regular) generalized spaces are paracompact.

The generalized spaces will be called *locales*. "Generalized" is imprecise, since arbitrary spaces are not determined by their lattices of open sets; but the "insertion" from spaces to locales is full and faithful on Hausdorff spaces. It preserves colimits and equalizers (subspaces with induced topology). It preserves products of compact spaces. For spaces A_i (or locales) having compactifications C_i, the locale product ΠA_i is embedded in ΠC_i and is the expected intersection of cylinders. I do not know if the cylinders $A_0 \times \Pi_{i>0} C_i$ are spaces for all spaces A_0; they are for paracompact A_0 and some others. At any rate, intersections are radically changed.

Lattices of sublocales cannot be said to be a technical improvement on Boolean algebras of subspaces, but they are very good lattices (and contain more information). Let us follow Benabou [1] in calling complete Brouwerian lattices *local lattices*; besides the awkwardness of the other term, Brouwerian people use different morphisms. A local lattice is a complete lattice in which finite meets distribute over all joins; a morphism is required to preserve finite meets and all joins. The lattice of sublocales (equalizers) in a locale A is colocal; it is the lattice of closed sublocales of a subobject $R(A)$. The map $R(A) \to A$ is merely monomorphic, not an embedding. The open sublocales are closed under the expected operations (joins and finite meets), and similarly for the closed sublocales. While the local law $x \wedge (\bigvee y_\alpha) = \bigvee (x \wedge y_\alpha)$ does not hold in general, it holds if x is open or closed.

This approach suffices for proving that finite space products of locally

Received March 15, 1971; revised April 29, 1971.

compact Hausdorff spaces are locale products, and a little on non-Hausdorff spaces. However, computing intersections of sublocales does not go far in this paper. The intersection of the rational and irrational numbers is computed for an illustration; it is the largest pointless sublocale of the rationals. Not every locale has a largest pointless sublocale, but a weak separation axiom suffices for that.

The results on paracompactness require use of uniform structure. Everything turns on the hyperspace HS(A) of closed non-empty sublocales of a uniform locale A. It is embarrassing that I cannot define a hyperlocale (which would inter al. contain A). However, knowing whether the mere hyperspace is complete determines whether A is absolutely closed. HS(A) governs my proof that the absolutely closed, or *hypercomplete*, uniform locales form a reflective subcategory. As in spaces [10], so here, A admits a hypercomplete uniformity if and only if it is paracompact. Therefore the paracompact locales are reflective, and therefore closed under products.

The problem, which space products of completely regular spaces are locale products, is related to questions in uniform spaces about which not much is known. But countable products of separable metric absolute G_δ's are preserved. For their uncountable products, the product space is not in general normal, but the product locale differs little, being the paracompact reflection of the product space.

So uniformity helps. But preservation of products is not such a delicate matter. $X \times Y$ is preserved if it is covered by the interiors of rectangles $x_i \times y_j$ which are preserved for all i and j.

It should be noted that all locales are embeddable in spaces. In fact, in spaces whose locale products are spaces, for free local lattices are topologies. Every locale has a smallest dense sublocale. That is "discrete", i.e. every monomorphism into it splits; we say *limitless*. They are doubly without limits, topologically and categorically, for they are dual to the category of complete Boolean algebras, which has limits but not colimits. A third limitation that they lack is sparseness; the Cantor set has a proper class of limitless subobjects. In general the limitless subobjects of a locale A form the lower class of a Dedekind cut. The upper class consists of the universal epic subobjects of A. The functor R (recall, $R(A)$ represents sublocales) is a subfunctor of 1 and thus can be iterated transfinitely; the lower bounds of all $R^\alpha(A)$ are the limitless subobjects. There is a smallest universal epic subobject only in the unusual case that there is a largest limitless one. It seems to be unusual also to have a smallest epic subobject, though most (not all) spaces have an obvious one, a discrete space. Oddly, every locale A has exactly one minimal epic subobject, viz. the dense part of $R(A)$.

The rings of real-valued continuous functions on locales are already known [9]. They have a natural lattice structure (positive = square) and are the archimedean homomorphs of rings of functions on spaces. They have a simpler characterization than is known, or seems possible, for the rings on spaces.

Another property besides paracompactness behaves better for locales than for topological spaces. This is quasi-compactness: each open covering has a finite subcovering. It is preserved under formation of directed inverse limits. For that, however, one need not go so far as to locales; it is true already for the maximal T_0-spaces with a given topology, sometimes called "sober" spaces, here called *primal* spaces.

1. Order.

A *local lattice* is a complete lattice satisfying $x \wedge (\bigvee y_\alpha) = \bigvee (x \wedge y_\alpha)$ for every family $\{y_\alpha\}$. A *morphism* of local lattices is a mapping preserving finite meets (including the empty meet 1) and arbitrary joins (including 0). Any category defined in this manner is called *equational* [14]. Equational categories in which free algebras exist (unlike the complete lattices and complete homomorphisms, for instance) are called *varietal* and are very good categories [14]. Benabou showed that free local lattices exist [1]. Indeed the word problem for free local lattices is easy, and they are topologies, the topologies of powers F^I of the two-point space F with one open point p. (Proof sketch. The generators are the subbasic open sets $U_i = \{x : x_i = p\}$. The basic meets $M_J = \bigcap \{U_i : i \in J\}$, J finite, are ordered in the obvious way, and their joins have unique irredundant forms. So, one readily calculates, they are free. This is in [1] except for the representation as topologies.)

The main points of the connection between topological spaces and local lattices are given in [15], but with more special treatment than is needed. The fact that it is a functorial Galois connection [12] sums it up. Every such connection between categories as good as these is representable, in this case as follows. Note that we have two categories over sets, Top (of spaces) and $\mathscr{L}$ (local lattices). On one set $\{0, 1\}$ one can put the topology of F, with 1 open, and the local lattice structure with $0 < 1$. This is a topological local lattice (though infinite meets are not continuous). Let F denote the set with both structures or either one. (Compare S^1 in the Pontrjagin connection.) Then taking the topology of a space,

$T\colon \mathrm{Top}^{\mathrm{op}} \to \mathscr{L}$, is homming into F; the continuous functions $X \to F$ are closed under the continuous local lattice operations (pointwise) and make a local lattice $T(X)$. Similarly for L in $\mathscr{L}$ we have a topological space $P(L)$ on $\mathrm{Hom}(L,F)$ in the topology of pointwise convergence, and a functor $P\colon \mathscr{L}^{\mathrm{op}} \to \mathrm{Top}$. An easy (and general) calculation shows that T and P are adjoint on the right. Now consider the form of P. The kernels of morphisms $L \to F$ are closed under joins, thus principal ideals. Their generators p must be $\wedge$-irreducible and distinct from 1; call these elements *primes* of L. Every prime yields a morphism to F. In a topology $T(X)$, every open set U is the intersection of the complements of the closures of single points not in U, thus a meet of primes. By easy calculation [15], $TPT(X)$ is isomorphic with $T(X)$; and the adjunction morphism is an isomorphism. Therefore [13] (or by parallel calculation) PTP is also naturally equivalent to P.

Consequently $T\colon \mathrm{Top}^{\mathrm{op}} \to \mathscr{L}$, regarded as a functor on Top to $\mathscr{L}^{\mathrm{op}}$, is full and faithful on *primal spaces* $P(L)$, T_0-spaces in which every prime open set is the complement of the closure of a point. Hausdorff spaces are primal. We call $\mathscr{L}^{\mathrm{op}}$ the category of *locales*, and the locales dual to topologies, *primal locales*. The functor $TP\colon \mathscr{L}^{\mathrm{op}} \to \mathscr{L}^{\mathrm{op}}$ is a coreflection taking each locale to its *primal part*.

Incidentally, though $T\colon \mathrm{Top} \to \mathscr{L}^{\mathrm{op}}$ does not preserve products, it preserves powers of F and hence takes F to a localic local lattice. That represents the duality between $\mathscr{L}$ and $\mathscr{L}^{\mathrm{op}}$, but not very usefully, since one has to associate to a locale A its set of points as underlying set—and this is not enough.

(Idly inquiring about other topological local lattices, one sees that every local lattice L has a unique compatible T_0 topology; a set $S \subset L$ is closed if and only if it is closed under taking $s' < s$ and directed joins. The topology is primal. I have no idea which L's make localic local lattices.)

Each locale A has several associated constructs, foremost being the local lattice by means of which it is defined. Officially that is A itself considered as an object of the dual category $\mathscr{L}$, but obviously one requires a different notation. We write it $T(A)$ and may call it the *dual* of A or the *lattice of open parts* or the *paratopology* of A (following [1], [15]). Another part, the primal part, was already noted; this will be fitted in with the general notion of a *part* or *sublocale* of A, defined as an equalizer $E \to A$ of $\mathscr{L}^{\mathrm{op}}$. (Equivalently, the dual of A maps to the dual of E surjectively; the paratopology is induced). Compare coequalizers or *quotient locales*, which generalize quotient spaces. (The dual map of local lattices must be an equalizer, not the same as injective; that would give only an epimorphism of locales.) The terminology clashes with the general categorical terms, sub- and quotient objects (originally sub- and quotient gadgets [8]). Those are of less topological interest since a discrete space D mapping bijectively to, say, a real line R exhibits D as a subobject and R as a quotient object. However, we are just starting in locales, and such subtopological matters as the construction of D will be of interest for a while. Equivalence classes of epics (quotient objects) will be called *images*. The single word *subobject* seems distinctive enough to retain.

An open part $x \in T(A)$ determines a sublocale (part) $X \to A$. Then $T(X)$ is the principal ideal generated by x, and the map $X \to A$ is defined by $T(A) \to T(X)$, where $y \mapsto y \wedge x$. Clearly this is a surjective morphism, so that X is really a part. There are also closed parts CX or $A - X$ indexed bijectively by the open parts X of A. Then $T(CX)$ is the principal dual ideal on x, and $T(A)$ maps to it by $y \mapsto y \vee x$.

Let us summarize the first facts about the sublocales of A; all nontrivial portions of the proofs will be given. They form a complete lattice, in fact a *colocal lattice* (an antiisomorph of a local lattice). The closed sublocales form a subobject, i.e. finite joins and all meets of closed parts are closed. Inserting the lattice of open parts preserves finite meets and all joins. Thus all sublocales have closures and interiors just as in spaces. Not all have complements; but closed CX is the complement of open X. What complements exist are unique, since the lattice is distributive.

Proving these assertions will precipitate us into constructions "refining the paratopology". Let us define a *basis* for A as a subset of $T(A)$ generating $T(A)$ by unrestricted joins. (A *sub-basis* is a local-lattice generating set, a *covering* is a subset with join 1, etc.) The locale A is *zero-dimensional* if it has a basis of open parts which are also closed. The locale A is *limitless* if every sublocale of A is open.

REMARK. In limitless A, trivially, closed sublocales are open. Hence $T(A)$ is a complemented lattice. Immediately, $T(A)$ is a complete Boolean algebra. Thence one may calculate that A is limitless, closing a loop. (1.6 improves it.)

The complemented local lattices are obviously closed under forming products and homomorphic images. Since complements in local lattices are unique, they are also closed under forming equalizers. Not just equalizers of pairs of morphisms $K \to L$ where K and L are complemented; it suffices that K should be complemented.

One expects a subcategory of a complete category which has these closure properties to be epireflective. More precisely, if the solution set condition holds it is reflective; in any case, each object that does have a reflection in it will map epimorphically to its reflection. The expectation fails here. The subcategory consists of the complete Boolean algebras, and the local lattice morphisms, preserving 0, complements, and arbitrary joins, are complete Boolean homomorphisms. In particular, a reflection of the free local lattice on $\aleph_0$ generators would be a free complete Boolean algebra on $\aleph_0$ generators, which does not exist. (This theorem of Gaifman and Hales has a short proof by Solovay [16].) To justify a statement in the introduction, reflecting the topology of the Cantor set X is the same unsolvable problem. Apropos of the question, what locales (containing no Cantor set) may have a limitless coreflection, a related problem is solved in [4].

Returning to the lattice $S(A)$ of sublocales of any locale A, joining them amounts to intersecting congruence relations, whence easily the insertion $i: T(A) \to S(A)$ preserves joins. As for meets, we want more. Let $i(x)$ be an open sublocale and s an arbitrary sublocale. A candidate for $s \wedge i(x)$ (a common lower bound) is the open part of s given by the image of x under $T(A) \to T(s)$. Moreover, that is the largest sublocale of s on which the image of x agrees with 1; so it is $s \wedge i(x)$. Explicitly, *u, v in $T(A)$ agree on $s \wedge i(x)$ if and only if $u \wedge x$ and $v \wedge x$ agree on s.*

In particular, i preserves joins and finite meets. For closed parts CX defined by x, we want: *u, v in $T(A)$ agree on $s \wedge CX$ if and only if $u \vee x$ and $v \vee x$ agree on s.* (In effect, the same proof.) Next, complements. If u, v agree on X and on CX, that is $u \wedge x = v \wedge x$ and $u \vee x = v \vee x$, then $u = v$ since $T(A)$ is distributive; $X \vee CX = 1$. As x agrees with 0 on CX and with 1 on X, $X \wedge CX = 0$. Next, on the meet of CX_i, 0 agrees with every finite join of the X_i and thus with $\bigvee X_i$. So the meet in $S(A)$ is all the way down to the closed meet. Joins are evident.

(1.1) In any complete lattice M call an element x *linear* if $x \wedge (\bigvee y_\alpha) = \bigvee (x \wedge y_\alpha)$ identically.

REMARK. Finite meets of linear elements are linear.

Call the joins of linear elements *smooth.*

REMARK. The smooth elements, as a partially ordered set, form a local lattice L.

Insertion $L \to M$ preserves joins, of course, including 0 and 1, but generally not meets. Thus M is local if and only if every element is linear, and if and only if every element is smooth.

(1.2) *In a complete lattice L, whenever $x > y$ there is a linear element t*

such that $t \wedge y = 0 \neq t \wedge x$, *if and only if* L *is colocal and its anti-isomorph is the dual of a zero-dimensional locale.*

PROOF. Among open parts of a zero-dimensional locale, $y > x$ means there is a complemented u under y but not under x; so $t \vee y = 1 \neq t \vee x$, where t is the complement. This t is antilinear, hence $t \vee (\bigwedge s_\alpha) = \bigwedge (t \vee s_\alpha)$. For the two sides have the same meet with t (viz. t) and with u (viz. $u \wedge (\bigwedge s_\alpha)$).

Conversely, the indicated condition on L makes every $x \vee (\bigwedge y_\alpha) = \bigwedge (x \vee y_\alpha)$. Otherwise the left side would be less. A linear t disjoint from the left side is disjoint from the summands, so $t \wedge (x \vee y_\alpha) = t \wedge y_\alpha$, and the meet of all these is 0. So L is the anti-dual of some locale. Every linear element e of L has a complement $e^\perp$, the join of all y disjoint from e. Here $e \wedge e^\perp = 0$ since e is linear; $e \vee e^\perp < 1$ would yield non-zero t disjoint from $e \vee e^\perp$, so from e, whence $t \leqq e^\perp$, which is absurd. Now in the local lattice L^{op} we have $y > x$ implying $t \vee y = 1 \neq t \vee x$ for some complemented t; hence $y \geqq t^\perp$, and y cannot strictly exceed the join of the complemented elements under it.

From the proof of 1.2, the linear elements of one of these lattices are the complemented elements. Looking ahead (of course, outside the present chain of proof), *the linear sublocales of a locale are the complemented sublocales.*

We need enough linear elements to apply 1.2; but we have them. For open x, any u, v in $T(A)$ agree on $x \wedge (\bigvee s_\alpha)$ if and only if $u \wedge x$ and $v \wedge x$ agree on $\bigvee s_\alpha$, that is on each s_α; thus if and only if u and v agree on $\bigvee (x \wedge s_\alpha)$. Similarly a closed part is linear in $S(A)$. As noted in 1.1, it follows that the meet of an open and a closed part is linear. For 1.2, if $x > y$ in $S(A)$, some u and v agree on y but not on x; then $u \Delta v$ (which is $i(u \vee v) \wedge CM$ where $m = u \wedge v$) is disjoint from y but not from x.

1.3 THEOREM. *For every locale A there exist a zero-dimensional locale $R(A)$ and a morphism $R(A) \to A$ inducing an isomorphism between the colocal lattices of closed sublocales of $R(A)$ and of sublocales of A by direct image and inverse image. This determines $R(A)$ up to a unique isomorphism.*

This is a routine translation of what we have proved. The routine begins: by defining $T(R(A))$ and $T(A) \to T(R(A))$, respectively as an anti-isomorph of $S(A)$ and as $x \mapsto CX$. The inverse-image map $I(f)$ on equalizers in any complete category, induced by $f: B \to A$, is completely meet-preserving; the direct-image map is completely join-preserving; there are inequalities and $IDI = I$, $DID = D$. When I is injective, as here, D inverts it.

1.4. *R is a subfunctor of* 1.

Proof. The map $R(A) \to A$ is monic. For suppose two maps f,g: $B \to R(A)$ are identified into A. Observe that the inverse-image map is functorial; $I(f)$ and $I(g)$ must agree on inverse images of parts of A, that is on all closed parts CX of $R(A)$. But $I(f)(CX)$ is the dual of the pushout of the quotient map $y \mapsto y \vee x$, which one finds to be $u \mapsto u \vee T(f)(x)$. Hence $T(f)=T(g)$ and $f=g$.

Given f: $B \to A$ and the natural maps r_B: $R(B) \to B$ and r_A, we wish to factor fr_B through r_A. This is equivalent to extending $T(fr_B)$ over $TR(A)$. This is equivalent, again, to extending the induced map from the colocal lattice of closed parts of A to $S(B)$ over $S(A)$. In fact $I(f)$ is the extension. We checked above that as a function it is an extension; it remains to prove that $I(f)$ is a morphism of colocal lattices, that is, preserving finite joins. (We remarked that it preserves all meets, hence l; observe that it also preserves 0 since a non-zero locale cannot map to 0.)

We have noted that $I(f)$ takes closed parts CX to $C[T(f)(x)]$. Similarly it agrees with $T(f)$ on open parts. Consider the Boolean algebra $W(A)$ generated (finitely) in $S(A)$ by the open and the closed parts. On it, $I(f)$ is a meet-preserving function into a distributive lattice with 0 and 1. We have a generating set $G \subset W(A)$ closed under complements, on which $I(f)$ preserves complements. Hence $I(f)$ on $W(A)$ is a Boolean homomorphism. (Having meets, it suffices to show that if $p,\bar{p}$ and $q,\bar{q}$ are preserved complementary pairs, the map h preserves also the pair $p \vee q$, $\bar{p} \wedge \bar{q}$. This is clear if

$$h(p) \vee h(q) \vee h(\bar{p} \wedge \bar{q}) = 1 .$$

That follows from

$$\begin{aligned} h(q) \vee h(\bar{p} \wedge \bar{q}) \geqq h(\bar{p} \wedge q) \vee h(\bar{p} \wedge \bar{q}) &= [h(\bar{p}) \wedge h(q)] \vee [h(\bar{p}) \wedge h(\bar{q})] \\ &= h(\bar{p}) \wedge [h(q) \vee h(\bar{q})] = h(\bar{p}).) \end{aligned}$$

Now in proving 1.3 we showed that x,y in $S(A)$ are meets of $\{x_\alpha\}$, $\{y_\beta\}$ in $W(A)$. Because $S(A)$ is colocal, $x \vee y$ is $\wedge(x_\alpha \vee y_\beta)$, expressed in terms of preserved operations; and the expression is correct in $S(B)$.

Applying 1.1 to $S(A)$ gives us another local lattice $T\Sigma(A)$ of smooth sublocales of A. (In incredible generality; the smooth elements of the lattice of equalizers or any other complete lattice associated with an object X define a locale associated with X. Not functorially; not in general and not here.) Moreover, open sublocales are smooth. Inserting them even into $S(A)$ preserves local-lattice operations; hence so does inserting them into $T\Sigma(A)$, and we have a locale morphism.

1.5 Theorem. *Every locale A has a smallest dense sublocale $D(A)$, which is limitless. The composite $DR(A) \to R(A) \to A$ is monomorphic and epimorphic. Also $DR(A) = \Sigma(A)$, the dual locale of the lattice of smooth parts of A, is a minimal epic subobject of A.*

Proof. $D(A)$ is just the familiar regular open sets. Precisely, define $T(A) \to TD(A)$ by identifying any two open parts which have the same closure. This is a congruence relation for finite meets and infinite joins, so $TD(A)$ is a local lattice and we have $D(A) \to A$, a sublocale. The relation turns pseudo-complements (the join of all y disjoint from x) into complements, so $D(A)$ is limitless. Now $\{0\}$ is a coset of the congruence, so $D(A)$ is dense in A. If $j: E \to A$ is a dense sublocale, and $T(j)$ identifies two open sets u, v, then $u \Delta v$ has interior disjoint from E, hence empty; thus $T(A) \to TD(A)$ factors across $T(j)$.

What we have connecting $T\Sigma(A)$ and $TR(A)$ is an insertion

$$k: T\Sigma(A) \to S(A);$$

but it looks right, since it inserts the smooth closed parts of $R(A)$, and as we have noted "linear" = "complemented" = "open-closed", hence "smooth closed" = "regular closed" in $R(A)$. In fact k followed by complementation is order-isomorphic into $TR(A)$ and is the standard cross-section of $TR(A) \to TDR(A)$. The map $T(A) \to T\Sigma(A)$ is injective, so $\Sigma(A) \to A$ is epic. It is monic since it is a composite of monics.

A proper *sublocale* of $\Sigma(A)$ would map to a non-dense sublocale of $R(A)$ and thus into a proper sublocale of A. So 1.5 will follow from

1.6. *A subobject of a limitless locale is a sublocale.*

Proof. If N is limitless and $L \to N$ is monic, introduce the image M of L and a limitless $K = \Sigma(L)$ mapping monically and epically to L. Then $K \to M$ is epic and monic; that is, $T(M) \to T(K)$ is epic and monic in complete Boolean algebras. But a (complete) subalgebra is an equalizer. (If $k \in T(K)$ is not in $T(M)$, subtract off the join of the elements of $T(M)$ below it, and look.) So $K \to L \to M$ is a string of isomorphisms.

Remarks. We have a minimal epic subobject $\Sigma(A)$. Not smallest epic; if A is the primal space with topology $[0,1]$, the primal part of $R(A)$ is a Sorgenfrey space and $\Sigma(A)$ has no points, but obviously there is a discrete epic subobject disjoint from $\Sigma(A)$. Clearly a space A whose points have complements in $S(A)$ has a smallest epic subobject, namely the discrete space; "epic" = "missing no points". If the points are closed they have complements; even if they are open in their closures they have complements. On the other hand, the smallest dense *sublocale* $D(A)$ is not a smallest dense subobject nor even under all epic subobjects; if A is a space without open points then $D(A)$ has no points and is disjoint from the discrete subobject.

REMARK. One would like to characterize the locales which can occur as $R(A)$. They are severely restricted. One can show that if $R(A)$ is primal, it (and A) has no non-empty pointless part; if $R(A)$ is primal Hausdorff, it is scattered.

To pull the constructions together we want the concept of a *universal epimorphism*: an epic $e: E \to A$ such that pulling back e along any morphism $t: T \to A$ yields an epic $\varepsilon: P \to T$. We won't be looking at pullbacks, which I cannot compute in $\mathscr{L}^{\mathrm{op}}$. Since the pullback is a universal commutative diagram, it is equivalent to say that there is always a diagram

$$\begin{array}{ccc} E & \longrightarrow & A \\ \uparrow & & \uparrow \\ X & \xrightarrow{\;f\;} & T \end{array}$$

with f epic. Further, I know very little of general universal epics. Quotient maps $E \to A$ need not be universal (although all epics in Top, and quotient maps in any variety, are universal). For any space A is a quotient of a scattered space E, which has no non-empty pointless sublocale (clearly enough). Choose A having a pointless sublocale T. The pullback $P \to E$ of an equalizer is an equalizer. Since P maps to T it is pointless, so it is empty. We shall consider universal epic subobjects. The subobjects of an object in any category are naturally partially ordered, and if there are only a set of them, in a complete category, they form a complete lattice. We know the Cantor set has more subobjects than that in $\mathscr{L}^{\mathrm{op}}$. Note that every set of subobjects has a meet: their intersection. Every set of subobjects represented by $m_\alpha: M_\alpha \to A$ has a join, obtained by mapping $\amalg M_\alpha$ to A by means of m_α on the αth summand and factoring across the smallest possible quotient locale. (Therefore the join of a set of limitless subobjects is limitless; this will also follow from 1.7.)

1.7 THEOREM. *The upper bounds of the limitless subobjects of a locale A are the universal epic subobjects. The map $R(A) \to A$ is a universal epic subobject. Thus transfinite iteration of R by $R^{\alpha+1}(A) = RR^{\alpha}(A)$ and inter-*

section at limit ordinals stays within the universal epic subobjects. The lower bounds of all $R^{\alpha}(A)$ *are limitless. Each one is embedded in some* $R^{\alpha}(A)$, *hence open-closed in* $R^{\alpha+2}(A)$.

PROOF. If a subobject $E \to A$ contains all limitless subobjects, consider any morphism $T \to A$. Let $X \to T$ be a limitless epic subobject of T. Factor $X \to T \to A$ across the quotient Q, which is limitless and a subobject of A. So $Q \to A$ factors through E; hence E is universal epic.

If $E \to A$ is monic and universal epic and $K \to A$ a limitless subobject, pull back to P. Since $P \to K$ is monic and epic, P is limitless by 1.6 and $P \to K$ is isomorphic; so $K \to A$ factors through E.

If $f: K \to A$ is a limitless subobject, then $K = R(K)$ and f factors through $R(A)$. So the indicated transfinite iteration stays within the upper bounds of all limitless subobjects. Suppose $E \to A$ is a subobject contained in all $R^{\alpha}(A)$. The images F_{α} of E in $R^{\alpha}(A)$ map to each other epically, and $E \to F_{\alpha}$ is epic. Applying T, we get a long expanding sequence of sublattices of $T(E)$; so for some α, the map $F_{\alpha+1} \to F_{\alpha}$ is invertible. Then F_{α} is a retract of its inverse image in $R^{\alpha+1}(A)$. Thus every part of F_{α} is an inverse image of a closed part of $R^{\alpha+1}(A)$, and is closed. Hence F_{α} is limitless; and the monic epic $E \to F_{\alpha}$ is invertible.

A curious corollary:

1.8. *The locale* $\Sigma(A)$ *is the only minimal epic subobject of* A.

PROOF SKETCH. One shows first that, if B has no dense open proper part, then B is limitless. One readily checks also that, if $E \to B$ is epic and $U \to B$ is an open part, then the pullback $P \to U$ is epic. Then look at E minimal epic in A. Evidently E is limitless. Factor $E \to A$ through $R(A)$, and apply the lemmas to the image B. Any dense part of B maps epically to A; so $E \to B$ is invertible.

In the way of extrema, one might expect a largest sublocale without points. Indeed, if each point is a complemented sublocale, this is immediate from the proof of 1.2. That is a condition inherited by sublocales; so *if* A *is embeddable in a space with complemented points,* A *has a largest pointless sublocale.* There is a primal space with just one non-closed point, for which the conclusion fails. Besides the bad point p take a sequence of real lines R_n. A non-empty open set consists of p, the complements of (ordinary) compact sets in each R_n for $n > k$, and an (ordinary) open set in R_k.

If each point is not merely complemented, but is the difference of two open parts, then pointless = without order-adjacent open parts; and it is easy to check that transfinitely iterated identification of neighbors in $T(A)$ will yield $T(P)$. Therefore in a Hausdorff space A, open sets U, V agree on P if and only if $U \Delta V$ is scattered.

As mentioned in the Introduction, in the sublocales of the real line R, the subspace J of irrationals contains the pointless part of its spatial complement Q; that is, if the difference of open U, V in R is contained in Q, it is scattered (since it is locally compact). By the way, this shows that Top $\to \mathscr{L}^{\mathrm{op}}$ does not preserve monomorphisms of fairly decent spaces; the natural map $\mathsf{J}+\mathsf{Q} \to \mathsf{R}$ is one-to-one, so monic in Top, but not monic in $\mathscr{L}^{\mathrm{op}}$. The insufficiency of points for testing whether $X \to Y$ is monic is the insufficiency of epics $D \to X$ from discrete D. It is easy, using 1.6, to show that a universal epic (monic) $E \to X$ suffices, $X \to Y$ being monic if (and only if) $E \to X \to Y$ is.

NOTE: the statement in the Introduction that Top $\to \mathscr{L}^{\mathrm{op}}$ preserves equalizers is in effect true topologically but not categorically. Being an equalizer is clearly preserved; equalizing a diagram, not.

2. Compactness.

The first main point about compact locales is that they are spaces. At least, this is a sound practical conclusion. To support and clarify it we need to define compactness, and before that, to treat separation. The second and third main points of this section of the paper are that Top $\to$ $\mathscr{L}^{\mathrm{op}}$ preserves products of compact spaces, and that the preservation of finite products is a local property.

The treatment of separation is fuller than I intended when planning this paper, because it turns out (it seems) that it is impossible to say "Hausdorff" for locales. We call a locale A *strongly Hausdorff* if the diagonal in $A \times A$ is closed. This is a very well behaved property, sufficient for the main results; but it is, as the name says, stronger than "Hausdorff" for spaces. Also it suggests the problem of a topological characterization of strongly Hausdorff spaces, not solved here. So two other separation properties will be introduced (besides regularity [15], which behaves more simply). Unfortunately neither is weaker than strong Hausdorffness for general locales; but in the compactness line, they extend the first main point to a class of locales containing those determined by all quasi-compact T_1-spaces (not necessarily primal) and the second to a class of quasi-compact spaces containing those which are locally Hausdorff.

A locale is called *quasi-compact* when each open covering has a finite subcovering, and *regular* when each open part is a join of open parts whose closures it contains [15]. We call a locale A *fit* if the colocal lattice $S(A)$ is generated by the open parts; *subfit* if every open part is a join of closed parts. Subfitness is not as well behaved as the other properties; we shall see that it is not hereditary nor productive. Its main virtue is in

2.1. *Quasi-compact subfit locales are primal.*

Proof. It suffices to show that there is a point in every non-zero difference of two open parts, i.e. in the meet of an open and a closed part. The closed part B is quasi-compact. A non-empty quasi-compact locale has a maximal open proper part u, by Zorn's lemma; u is prime and defines a point. Now a linear part B of a subfit locale is subfit, so its non-empty open parts contain non-empty closed parts and (here) points.

It is easy to see that not all quasi-compact locales are primal; indeed, any locale can be embedded in a quasi-compact one having just one more point.

Another exercise: A topological space, primal or not, defines a subfit locale if and only if every neighborhood of any point p contains a non-empty closed subset of the closure $\overline{p}$. So T_1-spaces are subfit. spec(Z) is an example of a subfit space that is not T_1. Its subspace on the points (0) and (2) is not subfit, so the property is not hereditary even for primal spaces. It is easily seen that it is productive for spaces.

"Fit" is to "regular" as "scattered" is to "discrete". Consider: The elements in $S(A)$ generated by the open parts—which we shall call the *fitted* parts—are of course the meets of finite joins of open parts, i.e. the meets of open parts. For fitness it suffices that every closed part is fitted, since as noted for 1.3, every part is a meet of parts $u \vee k$, where u is open and k closed. Let us display

2.2. *A locale is fit if and only if none of its closed parts B has a closed proper part C, every neighborhood of which in B is dense.*

Proof. For necessity, the indicated configuration would give C a non-zero complement in B, whose smallest dense sublocale D is non-zero and contained in every open part containing C. On the other hand, for any closed part C, consider the meet I of its open neighborhoods. It is contained in the meet M_1 of the closed neighborhoods of C. Every neighborhood N of C in M_1 is $M_1 \cap N^*$ for some neighborhood N^* of C in the original locale; so I is contained in the meet M_2 of the closed neighborhoods of C in M_1. Continuing, and intersecting at limit ordinals, we reach a closed part M_∞ containing C in which the only closed neighborhood of C is all of M_∞. The indicated condition implies $C = M_\infty = I$.

2.3. (1) *Regular locales are fit and strongly Hausdorff.*

(2) *A primal locale is regular if and only if its space of points is a regular space.*

(3) *The space of points of a strongly Hausdorff locale is a Hausdorff space, but not all Hausdorff spaces are strongly Hausdorff.*

(4) *For spaces, "fit" is independent of "strongly Hausdorff".*

(5) *Fit locales are subfit.*

(6) *The space of points of a fit locale is* T_1.

(7) *A strongly Hausdorff locale need not be subfit.*

(8) *A locale is fit if it is the join of a locally finite family of fitted fit parts.*

Summarizing, regular implies everything and fit implies subfit. There are no other implications except that for spaces one has strongly $\mathrm{T}_2 \Rightarrow \mathrm{T}_2 \Rightarrow \mathrm{T}_1 \Rightarrow$ subfit. By (8), fitness is nearly a local property. Only nearly:

(4′) *A strongly Hausdorff locally fit space need not be fit.*

Of course, for (3) of 2.3 we shall need a space whose square in $\mathscr{L}^{\mathrm{op}}$ is not a space. Let us note that if X and Y are disjoint dense sub-spaces of a Hausdorff space Z, then $X \times Y \subset Z \times Z$ contains some of the diagonal, at least its smallest dense part D. Whether the diagonal is closed or not, its closure contains no point of $X \times Y$; so $X \times Y$ has a non-zero part whose closure contains no point, and thus it is not primal.

Hence

2.4. *The functor* Top $\rightarrow \mathscr{L}^{\mathrm{op}}$ *does not preserve squares of quasi-compact primal* T_1*-spaces.*

Proof. Let $\alpha\mathrm{Q}$ be a one-point quasi-compactification of Q. (Neighborhoods of the new point are complements of compact parts of Q.) It is quasi-compact primal T_1 and has an open part Q whose square, by the preceding remarks, is not a space; so $\alpha\mathrm{Q} \times \alpha\mathrm{Q}$ is not a space.

Proof of 2.3. (1) Fitness. Assume that each open u is the join of open parts v_i whose closures c_i it contains. On the meet of the complements w_i of c_i, each v_i agrees with 0, so their join u agrees with 0. Also in the colocal lattice of parts,

$$u \vee (\wedge w_i) = \wedge (u \vee w_i) = 1 .$$

So the complement of u is $\wedge w_i$ and is fitted.

Strong Hausdorffness. Consider any sublocale B of $A \times A$ on which the two coordinate projections $f, g: B \to A$ do not agree. For some open u in A, we have that $T(f)(u)$ and $T(g)(u)$ are distinct open parts of B. We may choose notation so that the relative complement d of $T(g)(u)$ in $T(f)(u)$ is non-zero. Note that d is linear in $S(B)$. Now, as A is regular, u is a join of open parts v_i whose closures c_i it contains. Since $d \wedge T(f)(\vee v_i)$ is not zero and $T(f)$ preserves joins, some $d_i = d \wedge T(f)(v_i)$ is not zero. But d is disjoint from $T(g)(u)$. If w_i is the complement of c_i, we have $d_i \subset v_i \times w_i$; hence B is not contained in the closure of the diagonal.

Defer (2) and note that half of (3) is trivial. On the other hand, the real line with the topology consisting of sets $U \cup (V \cap \mathsf{Q})$, U and V open in the usual topology, is Hausdorff because the topology is stronger than the usual one. It contains the usual Q and J as parts. The dense part D of the usual real line is embedded in Q and in J and thus in $\mathsf{Q} \times \mathsf{J}$ in the square of the present example. One readily verifies that D is contained in the closure of the diagonal.

Next (4′). Tear up the real line more drastically; a deleted neighborhood of a point p is a set $N \cap \mathsf{Q}$, where N is a (deleted) neighborhood. This is a locally metrizable space X. Now $X \times X$ is the join of four complemented parts, $\mathsf{Q} \times \mathsf{Q}$ and three products with a discrete factor. Those three are primal, for the product of two primal locales is primal when one of them is discrete. (Exercise, or deduce from 2.10 below.) The closure of the diagonal contains no off-diagonal point since X is Hausdorff, and its intersection with the open part $\mathsf{Q} \times \mathsf{Q}$ is on the diagonal since Q is regular. So X is strongly Hausdorff. But the neighborhoods of J are as dense as ever.

For (5), given an open part u of a fit locale, its complement k is a meet of open parts v_i. Take their complements h_i. Since $k \leqq v_i$, we have $u \geqq h_i$; hence $u \geqq \vee h_i$, so $k \wedge (\vee h_i) = 0$. If $k \vee (\vee h_i)$ were not 1 it would be disjoint from some non-zero part p. Then p is disjoint from h_i, contained in v_i, contained in k, and so $p = 0$. So $\vee h_i = u$.

(6). The space being T_1 means that every prime open part u is maximal. Suppose on the contrary there were an open proper part v greater than u. Every open neighborhood w of the complement k of v satisfies $w \vee v = 1$, not $w \leqq u$; the one-point sublocale corresponding to u is contained in w, while it is not contained in k. So k is not fitted, the locale is not fit.

Using (6), (2) is an easy exercise. For (7), take part of the example X for (4′), the join of discrete J and the pointless part P of Q. Closed partx of X contained in Q are scattered; now the open part P contains no non-zero closed part.

(8). Let A have a locally finite family of fitted fit parts t_i with join 1. To show that a closed part h is the meet of its neighborhoods, it suffices to show it locally. There are finitely many nearby t_i, each a meet of open parts u_{ij}. Let r be the join of the other t's. The near $h \wedge t_i$ are meets of relatively open v_{ik}, that is there are open w_{ik} with $t_i \wedge (\bigwedge w_{ik}) = h \wedge t_i$. The parts

$$n_{jk} = r \vee \left(\bigvee_i (w_{ik} \wedge u_{ij})\right)$$

are neighborhoods of h, and nearby, their meet equals h by colocality.

For the remaining part of (4), a non-Hausdorff closed manifold, such as a circle with a doubled point, is an example by (8).

2.5. *A compact subset of a Hausdorff space is fitted.*

PROOF. The complement is the join of open parts whose closures it contains, by the usual compactness argument; so the proof of 2.3 (1) applies.

2.6 THEOREM. *Quasi-compact locally strongly Hausdorff locales are primal and fit.*

PROOF. Since each non-empty closed part has a point, no open proper part contains the primal part P. So P is quasi-compact. If P were not the whole locale, it would be disjoint from a non-zero part E. Let D be the smallest dense part of E and B the join of P and D. B is still quasi-compact (since no open proper part contains P) and covered by open strongly Hausdorff parts $u_1, \ldots, u_n$.

Let βD be the Stone space of the Boolean algebra $T(D)$. Since $T(D)$ is complete, βD is extremally disconnected and D is naturally identified with the dense part of βD. For each point x of βD let $\mathfrak{F}_x$ be the family of closed sets in P obtainable from neighborhoods U of x by inserting $U \wedge D$ in B, closing, and intersecting with P. It has the finite intersection property. Since disjoint (open) parts of D have disjoint closures in βD, the intersection $F(x)$ of $\mathfrak{F}_x$ has at most n points, one in each u_i. Now partition P into $2^n - 1$ subsets I_j, the equivalence classes of the relation of belonging to the same u's. A relatively closed set H of I_j and a point p of I_j not in H have disjoint neighborhoods in B. For H^- is disjoint from each u_i which does not contain p. Hence p and each point of H^- have disjoint neighborhoods, whence the assertion follows by quasi-compactness of $H^- \wedge P$.

Let K_j be the set of all points x of βD such that $F(x)$ meets I_j (necessarily in just one point). Then F induces single-valued functions $f_j\colon K_j \to I_j$, continuous because there are enough closed neighborhoods. Not all K_j are nowhere dense (=disjoint from D in $S(\beta D)$). So in some open set V of βD some K_j is dense. The restriction of f_j to $C = D \wedge V$ has a graph Γ in $D \times B \subset B \times B$ disjoint from the diagonal, being contained in $D \times P$. It is non-empty; graphs of morphisms are isomorphic with their domains. It is contained in the closure of the diagonal, in fact of the partial diagonal Δ in $D \times D$. For consider Δ and Γ in $\beta D \times B$. The graph of $f_j \mid K_j \cap V$ contains Γ; and every neighborhood of x in $K_j \cap V$ meets in D every neighborhood of $f_j(x)$, so $(x, f_j(x))$ is in the closure of Δ. The contradiction establishes primality.

P is the join of the finite family of parts I_j, which are regular, hence fit. To show that they are fitted it will suffice to show that the complements M_i of the u_i are fitted. M_i is covered by $n-1$ Hausdorff open sets, so 2.5 is the case $n=2$. Inductively we may assume that parts $M_i \cap M_k$ are fitted. M_i is the join of those and $M_i \cap W$, where W is the meet of the other u's. A point of $W - M_i$ is in every u and is Hausdorff separated from everybody; so it and M_i have disjoint neighborhoods, and $M_i \cap W$ is fitted in W. Since W is open, M_i is fitted and P is fit.

In particular, *compact* locales—quasi-compact strongly Hausdorff—are primal. There is additional information in the proof above.

2.7. *Quasi-compact locally Hausdorff spaces are primal and fit.*

PROOF. Locally Hausdorff spaces are primal by an elementary calculation. As for fitness, the proof of 2.6 used strong Hausdorffness in the second, third, and fourth paragraphs; given primality, the third paragraph is not needed and the rest uses only disjoint neighborhoods of points.

One would like to unite 2.6 and 2.7 in a common generalization. I have nothing to suggest, but note that one cannot intersect the hypotheses; a quasi-compact locale with a closed Hausdorff set of points might be a one-point quasi-compactification of almost anything.

2.8. *Regular, strongly Hausdorff, and fit locales form epireflective full subcategories.*

PROOF. Evidently, having a closed diagonal is inherited by sublocales and transmitted to products. Fitness is hereditary because it is determined by infinite meets in $S(A)$; regularity because it is determined by joins in $T(A)$ and order in $S(A)$. Similarly a product of fit locales ΠA_i has coordinate projections f inducing completely meet-preserving $I(f)$, which extends $T(f)$; so sub-basic closed parts are fitted, and thence all closed parts. Regularity of products is as easy. The solution set condition holds since morphisms factor through image parts and $\mathscr{L}^{\mathrm{op}}$, the dual of a variety, is co-well-powered.

2.9 THEOREM. *Product spaces of fit quasi-compact spaces are fit quasi-compact and are the product locales of their factors.*

PROOF. Of course the product space is quasi-compact by Tychonoff's theorem and easily shown to be fit. That is not part of the proof. The product locale is quasi-compact by [6], fit by 2.8, primal by 2.1. Hence it is the product in the category of primal spaces. In Top, that is reflective and closed under products.

2.10 THEOREM. *The product locale $W \times Y$ of two primal spaces is the product space if W is covered by the interiors of subspaces V_i for which $V_i \times Y$ is the product space, and also if W is a complemented sublocale of a primal space X for which $X \times Y$ is the product space.*

PROOF. First, a sublocale S of a space X having a complement T is a subspace. For each point is in S or T since $S \vee T = 1$; so $S \geqq S_0$, $T \geqq T_0$, where S_0 and T_0 are subspaces and $S_0 \vee T_0 = 1$. As $S_0 \wedge T_0 = S \wedge T = 0$ and complements are unique, T is T_0, and S is S_0. Now suppose $X \times Y$ is a space and consider projection $f: X \times Y \to X$. The locale $S \times Y$ is the equalizer $I(f)(S)$. As proved in 1.4, $I(f)$ preserves finite joins as well as meets. So it preserves complements. Now $S \times Y$, being complemented, is a subspace, and as $S \times Y$ and $T \times Y$ contain the space products, they are the space products.

If $\{u_i\}$ is an open covering of X, then $\{u_i \times Y\}$ is an open covering of $X \times Y$ since $T(f)$ is a morphism. Hence the hypothesis on W and $\{V_i\}$ implies the conclusion since the interior of V_i in W is open in V_i, and complemented.

Using 2.10 to extend the compact case of 2.9, the (second) "complement" clause is stronger than the "local" clause. For one readily verifies:

(i) Each compactifiable space X has a largest closed subspace N which is not locally compact at any point.

It is a three-line argument on the closure of the union of all such subspaces; however, for the next exercise, note that N is also the end of a transfinite process of removing locally compact open sets. This descent has the advantage of making sense in $S(X)$.

(ii) X is complemented in any and all compact spaces containing it if and only if N is empty. (The obstruction to existence of a complement has closure N.)

(iii) Emptiness of N is a local property of X.

Note also: For a sublocale B of a locale A, being complemented is a local property (by a short calculation, using "complemented" = "linear"). Finite product locales of locally compact spaces are spaces; locally compact topological groups are localic groups.

A different sort of preservation theorem:

2.11. *A directed inverse limit of quasi-compact locales is quasi-compact.*

Proof. First, while it is possible in principle to combine this result and the Tychonoff theorem for finite products of locales to get the general product theorem, one cannot use the present proof; it depends on the product theorem. Observe that an inverse limit of directed X_α's is a sublocale of ΠX_α which is the intersection of a directed family of parts P_α, each of which is a product; let P_α be the product of the X_β for which β does not strictly succeed α, inserted in the obvious way. It remains to show that quasi-compactness is preserved by intersecting directed families of parts P_α. It suffices to show that non-emptiness is preserved, for if $\{F_\gamma\}$ is a directed family of relatively closed non-empty parts of $\bigwedge P_\alpha$, then $\{\bar{F}_\gamma \wedge P_\alpha\}$ is a directed family of non-empty quasi-compact parts.

Given a downward directed family of non-empty quasi-compact parts P_δ, in a locale P which we may suppose quasi-compact, consider the closed parts F such that no $F \cap P_\delta$ is empty. Zorn's lemma applies to them for each δ, hence as a whole; there is a minimal such part M. Then M is not the join of two closed proper parts (each would miss some P_δ and M miss their meet). Thus M is the closure of a point $x \in S(P)$. The smallest dense part of M is not 0, so it is x. But each $P_\delta \wedge M$ is dense in M, for its closure certainly meets all P's. So $\bigwedge P_\delta$ contains x.

The three (equivalent) results just proved, on quasi-compactness and non-emptiness of intersections and inverse limits, hold by substantially the same proof in primal spaces. Prima facie, intersecting and taking limits are different operations there. I do not know if they have different values, i.e. whether directed inverse limits of quasi-compact primal locales are primal.

3. Uniformity.

The main results will require generalizing much of the first two chapters of "Uniform Spaces" [11]. Probably all that generalizes; in any case, it seems better to omit much detail but sketch the development first. We use the covering definition, and most of it is copied out below, e.g. to avoid confusion with the more general uniform locales of D. and S. Papert [15]. Entourages ought to work, but not in the present state of knowledge of product locales. We need pseudometrics and shall get them from spaces as follows. A uniform covering lies in a normal sequence, which is a basis of a preuniformity. That has a separated reflection (omitted exercise). That has a completion; the simplest way to do general completions as an addendum to this paper is to pick them out of hypercompletions (3.3). That is primal, an ordinary metric space (3.2). No further preliminaries seem necessary; it is not asserted that results in [11] not involved below can be generalized.

The *star* of a part y of a locale with respect to an open covering $\{x_i\}$ is the join of all x_i not disjoint from y. It contains the closure of y since the star and the other x_j cover. The covering $\{x_i\}$ is a *star-refinement* of $\{w_k\}$ if the star of each x_j with respect to $\{x_i\}$ is contained in some w_k. A *preuniformity* is a set μ of open coverings filtered by star-refinement. A *uniform neighborhood* of a part y with respect to μ is a part containing the star of y with respect to some covering in μ. In view of star-refinements, one can interpolate; each uniform neighborhood of y is a uniform neighborhood of a uniform neighborhood of y. Parts y, z are *far* if they have disjoint uniform neighborhoods. A preuniformity μ is a *uniformity* if each open part u is a join of parts y far from the complement of u.

Preuniform morphisms are those for which inverse images of uniform coverings are uniform. *(Pre-) uniform sublocales* or *parts* are sublocales $A \to B$ with the induced preuniformity. Morphisms inserting parts are called *embeddings*. The *image* of a morphism $C \to D$ is the smallest part of D through which it factors.

As in spaces, so in locales, a preuniformity on a dense part $A \to B$ has at most one extension. For the extension μ, it is not quite true (even in metric spaces) that an open covering of B uniform on A is uniform.

But if $\{V_j\}$ and $\{U_i\}$ are open coverings of B, if $\{V_j\}$ is a star-refinement of $\{U_i\}$, and the trace of $\{V_j\}$ on A is uniform, then $\{U_i\}$ is uniform. For when U_i contains the star of V_j it contains the closure and hence the largest open part of B whose trace on A is V_j; every open covering extending $\{V_j\}$ refines $\{U_i\}$.

3.1. *If $e: A \to B$ is a dense uniform embedding, C is a uniform locale, and $f: B \to C$ is a locale morphism for which fe is uniform, then f is uniform. If fe is a uniform embedding, so is f.*

PROOF. $T(f)$ preserves stars and star-refinements. For $\mathscr{V}$ star-refining $\mathscr{U}$ on C, the covering $f^{-1}(\mathscr{U})$ is refined by all open extensions of $fe^{-1}(\mathscr{V})$, and one of them is uniform. If fe is an embedding, then for $\mathscr{V}$ star-refining $\mathscr{U}$ on B, some $f^{-1}(\mathscr{W})$ is an open extension of the trace of $\mathscr{V}$ on A, and refines $\mathscr{U}$.

One defines convergent and Cauchy filters and (mere) completeness in the obvious way. A (pre-) uniform locale is *(pre-) metric* if its uniform coverings have a countable basis.

3.2. *A complete metric locale is primal.*

PROOF. As in 2.1, it suffices to produce one point in any non-empty complete part. (Open parts contain stars of parts, hence closures.) A descending sequence of elements of coverings from a countable basis is a Cauchy filter base and yields a point.

The *hyperspace* $\mathrm{HS}(A)$ of a uniform locale A is the uniform space whose points are the non-zero closed parts of A and whose entourages have a basis indexed by the uniform coverings $\mathscr{U}$ of A; the pair (x,y) is in the $\mathscr{U}$th entourage if the star of each with respect to $\mathscr{U}$ (its *$\mathscr{U}$-star*) contains the other. This is indeed a uniformity; if we admitted non-closed parts it would be preuniform and $\mathrm{HS}(A)$ the separated reflection. Incidentally, HS is a covariant functor. (Of course $\mathrm{HS}(f)(x)$ is the closure of the image.) A is *hypercomplete* if $\mathrm{HS}(A)$ is complete.

3.3 THEOREM. *The hypercomplete uniform locales form a reflective full subcategory.*

We shall prove this by constructing the reflection HA, so as to get also

3.4. *The reflection map r_A from a uniform locale A to its hypercompletion HA is a dense embedding.* HS(r_A) *is a completion.*

Consider arbitrary filters in $S(A)$, defined as subsets $\mathfrak{F}$ which are nonempty dual ideals ($1 \in \mathfrak{F}$, $G \geqq F \in \mathfrak{F}$ is in $\mathfrak{F}$, $F \wedge F'$ is in $\mathfrak{F}$). The *foot* of $\mathfrak{F}$ is a filter defined by a filter base Φ indexed by the uniform coverings $\mathscr{U}$; the $\mathscr{U}$th element $\Phi(\mathscr{U})$ of Φ is the meet of the $\mathscr{U}$-stars of the elements of $\mathfrak{F}$. Since $\mathscr{U}$-stars contain $\mathscr{V}$-stars when $\mathscr{V}$ refines $\mathscr{U}$, Φ is a filter base and the foot is well defined. It is its own foot, for the meet as $\mathscr{U}$ varies of the $\mathscr{V}$-star of $\Phi(\mathscr{U})$ is between $\Phi(\mathscr{V})$ and $\Phi(\mathscr{V}^*)$. More fully, $\mathfrak{F}$ is a foot if and only if each element of $\mathfrak{F}$ is a uniform neighborhood of an element of $\mathfrak{F}$ and for each $\mathscr{U}$ for some F for all F', the $\mathscr{U}$-star of F' contains F. (The latter condition defines *stable* filters [11]; stable filters are those which contain their feet.)

LEMMA. *The feet of a uniform locale, partially ordered by inclusion, form a local lattice.*

PROOF. If a filter $\mathfrak{G}$ contains $\mathfrak{F}$, each $\Gamma(\mathscr{U})$ is a subpart of $\Phi(\mathscr{U})$ and the foot of $\mathfrak{G}$ contains the foot of $\mathfrak{F}$. Then consider any set of feet $\mathfrak{F}_i$ of A, and their filter join $\#\mathfrak{F}_i$. The foot $\bigvee \mathfrak{F}_i$ of $\#\mathfrak{F}_i$ contains all $\mathfrak{F}_i$ and is contained in any foot containing all $\mathfrak{F}_i$; thus the feet form a complete lattice THA with join $\vee$.

It is easy to see that the intersection of two feet is a foot. Consider any $\mathfrak{F} \cap (\bigvee \mathfrak{G}_i)$, compared with $\bigvee(\mathfrak{F} \cap \mathfrak{G}_i)$. Unavoidably the first of these contains the second. Conversely, consider any part B belonging to the first foot. For some uniform covering $\mathscr{U}$, B contains the meet of the $\mathscr{U}$-stars of the elements of $\#\mathfrak{G}_i$. Also B contains the $\mathscr{V}$-star of an element C of $\mathfrak{F}$, for some uniform covering $\mathscr{V}$ which we may take finer than $\mathscr{U}$. For each element $E = G_1 \wedge \ldots \wedge G_k$ of $\#\mathfrak{G}_i$, the $\mathscr{V}$-star σ of $C \vee E$ is contained in the join of B and the $\mathscr{V}$-star of E; hence B contains their meet Σ as E varies, by colocality of $S(A)$. But $C \vee E$ is $\bigwedge(C \vee G_j)$, $\Sigma \in V(\mathfrak{F} \cap \mathfrak{G}_i)$, and the lemma holds.

Now putting $THA = T(HA)$, we have a locale HA. Map $\mathfrak{F} \in T(HA)$ to the join $\varrho\mathfrak{F}$ of all $v \in T(A)$ whose complements in $S(A)$ belong to $\mathfrak{F}$. This preserves finite meets since $T(A)$ is local. If v has its complement k in $\bigvee \mathfrak{F}_i$, then for some uniform covering $\mathscr{U}, k$ contains the meet of the $\mathscr{U}$-stars of the elements E of $\#\mathfrak{F}_i$. So v is contained in the join of their complements, and since the E's contain finite intersections we have $v \leqq \bigvee \varrho(\mathfrak{F}_i)$. Thus ϱ is a morphism and defines a map $r_A \colon A \to HA$. Visibly r_A is dense, that is $\mathfrak{F} \neq 0 \Rightarrow \varrho\mathfrak{F} \neq 0$.

For $u \in T(A)$, let u^* be the foot of uniform neighborhoods of the complement of u. Then $\varrho(u^*)$ is u since A is uniform (not preuniform). Thus ϱ is surjective; r_A is an embedding. Incidentally, one easily sees that u^* is the biggest foot whose trace on embedded A is u. For each uniform covering $\mathscr{U}=\{u_i\}$ of A, consider $\mathscr{U}^*=\{u^*_i\}$. It covers HA since ϱ is a morphism. Since ϱ is a morphism and u^* is the join of $\varrho^{-1}(u)$, the operation * preserves refinement and star-refinement and the $\mathscr{U}^*$ form a basis of a preuniformity. For each foot $\mathfrak{F} \in T(HA)$, each $x \in \mathfrak{F}$ is a uniform neighborhood of $z \in \mathfrak{F}$, and one can interpolate closed $y \in \mathfrak{F}$. Let v be the complement of y. Then $v^* \leqq \mathfrak{F}$. In $S(HA)$, v^* is far from the complement of $\mathfrak{F}$. The join of these v^* is plainly all of $\mathfrak{F}$. So we have a uniform locale HA, and r_A is a uniform embedding.

HS takes all embeddings $A \to B$ to embeddings; for any uniform covering $\mathscr{U}$ of A, refined by the trace of $\mathscr{V}$ covering B, elements of $\mathrm{HS}(A)$ which are $\mathscr{V}$-near are $\mathscr{U}$-near. Here, $\mathrm{HS}(A)$ is dense in $\mathrm{HS}(HA)$; a closed part K of HA is the complement of a foot $\mathfrak{F}$, and $\mathscr{U}^*$-near K in $\mathrm{HS}(HA)$ one finds the complement J of the foot of uniform neighborhoods of a closed part $F \in \mathfrak{F}$, by choosing a star-refinement $\mathscr{V}$ of $\mathscr{U}$ and such an F that for all $F' \in \mathfrak{F}$, the $\mathscr{V}$-star of F' contains F. This J is the closure $\mathrm{HS}(r_A)(F)$ of F in HA (which is the complementary form of a previous remark about open parts).

$\mathrm{HS}(HA)$ is complete. It suffices (fortunately for the notation) to show that Cauchy filters $\mathfrak{I}$ in $\mathrm{HS}(A)$ converge in $\mathrm{HS}(HA)$; and it is routine to check convergence to the complement of the foot consisting of all parts P of A such that for some $I \in \mathfrak{I}$, P is a uniform neighborhood of the join of all closed parts $i \in I$. It is a foot because $\mathfrak{I}$ is Cauchy.

It remains to show that r_A is a reflection map. For that, it suffices to show that H is functorial; for if A was already hypercomplete, r_A is an embedding for which the completion map $\mathrm{HS}(r_A)$ is bijective, whence $T(r_A)$ and r_A are invertible. Accordingly, given $f\colon A \to B$, define Hf by means of $T(Hf)$ as follows. Any foot $\mathfrak{G}$ of B has a filter base of open parts U; apply $T(f)$ to them. The resulting filter base need not be a foot base, but it defines a filter $f^{-1}(\mathfrak{G})$ which has a foot $T(Hf)(\mathfrak{G})$.

Checking, for the largest foot 1 of B, $f^{-1}(1)=1=T(Hf)$ (1). For two feet $\mathfrak{G}, \mathfrak{H}$ of B, $T(f)$ intersects open filter bases, and f^{-1} preserves $\cap$ since interiors are well defined. Taking the feet of filters also preserves $\cap$ since $S(A)$ is colocal. (That sounds backwards, as colocality is good behavior of finite joins; but a filter meet $\mathfrak{F} \cap \mathfrak{F}'$ is a filter of joins $F \vee F'$. Probably the locality of open parts and colocality of all parts are very little needed for this proof, because of the uniform-neighborhood spacing in a foot.) So $T(Hf)$ preserves finite meets.

For joins, we need to observe that the filters entering in the construction are not arbitrary. They are *spaced* filters $\mathfrak{F}$, i.e. each element of $\mathfrak{F}$ is a uniform neighborhood of an element of $\mathfrak{F}$. For a foot is spaced; f^{-1} preserves spacing when f is a uniform morphism; and spaced filters are closed under #. Now the foot of a spaced filter $\mathfrak{F}$ contains $\mathfrak{F}$, and is the least foot containing $\mathfrak{F}$ (since footing is monotone). Therefore footing preserves infinite joins (of spaced filters). So does f^{-1}, and $T(Hf)$ is a morphism of local lattices.

One calculates readily: $(Hf)r_A = r_B f$. By 3.1, Hf is uniform. Since maps are determined on dense parts, H is a functor, and 3.3 and 3.4 hold.

3.5. *A uniform space is the primal part of a hypercomplete uniform locale if and only if it is complete.*

PROOF. Indeed, the primal part of the hypercompletion of X is the completion; complete because the points form a closed set in HS(HX), and X dense because it is dense in all of HX.

3.6. THEOREM. *A uniform locale is closed wherever it is embedded if and only if it is hypercomplete.*

PROOF. Since "wherever it is embedded" is a part of a hypercomplete locale H, it suffices to identify the closed with the hypercomplete parts of H. Obviously for closed I in H, HS(I) is a principal ideal in HS(H) and is complete. Any part P is a dense part of some I. If P is a proper part of I, then there is a part $u \wedge k \neq 0$ in I disjoint from P, u open and k closed. Since u is a join of closed parts we get a non-zero closed part x of I disjoint from P. But, P being dense, one can approximate x with closed parts of P, showing that HS(P) is not complete.

3.7. COROLLARY. *The hypercompletion of a uniform locale A is the only hypercomplete locale in which A is densely embedded.*

Now we have pseudometrics; each uniform covering of a uniform locale is realized by a map into a (complete) metric space. Consequently the construction of locally fine coreflections [11] carries over. We also have the result that fully normal (subfit) locales are paracompact. (Without subfitness, the open coverings form a preuniformity μ and they all have locally finite open refinements. However, "paracompact" like "compact" should imply "separated". In a subfit fully normal locale, μ is a uniformity.) Without settling how separation axioms should be attached to the concepts, we can formulate:

3.8. *The paracompact regular locales are the fully normal subfit locales and are uniformizable.*

The proof is really unchanged from spaces, but the reader may welcome a sketch. Paracompact regular A is normal. For given disjoint closed parts j, k, their complements have open coverings whose elements u_i have closures not meeting both j and k. Well-order a locally finite refinement $\{v_\alpha\}$, and build up disjoint open J, K. Arrived at α, if v_α meets (say) j, add to J it minus the closed union of the locally finite family of closures of v_β's previously put in K.

Then any open covering $\mathscr{U}$ may be assumed locally finite, and it has an open closure-refinement $\mathscr{V}$ which has a locally finite refinement $\mathscr{W}$. The closures x_α of elements of $\mathscr{W}$ form a locally finite refinement of $\mathscr{U} = \{u_i\}$, and so do the joins y_i of all x_α contained in u_i. By Urysohn's lemma [15] there are locale maps $f_i \colon A \to [0,1]$ equal to 1 on y_i and 0 outside u_i, and the rest is clear.

3.9. THEOREM. *A uniform locale is hypercomplete if and only if it is paracompact and its locally fine coreflection is fine.*

PROOF. Locally fine coreflection λ by construction preserves the underlying locale and takes uniform embeddings to uniform embeddings. So if A is not hypercomplete, then r_A is a dense proper embedding and so is $\lambda(r_A)$; so λA is not hypercomplete. Therefore it is a dense proper part of $H\lambda A$. As in 3.6, there is a non-zero closed part x of $H\lambda A$ disjoint from λA. The complement of x is a join of open parts whose closures it contains. There is an open covering of λA which is not refined by the trace of any open covering of $H\lambda A$, so certainly not uniform; thus A does not satisfy the indicated conditions.

If A is hypercomplete, the functor H extends the uniform map $\lambda A \to A$, which is localically invertible, over $H\lambda A$; in locales, A is a retract of $H\lambda A$, thus closed, as well as dense. So λA is hypercomplete. Suppose some open covering $\mathscr{U}$ of λA were not uniform. Just as in spaces [11], form the filter N of all parts s which are uniform neighborhoods of closed parts in whose complement the trace of $\mathscr{U}$ is uniform. As in spaces, N is stable; since one can interpolate neighborhoods, it is a foot. Since $r_{\lambda A}$ is invertible, every foot has the form v^*, the uniform neighborhoods of a closed part k. Here $k \neq 0$ since $\mathscr{U}$ is not uniform. Hence k (being linear in $S(A)$) is not disjoint from every element u of $\mathscr{U}$, nor even from all closed parts y of u far from its complement. But this is absurd; the trace of $\mathscr{U}$ on u is trivially uniform, and by definition the complement of y belongs to N.

3.10 Corollary. *The paracompact regular locales form a reflective full subcategory.*

Proof. Evidently the uniformizable locales are closed under taking parts. They are also closed under products; whatever the detailed structure of $T(\Pi A_i)$ may be, the coordinate insertions of $T(A_i)$ give a subbasis, and the product preuniformity is a uniformity. Hence the uniformizable locales are epireflective. The category is equivalent to the category of fine uniform locales. The hypercompletion HA of a fine A must be fine because the map to the fine locale "homeomorphic" with HA (localically isomorphic) factors through it. Hence HA is a paracompact reflection.

Having checked first the uniformizable reflection, we can amplify; the forgetful functor from uniform locales to locales has an adjoint, taking each locale A to its uniformizable reflection with the fine uniformity. In particular, the forgetful functor preserves products.

3.11. *For locales or uniform locales, in a product of spaces the points are dense.*

Proof. In basic open sets there are points.

In either setting, the spatial product is the primal part of the locale product since coreflectors preserve products. Therefore:

3.12. Theorem. *The product locale of paracompact spaces X_i is the locale underlying the hypercompletion of their product space (for any hypercomplete uniformities on the X_i).*

Proof. The product space P is dense in the product locale Π by 3.11, and embedded as the primal part. Since Π is hypercomplete (with respect to the uniformities), 3.7 determines uniform Π as the hypercompletion of P, and the underlying locale is the product locale.

In particular, Top $\to \mathscr{L}^{\mathrm{op}}$ preserves the product of the X_i if and only if it is hypercomplete. It is known [11, Exercise VII.8] that this is a stringent condition. It holds for the product of a compact by a paracompact space, and for a countable product of completely metrizable spaces. From 2.6, one can step down (in finite products) to complemented parts and to locally preserved products. Note that this feeds back into the problem of hypercomplete product spaces; so far, it doesn't seem to feed back anything new.

A countable product locale of metrizable spaces X_i is a metrizable locale, embeddable in the product of any metric completions y_i of the X_i. At least in the separable case, it is the intersection of all such ΠY_i, or to clarify "intersection", the locale ΠX_i is the intersection of the spaces ΠZ_i over G_δ-sets Z_i containing X_i in some fixed completions Y_i. This follows from 2.10 and Exercise VII.8 (d) of [11].

3.13. *The product locale Π of uncountably many separable metrizable absolute G_δ's X_α is the paracompact reflection P^+ of their product space P.*

PROOF. That is, every morphism from P to a paracompact space extends over Π (uniquely, since P is dense). For P^+ is the hypercompletion of P in its fine uniformity. Every disjoint family of open subsets of P is countable [2], so every normal covering of P has a countable normal subcovering $\mathscr{U}$. Now $\mathscr{U}$ lies in a normal sequence of countable coverings (rather evidently here; true in general locales, by the same proof as in spaces [7]). So $\mathscr{U}$ is realized by a mapping f into a separable metric space. But f factors across a countable partial product of the X_α's [3]. That is the partial product locale; so f, and $\mathscr{U}$, extend over $\Pi, \mathscr{U}$ in the fine uniformity μ on Π. Since Π is paracompact, it is hypercomplete in μ. By 3.7, the proof is complete.

The real line R is an injective paracompact locale. One can almost say:

(3.14) R *is an injective cogenerator of the paracompact locales, and this follows from the results on rings now identifiable as* Hom$(A,$R$)$ *in* [9] *and the results on euclidean uniform coverings in* [7], *which generalize because the proofs are order-theoretic.*

This almost-assertion is numbered because it converts into a routine exercise when we recall from [7]: "*provided there are no measurable cardinal numbers*".

The fact that proofs from [7] generalize is no accident; that paper was originally about objects in a category $\mathscr{H}$ now visible as the hypercomplete uniform locales. Ginsburg and I could not find the paratopology, and therefore specialized back to spaces. The locally fine coreflection was an attempt to find "topology". Note that in $\mathscr{H}$, it does. In general a uniform locale A is locally fine if and only if HA is fine.

Should there be measurable cardinals, R would not cogenerate all paracompact locales A, in the original sense of "cogenerate": A has no proper image over which all morphisms from A to R factor [8]. Its injectiveness (with respect to equalizers in this category, i.e. closed parts) does not require such an excursion. That is just the Tietze–Urysohn extension theorem, which holds for normal locales (defined in [15], in the obvious way) by the same proof as in spaces. The cogenerating is Shirota's theorem, done order-theoretically in [7]. One need not do that either, since every uniform covering of a locale is realized by a map into a (metric) space. The crucial thing from [9] is an embedding theorem for several kinds of ringed quasi-spaces (so interpretable; stated as a dual embedding for rings). The case needed here spins off 3.13; the embedding is into P^+, the space used in [9] is P.

Comme je n'ai rien d'une troisième démonstration différente afin de tripler la délectation [5], c'est tout.

REFERENCES

1. J. Benabou, *Treillis locaux et paratopologies*, Séminaire C. Ehresmann, 1957/58. Fac· des Sciences de Paris, 1959.
2. M. Bockstein, *Un théoréme de séparabilité pour les produits topologiques*, Fund. Math. 35 (1948), 242–246.
3. H. H. Corson and J. R. Isbell, *Some properties of strong uniformities*, Quart. J. Math. 11 (1960), 17–33.
4. G. W. Day, *Free complete extensions of Boolean algebras*, Pacific J. Math. 15 (1965), 1145–1151.
5. J. Dieudonné, Math. Reviews 22 (1961), 1190–1191.
6. C. Ehresmann, *Gattungen von lokalen Strukturen*, Jber. Deutsch. Math.-Verein. 60 (1957/58), 59–77.
7. S. Ginsburg and J. Isbell, *Some operators on uniform spaces*, Trans. Amer. Math. Soc. 93 (1959), 145–168.
8. A. Grothendieck, *Sur quelques points d'algébre homologique*, Tôhoku Math. J. 9 (1957), 119–221.
9. M. Henriksen, J. Isbell, and D. Johnson, *Residue class fields of lattice-ordered algebras*, Fund. Math. 50 (1961), 107–117.
10. J. Isbell, *Supercomplete spaces*, Pacific J. Math. 12 (1962), 287–290.
11. J. Isbell, *Uniform Spaces* (Amer. Math. Soc. Math. Surveys 12), Providence, 1964.
12. J. Isbell, Top *and its adjoint relatives*, Proc. of the Kanpur Topology Conference (1968), Academia, Prague, 1971, 143–154.
13. J. Isbell, *General functorial semantics* I, Amer. J. Math. (to appear).
14. F. E. J. Linton, *Some aspects of equational categories*, Proc. Conference on Categorical Algebra (La Jolla 1965), Springer-Verlag, Berlin · Heidelberg · New York, 1966, 84–94.
15. D. Papert and S. Papert, *Sur les treillis des ouverts et les paratopologies*, Seminaire C. Ehresmann, 1957/58. Fac. des Sciences de Paris, 1959.
16. R. Solovay, *New proof of a theorem of Gaifman and Hales*, Bull. Amer. Math. Soc. 72 (1966), 282–284.

STATE UNIVERSITY OF NEW YORK, BUFFALO, N.Y., U.S.A.