ATTRACTORS IN RESTRICTED CELLULAR AUTOMATA

MIKE HURLEY

(Communicated by Kenneth R. Meyer)

ABSTRACT. The goal of this note is to extend previous results about the dynamics of cellular automata to "restricted cellular automata." Roughly speaking, a cellular automaton is a rule that updates a configuration of "states" that are arranged along the integer lattice in $\mathbb{R}$. In applications one often thinks of one of these states as "blank" or "quiescent," while the other "active" states evolve against a quiescent background. Often the physically relevant configurations are those with only a finite number of active states. If X_0 is the set of all such states, and if a cellular automaton maps X_0 to X_0, then its restriction to X_0 is a *restricted cellular automaton.* The main results show that there are rather strong constraints on the collection of attractors for any restricted cellular automaton. These constraints parallel those described in [H1] for the unrestricted case.

Cellular automata are a class of discrete dynamical systems on a Cantor set X. This Cantor set is defined as the bi-infinite product of a finite set S with itself: $X = \prod_{n \in \mathbb{Z}} S$. Giving S the discrete topology and X the product topology makes X a compact space homeomorphic to the usual Cantor middle-third set. One can consider elements of X as functions from $\mathbb{Z}$ to S, $X = \{x: \mathbb{Z} \to S\}$. The *shift map* is the homeomorphism σ on X that is defined by $(\sigma x)(n) = x(n + 1)$ for each n in $\mathbb{Z}$ and $x \in X$. A map $f: X \to X$ is a *cellular automaton* if it is continuous and commutes with σ: $f \circ \sigma = \sigma \circ f$.

The Curtis-Hedlund-Lyndon theorem [Hed] gives a characterization of cellular automata that makes them easy to construct: f is cellular automaton if and only if there is an integer $k \geq 0$ and a map $F: S^{2k+1} \to S$ with the property that

$$(fx)(n) = F(x(n - k), \dots, x(n + k))$$

for each $x \in X$ and integer n. F is called a *block map* that generates f.

Cellular automata were introduced by von Neumann as a model of biological self-reproduction. Since their introduction, and increasing in recent years, they have been proposed as models in a number of diverse areas (see, e.g., [FTS, W]). The general dynamical behavior of cellular automata is quite complex; in fact, a number of basic questions about cellular automata turn out to be formally undecidable [CHY]. On the other hand the dynamics of cellular automata are

rather severely restricted by the requirement that they commute with the shift; these constraints are due to the ergodic properties of σ [H1–3].

In using cellular automata as models, one can view a point $x \in X$ as the assignment of one of a finite number of states—the elements of S—to each site on the integer lattice in $\mathbb{R}$. Often one of the states is thought of as being "blank" or "quiescent," while the other states are thought of as "active." In this situation it may be natural to assume on physical grounds that only a finite number of sites are in active states. For definiteness suppose that $S = \{0, 1, 2, \ldots, N\}$ and let 0 denote the quiescent state. Thus attention can be restricted to the physically relevant subset of X,

$$X_0 = \{x \in X | x(n) = 0 \text{ for all but finitely many } n \in \mathbb{Z}\},$$

and to cellular automata that preserve X_0. X_0 is a countable dense subset of X that is invariant under the shift map. It is easy to see that a cellular automaton f on X will map X_0 into itself if and only if $f(z) = z$, where z is the bisequence consisting entirely of 0's, $z(n) = 0$ for all n. This will happen if and only if the generating block map F satisfies $F(0, 0, 0, \ldots, 0) = 0$. For the remainder of the paper we will consider only cellular automata that map X_0 into itself. This does not seriously limit the range of possible dynamics, for if f is any cellular automaton, then some iterate of f is conjugate to an automaton that preserves X_0. The reason for this is that f must permute the constant bisequences, so by replacing f with an appropriate iterate and relabeling the symbols we obtain a cellular automaton that has z as a fixed point.

The purpose of this note is to examine the dynamics of restricted cellular automata $f: X_0 \to X_0$. The main results will show that there are severe constraints on the collections of "attractors" that can occur for a map of this type. These results parallel results obtained in [H1] for unrestricted cellular automata.

References [H2–H3] establish analogous results where the notion of attractor is generalized using shift-invariant probability measures on X. Since X_0 is countable, any probability measure of X_0 must have an atom; that is, it must assign positive measure to some singleton. If the probability measure is also shift invariant then the orbit of any atom under the shift must be periodic. The only σ-periodic element of X_0 is the constant bisequence z consisting entirely of 0's. Consequently we will consider only topological attractors in this note.

1. DEFINITIONS AND BACKGROUND

We continue to assume that $f: X \to X$ is a cellular automaton that preserves the subset X_0. In what follows we will be using a metric d on X. Any metric that is compatible with the product topology will suffice, for example, $d(x, y) = 2^{-n}$, where $n = \min\{|j| \,|\, x(j) \neq y(j)\}$.

A subset A of X is an *attractor* of f is there is a nonempty open set U with the two properties: (i) $f(\text{clos}(U)) \subset U$, and (ii) $\bigcap_{n \geq 0} f^n(U) = A$. It follows from (i) that $\bigcap_{n \geq 0} f^n(U) = \bigcap_{n \geq 0} f^n(\text{clos}(U))$, and so the compactness of X ensures that A is compact and nonempty. If the forward orbit of a point p enters U, then it stays in U and moves asymptotically toward A under iteration by f. The set of all such points is the *basin* of A, $B(A) = \bigcup_{n \geq 0} f^{-n}(U)$. We will refer to a nonempty open set U satisfying (i) as an *absorbing set*; since X is compact, condition (i) is equivalent to the statement that U is a uniform neighborhood of $f(U)$.

There will always be at least one attractor, namely $A_{\max} = \bigcap_{n \geq 0} f^n(X)$. Any nonempty intersection of finitely many attractors is an attractor. An attractor A is *minimal* if no proper subset is also an attractor; there may be no minimal attractor. As an example, consider the identity map on X. Any nonempty set U that is both open and closed is an attractor for the identity, and since X is a Cantor set, any nonempty open and closed set has a proper subset that is also open and closed.

The following two results can be found in [H1].

1.0. Theorem. *Any cellular automaton $f: X \to X$ is of one of three mutually exclusive types*:

    (1) *f has a minimal attractor. In this case the minimal attractor A is unique, its basin $\mathrm{B}(A)$ is open and dense in X, and A shift invariant: $\sigma(A) = A$.*

    (2) *f has a pair of disjoint attractors. In this case each attractor of f contains a pair of disjoint attractors.*

    (3) *f has no minimal attractor, but the intersection of all the attractors of f is nonempty.*

1.1. Theorem. *If $Y \subset X$ is a finite set that is an attractor for the cellular automaton f, then there is a point $p \in Y$ that is fixed by f and is a minimal attractor. Moreover, p is also fixed by the shift (i.e., p is constant as a map from $\mathbb{Z}$ into the symbol set S.)*

In the next section we will show that these results remain true if X is replaced by X_0 and f by its restriction to X_0. Since X_0 is not compact, it will be necessary to generalize the definition of "attractor." The form of this generalization is primarily motivated by the desire to preserve the connection that exists in the compact case between the collection of attractors of f and the *chain recurrent set* of f. What follows is a brief outline of this connection. A point p is *chain recurrent* for f if for each $\varepsilon > 0$ there is a finite sequence $x_0, x_1, x_2, \ldots, x_{k-1}, x_k$ with the property that $d(f(x_j), x_{j+1}) < \varepsilon$ for $0 \leq j \leq k - 1$ and with $x_0 = x_k = p$. This definition is due to Conley, who also noted the following connection between the attractors of f and its chain recurrent set [C].

1.2. Theorem. *Suppose f is a continuous map from a compact space X to itself.*

    (i) *If A is an attractor and $p \in \mathrm{B}(A) - A$, then p is not chain recurrent.*

    (ii) *If p is not chain recurrent, then there is an attractor A with $p \in \mathrm{B}(A) - A$.*

The proof of this theorem can be found in [C]; [H1] contains a treatment in the context of cellular automata. Note that the theorem implies that any minimal attractor is contained in the chain recurrent set.

In extending these results to noncompact spaces it is necessary to make some changes in the definitions. In particular, the "attractors" in the compact setting will be replaced by "attractor-like sets." Examples motivating the details of the following definitions can be found in [H4].

Definitions. (i) A nonempty open subset U of X_0 is *absorbing* for f if there is a constant $\delta > 0$ with the property that U contains the δ-neighborhood of $f(U)$.

(ii) If U is absorbing then the set $A = \bigcap_{n \geq 0} \operatorname{clos}(f^n(U))$ is the *attractor-like set* determined by U. (It is possible that $\bar{A}$ is empty.) The set $\mathrm{B}(A\,;U) = \bigcup_{n \geq 0} f^{-n}(U)$ is called the *basin of A relative to U*.

(iii) The extended basin of A, $\mathrm{B}(A)$, is the union of the sets $\mathrm{B}(A\,;U)$ for all absorbing sets U that determine A.

In a compact space these definitions reduce to the definitions of absorbing set, attractor, and basin of attraction that were given earlier; see [H4] for details. The form of the definitions is motivated by the desire to obtain the analogue of Conley's theorem in noncompact spaces. The relevant result is

1.3. Theorem. *Let $f_0\colon X_0 \to X_0$ be the restriction of a cellular automaton f to X_0. Then*

   (i) *If A is an attractor-like set and $p \in \mathrm{B}(A) - A$, then p is not chain recurrent.*

   (ii) *If p is not chain recurrent, then there is an attractor-like set A with $p \in \mathrm{B}(A) - A$.*

Proof. In [H4] it is shown that conclusions (i) and (ii) will hold for any continuous map g of a metric space into itself, provided that g satisfies certain mild conditions, for example, if g is either uniformly continuous or a proper map. Since f_0 is the restriction of a continuous mapping of the compact set X, f_0 is uniformly continuous. $\square$

Theorem 1.3 can be viewed as some evidence that the above definition of attractor-like set is a dynamically natural one.

It is possible for an attractor-like set of the restricted automaton to be empty, even though we insist that absorbing sets be nonempty. This fact will complicate the rest of the paper, but it is a necessary complication. From the point of view of modeling, an attractor-like set that is empty can be viewed as "runaway growth:" the number of sites that are active (nonzero) grows without bound as the automaton is iterated, beginning at appropriate initial points. Examples of such automata are easy to construct.

The following observation will be used in the next section.

1.4. Remark. If U_1 and U_2 are absorbing sets, then so is their intersection, provided that it is nonempty.

2. THE MAIN RESULT

In this section we will show that the results of Theorem 1.0 carry over to the restricted case. The main result is the following theorem; it is phrased in terms of absorbing sets instead of attractor-like sets because the possiblity that attractor-like sets can be empty means that the notion "minimal attractor-like set" is not a particularly useful one.

2.0. Theorem. *Suppose $f_0\colon X_0 \to X_0$ is the restriction to X_0 of a cellular automaton $f\colon X \to X$. Then f_0 is of one of the following three mutually exclusive types; type j occurs here precisely when the unrestricted automaton f is of type j in Theorem 1.0.*

(1) *There is an absorbing set W^* that has the following "minimality" property: if V is any absorbing set then there is an $n \geq 0$ with $f^n(W^*) \subset V$. If A_0^* is the attractor-like set determined by W^*, then A_0^* is contained in every attractor-like set of f, A_0^* is invariant under the shift, and its extended basin is open and dense in X_0.*

(2) *There is a pair of disjoint absorbing sets. In this case any absorbing set contains a pair of disjoint absorbing sets.*

(3) *f_0 is not of type 1, but given any finite collection $W_1, \dots, W_n$ of absorbing sets, their intersection is nonempty and hence by Remark 1.4 is also absorbing.*

Theorem 2.0 will follow from Theorem 1.0 once we make the connection between attractors of f and attractor-like sets of f_0. Such a connection exists because X_0 is dense in X, and attractors in X are determined by absorbing sets, which are nonempty and open. This makes it possible to set up a correspondence between the absorbing sets of the unrestricted automaton and those of its restriction to X_0. The proof begins with three simple topological lemmas.

2.1. Lemma. *Suppose (Y, d) is a metric space and that $g\colon Y \to Y$ is continuous. If G is an absorbing set for g then so is $U = \operatorname{int}(\operatorname{clos}(G))$. Moreover G and U determine the same attractor-like set.*

Proof. Clearly $G \subset U \subset \operatorname{clos}(G)$. There is a $\delta > 0$ with the property that the distance from $g(y)$ to the complement of G is at least δ for any $y \in G$, and therefore the same is true for any $y \in U \subset \operatorname{clos}(G)$. If $y \in U$ then $d(f(y), Y - U) \geq d(f(y), Y - G) \geq \delta$, so U is absorbing. The fact that U and G determine the same attractor-like set follows from the fact that G is dense in U, so that $\operatorname{clos}(g^n(G)) = \operatorname{clos}(g^n(U))$ for each $n \geq 1$. □

2.2. Lemma. *Let U, G be as in the last lemma. Then $\operatorname{int}(\operatorname{clos}(U)) = U$.*

Proof. We want to show that the two sets $U = \operatorname{int}(\operatorname{clos}(G))$ and $P = \operatorname{int}(\operatorname{clos}(\operatorname{int}(\operatorname{clos}(G))))$ are the same. Working with P, if we delete the innermost closure operator we clearly get a subset of P; since G is open this new set is equal to U. Similarly if we delete the innermost interior operation in P we get a new set that contains P, and this new set is also equal to U. Thus $P = U$. □

In describing the relation between attractors of f on X and attractor-like sets of the restriction of f to X_0 we will be taking interiors and closures of sets Y with respect to X and to the subspace topology on X_0. We will denote these operations as $\operatorname{int}(Y ; X)$, $\operatorname{clos}(Y ; X_0)$, etc.

2.3. Lemma. (i) *If $Y \subset X_0$ then $\operatorname{clos}(Y ; X_0) = X_0 \cap \operatorname{clos}(Y ; X)$.*

(ii) *If Y is a closed subset of X, then $\operatorname{int}(Y \cap X_0 ; X_0) = X_0 \cap \operatorname{int}(Y ; X)$.*

Proof. (i) It is clear that the set on the right contains the set on the left. Conversely, there is a closed subset K of X with the property that $\operatorname{clos}(Y ; X_0) = K \cap X_0$. In particular $Y \subset K$, so that $\operatorname{clos}(Y ; X)$ is also contained in K. Thus

$$X_0 \cap \operatorname{clos}(Y ; X) \subset X_0 \cap K = \operatorname{clos}(Y ; X_0).$$

(ii) The set on the right is open in X_0 and contained in $Y \cap X_0$, so the set on the right is contained in the set on the left. To get the opposite inclusion, let O be an open set in X with the set on the left equal to $X_0 \cap O$. It suffices to show that $O \subset Y$, for then $O \subset \operatorname{int}(Y ; X)$ and $X_0 \cap O \subset X_0 \cap \operatorname{int}(Y ; X)$.

To see that $O \subset Y$ note that $O \cap X_0$ is both dense in O and contained in Y so that $O \subset \mathrm{clos}(O; X) \subset \mathrm{clos}(Y; X) = Y$. □

2.4. Proposition. *Let $f\colon X \to X$ be a cellular automaton that maps X_0 into itself, and let f_0 denote the restriction of f to X_0.*

2.4.1. If U is an absorbing set for f, then $W = U \cap X_0$ is an absorbing set for f_0. If A, A_0 are the attractor-like sets determined by U, W, then $A_0 = A \cap X_0$.

2.4.2. Suppose $V = G \cap X_0$ is an absorbing set for f_0, where G is open in X. Let $U = \mathrm{int}(\mathrm{clos}(G))$. Then U is absorbing for f, and if A, A_0 are the attractor-like sets determined by U, V, then $A_0 = A \cap X_0$. In particular, the absorbing sets V and $W = U \cap X_0$ determine the same attractor-like set of f_0.

2.4.3. In both 2.4.1 and 2.4.2, the extended basin of A_0, $\mathrm{B}(A_0)$, contains $\mathrm{B}(A) \cap X_0$.

Proof of 2.4.1. Clearly if U is absorbing relative to X then $X_0 \cap U$ is absorbing relative to X_0. It remains to show that $A_0 = A \cap X_0$. Suppose first that $p \in A_0$. Then for each $n \geq 0$, p is contained in $\mathrm{clos}(f^n(W); X_0)$. Since $X_0 \subset X$ and $W \subset U$, we have $\mathrm{clos}(f^n(W); X_0) \subset \mathrm{clos}(f^n(U); X)$. Thus p is contained in $\bigcap_{n \geq 0} \mathrm{clos}(f^n(U); X) = A$. Conversely, suppose $p \in A \cap X_0$. For each $n \geq 0$

$$p \in X_0 \cap \mathrm{clos}(f^n(U); X) = X_0 \cap \mathrm{clos}(f^n(W); X) = \mathrm{clos}(f^n(W); X_0),$$

where the first equality is due to the fact that W is dense in U and the second follows from Lemma 2.3(i). Taking the intersection over all $n \geq 0$ shows that $p \in A_0$.

Proof of 2.4.2. Since X is compact, to show U is absorbing we need only verify that $f(\mathrm{clos}(U)) \subset U$. Begin by noting that V, U, and G all have the same closures as subsets of X. Since V is absorbing for f_0, there is a positive constant δ with the property that if $x \in V$ then any point of X_0 that lies within δ of $f(x)$ must lie in $V \subset G \subset U$. The density of X_0 ensures that if p is any point of X that is within δ of $f(x)$ then $p \in \mathrm{clos}(U)$. The same argument shows that such a p has a neighborhood that is contained in $\mathrm{clos}(U)$. In short, $d(p, f(x)) < \delta$ implies that $p \in \mathrm{int}(\mathrm{clos}(U)) = U$, where the final equality follows from Lemma 2.2. To finish the proof that U is absorbing for f, suppose y is a point of $\mathrm{clos}(U)$. Since $\mathrm{clos}(U) = \mathrm{clos}(V)$, there is a point $x \in V$ with $d(f(x), f(y)) < \delta$, and so the preceding argument shows that $f(y) \in U$.

It remains to show that $A_0 = X_0 \cap A$. This is complicated by the fact that G might not be absorbing. The argument goes as follows: Lemma 2.1 shows that V and $\mathrm{int}(\mathrm{clos}(V; X_0); X_0)$ are absorbing sets for f_0 that determine the same attractor-like set, namely, A_0. On the other hand, 2.4.1 shows that $W = X_0 \cap U$ is also an absorbing set for f_0, and that W determines $X_0 \cap A$. To show that $A_0 = X_0 \cap A$ will show that the absorbing sets $\mathrm{int}(\mathrm{clos}(V; X_0); X_0)$ and $X_0 \cap U$ are equal. Let $Y = \mathrm{clos}(U; X)$ and observe

$$(*) \qquad \mathrm{clos}(V; X_0) = X_0 \cap \mathrm{clos}(V; X) = X_0 \cap \mathrm{clos}(U; X) = X_0 \cap Y,$$

where the first equality follows from Lemma 2.3(i) and the second from the fact that V is dense in U. It follows that

$$\mathrm{int}(\mathrm{clos}(V; X_0); X_0) = \mathrm{int}(X_0 \cap Y; X_0).$$

Since $Y = \text{clos}(U; X)$ is closed in X, we can apply Lemmas 2.3(ii) and 2.2 to obtain

$$\text{int}(X_0 \cap Y; X_0) = X_0 \cap \text{int}(Y; X) = X_0 \cap \text{int}(\text{clos}(U; X); X) = X_0 \cap U.$$

This completes the proof of 2.4.2.

Proof of 2.4.3. The extended basin of A_0 is the union of $\text{B}(A_0; N)$ as N ranges over the collection of absorbing sets that determine A_0, so it is sufficient to show that $\text{B}(A_0; W) = \text{B}(A) \cap X_0$. $\text{B}(A_0; W)$ is defined as $\bigcup_{n \geq 0} f_0^{-n}(W)$, and $f_0^{-n}(W) = f^{-n}(W) \cap X_0 = f^{-n}(U \cap X_0) \cap X_0$. Since X_0 is forward invariant under f, this last set is the same as $f^{-n}(U) \cap X_0$. The union of the sets $f^{-n}(U) \cap X_0$ is $\text{B}(A) \cap X_0$. □

We will refer to the following consequence of the proof of 2.4.2: for V, U as in 2.4.2, we have

$$(2.5) \qquad\qquad V \subset U \cap X_0 \subset \text{clos}(V; X_0).$$

The first inclusion is obvious and the second follows from $(*)$.

Proof of Theorem 2.0. We will show that for $j = 1, 2, 3$, if the unrestricted automaton f is of type j in Theorem 1.0, then its restriction is of type j in Theorem 2.0.

(1) Let A^* be the minimal attractor of f and let U^* be an absorbing set that determines A^*. Set $W^* = U^* \cap X_0$, and denote the attractor-like set determined by W^* by A_0^*. Suppose V is some absorbing set for f_0, and as in 2.4.2 let U be the corresponding absorbing set for f. If A is the attractor for f determined by U, then $A^* \subset A \subset U$. Since X is compact and $A^* = \bigcap_{n \geq 0} f^n(U^*)$, there is an integer n with $f^n(U^*) \subset U$. Thus

$$f^n(W^*) \subset f^n(U^*) \cap X_0 \subset U \cap X_0.$$

Combining this with (2.5) we obtain

$$f^{n+1}(W^*) \subset f(U \cap X_0) \subset f_0(\text{clos}(V; X_0)) \subset V,$$

where we use the assumption that V is absorbing for f_0 to obtain the last inclusion. Since the basin of A^* is open and dense in X, the extended basin of A_0^* will be open and dense in X_0 by 2.4.3. A_0^* is shift invariant because A_0^* is the intersection of A^* and X_0, both of which are shift invariant.

(2) Let A_1, A_2 be a disjoint pair of attractors for f, and let U_1, U_2 be associated absorbing sets. Since X is compact, the forward orbit of any point in the closure of U_i tends asymptotically toward A_i, so the closures of U_1 and U_2 must be disjoint. Let $V_i = U_i \cap X_0$ be the associated absorbing sets for f_0. V_1 and V_2 also have disjoint closures, so they determine disjoint attractor-like sets. This gives the first assertion of (2). To establish the second, consider some absorbing set V for f_0. Let G, A and U be as in 2.4.2. A is an attractor for f, so A contains a disjoint pair of attractors. Due to the compactness of X, this means that a disjoint pair of absorbing sets can be found in any given neighborhood of A; in particular, they can be found in U; call them U_1 and U_2. As above, if $V_i = U_i \cap X_0$ then V_1 and V_2 are disjoint absorbing sets for f_0. By Remark 1.4 the sets $V_i \cap V$ are absorbing if they are nonempty, so to finish the proof of (2) it is enough to show that $V \cap V_i \neq \varnothing$ for $i = 1, 2$. To

do this, recall that G is open in X with $V = G \cap X_0$ and $U = \operatorname{int}(\operatorname{clos}(G))$. Thus G is open and dense in U, so $G \cap U_i$ is open and nonempty. Since X_0 is dense in X, we conclude that $\varnothing \neq G \cap U_i \cap X_0 = (G \cap X_0) \cap (U_i \cap X_0) = V \cap V_i$.

(3) In this case f has no minimal attractor, but all the attractors of f intersect. First we show that if V_1, V_2 are absorbing sets for f_0, then $V_1 \cap V_2 \neq \varnothing$. For $i = 1, 2$ let G_i and U_i be the open subsets of X associated to V_i as in 2.4.2. Each U_i is absorbing for f and so determines an attractor. These attractors intersect, so the open set $U = U_1 \cap U_2$ is nonempty. $U_i = \operatorname{int}(\operatorname{clos}(G_i))$, so G_i is an open and dense subset of U_i, and thus $G_1 \cap G_2$ is open and dense in U. In particular, $G_1 \cap G_2$ is nonempty; since it is open, its intersection with X_0 is also nonempty. Thus $V_1 \cap V_2 = G_1 \cap G_2 \cap X_0 \neq \varnothing$. It remains to show that f_0 is not of type (1); to do this we will show that if V is any absorbing set for f_0 then there is another absorbing set W with $f^n(V)$ not contained in W for any n. Given V, let U be absorbing for f as in 2.4.2. Since the attractor determined by U is not minimal, there is another absorbing set for f, N, such that N does not contain $f^n(U)$ for any n. Let $W = N \cap X_0$. If $f^n(V) \subset W$, then

$$f^n(U) \subset \operatorname{clos}(f^n(U)) = \operatorname{clos}(f^n(V)) \subset \operatorname{clos}(W) \subset \operatorname{clos}(N)$$

where the closures are taken in X, and we have used the density of V in U. Applying f and using the assumption that N is absorbing gives the contradiction $f^{n+1}(U) \subset N$. $\square$

2.6. Corollary. *Using the notation of Theorem* 2.0, *suppose* f_0 *is a type* 1 *cellular automaton and that* A_0^* *is finite. Then either* A_0^* *is the empty set or else* $A_0^* = \{z\}$ *where* z *in* X *is the configuration that has the quiescent state at every site*: $z(n) = 0$ *for all* $n \in \mathbb{Z}$.

Proof. The theorem shows that A_0^* is shift-invariant; since it is also finite, each point in A_0^* must be periodic under the shift. If $x \in A_0^*$ is nonzero at some site n it will be nonzero at infinitely many sites, which contradicts the fact that $x \in X_0$. $\square$

Corollary 2.6 can be interpreted as saying that the only finite minimal attractor-like sets for a restricted cellular automaton correspond either to runaway growth for almost all initial conditions or to eventual extinction for almost all initial conditions.

3. n-DIMENSIONAL CELLULAR AUTOMATA

Strictly speaking the cellular automata described above are *one-dimensional* cellular automata. For each finite n there is a related concept of *n-dimensional cellular automata*. These are defined as follows: let S continue to denote a finite set of symbols, and let X be all maps from the integer lattice in $\mathbb{R}^n$ to S, $X = \{x : \mathbb{Z}^n \to S\}$. For each w in $\mathbb{Z}^n$ there is an associated shift on X, $\sigma_w : X \to X$, defined as $(\sigma_w x)(v) = x(v + w)$. An n-dimensional cellular automaton is a map $f : X \to X$ that commutes with all of these shifts.

Theorems 1.0 and 1.2 are true for n-dimensional cellular automata; see [H1]. The argument in §2 that derives Theorem 2.0 from Theorem 1.0 does not require that the cellular automata being considered to one-dimensional, so Theorem 2.0 and Corollary 2.6 also hold for any finite-dimensional cellular automaton.

References

[C]   C. Conley, *Isolated invariant sets and the Morse index*, CBMS, vol. 38, Amer. Math. Soc., Providence, RI, 1978.

[CHY]   K. Culik II, L. P. Hurd, and S. Yu, *Computation theoretic aspects of cellular automata*, Phys. D **46** (1990), 357–378.

[FTS]   D. Farmer, T. Toffoli, and S. Wolfram (eds.), *Cellular automata: proceedings of an interdisciplinary workshop*, Phys. D **10** (1984).

[Hed]   Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Math. Systems Theory **3** (1969), 320–375.

[H1]   M. Hurley, *Attractors in cellular automata*, Ergodic Theory and Dynamical Systems **10** (1990), 131–140.

[H2]   ____, *Ergodic aspects of cellular automata*, Ergodic Theory and Dynamical Systems **10** (1990), 671–685.

[H3]   ____, *Varieties of periodic attractor in cellular automata*, Trans. of the AMS, **326** (1991), 701–726.

[H4]   ____, *Chain recurrence and attraction in noncompact spaces*, Ergodic Theory and Dynamical Systems, to appear.

[W]   S. Wolfram, *Theory and applications of cellular automata*, World Scientific, Singapore, 1986.

DEPARTMENT OF MATHEMATICS AND STATISTICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106-7058

E-mail address: mgh3@po.CWRU.EDU