Attribute Implications in a Fuzzy Setting*

Radim Bělohlávek and Vilém Vychodil

Department of Computer Science, Palacky University, Olomouc,
Tomkova 40, CZ-779 00 Olomouc, Czech Republic
{radim.belohlavek, vilem.vychodil}@upol.cz

Abstract. The paper is an overview of recent developments concerning attribute implications in a fuzzy setting. Attribute implications are formulas of the form $A \Rightarrow B$, where A and B are collections of attributes, which describe dependencies between attributes. Attribute implications are studied in several areas of computer science and mathematics. We focus on two of them, namely, formal concept analysis and databases.

Keywords: attribute implication, fuzzy logic, functional dependency, concept lattice.

1 Introduction

Formulas of the form $A \Rightarrow B$ where A and B are collections of attributes have been studied for a long time in computer science and mathematics. In formal concept analysis (FCA), formulas $A \Rightarrow B$ are called attribute implications. Attribute implications are interpreted in formal contexts, i.e. in data tables with binary attributes, and have the following meaning: Each object having all attributes from A has also all attributes from B, see e.g. [22, 25]. In databases, formulas $A \Rightarrow B$ are called functional dependencies. Functional dependencies are interpreted in relations on relation schemes, i.e. in data tables with arbitrarily-valued attributes and have the following meaning: Any two objects which have the same values of attributes from A have also the same values of attributes from B, see e.g. [2, 29].

In what follows, we present an overview of some recent results on attribute implications and functional dependencies developed from the point of view of fuzzy logic. Section 2 provides an overview to some notions of fuzzy logic which will be needed. Section 3 deals with attribute implications in a fuzzy setting. Section 4 deals with functional dependencies in a fuzzy setting. Section 5 discusses Armstrong-like rules. Section 6 contains concluding remarks.

2 Preliminaries in Fuzzy Logic and Fuzzy Sets

Contrary to classical logic, fuzzy logic uses a scale L of truth degrees, the most favorite choice being $L = [0, 1]$ (real unit interval) or some subchain of $[0, 1]$.

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This enables to consider intermediate truth degrees of propositions, e.g. "object x has attribute y" has a truth degree 0.8 indicating that the proposition is almost true. In addition to L, one has to pick an appropriate collection of logical connectives (implication, conjunction, ...). A general choice of a set of truth degrees plus logical connectives is represented by so-called complete residuated lattices (equipped possibly with additional operations). The rest of this section presents an introduction to fuzzy logic notions we will need. Details can be found e.g. in [4, 24, 26], a good introduction to fuzzy logic and fuzzy sets is presented in [28].

A complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c$$

for each $a, b, c \in L$. Elements a of L are called truth degrees. Fuzzy logic is truth-functional and $\otimes$ and $\rightarrow$ are truth functions of ("fuzzy") conjunction and ("fuzzy") implication. That is, if $||\varphi||$ and $||\psi||$ are truth degrees of formulas φ and ψ then $||\varphi|| \otimes ||\psi||$ is a truth degree of formula $\varphi \& \psi$ (& is a symbol of conjunction connective); analogously for implication.

A useful connective is that of a truth-stressing hedge (shortly, a hedge) [26, 27]. A hedge is a unary function ${}^* : L \rightarrow L$ satisfying $1^* = 1$, $a^* \leq a$, $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, $a^{**} = a^*$, for each $a, b \in L$. Hedge * is a truth function of logical connective "very true", see [26, 27]. The properties of hedges have natural interpretations, see [27].

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval) or L being a finite chain. We refer to [4, 26] for details.

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ $(a \in L)$; (ii) globalization [34]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

A special case of a complete residuated lattice with hedge is the two-element Boolean algebra $\langle \{0, 1\}, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$. Note that if we prove an assertion for general $\mathbf{L}$, then, as a particular example, we get a "crisp version" of this assertion for $\mathbf{L}$ being $\mathbf{2}$.

Having $\mathbf{L}$, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". We say "fuzzy set" instead of "$\mathbf{L}$-set" if $\mathbf{L}$ is obvious. If $U = \{u_1, \ldots, u_n\}$ then A can be denoted by $A = \{{}^{a_1}/u_1, \ldots, {}^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i = 1, \ldots, n$. For brevity, we introduce the following convention:

we write $\{\ldots, u, \ldots\}$ instead of $\{\ldots, {}^{1}/u, \ldots\}$, and we also omit elements of U whose membership degree is zero. For example, we write $\{u, {}^{0.5}/v\}$ instead of $\{{}^{1}/u, {}^{0.5}/v, {}^{0}/w\}$, etc. Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in \mathbf{L}^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in \mathbf{L}^X$, $a \otimes A \in \mathbf{L}^X$ is defined by $(a \otimes A)(x) = a \otimes A(x)$.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \textstyle\bigwedge_{u \in U} \bigl(A(u) \to B(u)\bigr), \tag{1}$$

which generalizes the classical subsethood relation $\subseteq$. $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$ (A is fully contained in B). As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

A binary $\mathbf{L}$-relation $\approx$ in U (i.e., between U and U) is called reflexive if for each $u \in U$ we have $u \approx u = 1$; symmetric if for each $u, v \in U$ we have $u \approx v = v \approx u$; transitive if for each $u, v, w \in U$ we have $(u \approx v) \otimes (v \approx w) \leq (u \approx w)$; $\mathbf{L}$-equivalence if it is reflexive, symmetric, and transitive; $\mathbf{L}$-equality if it is an $\mathbf{L}$-equivalence for which $u \approx v = 1$ iff $u = v$.

In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [4, 26]. Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice and * (possibly with indices) denotes a hedge.

3 Attribute Implications

3.1 Attribute Implications, Validity, Theories and Models

We first introduce attribute implications. Suppose Y is a finite set (of attributes).

Definition 1. *A* (fuzzy) attribute implication (over Y) *is an expression* $A \Rightarrow B$, *where* $A, B \in \mathbf{L}^Y$ *(*A *and* B *are fuzzy sets of attributes).*

Fuzzy attribute implications are our basic formulas. The intended meaning of $A \Rightarrow B$ is: "if it is true that an object has all attributes from A, then it has also all attributes from B".

Remark 1. For an fuzzy attribute implication $A \Rightarrow B$, both A and B are fuzzy sets of attributes. Particularly, A and B can both be ordinary sets (i.e. $A(y) \in \{0, 1\}$ and $B(y) \in \{0, 1\}$ for each $y \in Y$), i.e. ordinary attribute implications are a special case of fuzzy attribute implications.

A fuzzy attribute implication does not have any kind of "validity" on its own—it is a syntactic notion. In order to consider validity, we introduce an interpretation of fuzzy attribute implications. Fuzzy attribute implications are meant to be interpreted in data tables with fuzzy attributes. A *data table with fuzzy attributes* (called also a *formal fuzzy context*) can be seen as a triplet $\langle X, Y, I\rangle$ where X is a set of objects, Y is a finite set of attributes (the same as above in the definition of a fuzzy attribute implication), and $I \in \mathbf{L}^{X\times Y}$ is a binary $\mathbf{L}$-relation between X and Y assigning to each object $x \in X$ and each attribute $y \in Y$ a degree $I(x, y)$ to which x has y. $\langle X, Y, I\rangle$ can be thought of as a table with rows and columns corresponding to objects $x \in X$ and attributes $y \in Y$, respectively, and table entries containing degrees $I(x, y)$. A row of a table $\langle X, Y, I\rangle$ corresponding to an object $x \in X$ can be seen as a fuzzy set I_x of attributes to which an attribute $y \in Y$ belongs to a degree $I_x(y) = I(x, y)$.

The basic step in the definition of a validity of a fuzzy attribute implication $A \Rightarrow B$ is its validity in a fuzzy set M of attributes.

Definition 2. *For a fuzzy attribute implication $A \Rightarrow B$ over Y and a fuzzy set $M \in \mathbf{L}^Y$ of attributes, we define a* degree $||A \Rightarrow B||_M \in L$ to which $A \Rightarrow B$ is valid in M *by*

$$||A \Rightarrow B||_M = S(A, M)^* \to S(B, M). \tag{2}$$

Remark 2. (1) $S(A, M)$ and $S(B, M)$ are the degrees to which A and B are contained in M, as defined by (1); * is a truth-stressing hedge; $\to$ is a truth function of implication. Therefore, it is easily seen that if M is a fuzzy set of attributes of some object x then $||A \Rightarrow B||_M$ is a truth degree of a proposition "if it is (very) true that x has all attributes from A then x has all attributes from B".

(2) A hedge * plays a role of a parameter controlling the semantics. Consider the particular forms of (2) for the boundary choices of *. First, if * is identity, (2) becomes

$$||A \Rightarrow B||_M = S(A, M) \to S(B, M).$$

In this case, $||A \Rightarrow B||_M$ is a truth degree of "if A is contained in M then B is contained in M". Second, if * is globalization, (2) becomes

$$||A \Rightarrow B||_M = \begin{cases} S(B, M) & \text{if } A \subseteq M, \\ 1 & \text{otherwise.} \end{cases}$$

In this case, $||A \Rightarrow B||_M$ is a truth degree of "B is contained in M" provided A is fully contained in M (i.e. $A(y) \leq M(y)$ for each $y \in Y$), and $||A \Rightarrow B||_M$ is 1 otherwise. Therefore, compared to the former case (* being identity), partial truth degrees of "A is contained in M" are disregarded for * being globalization.

(3) Consider now the case $\mathbf{L} = \mathbf{2}$ (i.e., the structure of truth degrees is a two-element Boolean algebra of classical logic). In this case, $||A \Rightarrow B||_M = 1$ iff we have that if $A \subseteq M$ then $B \subseteq M$. Hence, for $\mathbf{L} = \mathbf{2}$, Definition 2 yields the well-known definition of validity of an attribute implication in a set of attributes, cf. [22].

We now extend the definition of a validity of attribute implications to validity in systems of fuzzy sets of attributes and to validity in data tables with fuzzy attributes.

Definition 3. *For a system $\mathcal{M}$ of* **L***-sets in Y, define a* degree $||A \Rightarrow B||_{\mathcal{M}}$ to which $A \Rightarrow B$ is true in (each M from) $\mathcal{M}$ *by*

$$||A \Rightarrow B||_{\mathcal{M}} \;=\; \bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_{M}. \tag{3}$$

Given a data table $\langle X, Y, I\rangle$ with fuzzy attributes, define a degree $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ to which $A \Rightarrow B$ is true in $\langle X, Y, I\rangle$ *by*

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} \;=\; ||A \Rightarrow B||_{\{I_x \,|\, x \in X\}}. \tag{4}$$

Remark 3. Since I_x represents a row of table $\langle X, Y, I\rangle$ corresponding to x (recall that $I_x(y) = I(x, y)$ for each $y \in Y$), $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is, in fact, a degree to which $A \Rightarrow B$ is true in a system $\mathcal{M} = \{I_x \,|\, x \in X\}$ of all rows of table $\langle X, Y, I\rangle$.

Remark 4. For a fuzzy attribute implication $A \Rightarrow B$, degrees $A(y) \in L$ and $B(y) \in L$ can be seen as thresholds. This is best seen when * is globalization, i.e. $1^* = 1$ and $a^* = 0$ for $a < 1$. Since for $a, b \in L$ we have $a \leq b$ iff $a \to b = 1$, we have

$$(a \to b)^* = \begin{cases} 1 & \text{iff } a \leq b, \\ 0 & \text{iff } a \not\leq b. \end{cases}$$

Therefore, $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ means that a proposition "for each object $x \in X$: if for each attribute $y \in Y$, x has y in degree greater than or equal to (a threshold) $A(y)$, then for each $y \in Y$, x has y in degree at least $B(y)$" is true. In general, $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is a truth degree of the latter proposition. As a particular example, if $A(y) = a$ for $y \in Y_A \subseteq Y$ (and $A(y) = 0$ for $y \notin Y_A$) $B(y) = b$ for $y \in Y_B \subseteq Y$ (and $B(y) = 0$ for $y \notin Y_B$), the proposition says "for each object $x \in X$: if x has all attributes from Y_A in degree at least a, then x has all attributes from Y_B in degree at least b", etc. That is, having A and B fuzzy sets allows for a rich expressibility of relationships between attributes which is why we want A and B to be fuzzy sets in general.

Example 1. For illustration, consider Tab. 1, where table entries are taken from **L** defined on the real unit interval $L = [0, 1]$ with * being globalization. Consider now the following fuzzy attribute implications.

(1) $\{{}^{0.3}\!/y_3, {}^{0.7}\!/y_4\} \Rightarrow \{y_1, {}^{0.3}\!/y_2, {}^{0.8}\!/y_4, {}^{0.4}\!/y_6\}$ is true in degree 1 in data table from Tab. 1. On the other hand, implication $\{y_1, {}^{0.3}\!/y_3\} \Rightarrow \{{}^{0.1}\!/y_2, {}^{0.7}\!/y_5, {}^{0.4}\!/y_6\}$ is not true in degree 1 in Tab. 1—object x_2 can be taken as a counterexample: x_2 does not have attribute y_5 in degree greater than or equal to 0.7.

(2) $\{y_1, y_2\} \Rightarrow \{y_4, y_5\}$ is a crisp attribute implication which is true in degree 1 in the table. On the contrary, $\{y_5\} \Rightarrow \{y_4\}$ is also crisp but it is not true in degree 1 (object x_3 is a counterexample).

Table 1. Data table with fuzzy attributes

I	y_1	y_2	y_3	y_4	y_5	y_6
x_1	1.0	1.0	0.0	1.0	1.0	0.2
x_2	1.0	0.4	0.3	0.8	0.5	1.0
x_3	0.2	0.9	0.7	0.5	1.0	0.6
x_4	1.0	1.0	0.8	1.0	1.0	0.5

$X = \{x_1, \ldots, x_4\}$

$Y = \{y_1, \ldots, y_6\}$

(3) Implication $\{{}^{0.5}/y_5, {}^{0.5}/y_6\} \Rightarrow \{{}^{0.3}/y_2, {}^{0.3}/y_3\}$ is in the above-mentioned form for $Y_A = \{y_5, y_6\}$, $Y_B = \{y_2, y_3\}$, $a = 0.5$, and $b = 0.3$. The implication is true in data table in degree 1. $\{{}^{0.5}/y_5, {}^{0.5}/y_6\} \Rightarrow \{{}^{0.3}/y_1, {}^{0.3}/y_2\}$ is also in this form (for $Y_B = \{y_1, y_2\}$) but it is not true in the data table in degree 1 (again, take x_3 as a counterexample).

We now come to the notions of a *theory* and a *model.* In logic, a theory is considered as a collection of formulas. The formulas are considered as valid formulas we can use when making inferences. In fuzzy logic, a theory T can be considered as a fuzzy set of formulas, see [30] and also [23, 26]. Then, for a formula φ, a degree $T(\varphi)$ to which φ belongs to T can be seen as a degree to which we assume φ valid (think of φ as expressing "Mary likes John", "temperature is high", etc.). This will be also our approach. In general, we will deal with fuzzy sets T of attribute implications. Sometimes, we use only sets T of attribute implications (particularly when interested only in fully true implications). The following definition introduces the notion of a model.

Definition 4. *For a fuzzy set T of fuzzy attribute implications, the set* Mod(T) *of all* models *of T is defined by*

$$\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \textit{for each } A, B \in \mathbf{L}^Y : T(A \Rightarrow B) \leq ||A \Rightarrow B||_M\}.$$

That is, $M \in \mathrm{Mod}(T)$ means that for each attribute implication $A \Rightarrow B$, a degree to which $A \Rightarrow B$ holds in M is higher than or at least equal to a degree $T(A \Rightarrow B)$ prescribed by T. Particularly, for a crisp T, $\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A \Rightarrow B \in T : ||A \Rightarrow B||_M = 1\}$.

3.2 Relationship to Fuzzy Concept Lattices

Analogously as in the ordinary case, there is a close relationship between attribute implications and concept lattices. A useful structure derived from $\langle X, Y, I\rangle$ which is related to attribute implications is a so-called fuzzy concept lattice with hedges [10]. Let $*_X$ and $*_Y$ be hedges (their meaning will become apparent later). For **L**-sets $A \in \mathbf{L}^X$ (**L**-set of objects), $B \in \mathbf{L}^Y$ (**L**-set of attributes) we define **L**-sets $A^\uparrow \in \mathbf{L}^Y$ (**L**-set of attributes), $B^\downarrow \in \mathbf{L}^X$ (**L**-set of objects) by

$$A^\uparrow(y) = \textstyle\bigwedge_{x\in X}\big(A(x)^{*_X} \to I(x,y)\big) \quad \text{and} \quad B^\downarrow(x) = \textstyle\bigwedge_{y\in Y}\big(B(y)^{*_Y} \to I(x,y)\big).$$

We put $\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^\uparrow = B,\ B^\downarrow = A\}$. For $\langle A_1, B_1\rangle$, $\langle A_2, B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, put $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq$

B_1; both ways are equivalent). Operators $^{\downarrow},^{\uparrow}$ form a Galois connection with hedges [10]. $\langle \mathcal{B}(X^{*_X}, Y^{*_Y}, I), \leq \rangle$ is called a *fuzzy concept lattice* (*with hedges*) induced by $\langle X, Y, I \rangle$; $\langle A, B \rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ are called formal concepts. For $*_Y = \mathrm{id}_L$ (identity), we write only $\mathcal{B}(X^{*_X}, Y, I)$.

Remark 5. (1) Fuzzy concept lattices with hedges generalize some of the approaches to concept lattices from the point of view of a fuzzy approach, see [14] for details.

(2) Hedges can be seen as parameters which control the size of a fuzzy concept lattice (the stronger the hedges, the smaller $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$). See [10] for details.

(3) For $\mathbf{L} = \mathbf{2}$, a fuzzy concept lattice with hedges coincides with the ordinary concept lattice.

For each $\langle X, Y, I \rangle$ we consider a set $\mathrm{Int}(X^{*_X}, Y^{*_Y}, I) \subseteq \mathbf{L}^Y$ of all intents of concepts of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, i.e.

$$\mathrm{Int}(X^{*_X}, Y^{*_Y}, I) = \{B \in \mathbf{L}^Y \mid \langle A, B \rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I) \text{ for some } A \in \mathbf{L}^X\}.$$

For $^{*_X} = {}^*$ (the hedge used in (2)) and $^{*_Y} = \mathrm{id}_L$ (identity on L), $\mathcal{B}(X^*, Y, I)$ and $\mathrm{Int}(X^*, Y, I)$ play analogous roles for fuzzy attribute implications to the roles of ordinary concept lattices and systems of intents for ordinary attribute implications.

We close this section by a theorem showing some formulas expressing a degree $||A \Rightarrow B||_M$ in terms of fuzzy concept lattices with hedges and the operators $^{\uparrow}$ and $^{\downarrow}$. For hedges $\bullet, * : L \to L$ put $\bullet \leq *$ iff $a^\bullet \leq a^*$ for each $a \in L$.

Theorem 1 ([13]). *For a data table $\langle X, Y, I \rangle$ with fuzzy attributes, hedges $\bullet$ and $*$ with $\bullet \leq *$, and an attribute implication $A \Rightarrow B$, the following values are equal:*

$||A \Rightarrow B||_{\langle X,Y,I \rangle}$, $\quad ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}$, $\quad S(B, A^{\downarrow\uparrow})$,
$\bigwedge_{x \in X, a \in L} S(a^* \otimes A, \{{}^1/x\}^{\uparrow})^\bullet \to S(a^* \otimes B, \{{}^1/x\}^{\uparrow})$,
$\bigwedge_{x \in X, a \in L} S(A, \{{}^a/x\}^{\uparrow})^\bullet \to S(B, \{{}^a/x\}^{\uparrow})$,
$\bigwedge_{a \in L} ||a^* \otimes A \Rightarrow a^* \otimes B||_{\langle X,Y,I \rangle}$,
$\bigwedge_{M \in \mathrm{Int}(X^*,Y,I)} S(A, M)^\bullet \to S(B, M)$.

3.3 Complete Sets and Guigues-Duquenne Bases

We now turn our attention to the notions of semantic entailment, completeness in data tables, non-redundant basis, etc.

Definition 5. *A degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ semantically follows from a fuzzy set T of attribute implications is defined by*

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Mod}(T)}. \tag{5}$$

That is, $||A \Rightarrow B||_T$ can be seen as a degree to which $A \Rightarrow B$ is true in each model of T. From now on in this section, we will assume that T is an ordinary set of fuzzy attribute implications.

Definition 6. *A set* T *of attribute implications is called* complete (in $\langle X, Y, I\rangle$) *if* $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$ *for each* $A \Rightarrow B$. *If* T *is complete and no proper subset of* T *is complete, then* T *is called a* non-redundant basis (of $\langle X, Y, I\rangle$).

Note that both the notions of a complete set and a non-redundant basis refer to a given data table with fuzzy attributes.

Since we are primarily interested in implications which are fully true in $\langle X, Y, I\rangle$, the following notion seems to be of interest. Call T *1-complete in* $\langle X, Y, I\rangle$ if we have that $||A \Rightarrow B||_T = 1$ iff $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ for each $A \Rightarrow B$. Clearly, if T is complete then it is also 1-complete. Surprisingly, we have also

Theorem 2 ([12]). *T is 1-complete in $\langle X, Y, I\rangle$ iff T is complete in $\langle X, Y, I\rangle$.*

The following assertion shows that the models of a complete set of fuzzy attribute implications are exactly the intents of the corresponding fuzzy concept lattice.

Theorem 3 ([7]). *T is complete iff* $\mathrm{Mod}(T) = \mathrm{Int}(X^*, Y, I)$.

In the following, we focus on so-called Guigues-Duquenne basis, i.e. a non-redundant basis based on the notion of a pseudointent, see [21, 22, 25]. As we will see, the situation is somewhat different from what we know from the ordinary case. We start by the notion of a system of pseudointents.

Definition 7. *Given* $\langle X, Y, I\rangle$, $\mathcal{P} \subseteq \mathbf{L}^Y$ *(a system of fuzzy sets of attributes) is called a* system of pseudo-intents *of* $\langle X, Y, I\rangle$ *if for each* $P \in \mathbf{L}^Y$ *we have:*

$$P \in \mathcal{P} \quad \textit{iff} \quad P \neq P^{\downarrow\uparrow} \textit{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \textit{ for each } Q \in \mathcal{P} \textit{ with } Q \neq P.$$

It is easily seen that if $\mathbf{L}$ is a complete residuated lattice with globalization then $\mathcal{P}$ is a system of pseudo-intents of $\langle X, Y, I\rangle$ if for each $P \in \mathbf{L}^Y$ we have:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ for each } Q \in \mathcal{P} \text{ with } Q \subset P.$$

In addition to that, in case of finite $\mathbf{L}$, for each data table with finite set of attributes there is exactly one system of pseudo-intents which can be described recursively the same way as in the classical case [22, 25]:

Theorem 4 ([11]). *Let $\mathbf{L}$ be a finite residuated lattice with globalization. Then for each $\langle X, Y, I\rangle$ with finite Y there is a unique system of pseudo-intents $\mathcal{P}$ of $\langle X, Y, I\rangle$ and*

$$\mathcal{P} = \{P \in \mathbf{L}^Y \mid P \neq P^{\downarrow\uparrow} \textit{ and } Q^{\downarrow\uparrow} \subseteq P \textit{ holds for each } Q \in \mathcal{P} \textit{ such that } Q \subset P\}.$$

Remark 6. (1) Neither the uniqueness of $\mathcal{P}$ nor the existence of $\mathcal{P}$ is assured in general, see [11].

(2) For $\mathbf{L} = \mathbf{2}$, the system of pseudointents described by Theorem 4 coincides with the ordinary one.

The following theorem shows that each system of pseudointents induces a non-redundant basis.

Theorem 5 ([11]). *Let $\mathcal{P}$ be a system of pseudointents of $\langle X, Y, I\rangle$. Then $T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$ is a non-redundant basis of $\langle X, Y, I\rangle$ (so-called Guigues-Duquenne basis).*

Non-redundancy of T does not ensure that T is minimal in terms of its size. The following theorem shows a generalization of a well-known result saying that Guigues-Duquenne basis is minimal in terms of its size.

Theorem 6 ([11]). *Let $\mathbf{L}$ be a finite residuated lattice with * being the globalization, Y be finite. Let T be a Guigues-Duquenne basis of $\langle X, Y, I\rangle$, i.e. $T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$ where $\mathcal{P}$ is the system of pseudointents of $\langle X, Y, I\rangle$. If T' is complete in $\langle X, Y, I\rangle$ then $|T| \leq |T'|$.*

Remark 7. For hedges other than globalization we can have several systems of pseudointents. The systems of pseudointents may have different numbers of elements, see [11].

3.4 Algorithms for Generating Systems of Pseudointents

CASE 1: Finite $\mathbf{L}$ *and* * *being globalization.* If $\mathbf{L}$ is finite and * is globalization, there is a unique system $\mathcal{P}$ of pseudointents for $\langle X, Y, I\rangle$, see Theorem 4. In what follows we describe an algorithm for computing this $\mathcal{P}$. The algorithm is based on the ideas of Ganter's algorithm for computing ordinary pseudointents, see [21, 22]. Details can be found in [7].

For simplicity, let us assume that $\mathbf{L}$ is, moreover, linearly ordered. For $Z \in \mathbf{L}^Y$ put

$$\begin{aligned} Z^{T^*} &= Z \cup \bigcup\{B \otimes S(A, Z)^* \mid A \Rightarrow B \in T \text{ and } A \neq Z\}, \\ Z^{T^*_0} &= Z, \\ Z^{T^*_n} &= (Z^{T^*_{n-1}})^{T^*}, \quad \text{for } n \geq 1, \end{aligned}$$

and define an operator cl_{T^*} on $\mathbf{L}$-sets in Y by

$$cl_{T^*}(Z) = \textstyle\bigcup_{n=0}^{\infty} Z^{T^*_n}.$$

Theorem 7 ([7]). *cl_{T^*} is a fuzzy closure operator, and*

$$\{cl_{T^*}(Z) \mid Z \in \mathbf{L}^Y\} = \mathcal{P} \cup \mathrm{Int}(X^*, Y, I).$$

Using Theorem 7, we can get all intents and all pseudo-intents (of a given data table with fuzzy attributes) by computing the fixed points of cl_{T^*}. This can be done with polynomial time delay using a "fuzzy extension" of Ganter's algorithm for computing all fixed points of a closure operator, see [6]. We refer to [7] for details.

CASE 2: Finite $\mathbf{L}$ *and arbitrary* *. If $\mathbf{L}$ is finite and * is an arbitrary hedge (not necessarily globalization), the systems of pseudointents for $\langle X, Y, I\rangle$ can be computed using algorithms for generating maximal independent sets in graphs.

Namely, as we show in the following, systems of pseudointents in this case can be identified with particular maximal independent sets. (details can be found in [15]): For $\langle X, Y, I\rangle$ define a set V of fuzzy sets of attributes by

$$V = \{P \in \mathbf{L}^Y \mid P \neq P^{\downarrow\uparrow}\}. \tag{6}$$

If $V \neq \emptyset$, define a binary relation E on V by

$$E = \{\langle P, Q\rangle \in V \mid P \neq Q \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P \neq 1\}. \tag{7}$$

In this case, $\mathbf{G} = \langle V, E \cup E^{-1}\rangle$ is a graph. For any $Q \in V$ and $\mathcal{P} \subseteq V$ define the following subsets of V: $\mathrm{Pred}(Q) = \{P \in V \mid \langle P, Q\rangle \in E\}$, and $\mathrm{Pred}(\mathcal{P}) = \bigcup_{Q\in\mathcal{P}} \mathrm{Pred}(Q)$.

Theorem 8 ([15]). *Let* $\mathbf{L}$ *be finite,* * *be any hedge,* $\langle X, Y, I\rangle$ *be a data table with fuzzy attributes,* $\mathcal{P} \subseteq \mathbf{L}^Y$, V *and* E *be defined by* (6) *and* (7)*, respectively. Then the following statements are equivalent.*

(i) $\mathcal{P}$ *is a system of pseudo-intents;*
(ii) $V - \mathcal{P} = \mathrm{Pred}(\mathcal{P})$*;*
(iii) $\mathcal{P}$ *is a maximal independent set in* $\mathbf{G}$ *such that* $V - \mathcal{P} = \mathrm{Pred}(\mathcal{P})$.

The Theorem gives a way to compute systems of pseudo-intents. It suffices to find all maximal independent sets in $\mathbf{G}$ and check which of them satisfy additional condition $V - \mathcal{P} = \mathrm{Pred}(\mathcal{P})$.

4 Functional Dependencies over Domains with Similarity Relations

As we mentioned in Section 1, ordinary attribute implications have been used in databases under the name functional dependencies. Functional dependencies are interpreted in data tables with arbitrarily-valued attributes. A table entry corresponding to an object (row) x and an attribute (column) y contains an arbitrary value from a so-called domain D_y (set of all possible values for y). Then, $A \Rightarrow B$ is considered true in such a table if any two objects (rows) which agree in their values of attributes from A agree also in their values of attributes from B. In this section we consider functional dependencies from the point of view of a fuzzy approach. We show several relationships to fuzzy attribute implications. Most importantly, we argue that in a fuzzy setting, the concept of a functional dependence is an interesting one for the theory of databases.

Definition 8. *A* (fuzzy) functional dependence (over attributes Y) *is an expression* $A \Rightarrow B$*, where* $A, B \in \mathbf{L}^Y$ (A *and* B *are fuzzy sets of attributes*).

Therefore, the notion of a fuzzy functional dependence coincides with the notion of a fuzzy attribute implication. We prefer using both of the terms, depending on the context of usage. Fuzzy functional dependencies will be interpreted in data tables over domains with similarities.

Definition 9. *A* data table over domains with similarity relations *is a tuple* $\mathcal{D} = \langle X, Y, \{\langle D_y, \approx_y\rangle \,|\, y \in Y\}, T\rangle$ *where*

- X *is a non-empty set* (*of objects, table items*),
- Y *is a non-empty finite set* (*of attributes*),
- *for each* $y \in Y$, D_y *is a non-empty set* (*of values of attribute* y) *and* $\approx_y$ *is a binary fuzzy relation which is reflexive and symmetric,*
- T *is a mapping assigning to each* $x \in X$ *and* $y \in Y$ *a value* $T(x, y) \in D_y$ (*value of attribute* y *on object* x).

$\mathcal{D}$ will always denote some data table over domains with similarity relations with its components denoted as above.

Remark 8. (1) Consider $L = \{0, 1\}$ (case of classical logic). If each $\approx_y$ is an equality (i.e. $a \approx_y b = 1$ iff $a = b$), then $\mathcal{D}$ can be identified with what is called a relation on relation scheme Y with domains D_y ($y \in Y$) [29].

(2) For $x \in X$ and $Z \subseteq Y$, $x[Z]$ denotes a tuple of values $T(x, y)$ for $y \in Z$. We may assume that attribute from Y are numbered, i.e. $Y = \{y_1, \ldots, y_n\}$, and thus linearly ordered by this numbering, and assume that attributes in $x[Z]$ are ordered in this way. Particularly, $x[y]$ is $x[\{y\}]$ which can be identified with $T(x, y)$.

(3) $\mathcal{D}$ can be seen as a table with rows and columns corresponding to $x \in X$ and $y \in Y$, respectively, and with table entries containing values $T(x, y) \in D_y$. Moreover, each domain D_y is equipped with an additional information about similarity of elements from D_y.

Given a data table $\mathcal{D} = \langle X, Y, \{\langle D_y, \approx_y\rangle \,|\, y \in Y\}, T\rangle$, we want to introduce a condition for a functional dependence $A \Rightarrow B$ to be true in $\mathcal{D}$ which says basically the following: "for any two objects $x_1, x_2 \in X$: if x_1 and x_2 have similar values on attributes from A then x_1 and x_2 have similar values on attributes from A". Define first for a given $\mathcal{D}$, objects $x_1, x_2 \in X$, and a fuzzy set $C \in \mathbf{L}^Y$ of attributes a *degree* $x_1(C) \approx x_2(C)$ *to which* x_1 *and* x_2 *have similar values on attributes from* C (agree on attributes from C) by

$$x_1(C) \approx x_2(C) \;=\; \bigwedge_{y \in Y} \big(C(y) \to (x_1[y] \approx_y x_2[y])\big). \tag{8}$$

That is, $x_1(C) \approx x_2(C)$ is truth degree of proposition "for each attribute $y \in Y$: if y belongs to C then the value $x_1[y]$ of x_1 on y is similar to the value $x_2[y]$ of x_2 on y", which can be seen as a degree to which x_1 and x_2 have similar values on attributes from C. Then, the above idea of validity of a functional dependence is then captured by the following definition.

Definition 10. *A* degree $||A \Rightarrow B||_{\mathcal{D}}$ to which $A \Rightarrow B$ is true in $\mathcal{D}$ *is defined by*

$$||A \Rightarrow B||_{\mathcal{D}} = \bigwedge_{x_1, x_2 \in X} \big((x_1(A) \approx x_2(A))^* \to (x_1(B) \approx x_2(B))\big). \tag{9}$$

Remark 9. (1) If A and B are crisp sets then A and B may be considered as ordinary sets and $A \Rightarrow B$ may be seen as an ordinary functional dependence. Then, if $\approx_y$ is an ordinary equality for each $y \in Y$, we have that $||A \Rightarrow B||_{\mathcal{D}} = 1$ iff $A \Rightarrow B$ is true in $\mathcal{D}$ in the usual sense of validity of ordinary functional dependencies.

(2) For a functional dependence $A \Rightarrow B$, degrees $A(y) \in L$ and $B(y) \in L$ can be seen as thresholds. Namely, if * is globalization, $||A \Rightarrow B||_{\mathcal{D}} = 1$ means that a proposition "for any objects $x_1, x_2 \in X$: if for each attribute $y \in Y$, $A(y) \leq (x_1[y] \approx_y x_2[y])$, then for each attribute $y' \in Y$, $B(y') \leq (x_1[y'] \approx_y x_2[y'])$" is true. That is, having A and B fuzzy sets allows for a rich expressibility, cf. Remark 4.

We now have two ways to interpret a fuzzy attribute implication (fuzzy functional dependence) $A \Rightarrow B$. First, given a data table $\mathcal{T} = \langle X, Y, I \rangle$ with fuzzy attributes, we can consider a degree $||A \Rightarrow B||_{\mathcal{T}}$ to which $A \Rightarrow B$ is true in $\mathcal{T}$, see (4). Second, given a data table $\mathcal{D}$ over domains with similarities, we can consider a degree $||A \Rightarrow B||_{\mathcal{D}}$ to which $A \Rightarrow B$ is true in $\mathcal{D}$, see (9). In the rest of this section, we focus on presenting the following relationship between the two kinds of semantics for our formulas $A \Rightarrow B$: The notion of semantic entailment based on data tables with fuzzy attributes coincides with the notion of semantic entailment based on data tables over domains with similarity relations.

As in case of fuzzy attribute implications, we introduce the notions of a model and semantic entailment for functional dependencies. For a fuzzy set T of fuzzy functional dependencies, the set $\mathrm{Mod}^{\mathrm{FD}}(T)$ of all *models* of T is defined by

$$\mathrm{Mod}^{\mathrm{FD}}(T) = \{\mathcal{D} \,|\, \text{for each } A, B \in \mathbf{L}^Y : T(A \Rightarrow B) \leq ||A \Rightarrow B||_{\mathcal{D}}\},$$

where $\mathcal{D}$ stands for an arbitrary data table over domains with similarities. A degree $||A \Rightarrow B||_T^{\mathrm{FD}} \in L$ to which $A \Rightarrow B$ *semantically follows* from a fuzzy set T of functional dependencies is defined by

$$||A \Rightarrow B||_T^{\mathrm{FD}} = \textstyle\bigwedge_{\mathcal{D} \in \mathrm{Mod}^{\mathrm{FD}}(T)} ||A \Rightarrow B||_{\mathcal{D}}.$$

Denoting now $||A \Rightarrow B||_T$, see (5), by $||A \Rightarrow B||_T^{\mathrm{AI}}$, one can prove the following theorem.

Theorem 9 ([17]). *For any fuzzy set T of fuzzy attribute implications and any fuzzy attribute implication $A \Rightarrow B$ we have*

$$||A \Rightarrow B||_T^{\mathrm{FD}} = ||A \Rightarrow B||_T^{\mathrm{AI}}. \tag{10}$$

5 Armstrong Rules and Provability

In this section we present a system of Armstrong-like rules for reasoning with fuzzy attribute implications. Throughout this section we assume that L is finite. We show that the system is complete w.r.t. the semantics of fuzzy attribute implications based on data tables with fuzzy attributes. Due to Theorem 9, this is equivalent to completeness w.r.t. the semantics based on data tables over domains with similarities. In fact, we show two kinds of completeness. The first one is a usual one and concerns provability and entailment of $A \Rightarrow B$ from a set T of attribute implications. Provability and entailment remain bivalent: $A \Rightarrow B$ is provable from T iff $A \Rightarrow B$ semantically follows from T in degree 1. The second one (called also graded completeness or Pavelka-style completeness) concerns provability and entailment of $A \Rightarrow B$ from a fuzzy set T of attribute implications. Provability and entailment themselves become graded: A degree to which $A \Rightarrow B$ is provable from T equals a degree to which $A \Rightarrow B$ semantically follows from T. Details can be found in [12, 16].

Our axiomatic system consists of the following *deduction rules.*

(Ax) infer $A \cup B \Rightarrow A$,

(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$,

(Mul) from $A \Rightarrow B$ infer $c^* \otimes A \Rightarrow c^* \otimes B$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Rules (Ax)–(Mul) are to be understood as follows: having functional dependencies which are of the form of functional dependencies in the input part (the part preceding "infer") of a rule, a rule allows us to infer (in one step) the corresponding functional dependence in the output part (the part following "infer") of a rule.

Completeness. A fuzzy attribute implication $A \Rightarrow B$ is called *provable* from a set T of fuzzy attribute implications using (Ax)–(Mul), written $T \vdash A \Rightarrow B$, if there is a sequence $\varphi_1, \ldots, \varphi_n$ of fuzzy attribute implications such that φ_n is $A \Rightarrow B$ and for each φ_i we either have $\varphi_i \in T$ or φ_i is inferred (in one step) from some of the preceding formulas (i.e., $\varphi_1, \ldots, \varphi_{i-1}$) using some of deduction rules (Ax)–(Mul). To comply to the notation $T \vdash A \Rightarrow B$, we write $T \models A \Rightarrow B$ to denote that $||A \Rightarrow B||_T = 1$ ($A \Rightarrow B$ semantically follows from T in degree 1). Then we have the first kind of completeness:

Theorem 10 ([16]). *For any set T of fuzzy attribute implications and any fuzzy attribute implication $A \Rightarrow B$ we have*

$$T \vdash A \Rightarrow B \quad \textit{iff} \quad T \models A \Rightarrow B.$$

Graded completeness. Now, we are going to define a notion of a degree $|A \Rightarrow B|_T$ of provability of a functional dependence of a fuzzy set T of functional dependencies. Then, we show that $|A \Rightarrow B|_T = ||A \Rightarrow B||_T^{\mathrm{FD}}$ which can be understood as a graded completeness (completeness in degrees). Note that graded completeness was introduced by Pavelka [30], see also [23, 26] for further information.

For a fuzzy set T of fuzzy attribute implications and for $A \Rightarrow B$ we define a *degree $|A \Rightarrow B|_T \in L$ to which $A \Rightarrow B$ is provable from T* by

$$|A \Rightarrow B|_T = \bigvee\{c \in L \mid \mathrm{c}(T) \vdash A \Rightarrow c \otimes B\}, \tag{11}$$

where $\mathrm{c}(T)$ is an ordinary set of fuzzy attribute implications defined by

$$\mathrm{c}(T) = \{A \Rightarrow T(A \Rightarrow B) \otimes B \mid A, B \in \mathbf{L}^Y \text{ and } T(A \Rightarrow B) \otimes B \neq \emptyset\}. \tag{12}$$

Then we have the second kind of completeness:

Theorem 11 ([16]). *For any fuzzy set T of fuzzy attribute implications and any fuzzy attribute implication $A \Rightarrow B$ we have*

$$|A \Rightarrow B|_T = ||A \Rightarrow B||_T.$$

6 Concluding Remarks

6.1 Bibliographic Remarks

The first study on fuzzy attribute implications is S. Pollandt's [31]. Pollandt uses the same notion of a fuzzy attribute implication, i.e. $A \Rightarrow B$ where A, B are fuzzy sets, and obtains several results. Pollandt's notion of validity is a special case of ours, namely the one for * being identity on L. On the other hand, her notion of a pseudointent corresponds to * being globalization. That is why Pollandt did not get a proper generalization of results leading to Guigues-Duquenne basis. Pollandt's [31] contains some other results (proper premises, implications in fuzzy-valued contexts) which we did not discuss here. We will comment more on Pollandt's results elsewhere.

[19, 33, 35] are papers dealing with fuzzy functional dependencies. Our approach presented in this paper is more general. Namely, [33, 35] consider formulas $A \Rightarrow B$ with A and B being ordinary sets, i.e. A and B are not suitable for expressing thresholds. In [19], thresholds in A and B are present but are the same in A and the same in B. Furthermore, the degrees are restricted to values from $[0, 1]$ in [19, 33, 35].

Our paper is based on [4]–[17].

6.2 Further Issues

Due to a limited scope of this paper, we did not cover several interesting topics, some of which are still under investigation. For instance, it is shown in [9, 13] that a data table $\mathcal{T}$ with fuzzy attributes can be transformed to a data table $\mathcal{T}'$ with binary attributes in such a way that fuzzy attribute implications true in degree 1 in $\mathcal{T}$ correspond in a certain way to ordinary attribute implications which are true in $\mathcal{T}'$. The transformation of data tables and attribute implications makes it possible to obtain an ordinary non-redundant basis T' for $\mathcal{T}'$ and to obtain a corresponding set T of fuzzy attribute implications from T'. However, while T is always complete for $\mathcal{T}$, it may be redundant. Note that some results on transformations of data tables with fuzzy attributes to tables with binary attributes which are related to attribute implications are also present in [31].

Interesting open problems include: further study of relationships between attribute implications in a fuzzy setting and ordinary attribute implications (from both ordinary formal contexts and many-valued contexts); study of further problems of attribute implications in a fuzzy setting; further study of functional dependencies and other kinds of dependencies in databases in a fuzzy setting; development of agenda for databases where domains are equipped with similarity relations.

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